# Private Information Design*

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**Abstract**

We consider a setting where each player of a normal-form game privately designs an information structure before playing the game. One of these designs is chosen at random to determine the distribution of the private messages that players receive. These messages allow players to correlate their actions; however, private information design implies a push from correlated to Nash equilibria. Indeed, the sequential equilibrium payoffs of the private information design extensive-form game are correlated equilibrium payoffs of the underlying normal-form game, but not all correlated equilibrium payoffs are sequential equilibrium payoffs. In generic 2-player games, the latter are specific convex combinations of two Nash equilibrium payoffs. In any 2-player game, imposing optimal beliefs from private information design implies that mutual knowledge of rationality alone is a sufficient epistemic condition for Nash equilibrium.

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# 1 Introduction

Information matters for economic outcomes; thus, it is likely that individuals will try to design the information structure in an optimal way. This idea has motivated a large number of papers since the influential work of Kamenica and Gentzkow (2011), in which one party designs an information structure to persuade another. But in strategic settings, individuals typically provide information to each other and have conflicting incentives over the information they receive. For example, highlighting the importance of persuasion as an economic activity, McCloskey and Klamer (1995) write, “portfolio managers talk full-time to decide on buying or selling. Stockbrokers talk to clients and to each other.” In general, different portfolio managers and stockbrokers have conflicting interests and may try to hide or relay misleading information about the activities of the market. Thus, in strategic interactions, information design should take into account the possibility that individuals may covertly attempt to learn about and manipulate the information of others as well as choosing their own information in an optimal way.

We consider the simple case where information is provided before the play of a complete information normal-form game. In general, any correlated equilibrium payoff can be achieved, thus often improving the welfare of all participants in comparison to what they can achieve in Nash equilibrium. Achieving correlated equilibrium payoffs requires lotteries over a set of messages that are privately observed by the players and which can be thought of as being chosen by an outside mediator or generated by some randomization device agreed upon and/or designed by the players themselves. Our motivation is that any such mechanism can be manipulated: different players may attempt to bribe the mediator to make certain recommendations or secretly tamper with the randomization device.

In contrast to the case where an impartial mediator controls the provision of information, we show that the players’ ability to privately (i.e. secretly) manipulate the information structure implies that not all correlated equilibrium payoffs are achievable. In 2-player games with a unique Nash equilibrium, only the payoff of the unique Nash equilibrium can be achieved when the players design their information optimally. In generic 2-player games, the only achievable payoffs are specific convex combinations of two Nash equilibrium payoffs. In this sense, private information design provides a justification for Nash equilibrium.

Our characterization of the outcomes of private information design can also be used to provide sufficient epistemic conditions for Nash equilibrium that require only knowledge of a strong form of rationality — roughly requiring that if players' types reflect a common prior and their private information, then rational types of each player design information optimally — but no knowledge of players' actions or conjectures. This is appealing because players often do not know the actions of others in strategic situations.

Our model of private information design is guided by the observation that there are many actions that players can take to influence the information structure: for example, one player may anticipate that another will tamper with the randomization device and respond by including additional safeguards. The other may anticipate this and secretly hide backdoors in the device. It is difficult to model explicitly each possible manipulation and its effect on the resulting information structure. On the other hand, we do not wish to rule out any kind of manipulation by assumption. Thus, our aim is to provide a reduced form model that captures the idea that players are able to manipulate the information structure in any way they desire, and we characterize the equilibrium outcomes when all such manipulations are allowed.

Specifically, we introduce a setting without payoff uncertainty in which each player in a normal-form game privately designs an information structure before playing the normal-form game. Thus, each player chooses a set of message profiles, where each message profile contains one message for each player, and a probability distribution on the chosen set of messages profiles; the choice of such an information structure is not observed by the other players. This gives rises to an information structure chosen by each player; the prevailing information structure that actually determines the message each player receives is then chosen randomly and privately (i.e. no player is informed about it) from those chosen by the players, with each of them having a strictly positive probability of being chosen. After privately receiving a message, each player chooses an action of the normal-form game. By augmenting the normal-form game with a pre-play private information design stage, we create an extensive-form game that aims to captures, in a simple way, the idea that each player can attempt to manipulate the information structure however he wishes and with positive probability he is successful. In Section 2, we motivate our formulation with a simple example and in Section 6, we discuss conceptual issues surrounding our model in detail.[1]

If information is provided by an explicit designer who is not a player (and who cannot be manipulated), then any correlated equilibrium payoff of the normal-form game can be achieved as shown by Aumann (1987). In contrast, when information is designed by the players allowing for the possibility of covert manipulations, not all correlated equilibrium payoffs can be achieved but only those that rely on a correlation device in which each player sends optimal messages from his point of view. Indeed, this follows because sequential equilibrium outcomes of the private information design extensive-form game are characterized by the optimality of the actions each player plays after receiving a message and of the messages he sends with strictly positive probability in the information structure he designs.

The reduction on achievable payoffs is particularly striking for generic 2-player games: In such games, the equilibrium payoffs of the information-design extensive-form are convex combinations of two Nash equilibrium payoffs of the underlying normal-form game, one preferred by each player, with the weight of the Nash equilibrium preferred by player 1 (resp. 2) being equal to the probability that player 1's (resp. player 2's) information structure is chosen. In particular, in 2-player normal-form games with a unique Nash equilibrium, the information-design extensive-form has a unique equilibrium payoff equal to the payoff of the unique Nash equilibrium of the underlying normal-form game.

This conclusion bears some similarity to the rationalistic interpretation of Nash equilibrium in Nash (1950) and we use it as a starting point to obtain two epistemic

[1] See also Carmona and Laohakunakorn (2023) for a discussion on how our framework can be applied to repeated games to capture some features of real-world cartels.

results for Nash equilibrium. Following Aumann and Brandenburger's (1995) interpretation of the common prior assumption, according to which each player's theory about the others reflects a common prior and additional information that he receives, we consider interactive belief systems for normal-form games where each player's types correspond to messages received. We introduce the idea of endogenous information in interactive belief systems by specifying that (a) the common prior is the (convex) combination of individual players' probability distributions over message profiles and (b) the messages each player sends with strictly positive probability are optimal for him whenever he is rational in the standard sense. This yields a strong notion of rationality and a rationalistic formalization of endogenous information in the context of interactive belief systems which we use to obtain sufficient epistemic conditions for Nash equilibrium that require no (mutual or common) knowledge of players' actions or conjectures.

Indeed, we show that common knowledge of our strong form of rationality implies that the action distribution is that of the unique Nash equilibrium. In fact, the profile formed with each player's conjecture about the play of his opponent is, at each state, equal to the unique Nash equilibrium. If we further refine our notion of endogenous information in interactive belief systems by imposing a lexicographic preference for simplicity — so that when comparing two information designs with the same expected payoff, the one with fewer messages is preferred — then we obtain a local epistemic condition for any 2-player normal-form game (not necessarily with a unique Nash equilibrium): If at some state of an interactive belief system incorporating this refinement of endogenous information, it is mutually known that players are rational, then the action profile at that state is a Nash equilibrium of the game.

Summing up, regardless of whether one focuses on epistemic conditions or on characterizations of equilibrium outcomes and payoffs, this paper shows that when information is designed optimally by the individuals involved in a strategic situation, there is a push from correlated to Nash equilibria.

The paper is organized as follows. Section 2 contains a motivating example and a discussion of some conceptual issues surrounding our formalization. Section 3 introduces our model of private information design and characterizes the sequential equilibrium outcomes of the information design extensive form game. Section 4 specializes our characterization result to the case of 2-player games, and Section 5 contains our epistemic results. Related literature is discussed in Section 6, along with some concluding remarks. Proofs of our results and further examples can be found in the Appendix.

# 2 Motivating example

We motivate our model of private information design and discuss some conceptual issues in the context of the following game of "chicken":

| $1\backslash 2$ | $A$ | $B$ |
|---|---|---|
| $A$ | $6,6$ | $1,7$ |
| $B$ | $7,1$ | $0,0$ |

The Nash equilibria are $(A, B)$, $(B, A)$ and $(\frac{1}{2}1_A+\frac{1}{2}1_B, \frac{1}{2}1_A+\frac{1}{2}1_B)$ giving payoffs $(7, 1)$, $(1, 7)$ and $(\frac{7}{2}, \frac{7}{2})$ respectively. It is well-known that there are correlated equilibria with payoffs outside the convex hull of the Nash equilibrium payoffs. The usual story why such correlated equilibria might be played is that there is a mediator who makes private action recommendations to each player. For example, if the action profiles $(A, A)$, $(A, B)$ and $(B, A)$ are recommended, each with probability $\frac{1}{3}$, then each player will find it optimal to follow the recommendation and the outcome is a correlated equilibrium with payoffs $(4\frac{2}{3}, 4\frac{2}{3})$. According to this story, each player receives a private signal (action recommendation) $m_i \in \{A, B\}$; the profile of signals $m = (m_1, m_2)$ is drawn from an information structure $\phi \in \Delta(\{A, B\} \times \{A, B\})$ chosen by the mediator.

Consider a player who has received recommendation $A$. He finds it optimal to play $A$ because he believes that his opponent has received recommendations $A$ and $B$ with equal probability. But if he could acquire some additional information about the message of his opponent, then he will find it optimal to play $B$ after receiving any information that increases his belief that his opponent has received a recommendation to play $A$. Moreover, each player may wish to influence the mediator so that the recommendation for himself to play $B$ is sent more often; and if, instead of a mediator, the recommendations are generated by a randomization device designed by the players themselves, each player may wish to tamper with it. In other words, players may have incentives (i) to acquire information beyond that received from the mediator and (ii) to influence the recommendations of the mediator or tamper with the randomization device. Thus, it is natural to consider a richer signal space $M_i \supseteq \{A, B\}$ and ask what happens when players can manipulate the information structure $\phi \in \Delta(M_1 \times M_2)$.

More generally, we may ask why a mediator would send action recommendations according to some given distribution. Where does the information structure come from? Our view is that the information structure is something that the players must design themselves. Thus, each player chooses an information structure $\phi \in \Delta(M_1 \times M_2)$. Since for any given distribution of signals, players may wish to acquire additional information about the signal of the other player, we should specify a rich enough signal space for this to be possible. We can achieve this by letting the set of possible signals for each player be $M_i = \mathbb{N}$. We also do not wish to restrict the type of information each player can choose to receive, and thus we let each player choose any information structure $\phi \in \Delta(\mathbb{N} \times \mathbb{N})$; the only restriction we impose is that $\phi$ must be finitely supported.

One issue with the above formulation is that there will be in general two distributions, $\phi_1$ and $\phi_2$ (one chosen by each player), over the message space $M_1 \times M_2 = \mathbb{N}^2$, but a message profile cannot be drawn from two distributions at once. If players agree in the sense that $\phi_1 = \phi_2$, then each can effectively take the role of the mediator and send private messages according to his own distribution; when they disagree, then each player should be able to impact the distribution of messages. These two properties are satisfied by specifying that the message profiles are determined by the distribution $\beta_1\phi_1 + \beta_2\phi_2$ for some $\beta_1 > 0$ and $\beta_2 > 0$ such that $\beta_1 + \beta_2 = 1$. One interpretation is that each player chooses an information structure and nature chooses the one that is realized, with player $i$'s information structure being chosen with probability $\beta_i$. More concretely, this situation can arise if each player proposes a randomization device to an outside mediator, who being indifferent between the two proposals then picks player $i$'s proposal with probability $\beta_i$; it can also arise if $\beta_i$ denotes the probability of player $i$ being the last one to tamper with the randomization device.

Another feature we want to capture is the ability of players to privately manipulate the information structure. We achieve this property by specifying that the players' choices are private in the sense that player $i$ observes only $\phi_i$ and $m_i$, but not $\phi_{-i}$, $m_{-i}$ or the realized information structure $\beta_i\phi_i+\beta_{-i}\phi_{-i}$. On the other hand, the assumption that $\phi_i$ is finitely supported on $\mathbb{N}\times\mathbb{N}$ gives each player the option of choosing an information structure that puts positive probability on message profiles that arise with zero probability from the information structure chosen by the other player. We view this as a simple way of allowing the players to learn whose information structure was chosen if they wish to do so.

Thus, according to our formulation, each player chooses $(\phi_i, \pi_i)$, where $\phi_i \in S$ is $i$'s information design and $\pi_i : M_i \times S \to \Delta\{A, B\}$ is $i$'s choice of action as a function of the message that he receives and his own information design, and where $S$ denotes the set of finitely supported probability measures on $M_1 \times M_2$. This gives rise to an extensive-form game, and our results imply that the set of sequential equilibrium payoffs is:

$$\left\{(7,1),(1,7),(\tfrac{7}{2},\tfrac{7}{2}),\beta_1(7,1)+\beta_2(1,7),\beta_1(7,1)+\beta_2(\tfrac{7}{2},\tfrac{7}{2}),\beta_1(\tfrac{7}{2},\tfrac{7}{2})+\beta_2(1,7)\right\}.$$

In particular, $(4\frac{2}{3}, 4\frac{2}{3})$ is not a sequential equilibrium payoff and the action distribution $\frac{1}{3}1_{(A,A)}+\frac{1}{3}1_{(A,B)}+\frac{1}{3}1_{(B,A)}$ is not the action distribution of a sequential equilibrium of the information design extensive-form game. This payoff profile and action distribution could be obtained with $\phi_1 = \phi_2 = \frac{1}{3}1_{(1,1)} + \frac{1}{3}1_{(1,2)} + \frac{1}{3}1_{(2,1)}$ and $\pi_i(1, \phi_i) = A$ and $\pi_i(2, \phi_i) = B$ for each $i$. But then player 1 would gain by deviating to $\phi_1' = 1_{(2,1)}$ thereby increasing the probability that his preferred action profile, $(B, A)$, is played.

In contrast, the payoff profile $\beta_1(7,1)+\beta_2(1,7)$ and action distribution $\beta_1 1_{(B,A)} + \beta_2 1_{(A,B)}$ can be obtained in sequential equilibrium, namely with $\phi_1 = 1_{(1,1)}$, $\phi_2 = 1_{(2,2)}$,

$\pi_1(1, \phi_1) = B$, $\pi_1(2, \phi_1) = A$, $\pi_2(1, \phi_2) = A$ and $\pi_2(2, \phi_2) = B$. In this equilibrium, each player, prior to his choice of action, knows (in the sense of probability 1 belief) whose information design was chosen and the message that the other player received. The key feature of our framework is, however, that players cannot always detect deviations; for instance, if player 2 deviates to $\phi_2' = 1_{(1,2)}$ and $m = (1, 2)$ realizes, player 1 still thinks that his information design has been chosen and that player 2 will play her action after designing $\phi_2$ and observing $m_2 = 1$.

Correlated equilibrium is justified in Aumann (1987) as the result of Bayesian rationality — each player is maximizing his utility given his information. But where does this information come from? If the information is chosen optimally by players who have the ability to privately manipulate the information structure, then only a very specific subset of the convex hull of Nash payoffs can be achieved in the chicken game.

# 3 Information design without payoff uncertainty

Consider a normal-form game $G = (A_i, u_i)_{i \in N}$ where the set $N$ of players is finite and, for each $i \in N$, $A_i$ is a finite set of player $i$'s actions and $u_i : A \to \mathbb{R}$ is player $i$'s payoff function, where $A = \prod_{i \in N} A_i$. Let $N(G)$ denote the set of Nash equilibria of $G$ and $C(G)$ the set of correlated equilibria of $G$.

In the information design problems we consider, a designer sends messages to the players (who then choose actions). The set of messages each player $i \in N$ can potentially receive is $M_i = \mathbb{N}$. An *information design* consists of (i) a finite subset $K_i$ of $M_i$ for each $i \in N$ and (ii) $\phi \in \Delta(K)$, where $K = \prod_{i \in N} K_i$.[2] We dispense with the message sets $(K_i)_{i \in N}$ from our notation since these can be obtained from $\phi$ by letting $K_i = \text{supp}(\phi_{M_i})$ for each $i \in N$. Let $S$ be the set of finitely supported probability measures on $M = \prod_{i \in N} M_i = \mathbb{N}^n$, where $n = |N|$. Assuming that the support of $\phi$

[2]Given a metric space $X$, $\Delta(X)$ denotes the set of Borel probability measures on $X$. For each $\mu \in \Delta(X)$, $\text{supp}(\mu)$ denotes the support of $\mu$. When $X = \prod_{j \in J} X_j$ for some finite set $J$, $\mu_{X_j}$ denotes the marginal of $\mu$ on $X_j$ for each $j \in J$.

is finite avoids unnecessary technical complications without imposing a bound on its elements and, in particular, on their number.

Consider first the case of an *explicit information designer* who chooses $\phi \in S$. Assume that each player $i \in N$ observes the choice of $\phi$ by the designer and the realization of his own message $m_i$ and then chooses an action $a_i \in A_i$. Each $\phi$ defines a incomplete-information game $G(\phi)$; as Bergemann and Morris (2016) have shown, the set of equilibrium payoffs in the incomplete-information games obtained by $\phi \in S$ equals the set of correlated equilibrium payoffs of $G$:

$$\{u \in \mathbb{R}^n : u \text{ is an equilibrium payoff of } G(\phi) \text{ for some } \phi \in S\} = u(C(G)).$$

We will contrast the above case of an explicit information designer with the case of private information design, which is the focus of this paper. The setting with *private information design* is as follows. Each player $i \in N$ is a designer and, thus, chooses an information design. The resulting $n$ information designs, $\phi_1, \ldots, \phi_n$, are then aggregated into a single information design $\phi = \sum_i \beta_i \phi_i$ that determines the messages that players receive, where $\beta_i > 0$ for each $i \in N$ and $\sum_i \beta_i = 1$; one interpretation is that the information design of each $i \in N$ is chosen with probability $\beta_i$. The information design is private in the sense that (i) it is done by the players, (ii) each player's choice of information design is his own private information and (iii) no player observes the aggregate information design — the choice of the aggregate information design is made by nature and is nature's private information.

More generally, we allow for $\beta_i = 0$ for some $i \in N$, in which case only the players in $\mathrm{supp}(\beta) = \{i \in N : \beta_i > 0\}$ choose an information design $\phi_i \in S$. The players' interaction is then described by the following extensive-form game $G_{id}$. At the beginning of the game, each player $i \in \mathrm{supp}(\beta)$ chooses an information design $\phi_i \in S$. After all players in $\mathrm{supp}(\beta)$ have chosen their information design, a profile of signals $m \in M$ is realized according to $\phi \in \Delta(M)$ defined by setting, for each $m \in M$,

$$\phi[m] = \sum_{i \in \mathrm{supp}(\beta)} \beta_i \phi_i[m].$$

Each player $i \in N$ observes $m_i \in M_i$ and, if $i \in \mathrm{supp}(\beta)$, his choice $\phi_i \in S$, and then chooses an action $a_i \in A_i$. Player $i$'s payoff is then $u_i(a_1, \ldots, a_n)$.

A (behavioral) strategy for player $i \in \text{supp}(\beta)$ is $\pi_i = (\pi_i^1, \pi_i^2)$ such that $\pi_i^1 \in \Delta(S)$ and $\pi_i^2 : M_i \times S \to \Delta(A_i)$ is measurable;[3] and, for $i \in N \setminus \text{supp}(\beta)$, it is $\pi_i = \pi_i^2$ with $\pi_i^2 : M_i \to \Delta(A_i)$. A strategy is $\pi = (\pi_1, \ldots, \pi_n)$. The set $S$ is convex and, hence, there is no point in allowing players to mix in the choice of $\phi_i$. For this reason, let $\Pi$ be the set of strategies $\pi$ such that $\pi_i^1 \in S$ (i.e. $\pi_i^1$ is pure) for each $i \in \text{supp}(\beta)$ and we focus on $\pi \in \Pi$.

For strategies $\pi \in \Pi$ and for each $i \in N$, $m_i \in M_i$ and $\phi_i \in S$, we often write $\phi_i^* = \pi_i^1$, $\pi_i(m_i, \phi_i) = \pi_i^2(m_i, \phi_i)$ and $\pi_i(m_i) = \pi_i^2(m_i, \phi_i^*)$ if $i \in \text{supp}(\beta)$. For $\pi \in \Pi$, we also write $u_i(\pi) = \sum_{m \in M} \phi^*[m] u_i(\pi(m))$ for each $i \in N$, where $\phi^*[m] = \sum_{i \in \text{supp}(\beta)} \beta_i \phi_i^*[m]$, $\pi(m) \in \Delta(A)$ is defined by $\pi(m)[a] = \prod_{i \in N} \pi_i(m_i)[a_i]$ for each $a \in A$ and, for each $\sigma \in \Delta(A)$, $u_i(\sigma) = \sum_{a \in A} \sigma[a] u_i(a)$. We sometimes abuse notation and also let $\pi(m) = (\pi_1(m_1), \ldots, \pi_n(m_n))$.

Let $n' = |\text{supp}(\beta)|$ and, for each strategy $\pi$, $m \in M$ and $\phi \in S^{n'}$, we let $\pi^2(m, \phi)$ be defined by $\pi_i^2(m, \phi) = \pi_i^2(m_i, \phi_i)$ if $i \in \text{supp}(\beta)$ and $\pi_i^2(m, \phi) = \pi_i^2(m_i)$ if $i \in N \setminus \text{supp}(\beta)$; we also write $\pi(m, \phi)$ for $\pi^2(m, \phi)$ and use $\pi_{-i}^2(m_{-i}, \phi_{-i})$ and $\pi_{-i}(m_{-i}, \phi_{-i})$ for the vector of mixed actions $\pi(m, \phi)$ without the $i$th coordinate.

We use sequential equilibrium as solution concept, defined analogously to Myerson and Reny (2020) (MR henceforth). Formally, a strategy $\pi \in \Pi$ is a *sequential equilibrium* if it is a perfect conditional $\varepsilon$-equilibrium for each $\varepsilon > 0$.

For each strategy $\pi \in \Pi$, the *action distribution* of $\pi$ is $\sigma_\pi \in \Delta(A)$ such that, for each $a \in A$,

$$\sigma_\pi[a] = \sum_{m \in M^*} \phi^*[m] \pi(m)[a].$$

Let

$$U(G) = \{u(\pi) : \pi \in \Pi \text{ is a sequential equilibrium of } G_{id}\}$$

be the set of sequential equilibrium payoffs of $G_{id}$. Theorem 1 states that each sequential equilibrium payoff of the private information design game $G_{id}$ is the payoff of a correlated equilibrium of $G$. Indeed, private or otherwise, all that information design does in our setting is to allow players to reach correlated equilibrium payoffs.

[3]The set $S$ is endowed with the topology of the weak convergence of probability measures.

**Theorem 1.** *For each n-player game $G$, if $\pi \in \Pi$ is a sequential equilibrium of $G_{id}$, then $\sigma_\pi \in C(G)$. Thus, $U(G) \subseteq u(C(G))$.*

Recall that $u(C(G))$ is the set of payoffs that players can reach when the information design is made by an explicit designer. With private information design there will, in general, be a reduction on the payoffs that the players can reach. The reason is that the messages $m \in \mathrm{supp}(\phi_i^*)$ that each player $i \in \mathrm{supp}(\beta)$ sends must be optimal for player $i$. This is established in Theorem 2 which fully characterizes the set of sequential equilibrium outcomes of $G_{id}$.

The following notation is used in the statement of Theorem 2. The *outcome* of a strategy $\pi \in \Pi$ is $\left((\phi_i^*)_{i\in\mathrm{supp}(\beta)}, \left((\pi_i(m_i))_{m_i\in\mathrm{supp}(\phi^*_{M_i})}\right)_{i\in N}\right)$; it consists of the information design for each player in $\mathrm{supp}(\beta)$ and, for each player and each message that he may receive with strictly positive probability, the action he will choose in response. Let $M^* = \prod_{i\in N}\mathrm{supp}(\phi^*_{M_i})$ be the product of the set of messages that each player may receive with strictly positive probability. We use the convention that $\mathrm{supp}(\phi_i^*) = \emptyset$ for each $i \notin \mathrm{supp}(\beta)$ and let, for each $i \in N$, $\mathrm{supp}(\beta_{-i}) = \mathrm{supp}(\beta) \setminus \{i\}$. For each $i \in N$ and $\delta \in \Delta(A_{-i})$, let $v_i(\delta) = \max_{\alpha\in\Delta(A_i)} u_i(\alpha, \delta)$ and $BR_i(\delta) = \{\alpha \in \Delta(A_i) : u_i(\alpha,\delta) = v_i(\delta)\}$ be, respectively, player $i$'s value function and best-reply correspondence.

**Theorem 2.** *For each n-player game $G$, $\left((\phi_i^*)_{i\in\mathrm{supp}(\beta)}, \left((\pi_i(m_i))_{m_i\in\mathrm{supp}(\phi^*_{M_i})}\right)_{i\in N}\right)$ is the outcome of a sequential equilibrium of $G_{id}$ if and only if, for each $i \in N$,*

$$v_i(\pi_{-i}(m_{-i})) = \max_{m'_{-i}\in M^*_{-i}} v_i(\pi_{-i}(m'_{-i})) \text{ and } \pi_i(m_i) \in BR_i(\pi_{-i}(m_{-i})) \tag{1}$$

*for each $m \in \mathrm{supp}(\phi_i^*)$, and*

$$\pi_i(m_i) \text{ solves } \max_{\alpha_i\in\Delta(A_i)} \sum_{m_{-i}} \frac{\sum_{j\in\mathrm{supp}(\beta_{-i})} \beta_j \phi_j^*[m_i, m_{-i}]}{\sum_{j\in\mathrm{supp}(\beta_{-i})} \beta_j \phi^*_{j,M_i}[m_i]} u_i(\alpha_i, \pi_{-i}(m_{-i})) \tag{2}$$

*for each $m_i \in \cup_{j\in\mathrm{supp}(\beta_{-i})}\mathrm{supp}(\phi^*_{j,M_i})$.*

Theorem 2 shows that sequential equilibrium outcomes of the private information design game are characterized by the optimality of the actions that each player chooses and, for each player in $\mathrm{supp}(\beta)$, of the messages he sends. Each player's messages are optimal in the sense that the payoff of the action profile it induces is the highest amongst the action profiles belonging to the outcome. The optimality of the actions chosen by each player $i \in N$ consists of $\pi_i(m_i)$ maximizing his expected payoff conditional on his information design not being chosen when $m_i$ is a message that he receives with strictly positive probability from the information design of the other players. The two optimality conditions imply that, for each player $i \in \text{supp}(\beta)$, $\pi_i(m_i)$ maximizes his expected payoff conditional on his information design being chosen when $m_i \in \text{supp}(\phi^*_{i,M_i}) \setminus \cup_{j \in \text{supp}(\beta_{-i})}\text{supp}(\phi^*_{j,M_i})$ and maximizes his expected payoff conditional on his information design not being chosen when $m_i \in \cup_{j \in \text{supp}(\beta_{-i})}\text{supp}(\phi^*_{j,M_i}) \setminus \text{supp}(\phi^*_{i,M_i})$; indeed, in the former case, player $i$ can only have received message $m_i$ if $\phi^*_i$ has been chosen and, in the latter case, only if $\phi^*_i$ has not been chosen. Furthermore, in the remaining case where $m_i \in \text{supp}(\phi^*_{i,M_i}) \cap (\cup_{j \in \text{supp}(\beta_{-i})}\text{supp}(\phi^*_{j,M_i}))$, it turns out that $\pi_i(m_i)$ must satisfy the two criteria. Briefly, this happens because player $i$ can always make sure that the message he sends to himself is different from the ones he may receive from the other players (we will elaborate on conditions (1) and (2) in the proof of the necessity part of Theorem 2 in Section A.1.2). Since each player $i \in N \setminus \text{supp}(\beta)$ does not design information, the optimality of his actions means that $\pi_i(m_i)$ maximizes his expected payoff given the information design chosen by the players in $\text{supp}(\beta)$.

An easy consequence of Theorem 2 is that $u(N(G)) \subseteq U(G)$ for each $n$-player game $G$. Indeed, for each Nash equilibrium $\sigma$ of $G$, set $\phi^*_i = 1_{(1,\ldots,1)}$ for each $i \in \text{supp}(\beta)$ and $\pi_i(1) = \sigma_i$ for each $i \in N$ to see that conditions (1) and (2) in Theorem 2 hold.

**Corollary 1.** *For each $n$-player game $G$, if $\sigma \in N(G)$, then there is a sequential equilibrium $\pi \in \Pi$ of $G_{id}$ such that $\sigma_\pi = \sigma$. Thus, $u(N(G)) \subseteq U(G)$.*

The following example further illustrates Theorem 2.

**Example 1.** Consider the following game, Example 2.5 in Aumann (1974), where player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix ($A_3 = \{L, M, R\}$):

| 1\2 | $A$ | $B$ |
|---|---|---|
| $A$ | $0,0,3$ | $0,0,0$ |
| $B$ | $1,0,0$ | $0,0,0$ |

| 1\2 | $A$ | $B$ |
|---|---|---|
| $A$ | $2,2,2$ | $0,0,0$ |
| $B$ | $0,0,0$ | $2,2,2$ |

| 1\2 | $A$ | $B$ |
|---|---|---|
| $A$ | $0,0,0$ | $0,0,0$ |
| $B$ | $0,1,0$ | $0,0,3$ |

Assume first that $\text{supp}(\beta) = N$. We will show that when $\min\{2\beta_1, 2\beta_2\} \geq \beta_3$, $(1-\beta_3)(2,2,2) + \beta_3(0,0,3)$ is a sequential equilibrium payoff, even though $u_i \leq 1$ for each $u \in u(N(G))$. Thus, correlation of players' actions through private information design can still significantly improve the payoff to everybody relative to Nash equilibrium payoffs. However, not all correlated equilibrium payoffs can be achieved.

The latter claim can be easily seen by considering $(2,2,2) \in u(C(G))$; if $(2,2,2) \in U(G)$, then, for some sequential equilibrium $\pi \in \Pi$,

$$(2,2,2) = \sum_{m \in \text{supp}(\phi^*)} \phi^*[m] u(\pi(m))$$

and, thus, $\pi(m) = (A, A, M)$ or $\pi(m) = (B, B, M)$ for each $m \in \text{supp}(\phi^*)$. But then, for each $m \in \text{supp}(\phi_3^*)$, $\pi_3(m_3)$ is not a best-reply against $\pi_{-3}(m_{-3})$, contradicting Theorem 2.[4]

We now establish the former claim. Let $\phi_1^* = \phi_2^* = \frac{1}{2}1_{(m_1', m_2', \hat{m}_3)} + \frac{1}{2}1_{(m_1'', m_2'', \hat{m}_3)}$, $\phi_3^* = \frac{1}{2}1_{(m_1', m_2', \hat{m}_3')} + \frac{1}{2}1_{(m_1'', m_2'', \hat{m}_3'')}$ and

$$\begin{aligned}
&\pi_1(m_1') = A, \pi_1(m_1'') = B, \\
&\pi_2(m_2') = A, \pi_2(m_2'') = B, \\
&\pi_3(\hat{m}_3) = M, \pi_3(\hat{m}_3') = L \text{ and } \pi_3(\hat{m}_3'') = R.
\end{aligned}$$

Thus, $\pi(m_1', m_2', \hat{m}_3) = (A, A, M)$, $\pi(m_1'', m_2'', \hat{m}_3) = (B, B, M)$, $\pi(m_1', m_2', \hat{m}_3') = (A, A, L)$ and $\pi(m_1'', m_2'', \hat{m}_3'') = (B, B, R)$. As we show in Appendix A.2, the conditions in Theorem 2 are satisfied if $\min\{2\beta_1, 2\beta_2\} \geq \beta_3$ and, thus, it follows that $(1-\beta_3)(2,2,2) + \beta_3(0,0,3)$ is a sequential equilibrium payoff.

[4]In contrast to the results in Ben-Porath (1998), $(2,2,2) \in u(C(G))$ cannot be approximated by $u \in U(G)$. Indeed, to get close to $(2,2,2)$, $\phi^*$ must put small probability on $m$ such that $\pi(m) \notin \{(A, A, M), (B, B, M)\}$. Thus, $\phi_3^*$ must also put small probability on such $m$. But then there exists $m' \in \text{supp}(\phi_3^*)$ such that $\pi(m') \in \{(A, A, M), (B, B, M)\}$, which contradicts Theorem 2.

We also use Example 1 to compare our setting to the one surveyed in Bergemann and Morris (2019), in which there is a unique (explicit or metaphorical) designer. We let player 3 be the designer and, thus, let $\beta_3 = 1$. For each $\phi_3 \in S$, let $\Pi^*(\phi_3)$ be the set of $(\pi(m))_{m\in\mathrm{supp}(\phi_3)}$ such that, for each $i \in N$ and $m_i \in \mathrm{supp}(\phi_{3,M_i})$, $\pi_i(m_i)$ solves $\max_{\alpha_i\in\Delta(A_i)} \sum_{m_{-i}} \frac{\phi_3[m_i,m_{-i}]}{\phi_{3,M_i}[m_i]} u_i(\alpha_i, \pi_{-i}(m_{-i}))$. Bergemann and Morris (2019) focus on outcomes $\left(\phi_3^*, (\pi(m))_{m\in\mathrm{supp}(\phi_3^*)}\right)$ such that $(\pi(m))_{m\in\mathrm{supp}(\phi_3^*)} \in \Pi^*(\phi_3^*)$ and $\phi_3^*$ solves

$$\max_{\phi_3\in S} \max_{(\pi(m))_{m\in\mathrm{supp}(\phi_3)}\in\Pi^*(\phi_3)} \sum_{m\in\mathrm{supp}(\phi_3)} \phi_3[m]u_3(\pi(m)). \tag{3}$$

It follows by Bergemann and Morris (2019, Proposition 4) that the value of (3) is equal to $\max_{\sigma\in C(G)} u_3(\sigma)$. However, as we show in Appendix A.2, player 3 cannot obtain a payoff of $\max_{\sigma\in C(G)} u_3(\sigma)$ in $G_{id}$.

# 4 2-player games

The characterization of equilibrium outcomes in Theorem 2 implies that, in general, not all correlated equilibrium payoffs of $G$ can be achieved in $G_{id}$ when information is designed privately. This point is more striking in 2-person games; indeed, in such games, equilibrium payoffs of $G_{id}$ form a particular subset of the convex hull of the Nash equilibrium payoffs of $G$.

**Theorem 3.** *For each 2-player game $G$,*

$$\begin{aligned} U(G) = \Bigg\{ & \beta_1 u^1 + \beta_2 u^2 : \forall i \in \mathrm{supp}(\beta),\ \textit{there exists } L_i, (\alpha^{i,l})_{l=1}^{L_i}, (\sigma^{i,l})_{l=1}^{L_i} \textit{ such that} \\ & u^i = \sum_{l=1}^{L_i} \alpha^{i,l} u(\sigma^{i,l}), \alpha^i \geq 0, \sum_{l=1}^{L_i} \alpha^{i,l} = 1, \\ & \sigma^{i,l} \in N(G) \textit{ and } u_i(\sigma^{i,k}) = u_i(\sigma^{i,l}) \geq u_i(\sigma^{j,r}) \\ & \forall k, l \in \{1, \ldots, L_i\}, j \in \mathrm{supp}(\beta_{-i}) \textit{ and } r \in \{1, \ldots, L_j\} \Bigg\}. \end{aligned}$$

Theorem 3 characterizes the equilibrium payoffs of $G_{id}$ for 2-player games. It shows that when player $i$'s information design is chosen, then the resulting payoff $u^i$ is a convex combination of payoffs of Nash equilibria of $G$, all of which give the same payoff $u_i^i$ to player $i$. Furthermore, this common payoff $u_i^i$ is no less than the payoff player $i$ obtains in each of the Nash equilibria of $G$ used to obtain $u^j$ when $j$ also belongs to supp($\beta$). In other words, player $i$ prefers (weakly at least) any Nash equilibria of $G$ used to obtain $u^i$ to any of them used to obtain $u^j$.

For the battle of the sexes,

| $1\backslash 2$ | $A$ | $B$ |
|---|---|---|
| $A$ | $2,1$ | $0,0$ |
| $B$ | $0,0$ | $1,2$ |

Theorem 3 implies that $U(G) = u(N(G)) \cup \{\beta_1(2,1) + \beta_2(1,2)\}$.

Example 1 shows that the conclusion of Theorem 3 does not extend beyond 2-player games. The reason why there is a sharper characterization of equilibrium payoffs in 2-player games is that, in such games, the maximization problem in condition (2) becomes

$$\max_{\alpha_i \in \Delta(A_i)} \sum_{m_j} \frac{\phi_j^*[m_i, m_j]}{\phi_{j,M_i}^*[m_i]} u_i(\alpha_i, \pi_j(m_j)) = \max_{\alpha_i \in \Delta(A_i)} u_i \left( \alpha_i, \sum_{m_j} \frac{\phi_j^*[m_i, m_j]}{\phi_{j,M_i}^*[m_i]} \pi_j(m_j) \right)$$

with $i \in N$ and $j \neq i$ and $\sum_{m_j} \frac{\phi_j^*[m_i,m_j]}{\phi_{j,M_i}^*[m_i]} \pi_j(m_j)$ is a mixed strategy of player $j$.

The characterization of $U(G)$ is simpler in generic games, such as the battle of the sexes, since then the payoff resulting after each information design is chosen is that of a Nash equilibrium. Let $\mathcal{G}$ be the set of games such that, for each Nash equilibria $\sigma$ and $\sigma'$ of $G$, if $u_i(\sigma) = u_i(\sigma')$ for some $i \in N$, then $u_j(\sigma) = u_j(\sigma')$ for $j \neq i$ (equivalently, if $u_i(\sigma) \neq u_i(\sigma')$ for some $i \in N$ then $u_j(\sigma) \neq u_j(\sigma')$ for $j \neq i$). We regard $\mathcal{G}$ as a subset of $\mathbb{R}^{2|A|}$. A subset $B$ of an Euclidean space is *generic* if the closure of its complement has Lebesgue measure zero.

**Corollary 2.** *The set $\mathcal{G}$ is generic and, for each 2-player game $G \in \mathcal{G}$,*

$$U(G) = \{\beta_1 u(\sigma) + \beta_2 u(\sigma') : \sigma, \sigma' \in N(G), u_1(\sigma) \geq u_1(\sigma'), u_2(\sigma') \geq u_2(\sigma)\}.$$

The proof of Corollary 2 actually shows that the set of games such that $u_i(\sigma) \neq u_i(\sigma')$ for each $i \in N$ and $\sigma, \sigma' \in N(G)$ such that $\sigma \neq \sigma'$ is generic. This set is contained in $\mathcal{G}$ and contains all games with a unique equilibrium as well as both the battle of the sexes and the game of chicken. It is clear from Corollary 2 that $U(G) = u(N(G))$ for each 2-player game $G$ with a unique Nash equilibrium.

# 5 Epistemic conditions for Nash equilibrium

Theorem 3 implies that, for 2-player games $G$ with a unique Nash equilibrium, only the payoff of that Nash equilibrium can be achieved in the private information design game $G_{id}$. In this section we build on this result by operationalizing the intuition that, when players' information is endogenously determined via private information design, no knowledge of players' actions or conjectures is needed for epistemic conditions that are sufficient for Nash equilibrium in 2-player games with a unique Nash equilibrium. This is so because endogenous information embeds a stronger notion of rationality than that of e.g. Aumann and Brandenburger (1995).

The starting point of the analysis of this section is the notion of an interactive belief system introduced by Aumann and Brandenburger (1995). An *interactive belief system for $G$* with a common prior $\phi^*$ consists of, for each player $i \in N$, a finite set $M_i^*$ of types or messages and, for each $m_i \in M_i^*$, a mixed action $\pi_i(m_i) \in \Delta(A_i)$ and a theory about other players' messages $\phi^*(\cdot|m_i)$ such that $\phi^* \in \Delta(\prod_{j \in N} M_j^*)$ and $M_i^* = \text{supp}(\phi^*_{M_i^*})$. This notion differs from the one in Aumann and Brandenburger (1995) just because we assume that the game $G$ is common knowledge, which is done for simplicity.[5]

The goal is to impose conditions on interactive belief systems so that it may intuitively reflect that players' information is endogenous. In general, as Aumann (1987, Section 4 (e)) as pointed out, it is not easy to distinguish between endogenous and exogenous information; on this, we follow Aumann and Brandenburger's (1995, p. 1163) motivation for the common prior assumption, namely, we consider a situation "in which the players had the same information and probability assessment, and then

[5]Aumann and Brandenburger (1995) focus on the case where $\pi_i(m_i)$ is pure but also allow it to be mixed. They also consider interactive belief systems where there is no common prior.

got different information." In our setting, we think of the players starting with no information, then each designing an information structure $\phi_i^*$ and finally receiving a message $m_i$ drawn from $\phi^* = \sum_{i\in N} \beta_i \phi_i^*$, which serves as the common prior.

The above setting is only apparently more specific than the one in the definition of an interactive belief system since no distinction arises between the two when $\phi_i^*$ can be an arbitrary information structure for each $i \in N$. Our goal is precisely to impose conditions on $\phi_i^*$ for each $i \in N$ that reflect the idea that each player designs $\phi_i^*$ optimally by only sending messages that are optimal for him whenever he is rational in the standard sense. The conditions of Theorem 2 give a sense of what such optimality may mean and we impose them on each player's probability distribution $\phi_i^*$.

To summarize, our motivation in this section is to operationalize a notion of endogenous information in interactive belief systems that matters for epistemic conditions that are sufficient for Nash equilibrium. In particular, inspired by Aumann and Brandenburger (1995, Section 7 d), we aim in this way to find a situation of economic interest where there is enough knowledge about a strong form of players' rationality to imply that a Nash equilibrium must be played. The results of this section show that if each player designs information optimally when he is rational then knowledge about players' rationality alone is enough to lead to Nash equilibrium.

Let $G = (A_i, u_i)_{i\in N}$ be a normal-form game and $E$ be an interactive belief system for $G$. For each $i \in N$, let

$$R_i = \left\{ m_i \in M_i^* : \pi_i(m_i) \text{ solves } \max_{\alpha_i\in\Delta(A_i)} \sum_{m_{-i}} \frac{\phi^*[m_i, m_{-i}]}{\phi^*_{M_i^*}[m_i]} u_i(\alpha_i, \pi_{-i}(m_{-i})) \right\}$$

be the set of messages $m_i$ at which player $i$ is rational. We say that $E$ is an *endogenous interactive belief system for $G$* if, for each $i \in N$, there is $\phi_i^* \in \Delta(M^*)$ and $\beta_i > 0$ such that $\phi^* = \sum_{i\in N} \beta_i \phi_i^*$, $\sum_{i\in N} \beta_i = 1$,

$$v_i(\pi_{-i}(m_{-i})) = \max_{m'_{-i}\in M^*_{-i}} v_i(\pi_{-i}(m'_{-i})) \text{ and } \pi_i(m_i) \in BR_i(\pi_{-i}(m_{-i})) \tag{4}$$

for each $m \in \text{supp}(\phi_i^*) \cap (R_i \times M^*_{-i})$ and

$$\pi_i(m_i) \text{ solves } \max_{\alpha_i\in\Delta(A_i)} \sum_{m_{-i}} \frac{\sum_{j\neq i} \beta_j \phi_j^*[m_i, m_{-i}]}{\sum_{j\neq i} \beta_j \phi^*_{j,M_i}[m_i]} u_i(\alpha_i, \pi_{-i}(m_{-i})) \tag{5}$$

for each $m_i \in \text{supp}(\phi^*_{i,M^*_i}) \cap \left(\cup_{j\neq i}\text{supp}(\phi^*_{j,M^*_i})\right) \cap R_i$.

As noted, this notion of an endogenous interactive belief system for $G$ captures the idea that, whenever a player $i$ is rational, his information design is optimal for him in the sense that it only sends messages that are optimal for him. Interactive belief systems are useful, in particular, to analyse (implications of) how players think about their opponents. In endogenous interactive belief systems, we want to capture that players reason about what extra properties must be satisfied by the action and message of a player who sent a particular message. Given a message $m \in \text{supp}(\phi^*_i)$ that player $i$ sends with strictly positive probability, if player $i$ is rational at $m$, then $i$ and his opponents can reason that $m$ must be optimal for $i$ since otherwise he would not have sent it. Thus, (4) should hold for each $m \in \text{supp}(\phi^*_i) \cap (R_i \times M^*_{-i})$ and (5) for each $m_i \in \text{supp}(\phi^*_{i,M^*_i}) \cap \left(\cup_{j\neq i}\text{supp}(\phi^*_{j,M^*_i})\right) \cap R_i$. On the other hand, it is unclear why the message $m \in \text{supp}(\phi^*_i)$ must be optimal when player $i$ is irrational at $m$; thus, we do not require (4) and (5) when $m_i \notin R_i$.

It turns out that all the results in this section hold if condition (4) is weakened to $\pi_i(m_i) \in BR_i(\pi_{-i}(m_{-i}))$ for each $m \in \text{supp}(\phi^*_i) \cap (R_i \times M^*_{-i})$. We nevertheless impose (4) in the definition of an endogenous interactive belief system as it is (4) rather than its weakening that is part of the optimality of information we seek to capture.[6]

There is another weakening of the notion of endogenous interactive belief systems for which our results hold. Consider the 2-player case and let

$$\begin{aligned} S &= \{m \in M^* : (4) \text{ holds for each } i \in \{1,2\} \text{ such that } m \in \text{supp}(\phi^*_i)\}, \\ T &= \{m \in M^* : (5) \text{ holds for each } i \in \{1,2\} \text{ such that} \\ &\qquad m_i \in \text{supp}(\phi^*_{1,M^*_i}) \cap \text{supp}(\phi^*_{2,M^*_i})\} \end{aligned}$$

and $R = R_1 \times R_2$. If $E$ is an endogenous interactive belief system for $G$ then $E$ is an interactive belief system for $G$ with a common prior $\phi^* = \beta_1\phi^*_1 + \beta_2\phi^*_2$ and

[6]For each theorem in this section, we will provide examples showing that none of the epistemic conditions we impose in addition to the assumption of endogenous interactive belief system can be dropped. Thus, imposing the strong version of (4) also has the advantage of making these counter-examples to extensions of our results stronger.

$R \subseteq S \cap T$. The condition $R \subseteq S \cap T$ requires that message profiles at which all players are rational must be optimal for those players that send it; it is thus weaker than the requirement present in the definition of an endogenous interactive belief system, according to which message profiles at which the sender is rational must be optimal for him. While the latter is more intuitive, the former clarifies the sense in which rationality in a endogenous interactive belief system corresponds to a stronger form of rationality in an interactive belief system. Indeed, (common or mutual) knowledge that players are rational (i.e. of $R$) in the above weaker form of an endogenous interactive belief system is equivalent to knowledge of $R \cap S \cap T$ in an interactive belief system.

The *action distribution* of an endogenous interactive belief system $E$ for $G$ is $\mu \in \Delta(A)$ such that, for each $a \in A$,

$$\mu[a] = \sum_{m \in M^*} \phi^*[m]\pi(m)[a],$$

where $\pi(m)[a] = \prod_{i \in N} \pi_i(m_i)[a_i]$. For each $m \in \text{supp}(\phi^*)$, *each player is rational at* $m$ if $m \in \prod_{i \in N} R_i$. If $G$ is a 2-player game and $i \in \{1, 2\}$, *player $i$'s conjecture about $j \neq i$ at $m$* is

$$\xi_i(m) = \sum_{m_j \in \text{supp}(\phi^*_{M_j})} \frac{\phi^*[m_i, m_j]}{\phi^*_{M_i}[m_i]} \pi_j(m_j).$$

Theorem 4 states that rationality at each message profile implies that the action distribution is a Nash equilibrium of $G$ in 2-player games with a unique equilibrium. In fact, in this case, players' profile of conjectures $(\xi_2(m), \xi_1(m))$ at each message profile $m$ equals the Nash equilibrium of the game.

**Theorem 4.** *Let $G$ be a 2-player game with a unique Nash equilibrium $\sigma$ and $E$ be an endogenous interactive belief system for $G$. If each player is rational at each $m \in \text{supp}(\phi^*)$, then the action distribution of $E$ is $\sigma$ and $(\xi_2(m), \xi_1(m)) = \sigma$ for each $m \in \text{supp}(\phi^*)$.*

Theorem 4 is analogous to the main theorem of Aumann (1987) which states that if each player is rational at each state of the world, then the action distribution is a correlated equilibrium of $G$.[7] Theorem 4 strengthens the conclusion by replacing correlated equilibrium with Nash equilibrium in 2-player games with a unique Nash equilibrium whenever the players' information is endogenously designed in an optimal way.[8]

Theorem 4 is also related to Theorem A of Aumann and Brandenburger (1995). In fact, if a 2-player game with a unique Nash equilibrium and the rationality of players are commonly known and the players' information is endogenously designed in an optimal way, then players' conjectures are also commonly known since they equal the unique Nash equilibrium each state. Thus, the assumptions of Theorem A of Aumann and Brandenburger (1995) hold at each state.

There is a similarity between Theorem 4 and Nash's (1950) rationalistic interpretation of Nash equilibrium since both require uniqueness of equilibrium. To the best of our knowledge, this requirement features in no epistemic result for Nash equilibrium other than Theorem 4 and, as Example 5 in Appendix A.4 shows, the assumption of a unique Nash equilibrium cannot be dropped from Theorem 4.[9]

Both Theorem 4 and the main theorem of Aumann (1987) have a global viewpoint since its assumptions and conclusions pertain to the entire state space. As the following example shows, the assumption that each player is rational at each state cannot be dropped from Theorem 4.

**Example 2.** Consider the following game $G$:

| $1\backslash 2$ | $L$ | $R$ |
|---|---|---|
| $T$ | $4, 2$ | $2, 1$ |
| $B$ | $3, 4$ | $1, 3$ |

Then $T$ is a dominant strategy for player 1 and $L$ is a dominant strategy for player 2. For this game, consider $M_1^* = \{t, b\}$, $M_2^* = \{l\}$, $\phi_1^* = 1_{(t,l)}$, $\phi_2^* = 1_{(b,l)}$, $\pi_1(t) = T$,

[7]Elements of $M^*$ are, as in Aumann and Brandenburger (1995), called states of the world or just states.

[8]In Appendix A.3, we provide an example of a 2-player game with a unique Nash equilibrium but a continuum of correlated equilibria.

[9]We also provide in Appendix A.4 examples showing that the requirement of endogenous information cannot be dropped from Theorem 4.

$\pi_1(b) = B$ and $\pi_2(l) = L$ and the resulting interactive belief system depicted below as in Aumann and Brandenburger (1995):

| $1\backslash 2$ | $l$ |
|---|---|
| $t$ | $1, \beta_1$ |
| $b$ | $1, \beta_2$ |

Then both players are rational at $m = (t, l)$ but $(\xi_2(m), \xi_1(m)) = (\beta_1 1_T + \beta_2 1_B, L)$ which is not a Nash equilibrium of $G$.

In contrast to Theorem 4 and the main theorem of Aumann (1987), Theorem A in Aumann and Brandenburger (1995) has a local viewpoint by providing sufficient epistemic conditions for Nash equilibrium at any state at which they hold. While the above example shows that players' rationality at a state does not imply that players' conjectures are a Nash equilibrium of the game, it illustrates the case where players' action profiles are a Nash equilibrium in each state where both players are rational. As Theorem 5 below shows, this conclusion holds for "simple" interactive belief systems for any 2-player game at states where there is mutual knowledge of players' rationality.

The notion of simplicity we use is analogous to the one in Abreu and Rubinstein (1988) and consists of adding a lexicographic preference for smaller messages spaces in the information design game $G_{id}$. We then use the resulting characterization of equilibrium outcomes to define more stringent endogenous interactive belief systems as follows.

Let $G$ be a normal-form game and $G_{id}$ be as in Section 3. We say that $\pi \in \Pi$ is a *sequential equilibrium with complexity costs* of $G_{id}$ if (i) $\pi$ is a sequential equilibrium of $G_{id}$ and (ii) for each $i \in N$, there does not exist $\phi_i \in S$ such that $|\text{supp}(\phi_i)| < |\text{supp}(\phi_i^*)|$ and $\sum_m (\phi_i, \phi_{-i}^*)[m] u_i(\pi_i(m_i, \phi_i), \pi_{-i}(m_{-i})) = \sum_m \phi^*[m] u_i(\pi(m))$. Since all messages of each player $i$ yield the same payoff to him, it follows that outcomes of sequential equilibria with complexity costs of $G_{id}$ are characterized, in addition to (1) and (2), by $|\text{supp}(\phi_i^*)| = 1$ for each $i \in N$.

The above then yields a refined notion of an endogenous interactive belief system as follows. We say that $E$ is an *endogenous interactive belief system with complexity* *costs* if $E$ is an endogenous interactive belief system and $|(R_i \times M^*_{-i}) \cap \mathrm{supp}(\phi^*_i)| \leq 1$ for each $i \in N$.[10] For each $m \in \mathrm{supp}(\phi^*)$, it is *mutually known that the players are rational at* $m$ if $m_i \in R_i$ and $\frac{\phi^*[\{m_i\}\times R_{-i}]}{\phi^*[m_i]} = 1$ for each $i \in N$. Theorem 5 shows that if it is mutually known that players are rational in some state of an endogenous interactive belief system with complexity costs for a 2-player game (not necessarily with an unique Nash equilibrium), then the action profile at that state is a Nash equilibrium.

**Theorem 5.** *Let $G$ be a 2-player game and $E$ be an endogenous interactive belief system with complexity costs for $G$. If it is mutually known that the players are rational at some $m \in \mathrm{supp}(\phi^*)$, then $\pi(m)$ is a Nash equilibrium of $G$.*

The striking feature of Theorem 5 is that mutual knowledge of rationality alone at some state is enough imply that the (mixed) action profile is a Nash equilibrium at that state. This is in contrast to the preliminary observation in Aumann and Brandenburger (1995) which requires that the action profile is mutually known. That the latter need not hold under the assumptions of Theorem 5 is illustrated in the following example.

**Example 3.** Consider the game of chicken in Section 2 and let $M^*_1 = \{b, \alpha\}$, $M^*_2 = \{a, b\}$, $\phi^*_1 = 1_{(b,a)}$, $\phi^*_2 = \frac{1}{2}1_{(\alpha,a)} + \frac{1}{2}1_{(\alpha,b)}$, $\pi_1(b) = B$, $\pi_1(\alpha) = \alpha 1_A + (1-\alpha)1_B$, $\pi_2(a) = A$ and $\pi_2(b) = B$ with $0 < \alpha < \frac{1}{2}$. The corresponding interactive belief system is:

| $1\backslash 2$ | $a$ | $b$ |
|---|---|---|
| $b$ | $1, \frac{2\beta_1}{1+\beta_1}$ | $0, 0$ |
| $\alpha$ | $\frac{1}{2}, \frac{1-\beta_1}{1+\beta_1}$ | $\frac{1}{2}, 1$ |

We have that $R_1 = M^*_1$ and $R_2 = \{a\}$.[11] Thus, it is mutually known that the players are rational at $m = (b, a)$. The conditions for an endogenous interactive belief system

[10] Theorem 5 also holds under the weaker condition $|R \cap \mathrm{supp}(\phi^*_i)| \leq 1$ for each $i \in N$.

[11] To see this, note that $\pi_1(b) = B \in BR_1(A) = BR_1\left(\sum_{m_2} \frac{\phi^*[b,m_2]}{\phi^*_{M^*_1}[b]}\pi_2(m_2)\right)$, $\pi_1(\alpha) = \alpha 1_A + (1-\alpha)1_B \in BR_1\left(\frac{1}{2}1_A + \frac{1}{2}1_B\right) = BR_1\left(\sum_{m_2} \frac{\phi^*[\alpha,m_2]}{\phi^*_{M^*_1}[\alpha]}\pi_2(m_2)\right)$, $\pi_2(a) = A \in BR_2\left(\frac{\alpha(1-\beta_1)}{1+\beta_1}1_A + \left(1 - \frac{\alpha(1-\beta_1)}{1+\beta_1}\right)1_B\right) = BR_2\left(\sum_{m_1} \frac{\phi^*[m_1,a]}{\phi^*_{M^*_2}[a]}\pi_1(m_1)\right)$, and $\pi_2(b) = B \notin BR_2(\alpha 1_A + (1-\alpha)1_B) = BR_2\left(\sum_{m_1} \frac{\phi^*[m_1,b]}{\phi^*_{M^*_2}[b]}\pi_1(m_1)\right)$.

with complexity costs are satisfied. Regarding player 1, we have that $\text{supp}(\phi^*_{1,M_1^*}) \cap \text{supp}(\phi^*_{2,M_1^*}) = \emptyset$, $\text{supp}(\phi^*_1) = \{(b,a)\}$, $\pi_1(b) = B \in BR_1(A) = BR_1(\pi_2(a))$ and $v_1(\pi_2(a)) = 7 > 1 = v_1(\pi_2(b))$. As for player 2, $\text{supp}(\phi^*_2) \cap (M_1^* \times R_2) = \{(\alpha, a)\}$, $\pi_2(a) = A \in BR_2(\alpha 1_A + (1-\alpha)1_B) = BR_2(\pi_1(\alpha))$ and $v_2(\pi_1(\alpha)) = 1 + 5\alpha > 1 = v_2(\pi_1(b))$. Furthermore, $\text{supp}(\phi^*_{1,M_2^*}) \cap \text{supp}(\phi^*_{2,M_2^*}) = \{a\}$ and $\pi_2(a) = A$ solves

$$\max_{a_2 \in A_2} \sum_{m_1} \frac{\phi^*_1[m_1, a]}{\phi^*_{1,M_2^*}[a]} u_2(\pi_1(m_1), a_2) = \max_{a_2 \in A_2} u_2(B, a_2).$$

However, $\pi_1(b) \neq \pi_1(\alpha)$, hence, player 2 does not know the action profile at $m = (b, a)$ since $\beta_2 > 0 \Leftrightarrow \beta_1 < 1$.

Example 3 shows that both players may know that the other is rational at some state, and thus the action profile must be a Nash equilibrium at that state, but the players need not know the action profile. It also shows that players' rationality need not be commonly known: indeed, player 2 is irrational at $m_2 = b$, assigns strictly positive probability that player 1 is of type $m_1 = \alpha$, which in turn assigns strictly positive probability that player 2 is irrational.[12] The reason why we can dispense with the requirement of the mutual knowledge of the action profile from Aumann and Brandenburger's (1995) preliminary observation is that our notion of rationality is stronger than theirs since it requires not only that each player best-replies against his conjecture of his opponent's action but also that the messages that he sends with strictly positive probability are optimal for him in the sense of (4) and (5).

Finally, Example 3 shows that the assumption of mutual knowledge of rationality cannot be weakened to require only rationality: when $m = (\alpha, a) \in R_1 \times R_2$, both players are rational but $\pi(m) = (\alpha 1_A + (1-\alpha)1_B, A)$ is not a Nash equilibrium.

[12] More formally, common knowledge of rationality fails because $K^2(R_1 \times R_2) = \emptyset$ (see Aumann and Brandenburger (1995) for the definition of the knowledge operator $K$ and its iterates such as $K^2$).

# 6 Discussion

Many papers have considered whether correlated equilibrium payoffs can be sustained as the outcome of an extended-form game where players can take "cheap" pre-play actions. For 2-player games, we find that only a very restricted set of outcomes is achievable when each player has the ability to influence and manipulate the information structure in a general way. The distinguishing feature of our model is that we allow each player to choose any information structure he desires, and with some probability the information structure he chooses is the one that actually determines the joint distribution of the messages of all players. This section provides a discussion of these features and how they relate with alternative formalizations in the literature.

(a) *Aggregation of information designs.* Our specification that the information structure is a convex combination of those chosen by the players is, as pointed out, a simple way of obtaining that both players can manipulate it without imposing any restrictions on the kinds of manipulations that are allowed (in the sense that each player has the chance to choose whichever information structure he desires, and with positive probability this information structure is realized). A more general way of combining the two information designs is to postulate an abstract aggregation function $\alpha : S^2 \to S$ such that if player 1 chooses information structure $\phi_1 \in S$ and player 2 chooses information structure $\phi_2 \in S$, then the realized information structure is $\alpha(\phi_1, \phi_2) \in S$. Alternative formulations of information design in a setting without an explicit designer can then be obtained by specifying alternative aggregation functions $\alpha$.

One such alternative is for each player $i \in \{1, 2\}$ to choose $\phi_i \in S$ and then assume that each $i$ receives two messages $m_i^1$ and $m_i^2$, where $m^1 = (m_1^1, m_2^1)$ and $m^2 = (m_1^2, m_2^2)$ are independently drawn from $\phi_1$ and $\phi_2$ respectively. We note that this formulation can be embedded in our framework under an alternative aggregation function $\alpha$. Indeed, let $\psi : \mathbb{N}^2 \times \mathbb{N}^2 \to \mathbb{N}^2$ be a bijection and let $\alpha(\phi_1, \phi_2) = (\phi_1 \times \phi_2) \circ \psi^{-1}$. If we additionally impose the restriction that players are only allowed to choose $\phi_i$ such that $\phi_i[m_1^i, m_2^i] = 0$ whenever $m_1^i \neq m_2^i$, then this formulation captures exactly the model of Aumann and Hart (2003) when cheap talk is restricted to take place over a single period — we discuss this relationship further in the next subsection. According to this formulation, note that player $i$ has no ability to influence the distribution of $m^{-i}$. On the other hand, our specification of $\alpha$ does not restrict players from learning about or influencing any aspect of the information structure. To what extent do our results extend to a more general class of aggregation functions? We leave this question for future research.

(b) *Cheap talk.* In contrast to our results, the literature has found that specific communication protocols can greatly expand the set of equilibrium outcomes. For example, for 2-player games, Aumann and Hart's (2003) results imply that any payoff in the convex hull of the Nash equilibrium payoffs can be achieved as the outcome of an extended game where players talk for as long as they like before playing the game. In Aumann and Hart (2003), messages are common knowledge so there is no possibility of getting payoffs outside of the convex hull, but cheap talk is enough for players to reach any outcome achievable using publicly observed lotteries. On the other hand, in our model, there are privately observed lotteries but nevertheless players can only get payoffs in $\text{co}(u(N(G)))$ and not even all of those (even if we were to vary $\beta$).

To understand the relationship between our model and cheap talk, consider the version of Aumann and Hart's (2003) model specialised to complete information and assume that talk takes place over a single period only. This can be captured in our framework using the alternative aggregation function proposed in (a) above. For each $i \in \{1, 2\}$, let $m_i = (m_i^1, m_i^2)$ and restrict each player's choice of $\phi_i$ to distributions over $m^i = (m_1^i, m_2^i)$ such that $\phi_i[m_1^i, m_2^i] = 0$ whenever $m_1^i \neq m_2^i$. Recall that $m^1$ and $m^2$ are independently drawn from $\phi_1$ and $\phi_2$ respectively. Thus, the message $m_i^i$ just reveals to $i$ the message he sends to $j$, and $m^i$ can be identified with the message sent by player $i$ in Aumann and Hart (2003).

The key difference between our $\beta$ specification and the setting of Aumann and Hart (2003) is that each player in the latter is sure that his opponent receives the message he sends, he knows what this message is, and his opponent cannot do anything to influence this message. On the other hand, according to our $\beta$ specification, there is always a possibility that each player gets to determine the messages of both players. Even in the context of cheap talk where it seems natural that each player can say what he likes and knows what he says, one can imagine situations where players can take actions that influence the *meaning* of the words that are spoken. If we interpret the message received by each player as his understanding of the words spoken by his opponent, then players may be uncertain of the messages they send. Moreover, if player 2 benefits from player 1 sending some message $\bar{m}^1$, then player 2 may want to take certain (unmodelled) actions that increase the likelihood that player 1 will send message $\bar{m}^1$. In the broader context of information provision, such manipulations seem even more natural but are ruled out by aggregation functions where each player can only influence one dimension of the message profile. Our model with the $\beta$ aggregation function captures the idea that players can affect, to some degree, every dimension of the message profile.

(c) *Communication protocols.* Beyond Aumann and Hart (2003), the literature has focused on whether players can communicate in a more sophisticated manner to achieve correlated equilibrium payoffs. For instance, Ben-Porath (1998) shows that each correlated equilibrium can be approximated by the action distribution of a sequential equilibrium in a specific information design extensive-form game that includes the possibility of credibly revealing messages and (in the case of two players) ball and urns.[13] However, the specification of such extensive form games rules out the possibility of certain manipulations by assumption. In contrast, our results imply that when players can secretly manipulate the information structure in a general way, then in 2-player games, only specific convex combinations of Nash equilibria can be supported as sequential equilibrium outcomes of the information design extensive-form game.

For example, consider once again the chicken game from Section 2. In Ben-Porath (1998), the correlated equilibrium $\phi = \frac{1}{3}1_{(A,A)} + \frac{1}{3}1_{(A,B)} + \frac{1}{3}1_{(B,A)}$ is close to the

[13]Gerardi (2004) obtains a stronger result for games with at least five players. See also Urbano and Vila (2002) for 2-player games where players are boundedly rational.

action distribution of a sequential equilibrium of his information design extensive-form, which works as follows. Player 2 lets player 1 choose a ball from an urn $U_1$ and the ball which player 1 draws from $U_1$ determines an action $a_1$; the induced distribution of player 1's action is $\phi_{A_1} = \frac{2}{3}1_A + \frac{1}{3}1_B$. Player 1 then gives player 2 an urn $U_2(a_1)$ inducing on $A_2$ the distribution $\frac{1}{2}1_A + \frac{1}{2}1_B$ if $a_1 = A$ and $1_A$ if $a_1 = B$. After this has occurred, and without going into too much detail about Ben-Porath's (1998) information design extensive-form game, there is a sufficiently high probability that the contents of the urns $U_1$ and $U_2(a_1)$ are revealed as well as the ball that was chosen by player 1 from $U_1$. Our point is that there is a (unmodelled) possibility of manipulation by one player in this extensive-form. Specifically, player 1 can send urn $U_2(B)$ to player 2 when the ball he draws from $U_1$ indicates that he should play $A$; in addition, he could take all the balls from $U_1$ and put them all inside again, except one ball indicating that he should play $B$. In this way, he obtains a payoff of 7 instead of $4\frac{2}{3}$.

(d) *Manipulability.* Ben-Porath (1998) is motivated by the idea that a reliable mediator who is immune to manipulation by the players is not always available, and he asks whether it is possible to achieve correlated equilibrium payoffs without using a mediator. However, players may wish to manipulate the information structure regardless of whether it is the result of some procedure designed by the players themselves or if it comes from a mediator. As the previous example shows, if players are able to manipulate the communication protocol, then they will do so as well, i.e. Ben-Porath's (1998) results require that certain manipulations are ruled out by assumption. Our model is a reduced form attempt to capture the idea that players can manipulate the information structure however they wish by specifying that with probability $\beta_i$, player $i$ can choose any information structure he desires, but with probability $(1 - \beta_i)$, he can do nothing.

(e) *Privacy.* An alternative to our assumption that the information design choices are made privately is to assume that information design is public so that, for example, each player observes the information structure $\beta_1\phi_1 + \beta_2\phi_2$ chosen by nature. To model this, we can let, for each player $i$, $i$'s action be a function of the message that he receives, his own information design and the information structure $\beta_1\phi_1 + \beta_2\phi_2$, i.e. $\pi_i : M_i \times S \times S \to \Delta(\{A, B\})$. Under this assumption, the payoff $(4\frac{2}{3}, 4\frac{2}{3})$ can be achieved by specifying that $\phi_1 = \phi_2 = \phi^* = \frac{1}{3}1_{(A,B)} + \frac{1}{3}1_{(B,A)} + \frac{1}{3}1_{(A,A)}$, $\pi_i(m_i, \phi_i, \phi^*) = m_i$ for each $\phi_i \in S$ and $m_i \in \text{supp}(\phi^*_{M_i})$; and $\pi_i(m_i, \phi_i, \hat{\phi}) = \frac{1}{2}1_A + \frac{1}{2}1_B$ for each $\phi_i \in S$, $\hat{\phi} \neq \phi^*$ and $m_i \in \text{supp}(\hat{\phi}_{M_i})$. Intuitively, deviations from $\phi^*$ can be deterred by the threat of reverting to the mixed strategy Nash equilibrium whenever some alternative information structure is realized. The reason we assume that information design is private is because we are interested in how the ability to manipulate the information structure affects the outcomes of the game. When the choice of information is observed, certain information structures can be sustained by the threat of punishment. Our aim is instead to ask which outcomes can arise abstracting away from the possibility of such threats.

(f) *Information design literature.* Our formalization follows closely the one used in recent information design papers. Bergemann and Morris (2019) provide a unified perspective on them with a general framework which features payoff uncertainty and underlying private information. Our setting contains none of these features but, in contrast, allows for multiple designers and for the designer to be a player of the game played after the information has been designed. In particular, if in some papers in the information design literature, one shuts down payoff uncertainty, underlying private information or both, then one obtains a particular case of the setting of those papers which can also be represented as a particular case of our framework.[14]

A more interesting exercise consists in extending our framework to include payoff

[14] For instance, the setting in Crawford and Sobel (1982) without private information is obtained by specifying, e.g., $\beta_1 = 1$ so that only player 1 (the sender) designs information and that both players' payoff functions be independent of player 1's action so that only the action of player 2 (the receiver) matters. This particular case of our framework is also a special case of the setting in Kamenica and Gentzkow (2011), namely one in which there is no payoff uncertainty. Without payoff uncertainty, a similar conclusion holds in the case where there are multiple senders as in Gentzkow and Kamenica (2017), formalized in our setting as follows: Letting now player 1 denote the receiver, we would have $\text{supp}(\beta) = N \setminus \{1\}$ and $u_i(a_1, a_{-1}) = u_i(a_1, a'_{-1})$ for each $i \in N$, $a_1 \in A_1$ and $a_{-1}, a'_{-1} \in A_{-1}$.

uncertainty and underlying private information. To see what can be gained by such extension, consider the question of what combinations of consumer and producer surplus can arise in a monopolistic market where the buyer's valuation is random. If the buyer is fully informed and the seller observes a signal about the valuation, Bergemann, Brooks, and Morris (2015) characterize the outcomes that can arise under some information structure for the seller. On the other hand, if the buyer is not fully informed, then she must learn about her own valuation. Under the assumption that the buyer (but not the seller) observes a signal about the valuation, Roesler and Szentes (2017) characterize the outcomes that can arise under some information structure for the buyer; and in particular, they identify an information structure that is optimal for the buyer. However, the seller may also have incentives to learn about and/or influence the signals received by the buyer. If both the buyer and seller are able to manipulate the information structure, what will be the resulting information structure? We address this question in another paper.

(g) *Endogenous information in interactive belief systems.* It is not obvious how the notion of endogenous information should be defined in interactive belief systems and, as Aumann (1987, Section 4 (e)) pointed out, it is not easy to distinguish between endogenous and exogenous information. For Aumann's (1987) main result that common knowledge of rationality implies correlated equilibrium, this distinction does not matter; the point of this paper is that it matters for its converse. This has been established in the most transparent way for 2-player games via Theorem 3 and Corollaries 2 and 3, the latter in Appendix A.5, through a characterization of the sequential equilibrium payoffs of the information design extensive-form. These results build on the characterization of sequential equilibrium outcomes provided by Theorem 2 which reveal what properties must the messages that each player sends with strictly positive probability satisfy.

Our epistemic analysis takes place in an interim stage, after the information design has been made but before the messages have been received by the players. Our formalization of endogenous information then uses those properties of messages that each player sends with strictly positive probability as the defining feature of endogenous information. Very roughly, the idea is that the messages that define players' types in the interactive belief system are not arbitrary but rather designed in an optimal way by each player and, thus, must satisfy those properties whenever the player is rational. As with other aspects of the model, this definition of endogenous information should not be taken literally. We view it as a way to make the point that endogenous information matters for epistemic conditions; our goal is to make this point in the strongest possible (but still in a non-trivial) form. Now that this point has been made, one can think about more "realistic" ways of adding endogenous information to interactive belief systems.

# A Appendix

## A.1 Proofs

We start by noting the properties that sequential equilibrium imposes on the equilibrium outcome. Namely, for each sequential equilibrium $\pi \in \Pi$,

$$\sum_m \phi^*[m] u_i(\pi(m)) \geq \sum_m (\phi_i', \phi_{-i}^*)[m] u_i(\pi_i'(m_i, \phi_i'), \pi_{-i}(m_{-i})), \tag{6}$$

for each $i \in \mathrm{supp}(\beta)$, $\phi_i' \in S$ and $\pi_i' : M_i \times S \to \Delta(A_i)$, where $(\phi_i', \phi_{-i}^*) = \beta_i \phi_i' + \sum_{j \in \mathrm{supp}(\beta) \setminus \{i\}} \beta_j \phi_j^*$, and

$$\sum_{m_{-i}} \frac{\phi^*[m]}{\phi_{M_i}^*[m_i]} u_i(\pi(m)) \geq \sum_{m_{-i}} \frac{\phi^*[m]}{\phi_{M_i}^*[m_i]} u_i(a_i, \pi_{-i}(m_{-i})) \tag{7}$$

for each $i \in N$, $m_i \in \mathrm{supp}(\phi_{M_i}^*)$ and $a_i \in A_i$.

### A.1.1 Proof of Theorem 1

Let $\pi \in \Pi$ be a sequential equilibrium of $G_{id}$, $\sigma = \sigma_\pi$, $i \in N$ and $a_i^*, a_i' \in A_i$. We will show that $\sum_{a_{-i}} \sigma[a_i^*, a_{-i}]\big(u_i(a_i^*, a_{-i}) - u_i(a_i', a_{-i})\big) \geq 0$.

We may assume that $a_i^* \in \mathrm{supp}(\sigma_{A_i})$. Thus, there exists $m_i \in \mathrm{supp}(\phi_{M_i}^*)$ such that $a_i^* \in \mathrm{supp}(\pi_i(m_i, \phi_i^*))$. The optimality of $a_i^*$, i.e. (7), implies that

$$\sum_{m_{-i}, a_{-i}} \phi^*[m] \pi(m, \phi^*)[a_i^*, a_{-i}]\big(u_i(a_i^*, a_{-i}) - u_i(a_i', a_{-i})\big) \geq 0.$$

This inequality holds for each $m_i$ satisfying the property that $m_i \in \text{supp}(\phi^*_{M_i})$ and $a_i^* \in \text{supp}(\pi_i(m_i, \phi_i^*))$. It also holds for each $m_i$ such that this property does not hold. Thus,

$$\sum_{m,a_{-i}} \phi^*[m]\pi(m,\phi^*)[a_i^*, a_{-i}]\big(u_i(a_i^*, a_{-i}) - u_i(a_i', a_{-i})\big) \geq 0.$$

Since

$$\begin{aligned}&\sum_{m,a_{-i}} \phi^*[m]\pi(m,\phi^*)[a_i^*, a_{-i}]\big(u_i(a_i^*, a_{-i}) - u_i(a_i', a_{-i})\big) = \\ &\sum_{a_{-i}} \big(u_i(a_i^*, a_{-i}) - u_i(a_i', a_{-i})\big) \sum_{m} \phi^*[m]\pi(m,\phi^*)[a_i^*, a_{-i}] = \\ &\sum_{a_{-i}} \big(u_i(a_i^*, a_{-i}) - u_i(a_i', a_{-i})\big)\sigma[a_i^*, a_{-i}],\end{aligned}$$

it follows that $\sum_{a_{-i}} \sigma[a_i^*, a_{-i}]\big(u_i(a_i^*, a_{-i}) - u_i(a_i', a_{-i})\big) \geq 0$. Thus, $\sigma \in C(G)$.

### A.1.2 Proof of the necessity part of Theorem 2

In each sequential equilibrium of $G_{id}$, any player $i \in \text{supp}(\beta)$ must send optimal messages $m$ in the sense that they induce an action profile $\pi(m)$ that maximizes $i$'s payoff function. This is stated in Lemma 1 which is a preliminary result for condition (1).

**Lemma 1.** *If $G$ is an $n$-player game and $\pi$ is a sequential equilibrium of $G_{id}$, then* $\text{supp}(\phi_i^*) \subseteq \{m \in M : u_i(\pi(m)) = \sup_{m' \in M} u_i(\pi(m'))\}$ *for each* $i \in \text{supp}(\beta)$.

**Proof.** Suppose not; then there is $i \in \text{supp}(\beta)$, $m' \in \text{supp}(\phi_i^*)$ and $m^* \in M$ such that $u_i(\pi(m^*)) > u_i(\pi(m'))$. Define $\phi_i'$ by setting, for each $m \in \text{supp}(\phi_i^*)$,

$$\phi_i'[m] = \begin{cases} 0 & \text{if } m = m', \\ \phi_i^*[m^*] + \phi_i^*[m'] & \text{if } m = m^*, \\ \phi_i^*[m] & \text{otherwise,} \end{cases}$$

and let $\pi_i' : M_i \times S \to \Delta(A_i)$ be such that $\pi_i'(m_i, \phi_i') = \pi_i(m_i, \phi_i^*)$ for each $m_i \in M_i$.

Then

$$
\begin{aligned}
&\sum_m (\phi'_i, \phi^*_{-i})[m] u_i(\pi'_i(m_i, \phi'_i), \pi_{-i}(m_{-i})) - \sum_m \phi^*[m] u_i(\pi(m)) \\
&= \sum_m (\phi'_i, \phi^*_{-i})[m] u_i(\pi(m)) - \sum_m \phi^*[m] u_i(\pi(m)) \\
&= \sum_m \beta_i \left(\phi'_i[m] - \phi^*_i[m]\right) u_i(\pi(m)) \\
&= \beta_i \phi^*_i[m'] \Big( u_i(\pi(m^*)) - u_i(\pi(m')) \Big) > 0.
\end{aligned}
$$

But this is a contradiction to (6) since $\pi$ is a sequential equilibrium of $G_{id}$. ■

The conclusion of Lemma 1 can be strengthened: for a message $m$ to be optimal, $u_i(\pi(m))$ must achieve $\max_{m'_{-i}} v_i(\pi_{-i}(m'_{-i}))$ and, thus, $\pi_i(m_i)$ be a best-reply to $\pi_{-i}(m_{-i})$.

**Lemma 2.** *If $G$ is an $n$-player game and $\pi$ is a sequential equilibrium of $G_{id}$, then*

$$
\begin{aligned}
\operatorname{supp}(\phi^*_i) \subseteq \{m \in M : \quad & v_i(\pi_{-i}(m_{-i})) = \sup_{m'_{-i} \in M_{-i}} v_i(\pi_{-i}(m'_{-i})) \\
& \textit{and } \pi_i(m_i) \in BR_i(\pi_{-i}(m_{-i}))\}
\end{aligned}
$$

*for each $i \in \operatorname{supp}(\beta)$.*

**Proof.** Suppose not; then there is $i \in \operatorname{supp}(\beta)$, $m' \in \operatorname{supp}(\phi^*_i)$ and $m^* \in M$ such that (i) $v_i(\pi_{-i}(m^*_{-i})) > v_i(\pi_{-i}(m'_{-i}))$ or (ii) $v_i(\pi_{-i}(m'_{-i})) = \sup_{\hat{m}_{-i} \in M_{-i}} v_i(\pi_{-i}(\hat{m}_{-i}))$ and $\pi_i(m'_i) \notin BR_i(\pi_{-i}(m'_{-i}))$; in case (ii), let $m^* = m'$. Let $a^*_i \in BR_i(\pi_{-i}(m^*_{-i}))$, $\bar{m}_i \notin \operatorname{supp}(\phi^*_{M_i})$, $\phi'_i = 1_{(\bar{m}_i, m^*_{-i})}$ and $\pi'_i : M_i \times S \to \Delta(A_i)$ be such that $\pi'_i(\bar{m}_i, \phi'_i) = a^*_i$ and $\pi'_i(m_i, \phi'_i) = \pi_i(m_i, \phi^*_i)$ for each $m_i \neq \bar{m}_i$. Then

$$
\begin{aligned}
&\sum_m (\phi'_i, \phi^*_{-i})[m] u_i(\pi'_i(m_i, \phi'_i), \pi_{-i}(m_{-i})) - \sum_m \phi^*[m] u_i(\pi(m)) \\
&= \sum_m \beta_i \phi'_i[m] u_i(\pi'_i(m_i, \phi'_i), \pi_{-i}(m_{-i})) - \sum_m \beta_i \phi^*_i[m] u_i(\pi(m)) \\
&= \beta_i \Big( u_i(a^*_i, \pi_{-i}(m^*_{-i})) - \sum_{m \in \operatorname{supp}(\phi^*_i)} \phi^*_i[m] u_i(\pi(m)) \Big) \\
&= \beta_i \Big( v_i(\pi_{-i}(m^*_{-i})) - u_i(\pi(m')) \Big)
\end{aligned}
$$

because $u_i(\pi(m)) = u_i(\pi(m'))$ for each $m \in \operatorname{supp}(\phi^*_i)$ by Lemma 1 as $m' \in \operatorname{supp}(\phi^*_i)$. Thus, if $v_i(\pi_{-i}(m^*_{-i})) > v_i(\pi_{-i}(m'_{-i}))$, then $v_i(\pi_{-i}(m^*_{-i})) - u_i(\pi(m')) \geq v_i(\pi_{-i}(m^*_{-i})) -$

$v_i(\pi_{-i}(m'_{-i})) > 0$; if $v_i(\pi_{-i}(m^*_{-i})) = v_i(\pi_{-i}(m'_{-i}))$, then $\pi_i(m'_i) \notin BR_i(\pi_{-i}(m'_{-i}))$ and $v_i(\pi_{-i}(m^*_{-i})) - u_i(\pi(m')) > v_i(\pi_{-i}(m^*_{-i})) - v_i(\pi_{-i}(m'_{-i})) \geq 0$. In either case, it follows that $\sum_m (\phi'_i, \phi^*_{-i})[m] u_i(\pi'_i(m_i, \phi'_i), \pi_{-i}(m_{-i})) - \sum_m \phi^*[m] u_i(\pi(m)) > 0$. But this is a contradiction to (6) since $\pi$ is a sequential equilibrium. ■

Lemma 2 implies that $\pi_i(m_i)$ is a best-reply against $\pi_{-i}(m_{-i})$ whenever $m \in \text{supp}(\phi^*_i)$ and $i \in \text{supp}(\beta)$. We will now show that if, in addition,

$$m_i \in \cup_{j \in \text{supp}(\beta_{-i})} \text{supp}(\phi^*_{j,M_i}),$$

then $\pi_i(m_i)$ solves

$$\max_{\alpha_i \in \Delta(A_i)} \sum_{m_{-i}} \frac{\sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi^*_j [m_i, m_{-i}]}{\sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi^*_{j,M_i}[m_i]} u_i(\alpha_i, \pi_{-i}(m_{-i})).$$

Thus, whenever $m_i \in \text{supp}(\phi^*_i) \cap (\cup_{j \in \text{supp}(\beta_{-i})} \text{supp}(\phi^*_{j,M_i}))$, $\pi_i(m_i)$ solves player $i$'s expected payoff conditional on his information design $\phi^*_i$ being chosen and also conditional on it not being chosen. The reason for this is that player $i$ can always differentiate the messages he receives from himself from those that he receives from the other players: if $m \in \text{supp}(\phi^*_i)$ is such that $\pi_i(m_i)$ does not maximize $i$'s expected payoff conditional on his information design $\phi^*_i$ not being chosen, then player $i$ would gain by deviating from $\phi^*_i$ by simply sending a message $(\bar{m}_i, m_{-i})$ with probability one for some $\bar{m}_i \notin \text{supp}(\phi^*_{M_i})$. If he receives message $m_i$, then he can be sure that his information design has not been chosen and can choose a solution to that problem in response to $m_i$; if he receives message $\bar{m}_i$, then the can be sure that his information design has been chosen and choose $\pi_i(m_i)$, which is a best-reply against $m_{-i}$, in response to $\bar{m}_i$.

**Lemma 3.** *If $G$ is an $n$-player game and $\pi$ is a sequential equilibrium of $G_{id}$, then*

$$\text{supp}(\phi^*_i) \subseteq \Big\{ m \in M : m_i \notin \cup_{j \in \text{supp}(\beta_{-i})} \text{supp}(\phi^*_{j,M_i}) \text{ or } \pi_i(m_i) \text{ solves } \max_{\alpha_i \in \Delta(A_i)} \sum_{m_{-i}} \frac{\sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi^*_j [m_i, m_{-i}]}{\sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi^*_{j,M_i}[m_i]} u_i(\alpha_i, \pi_{-i}(m_{-i})) \Big\}$$

*for each $i \in \text{supp}(\beta)$.*

**Proof.** Suppose not; then there is $i \in \text{supp}(\beta)$ and $m' \in \text{supp}(\phi_i^*)$ such that $m_i' \in \cup_{j \in \text{supp}(\beta_{-i})} \text{supp}(\phi_{j,M_i}^*)$ and $\pi_i(m_i')$ does not solve

$$\max_{\alpha_i \in \Delta(A_i)} \sum_{m_{-i}} \frac{\sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_j^*[m_i', m_{-i}]}{\sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_{j,M_i}^*[m_i']} u_i(\alpha_i, \pi_{-i}(m_{-i})). \tag{8}$$

Let $a_i^*$ be a solution to problem (8), $\bar{m}_i \notin \text{supp}(\phi_{M_i}^*)$, $\phi_i' = 1_{(\bar{m}_i, m_{-i}')}$ and $\pi_i' : M_i \times S \to \Delta(A_i)$ be such that

$$\pi_i'(m_i, \phi_i') = \begin{cases} a_i^* & \text{if } m_i = m_i', \\ \pi_i(m_i') & \text{if } m_i = \bar{m}_i, \\ \pi_i(m_i) & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}
&\sum_m (\phi_i', \phi_{-i}^*)[m] u_i(\pi_i'(m_i, \phi_i'), \pi_{-i}(m_{-i})) - \sum_m \phi^*[m] u_i(\pi(m)) \\
&= \beta_i \Big( u_i(\pi(m')) - \sum_{m \in \text{supp}(\phi_i^*)} \phi_i^*[m] u_i(\pi(m)) \Big) \\
&+ \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \sum_{m_{-i}} \phi_j^*[m_i', m_{-i}] \Big( u_i(a_i^*, \pi_{-i}(m_{-i})) - u_i(\pi_i(m_i'), \pi_{-i}(m_{-i})) \Big) \\
&= \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \sum_{m_{-i}} \phi_j^*[m_i', m_{-i}] \Big( u_i(a_i^*, \pi_{-i}(m_{-i})) - u_i(\pi_i(m_i'), \pi_{-i}(m_{-i})) \Big)
\end{aligned}$$

where the last equality follows by Lemma 1 since $m' \in \text{supp}(\phi_i^*)$. Since $\pi_i(m_i')$ does not solve problem (8) but $a_i^*$ does, it follows that

$$\sum_{m_{-i}} \frac{\sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_j^*[m_i', m_{-i}]}{\sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_{j,M_i}^*[m_i']} \Big( u_i(a_i^*, \pi_{-i}(m_{-i})) - u_i(\pi_i(m_i'), \pi_{-i}(m_{-i})) \Big) > 0$$

and, since $m_i' \in \cup_{j \in \text{supp}(\beta_{-i})} \text{supp}(\phi_{j,M_i}^*)$,

$$\sum_{j \in \text{supp}(\beta_{-i})} \beta_j \sum_{m_{-i}} \phi_j^*[m_i', m_{-i}] \Big( u_i(a_i^*, \pi_{-i}(m_{-i})) - u_i(\pi_i(m_i'), \pi_{-i}(m_{-i})) \Big) > 0.$$

Hence, $\sum_m (\phi_i', \phi_{-i}^*)[m] u_i(\pi_i'(m_i, \phi_i'), \pi_{-i}(m_{-i})) - \sum_m \phi^*[m] u_i(\pi(m)) > 0$. But this is a contradiction to (6) since $\pi$ is a sequential equilibrium of $G_{id}$. ■

It follows by Lemmas 2 and 3 that, for each sequential equilibrium outcome, $i \in N$ and $m \in \text{supp}(\phi_i^*)$, condition (1) in Theorem 2 holds and $\pi_i(m_i)$ solves

$$\max_{\alpha_i \in \Delta(A_i)} \sum_{m_{-i}} \frac{\sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_j^*[m_i, m_{-i}]}{\sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_{j,M_i}^*[m_i]} u_i(\alpha_i, \pi_{-i}(m_{-i}))$$

whenever $m_i \in \cup_{j \in \text{supp}(\beta_{-i})} \text{supp}(\phi^*_{j,M_i})$ and, hence,

$$m_i \in \text{supp}(\phi^*_i) \cap (\cup_{j \in \text{supp}(\beta_{-i})} \text{supp}(\phi^*_{j,M_i})).$$

In fact, regarding (1), note that if $i \in \text{supp}(\beta)$ and $m \in \text{supp}(\phi^*_i)$, then $m_j \in \text{supp}(\phi^*_{M_j})$ for each $j \in N$ and, thus, $m \in M^*$. Hence,

$$v_i(\pi_{-i}(m_{-i})) \leq \max_{m'_{-i} \in M^*_{-i}} v_i(\pi_{-i}(m'_{-i})) \leq \sup_{m'_{-i} \in M_{-i}} v_i(\pi_{-i}(m'_{-i})) = v_i(\pi_{-i}(m_{-i})).$$

Condition (7) implies that, for each $i \in N$, $\pi_i(m_i)$ solves

$$\max_{\alpha_i \in \Delta(A_i)} \sum_{m_{-i}} \frac{\sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi^*_j[m_i, m_{-i}]}{\sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi^*_{j,M_i}[m_i]} u_i(\alpha_i, \pi_{-i}(m_{-i}))$$

whenever $m_i \in \cup_{j \in \text{supp}(\beta_{-i})} \text{supp}(\phi^*_{j,M_i}) \setminus \text{supp}(\phi^*_i)$. This, together with what has been shown in the previous paragraph, shows that condition (2) in Theorem 2 holds.

### A.1.3 Proof of the sufficiency part of Theorem 2

Let $\left((\phi^*_i)_{i \in \text{supp}(\beta)}, \left((\pi_i(m_i))_{m_i \in \text{supp}(\phi^*_{M_i})}\right)_{i \in N}\right)$ be such that conditions (1) and (2) in Theorem 2 hold; we will show that it is the outcome of a sequential equilibrium.

We will construct a sequential equilibrium $\pi$ with the desired outcome. To this end, consider $\{\pi^\alpha, p^\alpha\}_\alpha$ defined as follows: The index set consists of $\alpha = (k, F, \hat{F})$ such that $k \in \mathbb{N}$, $F$ is a finite subset of $\mathbb{N}$ and $\hat{F}$ is a finite subset of $S$; this set is partially ordered by defining $(k', F', \hat{F}') \geq (k, F, \hat{F})$ if $k' \geq k$, $F \subseteq F'$ and $\hat{F} \subseteq \hat{F}'$. If $X$ is a finite set, let $\upsilon_X \in \Delta(X)$ be uniform on $X$. For each $i \in N$, let

$$\bar{m}_i \in \begin{cases} \text{supp}(\phi^*_{i,M_i}) & \text{if } i \in \text{supp}(\beta), \\ \text{supp}(\phi^*_{M_i}) & \text{if } i \notin \text{supp}(\beta), \end{cases}$$

$$\bar{q}_i[m_{-i}] = \begin{cases} \frac{\phi^*_i[\bar{m}_i, m_{-i}]}{\phi^*_{i,M_i}[\bar{m}_i]} & \text{if } i \in \text{supp}(\beta) \\ \frac{\sum_{j \in \text{supp}(\beta)} \beta_j \phi^*_j[\bar{m}_i, m_{-i}]}{\sum_{j \in \text{supp}(\beta)} \beta_j \phi^*_{j,M_i}[\bar{m}_i]} & \text{if } i \notin \text{supp}(\beta), \end{cases}$$

for each $m_{-i} \in M_{-i}$, and for each $\alpha = (k, F, \hat{F})$, let

$$\begin{aligned}
\tau_i^\alpha &= \frac{\sum_{l \in F \cup \left(\cup_{\phi \in \hat{F}} \mathrm{supp}(\phi_{M_i})\right)} 2^{-l} 1_l}{\sum_{l \in F \cup \left(\cup_{\phi \in \hat{F}} \mathrm{supp}(\phi_{M_i})\right)} 2^{-l}}, \\
q_i^\alpha &= \tau_i^\alpha \times \bar{q}_i, \\
\tau^\alpha &= \prod_{j \in N} \tau_j^\alpha, \\
q^\alpha &= (n')^{-1} \sum_{j \in \mathrm{supp}(\beta)} q_j^\alpha, \\
\hat{q}^\alpha &= n^{-1} \sum_{j \in N} q_j^\alpha, \\
\mu^\alpha &= (1 - k^{-1} - k^{-2}) q^\alpha + k^{-1} \hat{q}^\alpha + k^{-2} \tau^\alpha, \text{ and} \\
p^\alpha(\phi) &= (1 - k^{-1}) \sum_{j \in \mathrm{supp}(\beta)} \beta_j \phi_j + k^{-1} \mu^\alpha.
\end{aligned}$$

For each $m_i \notin \mathrm{supp}(\phi^*_{M_i})$, set $\pi_i(m_i, \phi_i^*) = \pi_i(\bar{m}_i)$ if $i \in \mathrm{supp}(\beta)$ and $\pi_i(m_i) = \pi_i(\bar{m}_i)$ if $i \notin \mathrm{supp}(\beta)$; hence, $\pi_i(m_i)$ is defined for each $i \in N$ and $m_i \in M_i$.

For each $i \in \mathrm{supp}(\beta)$, $m_i \in M_i$ and $\phi_i \neq \phi_i^*$ such that

$$\beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \mathrm{supp}(\beta_{-i})} \beta_j \phi^*_{j,M_i}[m_i] = 0,$$

let $\pi_i(m_i, \phi_i) = \pi_i(\bar{m}_i)$.

For each $i \in \mathrm{supp}(\beta)$, $m_i \in M_i$ and $\phi_i \neq \phi_i^*$ such that

$$\beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \mathrm{supp}(\beta_{-i})} \beta_j \phi^*_{j,M_i}[m_i] > 0,$$

let $\pi_i(m_i, \phi_i)$ be a best-reply against

$$\sum_{m_{-i}} \frac{\beta_i \phi_i[m_i, m_{-i}] + \sum_{j \in \mathrm{supp}(\beta_{-i})} \beta_j \phi_j^*[m_i, m_{-i}]}{\beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \mathrm{supp}(\beta_{-i})} \beta_j \phi^*_{j,M_i}[m_i]} \pi_{-i}(m_{-i}).$$

We may assume that $\pi_i : M_i \times S \to \Delta(A_i)$ is measurable. Note first that $M_i \times S = \cup_{r=1}^3 B_r$ with

$$\begin{aligned}
B_1 &= \{(m_i, \phi_i) : \phi_i = \phi_i^*\}, \\
B_2 &= \{(m_i, \phi_i) : \phi_i \neq \phi_i^* \text{ and } \beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \mathrm{supp}(\beta_{-i})} \beta_j \phi^*_{j,M_i}[m_i] = 0\} \text{ and} \\
B_3 &= \{(m_i, \phi_i) : \phi_i \neq \phi_i^* \text{ and } \beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \mathrm{supp}(\beta_{-i})} \beta_j \phi^*_{j,M_i}[m_i] > 0\}.
\end{aligned}$$

For each $r \in \{1, 2, 3\}$, $B_r$ is measurable. Indeed, $B_1$ is closed, $B_2$ is the intersection of an open set, $\{(m_i, \phi_i) : \phi_i \neq \phi_i^*\}$, with a closed set, $\{(m_i, \phi_i) : \beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi^*_{j,M_i}[m_i] = 0\}$, and $B_3$ is open. Then, for each measurable $B \subseteq \Delta(A_i)$, $\pi_i^{-1}(B) \cap B_1$ is measurable since $\pi_i^{-1}(B) \cap B_1$ is countable. Regarding $\pi_i^{-1}(B) \cap B_3$: Let $f : M_i \times S \to \Delta(A_{-i})$ be defined by setting, for each $(m_i, \phi_i) \in B_3$, $f(m_i, \phi_i) = \sum_{m_{-i}} \frac{\beta_i \phi_i[m_i, m_{-i}] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi^*_j[m_i, m_{-i}]}{\beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi^*_{j,M_i}[m_i]} \pi_{-i}(m_{-i})$. Letting $BR_i : \Delta(A_{-i}) \rightrightarrows \Delta(A_i)$ be player $i$'s best-reply correspondence in $G$, define $\Psi : M_i \times S \rightrightarrows \Delta(A_i)$ by setting, for each $(m_i, \phi_i) \in B_3$, $\Psi(m_i, \phi_i) = BR_i(f(m_i, \phi_i))$. Since $\Delta(A_i)$ is compact, $f$ is continuous and $BR_i$ is upper hemicontinuous, it follows that $\Psi$ is upper hemicontinuous and, hence, measurable (and, thus, weakly measurable). Hence, $\Psi$ has a measurable selection by the Kuratowski-Ryll-Nardzewski Selection Theorem (e.g. Aliprantis and Border (2006, Theorem 18.13, p. 600)). Finally, for each measurable $B \subseteq \Delta(A_i)$, $\pi_i^{-1}(B) = B_2$ if $\pi_i(\bar{m}_i) \in B$ and $\pi_i^{-1}(B) = \emptyset$ otherwise; thus $\pi_i^{-1}(B) \cap B_2$ is measurable.

Furthermore, let

$$\pi_i^{1,\alpha} = (1 - k^{-3}) 1_{\phi_i^*} + k^{-3} \upsilon_{\hat{F}} \text{ and } \pi_i^{2,\alpha}(m_i, \phi_i) = (1 - k^{-1}) \pi_i(m_i, \phi_i) + k^{-1} \upsilon_{A_i}$$

if $i \in \text{supp}(\beta)$. For each $i \notin \text{supp}(\beta)$, let

$$\pi_i^{2,\alpha}(m_i) = (1 - k^{-1}) \pi_i(m_i) + k^{-1} \upsilon_{A_i}.$$

Let $\varepsilon > 0$. We have that the following conditions in the definition of perfect conditional $\varepsilon$-equilibrium hold by construction:

1. For each $\alpha$, $\pi^\alpha$ is a strategy and $p^\alpha : S^{n'} \to \Delta(M)$ is measurable,

2. For each $i \in \text{supp}(\beta)$, $\sup_{B \in \mathcal{B}(S)} |\pi_i^{1,\alpha}[B] - 1_{\phi_i^*}[B]| \to 0$ and
$$\sup_{(m_i, \phi_i) \in M_i \times S, a_i \in A_i} |\pi_i^{2,\alpha}(m_i, \phi_i)[a_i] - \pi_i(m_i, \phi_i)[a_i]| \to 0, \text{[15]}$$

3. For each $i \in \text{supp}(\beta)$, $m_i \in M_i$, $\phi_i \in S$ and $a_i \in A_i$, there is $\bar{\alpha}$ such that $\pi_i^{1,\alpha}[\phi_i] > 0$ and $\pi_i^{2,\alpha}(m_i, \phi_i)[a_i] > 0$ for each $\alpha \geq \bar{\alpha}$,

[15] We let $\mathcal{B}(S)$ denote the class of Borel measurable subsets of $S$ and, for each $\phi \in S$, $1_\phi$ denote the probability measure on $S$ degenerate at $\phi$.

4. For each $i \in N \setminus \text{supp}(\beta)$, $\sup_{m_i \in M_i, a_i \in A_i} |\pi_i^{2,\alpha}(m_i)[a_i] - \pi_i(m_i)[a_i]| \to 0$,

5. For each $i \in N \setminus \text{supp}(\beta)$, $m_i \in M_i$ and $a_i \in A_i$, there is $\bar{\alpha}$ such that $\pi_i^{2,\alpha}(m_i)[a_i] > 0$ for each $\alpha \geq \bar{\alpha}$,

6. $\sup_{\phi \in S^{n'}, B \subseteq M} |p^\alpha(\phi)[B] - \sum_{i \in \text{supp}(\beta)} \beta_i \phi_i[B]| \to 0$, and

7. For each $\phi \in S^{n'}$ and $m \in M$, there is $\bar{\alpha}$ such that $p^\alpha(\phi)[m] > 0$ for each $\alpha \geq \bar{\alpha}$.

Note also that, for each $\alpha$, $\text{supp}(\pi^{1,\alpha})$ and $\text{supp}(p^\alpha)$ are finite. We define

$$S_i(F, \hat{F}) = \Big(\Big(F \cup (\cup_{\phi \in \hat{F}} \text{supp}(\phi_{M_i})) \cup (\cup_{j \in \text{supp}(\beta_{-i})} \text{supp}(\phi^*_{j,M_i}))\Big) \times \hat{F}\Big)$$
$$\cup \Big(\Big(F \cup (\cup_{\phi \in \hat{F}} \text{supp}(\phi_{M_i})) \cup (\cup_{j \in \text{supp}(\beta)} \text{supp}(\phi^*_{j,M_i}))\Big) \times \{\phi_i^*\}\Big)$$

for each $i \in \text{supp}(\beta)$ and

$$S_i(F, \hat{F}) = F \cup (\cup_{\phi \in \hat{F}} \text{supp}(\phi_{M_i})) \cup (\cup_{j \in \text{supp}(\beta)} \text{supp}(\phi^*_{j,M_i}))$$

for each $i \in N \setminus \text{supp}(\beta)$. If $(m, \phi) \in \mathbb{N}^n \times S^{n'}$ is such that $\pi^{1,\alpha}[\phi] > 0$ and $\sum_{\phi' \in \text{supp}(\pi^{1,\alpha})} p^\alpha(\phi')[m] > 0$, then $(m_i, \phi_i) \in S_i(F, \hat{F})$ for each $i \in \text{supp}(\beta)$ and $m_i \in S_i(F, \hat{F})$ for each $i \in N \setminus \text{supp}(\beta)$.

Thus, to show that $\pi$ is a perfect conditional $\varepsilon$-equilibrium, it remains to show that

8. for each $\alpha$,

   (a) For each $i \in \text{supp}(\beta)$ and $\phi'_i \in S$,

   $$\sum_{\phi \in \text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] \left(\sum_m p^\alpha(\phi)[m] u_i(\pi^{2,\alpha}(m, \phi))\right) \geq$$
   $$\sum_{\phi \in \text{supp}(1_{\phi'_i} \times \pi^{1,\alpha}_{-i})} (1_{\phi'_i} \times \pi^{1,\alpha}_{-i})[\phi] \left(\sum_m p^\alpha(\phi)[m] u_i(\pi^{2,\alpha}(m, \phi))\right) - \varepsilon,$$

   where $\pi^{1,\alpha} = \prod_{i \in \text{supp}(\beta)} \pi_i^{1,\alpha}$ and $1_{\phi'_i} \times \pi^{1,\alpha}_{-i} = 1_{\phi'_i} \times \prod_{j \in \text{supp}(\beta) \setminus \{i\}} \pi_j^{1,\alpha}$,

   (b) For each $i \in \text{supp}(\beta)$, $(m_i, \phi_i) \in M_i \times S$ such that

   $$\pi_i^{1,\alpha}[\phi_i] \sum_{\phi_{-i} \in \text{supp}(\pi^{1,\alpha}_{-i})} \pi^{1,\alpha}_{-i}[\phi_{-i}] p^\alpha_{M_i}(\phi_i, \phi_{-i})[m_i] > 0$$

and $a_i \in A_i$,

$$\frac{\sum_{\phi_{-i}\in \mathrm{supp}(\pi^{1,\alpha}_{-i})} \pi^{1,\alpha}_{-i}[\phi_{-i}] \left(\sum_{m_{-i}} p^\alpha(\phi_i,\phi_{-i})[m_i,m_{-i}]u_i(\pi^{2,\alpha}(m,\phi))\right)}{\sum_{\phi_{-i}\in \mathrm{supp}(\pi^{1,\alpha}_{-i})} \pi^{1,\alpha}_{-i}[\phi_{-i}]p^\alpha_{M_i}(\phi_i,\phi_{-i})[m_i]} \geq$$
$$\frac{\sum_{\phi_{-i}\in \mathrm{supp}(\pi^{1,\alpha}_{-i})} \pi^{1,\alpha}_{-i}[\phi_{-i}] \left(\sum_{m_{-i}} p^\alpha(\phi_i,\phi_{-i})[m_i,m_{-i}]u_i(a_i,\pi^{2,\alpha}_{-i}(m_{-i},\phi_{-i}))\right)}{\sum_{\phi_{-i}\in \mathrm{supp}(\pi^{1,\alpha}_{-i})} \pi^{1,\alpha}_{-i}[\phi_{-i}]p^\alpha_{M_i}(\phi_i,\phi_{-i})[m_i]}$$
$$-\varepsilon,$$

(c) For each $i \in N \setminus \mathrm{supp}(\beta)$, $m_i \in M_i$ such that

$$\sum_{\phi\in \mathrm{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi]p^\alpha_{M_i}(\phi)[m_i] > 0$$

and $a_i \in A_i$,

$$\frac{\sum_{\phi\in \mathrm{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] \left(\sum_{m_{-i}} p^\alpha(\phi)[m_i,m_{-i}]u_i(\pi^{2,\alpha}(m,\phi))\right)}{\sum_{\phi\in \mathrm{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi]p^\alpha_{M_i}(\phi)[m_i]} \geq$$
$$\frac{\sum_{\phi\in \mathrm{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] \left(\sum_{m_{-i}} p^\alpha(\phi)[m_i,m_{-i}]u_i(a_i,\pi^{2,\alpha}_{-i}(m_{-i},\phi_{-i}))\right)}{\sum_{\phi\in \mathrm{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi]p^\alpha_{M_i}(\phi)[m_i]} - \varepsilon.$$

We will show that condition 8 holds for some subnet of $\{\pi^\alpha, p^\alpha\}_\alpha$. Recall that $\alpha = (k, F, \hat{F})$. In what follows, we will often fix $F$ and $\hat{F}$ and take limits as $k \to \infty$.

Regarding condition 8 (a), let $i \in \mathrm{supp}(\beta)$ and $\phi'_i \in S$. We have that, for each finite subsets $F$ and $\hat{F}$ of $\mathbb{N}$ and $S$, respectively,

$$\lim_k \sum_{\phi\in \mathrm{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] \left(\sum_m p^\alpha(\phi)[m]u_i(\pi^{2,\alpha}(m,\phi))\right) = \sum_m \phi^*[m]u_i(\pi(m))$$

and that

$$\lim_k \sum_{\phi\in \mathrm{supp}(1_{\phi'_i}\times\pi^{1,\alpha}_{-i})} (1_{\phi'_i}\times\pi^{1,\alpha}_{-i})[\phi] \left(\sum_m p^\alpha(\phi)[m]u_i(\pi^{2,\alpha}(m,\phi))\right) =$$
$$\sum_m (\phi'_i,\phi^*_{-i})[m]u_i(\pi_i(m_i,\phi'_i),\pi_{-i}(m_{-i})).$$

Hence, by considering $\alpha$ such that $k \geq k_0$ for some $k_0 \in \mathbb{N}$, it is enough to show that

$$\sum_m \phi^*[m]u_i(\pi(m)) \geq \sum_m (\phi'_i,\phi^*_{-i})[m]u_i(\pi_i(m_i,\phi'_i),\pi_{-i}(m_{-i})),$$

which is equivalent to

$$\sum_m \phi_i^*[m]u_i(\pi(m)) \geq \sum_m \phi_i'[m]u_i(\pi_i(m_i,\phi_i'),\pi_{-i}(m_{-i})). \tag{9}$$

For each $j \in N$ and $m_j \in M_j$, $\pi_j(m_j) \in \{\pi_j(m_j') : m_j' \in \text{supp}(\phi^*_{M_j})\}$ since $\pi_j(m_j) = \pi_j(\bar{m}_j)$ whenever $m_j \notin \text{supp}(\phi^*_{M_j})$. Thus, by (1),

$$\sum_m \phi_i'[m]u_i(\pi_i(m_i,\phi_i'),\pi_{-i}(m_{-i})) \leq \sum_m \phi_i'[m]v_i(\pi_{-i}(m_{-i}))$$
$$\leq \max_{m_{-i}\in M^*_{-i}} v_i(\pi_{-i}(m_{-i})) = \sum_m \phi_i^*[m]u_i(\pi(m))$$

and, hence, (9) holds. It then follows that condition 8 (a) also holds.

Consider condition 8 (b) and (c). For each $i \in \text{supp}(\beta)$, finite subset $F$ of $\mathbb{N}$, finite subset $\hat{F}$ of $S$, $(m_i,\phi_i) \in S_i(F,\hat{F})$ and $\gamma_i \in \Delta(A_i)$, we have that

$$\lim_k \frac{\sum_{\phi_{-i}\in\text{supp}(\pi^{1,\alpha}_{-i})} \pi^{1,\alpha}_{-i}[\phi_{-i}]\left(\sum_{m_{-i}} p^\alpha(\phi_i,\phi_{-i})[m_i,m_{-i}]u_i(\gamma_i,\pi^{2,\alpha}_{-i}(m_{-i},\phi_{-i}))\right)}{\sum_{\phi_{-i}\in\text{supp}(\pi^{1,\alpha}_{-i})} \pi^{1,\alpha}_{-i}[\phi_{-i}]p^\alpha_{M_i}(\phi_i,\phi_{-i})[m_i]}$$
$$= \sum_{m_{-i}} \frac{\phi_i^*[\bar{m}_i,m_{-i}]}{\phi^*_{i,M_i}[\bar{m}_i]}u_i(\gamma_i,\pi_{-i}(m_{-i}))$$

if $\beta_i\phi_{i,M_i}[m_i] + \sum_{j\in\text{supp}(\beta_{-i})}\beta_j\phi^*_{j,M_i}[m_i] = 0$, and

$$\lim_k \frac{\sum_{\phi_{-i}\in\text{supp}(\pi^{1,\alpha}_{-i})} \pi^{1,\alpha}_{-i}[\phi_{-i}]\left(\sum_{m_{-i}} p^\alpha(\phi_i,\phi_{-i})[m_i,m_{-i}]u_i(\gamma_i,\pi^{2,\alpha}_{-i}(m_{-i},\phi_{-i}))\right)}{\sum_{\phi_{-i}\in\text{supp}(\pi^{1,\alpha}_{-i})} \pi^{1,\alpha}_{-i}[\phi_{-i}]p^\alpha_{M_i}(\phi_i,\phi_{-i})[m_i]} =$$
$$\sum_{m_{-i}} \frac{\beta_i\phi_i[m_i,m_{-i}] + \sum_{j\in\text{supp}(\beta_{-i})}\beta_j\phi_j^*[m_i,m_{-i}]}{\beta_i\phi_{i,M_i}[m_i] + \sum_{j\in\text{supp}(\beta_{-i})}\beta_j\phi^*_{j,M_i}[m_i]}u_i(\gamma_i,\pi_{-i}(m_{-i}))$$

if $\beta_i\phi_{i,M_i}[m_i] + \sum_{j\in\text{supp}(\beta_{-i})}\beta_j\phi^*_{j,M_i}[m_i] > 0$. The latter case is clear since all terms in the denominator of the fraction converge to zero except the one that converges to $\beta_i\phi_{i,M_i}[m_i] + \sum_{j\in\text{supp}(\beta_{-i})}\beta_j\phi^*_{j,M_i}[m_i]$ and similarly regarding the numerator.

In the former case, both the numerator and the denominator converge to zero since $\beta_i\phi_{i,M_i}[m_i] + \sum_{j\in\text{supp}(\beta_{-i})}\beta_j\phi^*_{j,M_i}[m_i] = 0$. Multiplying each by $k$, it follows that all terms converge to zero except the ones corresponding to the case where $\pi^{1,\alpha}_j = \phi^*_j$ for each $j \neq i$ and $p^\alpha(\phi_i,\phi^*_{-i}) = q^\alpha$. Furthermore, for each $m_{-i} \in M_{-i}$,

$$q^\alpha[m_i,m_{-i}] = (n')^{-1}(q_i^\alpha[m_i,m_{-i}] + \sum_{j\in\text{supp}(\beta)\setminus\{i\}} q_j^\alpha[m_i,m_{-i}]),$$
$$q_i^\alpha[m_i,m_{-i}] = \tau_i^\alpha[m_i]\bar{q}_i[m_{-i}] \text{ and}$$
$$q_j^\alpha[m_i,m_{-i}] = 0 \text{ for each } j \in \text{supp}(\beta)\setminus\{i\},$$

the latter since $m_i \notin \text{supp}(\phi^*_{j,M_i})$. Hence, $q^\alpha[m_i, m_{-i}] = (n')^{-1}\tau_i^\alpha[m_i]\bar{q}_i[m_{-i}]$ and $q^\alpha_{M_i}[m_i] = (n')^{-1}\tau_i^\alpha[m_i]$. Thus,

$$\frac{q^\alpha[m_i, m_{-i}]}{q^\alpha_{M_i}[m_i]} = \bar{q}_i[m_{-i}] = \frac{\phi_i^*[\bar{m}_i, m_{-i}]}{\phi^*_{i,M_i}[\bar{m}_i]}.$$

Similarly, for each $i \notin \text{supp}(\beta)$, finite subset $F$ of $\mathbb{N}$, finite subset $\hat{F}$ of $S$, $m_i \in S_i(F, \hat{F})$ and $\gamma_i \in \Delta(A_i)$, we have that

$$\lim_k \frac{\sum_{\phi \in \text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] \left(\sum_{m_{-i}} p^\alpha(\phi)[m_i, m_{-i}] u_i(\gamma_i, \pi^{2,\alpha}_{-i}(m_{-i}, \phi_{-i}))\right)}{\sum_{\phi \in \text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] p^\alpha_{M_i}(\phi)[m_i]} =$$
$$\sum_{m_{-i}} \frac{\sum_{j \in \text{supp}(\beta)} \beta_j \phi_j^*[\bar{m}_i, m_{-i}]}{\sum_{j \in \text{supp}(\beta)} \beta_j \phi^*_{j,M_i}[\bar{m}_i]} u_i(\gamma_i, \pi_{-i}(m_{-i}))$$

if $\sum_{j \in \text{supp}(\beta)} \beta_j \phi^*_{j,M_i}[m_i] = 0$, and

$$\lim_k \frac{\sum_{\phi \in \text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] \left(\sum_{m_{-i}} p^\alpha(\phi)[m_i, m_{-i}] u_i(\gamma_i, \pi^{2,\alpha}_{-i}(m_{-i}, \phi_{-i}))\right)}{\sum_{\phi \in \text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] p^\alpha_{M_i}(\phi)[m_i]} =$$
$$\sum_{m_{-i}} \frac{\sum_{j \in \text{supp}(\beta)} \beta_j \phi_j^*[m_i, m_{-i}]}{\sum_{j \in \text{supp}(\beta)} \beta_j \phi^*_{j,M_i}[m_i]} u_i(\gamma_i, \pi_{-i}(m_{-i}))$$

if $\sum_{j \in \text{supp}(\beta)} \beta_j \phi^*_{j,M_i}[m_i] > 0$. The latter case is as in the case $i \in \text{supp}(\beta)$. In the former case, both the numerator and the denominator converge to zero since $\sum_{j \in \text{supp}(\beta)} \beta_j \phi^*_{j,M_i}[m_i] = 0$; furthermore, $q^\alpha_{M_i}[m_i] = 0$ for the same reason. Multiplying each by $k^2$, it follows that all terms converge to zero except the ones corresponding to the case where $\pi_j^{1,\alpha} = \phi_j^*$ for each $j \neq i$ and $p^\alpha(\phi_i, \phi^*_{-i}) = \hat{q}^\alpha$. Furthermore, for each $m_{-i} \in M_{-i}$,

$$\hat{q}^\alpha[m_i, m_{-i}] = n^{-1}(q_i^\alpha[m_i, m_{-i}] + \sum_{j \in N} q_j^\alpha[m_i, m_{-i}]),$$
$$q_i^\alpha[m_i, m_{-i}] = \tau_i^\alpha[m_i]\bar{q}_i[m_{-i}] \text{ and}$$
$$q_j^\alpha[m_i, m_{-i}] = 0 \text{ for each } j \neq i,$$

the latter since $m_i \notin \text{supp}(\phi^*_{M_i})$. Thus,

$$\frac{\hat{q}^\alpha[m_i, m_{-i}]}{\hat{q}^\alpha_{M_i}[m_i]} = \bar{q}_i[m_{-i}] = \frac{\sum_{j \in \text{supp}(\beta)} \beta_j \phi_j^*[\bar{m}_i, m_{-i}]}{\sum_{j \in \text{supp}(\beta)} \beta_j \phi^*_{j,M_i}[\bar{m}_i]}.$$

We will next show that $\pi_i(m_i, \phi_i)$ solves

$$\max_{\gamma_i \in \Delta(A_i)} \lim_k \frac{\sum_{\phi_{-i} \in \text{supp}(\pi_{-i}^{1,\alpha})} \pi_{-i}^{1,\alpha}[\phi_{-i}] \left( \sum_{m_{-i}} p^\alpha(\phi_i, \phi_{-i})[m_i, m_{-i}] u_i(\gamma_i, \pi_{-i}^{2,\alpha}(m_{-i}, \phi_{-i})) \right)}{\sum_{\phi_{-i} \in \text{supp}(\pi_{-i}^{1,\alpha})} \pi_{-i}^{1,\alpha}[\phi_{-i}] p_{M_i}^\alpha(\phi_i, \phi_{-i})[m_i]} \tag{10}$$

for each $i \in \text{supp}(\beta)$, $m_i \in M_i$, $\phi_i \in S$, and $\pi_i(m_i)$ solves

$$\max_{\gamma_i \in \Delta(A_i)} \lim_k \frac{\sum_{\phi \in \text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] \left( \sum_{m_{-i}} p^\alpha(\phi)[m_i, m_{-i}] u_i(\gamma_i, \pi_{-i}^{2,\alpha}(m_{-i}, \phi_{-i})) \right)}{\sum_{\phi \in \text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] p_{M_i}^\alpha(\phi)[m_i]} \tag{11}$$

for each $i \notin \text{supp}(\beta)$ and $m_i \in M_i$.

We first establish (10). If $\beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_{j,M_i}^*[m_i] = 0$, then

$$\begin{aligned}
&\lim_k \frac{\sum_{\phi_{-i} \in \text{supp}(\pi_{-i}^{1,\alpha})} \pi_{-i}^{1,\alpha}[\phi_{-i}] \left( \sum_{m_{-i}} p^\alpha(\phi_i, \phi_{-i})[m_i, m_{-i}] u_i(\gamma_i, \pi_{-i}^{2,\alpha}(m_{-i}, \phi_{-i})) \right)}{\sum_{\phi_{-i} \in \text{supp}(\pi_{-i}^{1,\alpha})} \pi_{-i}^{1,\alpha}[\phi_{-i}] p_{M_i}^\alpha(\phi_i, \phi_{-i})[m_i]} \\
&= \sum_{m_{-i}} \frac{\phi_i^*[\bar{m}_i, m_{-i}]}{\phi_{i,M_i}^*[\bar{m}_i]} u_i(\gamma_i, \pi_{-i}(m_{-i})).
\end{aligned}$$

Since $\pi_i(m_i, \phi_i) = \pi_i(\bar{m}_i)$ and $\pi_i(\bar{m}_i) \in BR_i(\pi_{-i}(m_{-i}))$ for each $m_{-i} \in M_{-i}$ such that $(\bar{m}_i, m_{-i}) \in \text{supp}(\phi_i^*)$ by (1), it follows that (10) holds in this case.

If $\beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_{j,M_i}^*[m_i] > 0$ and $\phi_i \neq \phi_i^*$, then

$$\begin{aligned}
&\lim_k \frac{\sum_{\phi_{-i} \in \text{supp}(\pi_{-i}^{1,\alpha})} \pi_{-i}^{1,\alpha}[\phi_{-i}] \left( \sum_{m_{-i}} p^\alpha(\phi_i, \phi_{-i})[m_i, m_{-i}] u_i(\gamma_i, \pi_{-i}^{2,\alpha}(m_{-i}, \phi_{-i})) \right)}{\sum_{\phi_{-i} \in \text{supp}(\pi_{-i}^{1,\alpha})} \pi_{-i}^{1,\alpha}[\phi_{-i}] p_{M_i}^\alpha(\phi_i, \phi_{-i})[m_i]} \\
&= \sum_{m_{-i}} \frac{\beta_i \phi_i[m_i, m_{-i}] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_j^*[m_i, m_{-i}]}{\beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_{j,M_i}^*[m_i]} u_i(\gamma_i, \pi_{-i}(m_{-i})) \\
&= u_i \left( \gamma_i, \sum_{m_{-i}} \frac{\beta_i \phi_i[m_i, m_{-i}] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_j^*[m_i, m_{-i}]}{\beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_{j,M_i}^*[m_i]} \pi_{-i}(m_{-i}) \right).
\end{aligned}$$

Since $\pi_i(m_i, \phi_i)$ is optimal against $\sum_{m_{-i}} \frac{\beta_i \phi_i[m_i, m_{-i}] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_j^*[m_i, m_{-i}]}{\beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_{j,M_i}^*[m_i]} \pi_{-i}(m_{-i})$, it follows that (10) holds in this case.

Finally, consider the case where $\phi_i = \phi_i^*$ and

$$\beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_{j,M_i}^*[m_i] > 0.$$

Note that it is enough to show that

$$\sum_{m_{-i}} \phi^*[m]\big(u_i(\pi(m)) - u_i(a_i, \pi_{-i}(m_{-i}))\big) \geq 0 \tag{12}$$

for each $a_i \in A_i$ and that

$$\begin{aligned}&\sum_{m_{-i}} \phi^*[m]\big(u_i(\pi(m)) - u_i(a_i, \pi_{-i}(m_{-i}))\big)\\ &= \sum_{m_{-i}} \beta_i\phi_i^*[m]\big(u_i(\pi(m)) - u_i(a_i, \pi_{-i}(m_{-i}))\big)\\ &+ \sum_{m_{-i}} \sum_{j\in \text{supp}(\beta_{-i})} \beta_j\phi_j^*[m]\big(u_i(\pi(m)) - u_i(a_i, \pi_{-i}(m_{-i}))\big).\end{aligned}$$

We have that $u_i(\pi(m)) \geq u_i(a_i, \pi_{-i}(m_{-i}))$ for each $m_{-i}$ such that $\phi_i^*[m] > 0$ by (1); moreover, for each $m_{-i}$ such that $\phi_j^*[m] > 0$ for some $j \in \text{supp}(\beta_{-i})$, then

$$m_i \in \cup_{j\in\text{supp}(\beta_{-i})}\text{supp}(\phi^*_{j,M_i})$$

and, hence, $\sum_{m_{-i}} \sum_{j\in\text{supp}(\beta_{-i})} \beta_j\phi_j^*[m]\big(u_i(\pi(m)) - u_i(a_i, \pi_{-i}(m_{-i}))\big) \geq 0$ by (2). Thus, (12) holds and so does (10).

We next establish (11). If $\sum_{j\in\text{supp}(\beta)} \beta_j\phi^*_{j,M_i}[m_i] = 0$, then it follows that

$$\begin{aligned}&\lim_k \frac{\sum_{\phi\in\text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi]\left(\sum_{m_{-i}} p^\alpha(\phi)[m_i, m_{-i}]u_i(a_i, \pi^{2,\alpha}_{-i}(m_{-i}, \phi_{-i}))\right)}{\sum_{\phi\in\text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi]p^\alpha_{M_i}(\phi)[m_i]}\\ &= \sum_{m_{-i}} \frac{\sum_{j\in\text{supp}(\beta)} \beta_j\phi_j^*[\bar{m}_i, m_{-i}]}{\sum_{j\in\text{supp}(\beta)} \beta_j\phi^*_{j,M_i}[\bar{m}_i]} u_i(a_i, \pi_{-i}(m_{-i})).\end{aligned}$$

Since $\pi_i(m_i) = \pi_i(\bar{m}_i)$, it follows by (2) that (11) holds in this case.

If $\sum_{j\in\text{supp}(\beta)} \beta_j\phi^*_{j,M_i}[m_i] > 0$, then it is enough to establish (12). For each $a_i \in A_i$, we have that

$$\begin{aligned}&\sum_{m_{-i}} \phi^*[m]\big(u_i(\pi(m)) - u_i(a_i, \pi_{-i}(m_{-i}))\big)\\ &= \sum_{m_{-i}} \sum_{j\in\text{supp}(\beta)} \beta_j\phi_j^*[m]\big(u_i(\pi(m)) - u_i(a_i, \pi_{-i}(m_{-i}))\big) \geq 0\end{aligned}$$

by (2). Thus, (12) holds and so does (11).

The above arguments show that, for each finite subsets $F$ of $\mathbb{N}$ and $\hat{F}$ of $S$, condition 8 holds whenever $k$ is sufficiently high. Specifically, condition 8 (a) holds for each $i \in N$ whenever $k \geq k_0$. For each $i \in \text{supp}(\beta)$ and $(m_i, \phi_i) \in S_i(F, \hat{F})$, there is $k(m_i, \phi_i)$ such that condition 8 (b) holds whenever $k \geq k(m_i, \phi_i)$. For each $i \in N \setminus \text{supp}(\beta)$ and $m_i \in S_i(F, \hat{F})$, there is $k(m_i)$ such that condition 8 (c) holds whenever $k \geq k(m_i)$. Thus, let

$$k(F, \hat{F}) = \max \left\{ k_0, \max_{i \in \text{supp}(\beta)} \max_{(m_i, \phi_i) \in S_i(F, \hat{F})} k(m_i, \phi_i), \max_{i \in N \setminus \text{supp}(\beta)} \max_{m_i \in S_i(F, \hat{F})} k(m_i) \right\}.$$

Since condition 8 (b) is trivially satisfied when

$$\pi_i^{1,\alpha}[\phi_i] \sum_{\phi_{-i} \in \text{supp}(\pi_{-i}^{1,\alpha})} \pi_{-i}^{1,\alpha}[\phi_{-i}] p_{M_i}^{\alpha}(\phi_i, \phi_{-i})[m_i] = 0,$$

i.e. when $i \in \text{supp}(\beta)$ and $(m_i, \phi_i) \notin S_i(F, \hat{F})$, and that condition 8 (c) is trivially satisfied when $\sum_{\phi \in \text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] p_{M_i}^{\alpha}(\phi)[m_i] = 0$, i.e. when $i \in N \setminus \text{supp}(\beta)$ and $m_i \notin S_i(F, \hat{F})$, it follows that condition 8 holds whenever $k \geq k(F, \hat{F})$. This allows us to define the following subnet $\{\pi^{\varphi(\eta)}, p^{\varphi(\eta)}\}_\eta$ of $\{\pi^\alpha, p^\alpha\}_\alpha$ such that condition 8 holds.

The index set of the subnet $\{\pi^{\varphi(\eta)}, p^{\varphi(\eta)}\}_\eta$ is the same as the one in the net $\{\pi^\alpha, p^\alpha\}_\alpha$. The function $\varphi : \eta \mapsto \alpha$ is defined by setting, for each $\eta = (k, F, \hat{F})$,

$$\varphi(\eta) = \left( \max \left\{ k, k(F, \hat{F}) \right\}, F, \hat{F} \right).$$

It is then clear that condition 8 holds and that, as required by the definition of a subnet, for each $\alpha_0$, there exists $\eta_0$, e.g. $\eta_0 = \alpha_0$, such that $\varphi(\eta) \geq \alpha_0$ for each $\eta \geq \eta_0$.

### A.1.4 Proof of Theorem 3

Let $\mathcal{U}$ be the set in the right-hand side of the equality in the statement of the theorem. We start by showing that $U(G) \subseteq \mathcal{U}$.

Let $i, j \in N$, $i \neq j$ and $i \in \text{supp}(\beta)$. We then have that, for each $m_j \in \text{supp}(\phi_{i,M_j}^*)$,

$$\left( \sum_{m_i} \frac{\phi_i^*[m_i, m_j]}{\phi_{i,M_j}^*[m_j]} \pi_i(m_i), \pi_j(m_j) \right) \text{ is a Nash equilibrium of } G.$$

Indeed, it follows by (2) that $\pi_j(m_j) \in BR_j \left( \sum_{m_i} \frac{\phi_i^*[m_i, m_j]}{\phi_{i,M_j}^*[m_j]} \pi_i(m_i) \right)$. Furthermore, for each $m_i \in M_i$ such that $(m_i, m_j) \in \text{supp}(\phi_i^*)$ and $a_i \in A_i$, $u_i(\pi_i(m_i), \pi_j(m_j)) \geq$

$u_i(a_i, \pi_j(m_j))$ by (1). Thus,

$$\sum_{m_i} \frac{\phi_i^*[m_i, m_j]}{\phi_{i,M_j}^*[m_j]} u_i(\pi_i(m_i), \pi_j(m_j)) \geq \sum_{m_i} \frac{\phi_i^*[m_i, m_j]}{\phi_{i,M_j}^*[m_j]} u_i(a_i, \pi_j(m_j)), \text{ i.e.}$$

$$u_i\left(\sum_{m_i} \frac{\phi_i^*[m_i, m_j]}{\phi_{i,M_j}^*[m_j]} \pi_i(m_i), \pi_j(m_j)\right) \geq u_i(a_i, \pi_j(m_j)).$$

We have that

$$u(\pi) = \beta_1 \sum_{m \in \text{supp}(\phi_1^*)} \phi_1^*[m] u(\pi(m)) + \beta_2 \sum_{m \in \text{supp}(\phi_2^*)} \phi_2^*[m] u(\pi(m)).$$

Hence, we compute $u^i := \sum_{m \in \text{supp}(\phi_i^*)} \phi_i^*[m] u(\pi(m))$ for each $i \in \text{supp}(\beta)$. Let $i, k \in N$. Then

$$\begin{aligned} u_k^i &= \sum_{m_j} \phi_{i,M_j}^*[m_j] \sum_{m_i} \frac{\phi_i^*[m_i, m_j]}{\phi_{i,M_j}^*[m_j]} u_k(\pi_i(m_i), \pi_j(m_j)) \\ &= \sum_{m_j} \phi_{i,M_j}^*[m_j] u_k\left(\sum_{m_i} \frac{\phi_i^*[m_i, m_j]}{\phi_{i,M_j}^*[m_j]} \pi_i(m_i), \pi_j(m_j)\right). \end{aligned}$$

Thus,

$$u^i = \sum_{m_j} \phi_{i,M_j}^*[m_j] u\left(\sum_{m_i} \frac{\phi_i^*[m_i, m_j]}{\phi_{i,M_j}^*[m_j]} \pi_i(m_i), \pi_j(m_j)\right).$$

Hence, for each $m_j \in \text{supp}(\phi_{i,M_j}^*)$, there is a Nash equilibrium

$$\sigma^{i,m_j} = \left(\sum_{m_i} \frac{\phi_i^*[m_i, m_j]}{\phi_{i,M_j}^*[m_j]} \pi_i(m_i), \pi_j(m_j)\right)$$

of $G$ such that $u^i = \sum_{m_j} \alpha^{i,m_j} u(\sigma^{i,m_j})$ with $\alpha^{i,m_j} = \phi_{i,M_j}^*[m_j]$. Then let $L_i = |\text{supp}(\phi_{i,M_j}^*)|$ and, writing $\text{supp}(\phi_{i,M_j}^*) = \{m_j^1, \ldots, m_j^{L_i}\}$, let $\alpha^{i,l} = \phi_{i,M_j}^*[m_j^l]$ and $\sigma^{i,l} = \sigma^{i,m_j^l}$ for each $l \in \{1, \ldots, L_i\}$.

For each $m_j \in \text{supp}(\phi_{i,M_j}^*)$, it follows by (1) that

$$u_i(\sigma^{i,m_j}) = \sum_{m_i'} \frac{\phi_i^*[m_i', m_j]}{\phi_{i,M_j}^*[m_j]} u_i(\pi_i(m_i'), \pi_j(m_j)) = \max_{m^* \in M} u_i(\pi(m^*)).$$

Thus, $u_i(\sigma^{i,m_j}) = u_i(\sigma^{i,m_j'})$ for each $m_j' \in \text{supp}(\phi_{i,M_j}^*)$. If $j \in \text{supp}(\beta)$, it then follows that, for each $m_i \in \text{supp}(\phi_{j,M_i}^*)$,

$$u_i(\sigma^{j,m_i}) = \sum_{m_j'} \frac{\phi_j^*[m_i, m_j']}{\phi_{j,M_i}^*[m_i]} u_i(\pi_i(m_i), \pi_j(m_j')) \leq \max_{m^* \in M} u_i(\pi(m^*));$$

thus, $u_i(\sigma^{i,m_j}) \geq u_i(\sigma^{j,m_i})$. This completes the proof that $U(G) \subseteq \mathcal{U}$.

We now show that $\mathcal{U} \subseteq U(G)$. Let $u \in \mathcal{U}$. Assume first that $\text{supp}(\beta) = N$ and let $u = \beta_1 u^1 + \beta_2 u^2$ with $u^i = \sum_{l=1}^{L_i} \alpha^{i,l} u(\sigma^{i,l})$, $\sum_{l=1}^{L_i} \alpha^{i,l} = 1$, $\alpha^{i,l} \geq 0$, $\sigma^{i,l} \in N(G)$, $u_i(\sigma^{i,l}) = u_i(\sigma^{i,k}) \geq u_j(\sigma^{j,r})$ for each $i, j \in N$ with $i \neq j$, $l, k \in \{1, \ldots, L_i\}$ and $r \in \{1, \ldots, L_j\}$.

For each $i \in N$, $l \in \{1, \ldots, L_1\}$ and $k \in \{1, \ldots, L_2\}$, pick $m_i^{1,l}$ and $m_i^{2,k}$ in $M_i$ such that $m_i^{1,l} \neq m_i^{1,r}$, $m_i^{2,k} \neq m_i^{2,s}$ and $m_i^{1,l} \neq m_i^{2,k}$ for each $l, r \in \{1, \ldots, L_1\}$ and $k, s \in \{1, \ldots, L_2\}$. Set $\phi_1^* = \sum_{l=1}^{L_1} \alpha^{1,l} 1_{m^{1,l}}$, $\phi_2^* = \sum_{l=1}^{L_2} \alpha^{2,l} 1_{m^{2,l}}$ and, for each $i \in N$, $j \in N$ and $l \in \{1, \ldots, L_j\}$, $\pi_i(m_i^{j,l}) = \sigma_i^{j,l}$.

Fix $i \in N$ and let $j \neq i$. For each $m_j \in M_j^*$, $\pi_j(m_j) = \sigma_j^{k,l}$ for some $k \in \{i, j\}$ and $l \in \{1, \ldots, L_k\}$. Since $v_i(\sigma_j^{i,l}) = u_i(\sigma^{i,l}) \geq u_i(\sigma^{j,r}) = v_i(\sigma_j^{j,r})$ for each $l \in \{1, \ldots, L_i\}$ and $r \in \{1, \ldots, L_j\}$, it follows that $\max_{m_j \in M_j^*} v_i(\pi_j(m_j)) = u_i(\sigma^{i,l})$ for each $l \in \{1, \ldots, L_i\}$. Since, for each $m \in \text{supp}(\phi_i^*)$, there exists $l \in \{1, \ldots, L_i\}$ such that $\pi_i(m_i) = \sigma_i^{i,l}$ and $\pi_j(m_j) = \sigma_j^{i,l}$, it follows that $\text{supp}(\phi_i^*) \subseteq \{m \in M : v_i(\pi_{-i}(m_{-i})) = \max_{m'_{-i} \in M_{-i}} v_i(\pi_{-i}(m'_{-i}))$ and $\pi_i(m_i) \in BR_i(\pi_{-i}(m_{-i}))\}$. Thus, (1) holds.

Moreover, for each $m_i \in \text{supp}(\phi_{j,M_i}^*)$, $m_i = m_i^{j,l}$ for some $l \in \{1, \ldots, L_j\}$ and hence $\pi_i(m_i) = \sigma_i^{j,l}$ solves

$$\max_{\alpha_i \in \Delta(A_i)} \sum_{m_j} \frac{\phi_j^*[m_i, m_j]}{\phi_{j,M_i}^*[m_i]} u_i(\alpha_i, \pi_j(m_j)) = \max_{\alpha_i \in \Delta(A_i)} u_i(\alpha_i, \sigma_j^{j,l}).$$

Thus, (2) holds. It then follows by Theorem 2 that $(\phi_i^*, (\pi_i(m_i))_{m_i \in \text{supp}(\phi_{M_i}^*)})_{i \in N}$ is the outcome of a sequential equilibrium of $G_{id}$ and, thus, that $u \in U(G)$.

If $\text{supp}(\beta) \neq N$, let $i \in \text{supp}(\beta)$ and pick $m_j^l \in M_j$ such that $m_j^l \neq m_j^r$ for each $j \in N$ and $l, r \in \{1, \ldots, L_i\}$. Define $\phi_i^* = \sum_{l=1}^{L_i} \alpha^{i,l} 1_{m^l}$ and, for each $j \in N$ and $l \in \{1, \ldots, L_i\}$, $\pi_j(m_j^l) = \sigma_j^{i,l}$. The remainder of the argument is now analogous to the one above.

### A.1.5 Proof of Corollary 2

The characterization of $U(G)$ follows from the definition of $\mathcal{G}$ and Theorem 3.

Standard results (e.g. Theorems 2.5.5 and 2.6.1 in van Damme (1991) and their proofs) imply that there is an open set $O$ of $\mathbb{R}^{2|A|}$ such that its complement has

Lebesgue measure zero and, for each $u \in O$, there is an open neighborhood $V_u$ of $u$ and $|N(u)|$ continuous functions, $f_k : V_u \to \Delta(A_1) \times \Delta(A_2)$ with $k \in \{1, \ldots, |N(u)|\}$ such that, for each $u' \in V_u$, $N(u') = \big\{f_k(u') : k \in \{1, \ldots, |N(u)|\}\big\}$ and $f_k(u) \neq f_l(u)$ for each $k, l \in \{1, \ldots, |N(u)|\}$ with $k \neq l$.[16] Shrinking $V_u$ if needed, we may assume that, for each $a \in A$, $k, l \in \{1, \ldots, |N(u)|\}$ and $u' \in V_u$, $f_k(u')[a] \neq f_l(u')[a]$ if $f_k(u)[a] \neq f_l(u)[a]$.

We have that $\mathbb{R}^{2|A|}$ is separable, hence, there is a countable collection $\{V_{u_j}\}_{j=1}^{\infty}$ such that $O = \cup_{j=1}^{\infty} V_{u_j}$. Define, for each $j \in \mathbb{N}$, $I_j = \{1, \ldots, |N(u_j)|\}$ and

$$O_j = \cap_{(i,k,l) \in N \times I_j^2 : k \neq l} \{u \in V_{u_j} : u_i(f_k(u)) \neq u_i(f_l(u))\}.$$

Then $O_j$ is open and $\cup_{j=1}^{\infty} O_j \subseteq \mathcal{G}$. It thus suffices to show that $C_{j,i,k,l} = \{u \in V_{u_j} : u_i(f_k(u)) = u_i(f_l(u))\}$ has Lebesgue measure zero for each $j \in \mathbb{N}$ and $(i, k, l) \in N \times I_j^2$ such that $k \neq l$.

Let $j \in \mathbb{N}$ and $(i, k, l) \in N \times I_j^2$ be such that $k \neq l$. Since $f_k(u_j) \neq f_l(u_j)$, let $a \in A$ be such that $f_k(u)[a] \neq f_l(u)[a]$ for each $u \in V_{u_j}$. Then

$$C_{j,i,k,l} \subseteq \left\{ u \in V_{u_j} : u_i(a) = \frac{\sum_{a' \neq a} u_i(a')\big(f_l(u)[a'] - f_k(u)[a']\big)}{f_k(u)[a] - f_l(u)[a]} \right\}.$$

It then follows by Tonelli's Theorem (e.g. Wheeden and Zygmund (1977, Theorem 6.10, p. 92)) that $C_{j,i,k,l}$ has Lesbegue measure zero.

### A.1.6 Proof of Theorem 4

Note first that $R_1 \times R_2 = M^*$. Let $i, j \in N$ be such that $i \neq j$. As in the proof of Theorem 3, we have that, for each $m_j \in \operatorname{supp}(\phi^*_{i,M_j})$,

$$\left( \sum_{m_i} \frac{\phi^*_i[m_i, m_j]}{\phi^*_{i,M_j}[m_j]} \pi_i(m_i), \pi_j(m_j) \right) \text{ is a Nash equilibrium of } G.$$

Indeed, if $m_j \notin \operatorname{supp}(\phi^*_{j,M_j})$, it follows by rationality of $j$ that

$$\pi_j(m_j) \in BR_j \left( \sum_{m_i} \frac{\phi^*_i[m_i, m_j]}{\phi^*_{i,M_j}[m_j]} \pi_i(m_i) \right).$$

[16] The set $N(u)$ denotes the set of Nash equilibria of the game with payoff function $u$.

It follows by the definition of $E$ that this conclusion also holds if $m_j \in \text{supp}(\phi^*_{j,M_j})$ (recall that $m_j \in \text{supp}(\phi^*_{i,M_j})$) and that $u_i(\pi_i(m_i), \pi_j(m_j)) \geq u_i(a_i, \pi_j(m_j))$ for each $m_i \in M_i$ such that $(m_i, m_j) \in \text{supp}(\phi^*_i)$ and $a_i \in A_i$. Thus, the claim follows as in the proof of Theorem 3.

It then follows that, for each $m_j \in \text{supp}(\phi^*_{i,M_j})$,

$$\left(\sum_{m_i} \frac{\phi^*_i[m_i, m_j]}{\phi^*_{i,M_j}[m_j]} \pi_i(m_i), \pi_j(m_j)\right) = (\sigma_i, \sigma_j)$$

since $G$ has a unique Nash equilibrium.

We have that, for each $a \in A$,

$$\mu[a] = \beta_1 \sum_{m \in \text{supp}(\phi^*_1)} \phi^*_1[m]\pi(m)[a] + \beta_2 \sum_{m \in \text{supp}(\phi^*_2)} \phi^*_2[m]\pi(m)[a]$$

and, for each $i \in N$,

$$\sum_{m \in \text{supp}(\phi^*_i)} \phi^*_i[m]\pi(m)[a] = \sum_{m_j} \phi^*_{i,M_j}[m_j] \sum_{m_i} \frac{\phi^*_i[m_i, m_j]}{\phi^*_{i,M_j}[m_j]} \pi_i(m_i)[a_i]\pi_j(m_j)[a_j]$$
$$= \sum_{m_j} \phi^*_{i,M_j}[m_j] \left(\pi_j(m_j)[a_j] \sum_{m_i} \frac{\phi^*_i[m_i, m_j]}{\phi^*_{i,M_j}[m_j]} \pi_i(m_i)[a_i]\right) = \sigma_i[a_i]\sigma_j[a_j].$$

Thus, $\mu = \sigma$.

Furthermore, for each $m \in \text{supp}(\phi^*)$ and player $j$,

$$\begin{aligned}
\xi_j(m) &= \sum_{m'_i} \frac{\phi^*[m'_i, m_j]}{\phi^*_{M_j}[m_j]} \pi_i(m'_i) \\
&= \frac{\beta_i \phi^*_{i,M_j}[m_j] \sum_{m'_i} \frac{\phi^*_i[m'_i, m_j]}{\phi^*_{i,M_j}[m_j]} \pi_i(m'_i) + \beta_j \sum_{m'_i} \phi^*_j[m'_i, m_j]\pi_i(m'_i)}{\phi^*_{M_j}[m_j]} \\
&= \sigma_i
\end{aligned}$$

since $m_j \in \text{supp}(\phi^*_{i,M_j})$ implies that $\sum_{m'_i} \frac{\phi^*_i[m'_i, m_j]}{\phi^*_{i,M_j}[m_j]} \pi_i(m'_i) = \sigma_i$ and $m'_i \in \text{supp}(\phi^*_{j,M_i})$ implies that $\pi_i(m'_i) = \sigma_i$.

### A.1.7 Proof of Theorem 5

Let $m \in \text{supp}(\phi^*)$ be such that it is mutually known that the players are rational at $m$. Thus, $m \in R_1 \times R_2$. Let $i \in N$ be such that $m \in \text{supp}(\phi^*_i)$ and $j \neq i$. Then

$\pi_i(m_i) \in BR_i(\pi_j(m_j))$ by the definition of $E$. Furthermore,

$$\pi_j(m_j) \in BR_j\left(\sum_{m_i'} \frac{\phi_i^*[m_i', m_j]}{\phi_{i,M_j}^*[m_j]}\pi_i(m_i')\right);$$

indeed, this follows by the rationality of player $j$ at $m$ if $m_j \notin \mathrm{supp}(\phi_{j,M_j}^*)$ and by the definition of $E$ if $m_j \in \mathrm{supp}(\phi_{j,M_j}^*)$ (recall that $m_j \in \mathrm{supp}(\phi_{i,M_j}^*)$). Since $m \in \mathrm{supp}(\phi_i^*) \cap (R_i \times M_j^*)$ and $|\mathrm{supp}(\phi_i^*) \cap (R_i \times M_j^*)| \leq 1$, it follows that $\mathrm{supp}(\phi_i^*) \cap (R_i \times M_j^*) = \{m\}$. Since player $i$'s rationality is known by player $j$ at $m$, we have $\phi^*[m_i', m_j] = 0$ for each $m_i' \notin R_i$ and, therefore, $\sum_{m_i' \in M_i^*} \frac{\phi_i^*[m_i', m_j]}{\phi_{i,M_j}^*[m_j]}\pi_i(m_i') = \sum_{m_i' \in R_i} \frac{\phi_i^*[m_i', m_j]}{\phi_{i,M_j}^*[m_j]}\pi_i(m_i') = \pi_i(m_i)$. Thus, $\pi_j(m_j) \in BR_j(\pi_i(m_i))$ which, in addition to $\pi_i(m_i) \in BR_i(\pi_j(m_j))$, shows that $\pi(m) = (\pi_i(m_i), \pi_j(m_j))$ is a Nash equilibrium of $G$.

## A.2 Details for Example 1

In this section we provide the details for Example 1. We first conclude the argument showing that $(1 - \beta_3)(2, 2, 2) + \beta_3(0, 0, 3)$ is a sequential equilibrium payoff when $\min\{2\beta_1, 2\beta_2\} \geq \beta_3$.

Let $i \in \{1, 2\}$ and $m \in \mathrm{supp}(\phi_i^*)$. Then $\pi_i(m_i) = A$ and $\pi_{-i}(m_{-i}) = (A, M)$ or $\pi_i(m_i) = B$ and $\pi_{-i}(m_{-i}) = (B, M)$. In either case, $\pi_i(m_i) \in BR_i(\pi_{-i}(m_{-i}))$ and $v_i(\pi_{-i}(m_{-i})) = 2 \geq v_i(\pi_{-i}(m_{-i}'))$ for each $m_{-i}' \in M_{-i}^*$.

Furthermore, for each $m_i \in \cup_{j \in \mathrm{supp}(\beta_{-i})}\mathrm{supp}(\phi_{j,M_i}^*) = \{m_i', m_i''\}$, $\pi_i(m_i)$ solves

$$\max_{a_i \in A_i} \sum_{m_{-i}} \frac{\sum_{j \neq i} \beta_j \phi_j^*[m_i, m_{-i}]}{\sum_{j \neq i} \beta_j \phi_{j,M_i}^*[m_i]} u_i(a_i, \pi_{-i}(m_{-i})).$$

Indeed, if $m_i = m_i'$, then $\pi_i(m_i) = A$ and, letting $j \in \{1, 2\}$ with $j \neq i$, the maximization problem is

$$\max_{a_i \in A_i} \frac{\beta_j u_i(a_i, (A, M)) + \beta_3 u_i(a_i, (A, L))}{\beta_j + \beta_3};$$

if $i = 1$, $a_i = A$ yields $\frac{2\beta_2}{\beta_2+\beta_3}$ whereas $a_i = B$ yields $\frac{\beta_3}{\beta_2+\beta_3}$; thus, $\pi_1(m_1')$ solves the maximization problem since $2\beta_2 \geq \beta_3$; if $i = 2$, then $a_i = A$ yields $\frac{2\beta_1}{\beta_1+\beta_3}$ whereas

$a_i = B$ yields 0; thus, $\pi_2(m_2')$ solves the maximization problem. If $m_i = m_i''$, the maximization problem is

$$\max_{a_i \in A_i} \frac{\beta_j u_i(a_i,(B,M)) + \beta_3 u_i(a_i,(B,R))}{\beta_j + \beta_3};$$

if $i = 1$, $a_i = \pi_i(m_i'') = B$ yields $\frac{2\beta_2}{\beta_2+\beta_3}$ whereas $a_i = A$ yields 0; thus $\pi_i(m_i'')$ solves the maximization problem; if $i = 2$, then $a_i = \pi_i(m_i'') = B$ yields $\frac{2\beta_1}{\beta_1+\beta_3}$ whereas $a_i = A$ yields $\frac{\beta_3}{\beta_1+\beta_3}$; thus $\pi_i(m_i'')$ solves the maximization problem since $2\beta_1 \geq \beta_3$.

Consider next $m \in \text{supp}(\phi_3^*)$. Then $\pi_3(m_3) = L$ and $\pi_{-3}(m_{-3}) = (A, A)$ or $\pi_3(m_3) = R$ and $\pi_{-3}(m_{-3}) = (B, B)$. In either case, $\pi_3(m_3) \in BR_3(\pi_{-3}(m_{-3}))$ and $v_3(\pi_{-3}(m_{-3})) = 3 \geq v_3(\pi_{-3}(m_{-3}'))$ for each $m_{-3}' \in M_{-3}^*$. It follows that condition (1) in Theorem 2 is satisfied.

Finally, note that $\cup_{j\in \text{supp}(\beta_{-3})}\text{supp}(\phi_{j,M_3}^*) = \{\hat{m}_3\}$ and that $\pi_3(\hat{m}_3) = M$ solves

$$\begin{aligned}
&\max_{a_3 \in A_3} \sum_{m_{-3}} \frac{\sum_{j\neq 3} \beta_j \phi_j^*[\hat{m}_3, m_{-3}]}{\sum_{j\neq 3} \beta_j \phi_{j,M_3}^*[\hat{m}_3]} u_3(a_3, \pi_{-3}(m_{-3})) \\
&\qquad = \max_{a_3 \in A_3} \frac{u_3(A,A,a_3) + u_3(B,B,a_3)}{2}.
\end{aligned}$$

Thus, condition (2) in Theorem 2 is also satisfied. Hence, it follows by Theorem 2 that $(1-\beta_3)(2,2,2)+\beta_3(0,0,3)$ is a sequential equilibrium payoff when $\min\{2\beta_1, 2\beta_2\} \geq \beta_3$.

We conclude this section by showing that player 3 cannot obtain a payoff of $\max_{\sigma\in C(G)} u_3(\sigma)$ in $G_{id}$ when $\beta_3 = 1$. First note (we will show this below) that $\max_{\sigma\in C(G)} u_3(\sigma) = 8/3$ and that $\sigma \in C(G)$ is a solution to this problem if and only if $\sigma(A,B,L) = \sigma(B,B,L) = \sigma(A,B,M) = \sigma(B,A,M) = \sigma(A,A,R) = \sigma(A,B,R) = \sigma(B,A,L) = \sigma(B,A,R) = 0$,

$$\begin{aligned}
&\sigma(A,A,L) = 2\sigma(A,A,M), \\
&\sigma(B,B,R) = 2\sigma(B,B,M), \\
&\frac{\sigma(B,B,M)}{2} \leq \sigma(A,A,M) \leq 2\sigma(B,B,M), \text{ and} \\
&\sigma(A,A,M) + \sigma(B,B,M) = \frac{1}{3}.
\end{aligned}$$

We use the above to show that player 3 cannot obtain the payoff $\max_{\sigma\in C(G)} u_3(\sigma) = 8/3$ in $G_{id}$. Indeed, suppose that there is a sequential equilibrium $\pi \in \Pi$ of $G_{id}$ such that $u_3(\pi) = 8/3$. Since $\sigma_\pi \in C(G)$ by Theorem 1 and $u(\pi) = u(\sigma_\pi)$, it follows that $\sigma_\pi$ solves $\max_{\sigma \in C(G)} u_3(\sigma) = 8/3$. It follows by the above characterization of the correlated equilibria of $G$ that achieve a payoff of $8/3$ to player 3 that (i) $\sigma_\pi(A, A, M) > 0$ and (ii) $\sigma_\pi(\{a \in A : a_1 = a_2\}) = 1$. The latter implies that, for each $m \in \text{supp}(\phi_3^*)$, $(\pi_1(m_1), \pi_2(m_2)) \in \{(A, A), (B, B)\}$. Since $\pi_3(m_3) \in BR_3((\pi_1(m_1), \pi_2(m_2)))$ by Theorem 2, it follows that $\pi_3(m_3) \in \{L, R\}$. In particular, $\pi_3(m_3)[M] = 0$ for each $m_3 \in \text{supp}(\phi_{3,M_3}^*)$ and, thus, $\sigma_\pi(A, A, M) = 0$. But this contradicts (i).

We finally show that $\max_{\sigma \in C(G)} u_3(\sigma) = 8/3$ and characterize the solutions to this problem. The maximization problem is:

$$\max_{\sigma \in \Delta(A)} 2\sigma(A, A, M) + 2\sigma(B, B, M) + 3\sigma(A, A, L) + 3\sigma(B, B, R) \tag{13}$$

$$\text{subject to } 2\sigma(A, A, M) \geq \sigma(A, A, L) + 2\sigma(A, B, M) \tag{14}$$

$$\sigma(B, A, L) + 2\sigma(B, B, M) \geq 2\sigma(B, A, M) \tag{15}$$

$$2\sigma(B, B, M) \geq \sigma(B, B, R) + 2\sigma(A, B, M) \tag{16}$$

$$\sigma(B, A, R) + 2\sigma(A, A, M) \geq 2\sigma(B, A, M) \tag{17}$$

$$\sigma(A, A, L) \geq 2\sigma(B, B, L) \tag{18}$$

$$2\sigma(B, B, M) \geq \sigma(A, A, M) \tag{19}$$

$$2\sigma(A, A, M) \geq \sigma(B, B, M) \tag{20}$$

$$\sigma(B, B, R) \geq 2\sigma(A, A, R). \tag{21}$$

These expressions consist of the payoff of player 3, followed by the following obedience conditions for action $a_i$ of player $i$, denoted $(i, a_i)$ : $(1, A)$, $(1, B)$, $(2, B)$, $(2, A)$, $(3, L)$ (this is only (18) and consists of a deviation to $M$ since for a deviation to $R$ not to be profitable requires $\sigma(A, A, L) \geq \sigma(B, B, L)$ which is implied by (18)), $(3, M)$ (this is (19) and (20) consisting of deviations to $L$ and $R$ respectively) and $(3, R)$ (this is only (21) and consists of a deviation to $M$ since for a deviation to $L$ not to be profitable requires $\sigma(B, B, R) \geq \sigma(A, A, R)$ which is implied by (21)).

The solution has

$$\begin{aligned}\sigma(A,B,L)=\sigma(B,B,L)=\sigma(A,B,M)=\sigma(B,A,M)&\\ =\sigma(A,A,R)=\sigma(A,B,R)=0&\end{aligned}$$

since these variables appear neither in the objective function nor on the left-hand side of the inequalities. Then (15), (17), (18) and (21) are satisfied and, thus,

$$\sigma(B,A,L)=\sigma(B,A,R)=0$$

since these variables appear neither in the objective function nor on the left-hand side of the remaining inequalities.

Letting $A'=\{(A,A,M),(B,B,M),(A,A,L),(B,B,R)\}$, the problem reduces to

$$\max_{\sigma\in\Delta(A')} 2\sigma(A,A,M)+2\sigma(B,B,M)+3\sigma(A,A,L)+3\sigma(B,B,R) \tag{22}$$

$$\text{subject to } 2\sigma(A,A,M)\geq\sigma(A,A,L) \tag{23}$$

$$2\sigma(B,B,M)\geq\sigma(B,B,R) \tag{24}$$

$$2\sigma(B,B,M)\geq\sigma(A,A,M) \tag{25}$$

$$2\sigma(A,A,M)\geq\sigma(B,B,M). \tag{26}$$

Conditions (25) and (26) are equivalent to

$$\frac{\sigma(B,B,M)}{2}\leq\sigma(A,A,M)\leq 2\sigma(B,B,M).$$

We claim that (23) and (24) are satisfied with equality if $\sigma$ is a solution to this problem. Suppose not; then $2\sigma(A,A,M)>\sigma(A,A,L)$ or $2\sigma(B,B,M)>\sigma(B,B,R)$; for concreteness, assume the former case. Define

$$\sigma'(A,A,M)=\lambda\sigma(A,A,M),$$

$$\sigma'(B,B,M)=\lambda\sigma(B,B,M),$$

$$\sigma'(B,B,R)=\lambda\sigma(B,B,R) \text{ and}$$

$$\sigma'(A,A,L)=1-\sigma'(A,A,M)-\sigma'(B,B,M)-\sigma'(B,B,R)$$

for some $\lambda \in (0,1)$ sufficiently close to 1 such that $2\sigma'(A,A,M) \geq \sigma'(A,A,L)$. Thus, $\sigma' \in \Delta(A')$ and satisfies (23)–(26). Furthermore,

$$2\sigma(A,A,M) + 2\sigma(B,B,M) + 3\sigma(A,A,L) + 3\sigma(B,B,R) =$$
$$3 - \sigma(A,A,M) - \sigma(B,B,M) < 3 - \sigma'(A,A,M) - \sigma'(B,B,M) =$$
$$2\sigma'(A,A,M) + 2\sigma'(B,B,M) + 3\sigma'(A,A,L) + 3\sigma'(B,B,R),$$

contradicting the assumption that $\sigma$ is a solution to this problem. This shows that

$$\sigma(A,A,L) = 2\sigma(A,A,M) \text{ and } \sigma(B,B,R) = 2\sigma(B,B,M).$$

Then $\sum_a \sigma(a) = 1$ is equivalent to

$$\sigma(A,A,M) + \sigma(B,B,M) = \frac{1}{3}.$$

Finally, (22) is equal to

$$8(\sigma(A,A,M) + \sigma(B,B,M)) = \frac{8}{3}.$$

## A.3 A 2-player game with unique Nash but multiple correlated equilibria

In this section, we show that the following game has a unique Nash but multiple correlated equilibria.

| $1\backslash 2$ | $H$ | $T$ | $R$ |
|---|---|---|---|
| $H$ | $1,-1$ | $-1,1$ | $-1,-1$ |
| $T$ | $-1,1$ | $1,-1$ | $1,-1$ |
| $B$ | $2,0$ | $-3,0$ | $-3,1$ |

Consider the action distribution that puts the following probabilities on each action profile:

| $1\backslash 2$ | $H$ | $T$ | $R$ |
|---|---|---|---|
| $H$ | $\alpha$ | $\alpha$ | $0$ |
| $T$ | $\alpha$ | $\alpha$ | $0$ |
| $B$ | $\beta$ | $0$ | $0$ |

where $4\alpha + \beta = 1$.

It is easy to see that the obedience conditions hold for player 1 and, after recommendation $T$, for player 2. After recommendation $H$, player 2 is indifferent between $H$ and $T$. Thus, this is a correlated equilibrium as long as player 2 prefers $H$ to $R$, which is the case if $0 \geq -2\alpha + \beta$, or equivalently $\beta \leq 2\alpha$. E.g. $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{3}$ is a correlated equilibrium distribution.

On the other hand, $\alpha = \frac{1}{4}$ and $\beta = 0$ is the unique Nash equilibrium distribution. To see this, first note that Player 1 cannot play a pure strategy in equilibrium. Now suppose that player 1 plays $H$ and $B$ with positive probability in equilibrium. Let $q$ be the probability that player 2 plays $H$. For player 1 to be indifferent between $H$ and $B$, we need $q = \frac{2}{3}$, in which case player 1 gets $\frac{1}{3}$ from playing either $H$ or $B$. Also in this case, player 1's payoff from playing $T$ is $-\frac{1}{3}$ and so player 1 must play $T$ with zero probability (this argument also implies that there is no equilibrium where player 1 plays all three actions with positive probability). But if player 1 does not play $T$, $H$ becomes strictly dominated (by a mixture of $T$ and $R$) for player 2, so player 2 must play $H$ with zero probability, contradicting $q = \frac{2}{3}$.

Next suppose that player 1 plays $T$ and $B$ with positive probability in equilibrium. In this case, we must have $q = \frac{4}{7}$ and player 1 gets payoff $-\frac{1}{7}$ from either playing $T$ and $B$. But when $q = \frac{4}{7}$, player 1 gets payoff $\frac{1}{7}$ from playing $H$, a contradiction to $T$ and $B$ being played with positive probability.

Thus, Player 1 must play $H$ and $T$ with positive probability in any equilibrium. In this case, $q = \frac{1}{2}$, which implies that player 1 must play $B$ with zero probability. Thus, $R$ becomes strictly dominated for player 2 and $P(H) = P(T) = \frac{1}{2}$. For player 2 to be indifferent between $H$ and $T$, we must also have $P(H) = P(T) = \frac{1}{2}$ for player 1, giving us the unique Nash equilibrium.

## A.4 Tightness of the Epistemic Results

We show that none of the assumptions of Theorems 4 and 5 can be dropped. Example 2 in Section 5 already showed this for the assumption in Theorem 4 that each player is rational at each state, and Example 3 in Section 5 showed that the assumption of mutual knowledge of rationality in Theorem 5 cannot be weakened.

The following example shows that the assumption of endogenous information cannot be dropped from Theorem 4.

**Example 4.** Consider the example in Section A.3 with $M_1^* = \{h, t, b\}$, $M_2^* = \{h, t\}$, $\pi_1(h) = H$, $\pi_1(t) = T$, $\pi_1(b) = B$, $\pi_2(h) = H$, $\pi_2(t) = T$, $\phi^*$ given as in the following table and with $\frac{1}{6} < \alpha < \frac{1}{4}$:

| $1\backslash 2$ | $h$ | $t$ |
|---|---|---|
| $h$ | $\alpha$ | $\alpha$ |
| $t$ | $\alpha$ | $\alpha$ |
| $b$ | $\beta$ | $0$ |

Then $\beta > 0$ and the corresponding interactive belief system is:

| $1\backslash 2$ | $h$ | $t$ |
|---|---|---|
| $h$ | $\frac{1}{2}, \frac{\alpha}{2\alpha+\beta}$ | $\frac{1}{2}, \frac{1}{2}$ |
| $t$ | $\frac{1}{2}, \frac{\alpha}{2\alpha+\beta}$ | $\frac{1}{2}, \frac{1}{2}$ |
| $b$ | $1, \frac{\beta}{2\alpha+\beta}$ | $0, 0$ |

Then players are rational at each state but, for each $m_1$, players' conjectures at states $(m_1, h)$ are not equal to the Nash equilibrium of game.

The next example shows that the assumption of uniqueness of Nash equilibrium cannot be dropped from Theorem 4.

**Example 5.** Consider the game of chicken in Section 2 and let $M_1^* = \{a, b\}$, $M_2^* = \{\frac{1}{2}, b\}$, $\phi_1^* = \frac{1}{2}1_{(a,\frac{1}{2})} + \frac{1}{2}1_{(b,\frac{1}{2})}$, $\phi_2^* = 1_{(a,b)}$ and $\pi_i(m_i)$ be the action corresponding to the message $m_i$ for each $m_i \in M_i^*$ and $i \in \{1, 2\}$: $\pi_1(a) = A$, $\pi_1(b) = B$, $\pi_2(\frac{1}{2}) = \frac{1}{2}1_A + \frac{1}{2}1_B$ and $\pi_2(b) = B$. The corresponding interactive belief system is:

| $1\backslash 2$ | $\frac{1}{2}$ | $b$ |
|---|---|---|
| $a$ | $\frac{\beta_1}{\beta_1+2\beta_2}, \frac{1}{2}$ | $\frac{2\beta_2}{\beta_1+2\beta_2}, 1$ |
| $b$ | $1, \frac{1}{2}$ | $0, 0$ |

Players are rational at each state: Player 1, at $m_1 = a$, assigns probability $\frac{\beta_1}{\beta_1+2\beta_2}\frac{1}{2} + \left(1 - \frac{\beta_1}{\beta_1+2\beta_2}\right) 0 < \frac{1}{2}$ to player 2 playing $A$ and, thus, $\pi_1(a) = A$ is optimal; at $m_1 = b$, player 1 assigns probability $\frac{1}{2}$ to player 2 playing $A$ and, thus, $\pi_1(b) = B$ is optimal. Player 2 assigns probability $\frac{1}{2}$ (resp. 1) to player 1 playing $A$ at $m_2 = \frac{1}{2}$ (resp. $m_2 = b$), hence $\pi_2(\frac{1}{2}) = \frac{1}{2}1_A + \frac{1}{2}1_B$ (resp. $\pi_2(b) = B$) is optimal.

The conditions for an endogenous interactive belief system are satisfied. Indeed, $\mathrm{supp}(\phi_1^*) = \{(a, \frac{1}{2}), (b, \frac{1}{2})\}$, $\pi_1(a), \pi_1(b) \in BR_1(\pi_2(\frac{1}{2}))$ and $v_1(\pi_2(\frac{1}{2})) = \frac{7}{2} > 1 = v_1(\pi_2(b))$. Furthermore, $\mathrm{supp}(\phi^*_{1,M_1^*}) \cap \mathrm{supp}(\phi^*_{2,M_1^*}) = \{a\}$ and $\pi_1(a) = A$ solves $\max_{a_1\in A_1} \sum_{m_2} \frac{\phi_2^*[a,m_2]}{\phi^*_{2,M_1^*}[a]} u_1(a_1, \pi_2(m_2)) = \max_{a_1\in A_1} u_1(a_1, B)$. Regarding player 2, $\mathrm{supp}(\phi^*_{1,M_2^*}) \cap \mathrm{supp}(\phi^*_{2,M_2^*}) = \emptyset$, $\mathrm{supp}(\phi_2^*) = \{(a, b)\}$, $\pi_2(b) \in BR_2(\pi_1(a))$ and $v_2(\pi_1(a)) = 7 > 1 = v_2(\pi_1(b))$.

However, when $m = (a, b)$,

$$(\xi_2(m), \xi_1(m)) = \left(A, \frac{\beta_1}{2\beta_1 + 4\beta_2} 1_A + \left(1 - \frac{\beta_1}{2\beta_1 + 4\beta_2}\right) 1_B\right)$$

is not a Nash equilibrium.

Example 5 also shows that the assumption that $E$ is such that $|(R_i \times M_{-i}^*) \cap \mathrm{supp}(\phi_i^*)| \leq 1$ for each $i \in N$ cannot be dropped from Theorem 5. Indeed, $E$ is an endogenous belief system and players are rational at each state but, when $m = (b, \frac{1}{2})$, $\pi(m) = (B, \frac{1}{2}1_A + \frac{1}{2}1_B)$ is not a Nash equilibrium of the game.

The following example shows that the requirement that $E$ be an endogenous belief system cannot be dropped from Theorem 5.

**Example 6.** Consider the game of chicken in Section 2 (or matching pennies) and let $M_1^* = \{\frac{1}{2}\}$, $M_2^* = \{a, b\}$, $\phi_1^* = 1_{(\frac{1}{2},a)}$, $\phi_2^* = 1_{(\frac{1}{2},b)}$, $\pi_1(\frac{1}{2}) = \frac{1}{2}1_A + \frac{1}{2}1_B$, $\pi_2(a) = A$, $\pi_2(b) = B$ and $\beta_1 = \frac{1}{2}$. The corresponding interactive belief system is:

| $1\backslash 2$ | $a$ | $b$ |
|---|---|---|
| $\frac{1}{2}$ | $\frac{1}{2}, 1$ | $\frac{1}{2}, 1$ |

Then players are rational at each state and $|(R_i \times M_{-i}^*) \cap \mathrm{supp}(\phi_i^*)| \leq 1$ for each $i \in \{1, 2\}$ holds. But, for each $m \in M^*$, $\pi(m)$ is not a Nash equilibrium.

## A.5 Equilibrium payoffs with complexity costs

We consider the set $U_c(G)$ of payoffs of sequential equilibria with complexity costs of $G_{id}$ for a given normal-form game $G$. In general, $U_c(G) \subseteq U(G)$ since every outcome of a sequential equilibrium with complexity costs is the outcome of a sequential equilibrium (without complexity costs).

For 2-player games $G$, using the restriction that each player uses only one message in the proof of Theorem 3, we obtain:

**Corollary 3.** *For each 2-player game $G$,*

$$U_c(G) = \{\beta_1 u(\sigma) + \beta_2 u(\sigma') : \sigma, \sigma' \in N(G), u_1(\sigma) \geq u_1(\sigma'), u_2(\sigma') \geq u_2(\sigma)\}.$$

In particular, $U_c(G) = U(G)$ for any 2-player game $G \in \mathcal{G}$.

When there are more than two players, the relationship $U_c(G) \subseteq U(G)$, together with the example in Section 3, implies that not all correlated equilibrium payoffs can be achieved, i.e. $u(C(G)) \setminus U_c(G) \neq \emptyset$ is possible. It is also still the case that $U_c(G) \setminus u(N(G)) \neq \emptyset$. The latter conclusion is established by the following example, analogous to Example 2.5 in Aumann (1974). As before, player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix ($A_3 = \{L, M, R\}$):

| $1\backslash 2$ | $A$ | $B$ |
|---|---|---|
| $A$ | $0,0,3$ | $0,0,0$ |
| $B$ | $0,0,0$ | $0,0,0$ |

| $1\backslash 2$ | $A$ | $B$ |
|---|---|---|
| $A$ | $2,2,2$ | $0,0,0$ |
| $B$ | $0,0,0$ | $2,2,2$ |

| $1\backslash 2$ | $A$ | $B$ |
|---|---|---|
| $A$ | $0,0,0$ | $0,0,0$ |
| $B$ | $0,0,0$ | $0,0,3$ |

We have that if $u \in u(N(G))$, then $u = (1,1,1)$ (corresponding to the Nash equilibrium where player 3 plays $M$ and each of the remaining players plays each of his actions with probability equal to 1/2) or $u_1 = u_2 = 0$ (corresponding to the Nash equilibria where player 3 plays $M$ with zero probability e.g. $(A, A, A)$). We will show that when $\frac{\beta_2}{2} \leq \beta_1 \leq 2\beta_2$, $(1 - \beta_3)(2, 2, 2) + \beta_3(0, 0, 3)$ is the payoff of a sequential equilibrium with complexity costs.

To see the above, let $\phi_1^* = 1_{(m_1', m_2', \hat{m}_3)}$, $\phi_2^* = 1_{(m_1'', m_2'', \hat{m}_3)}$, $\phi_3^* = 1_{(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3)}$ and

$$\pi_1(m_1') = \pi_1(\tilde{m}_1) = A, \pi_1(m_1'') = B,$$
$$\pi_2(m_2') = \pi_2(\tilde{m}_2) = A, \pi_2(m_2'') = B,$$
$$\pi_3(\hat{m}_3) = M \text{ and } \pi_3(\tilde{m}_3) = L.$$

Thus, $\pi(m_1', m_2', \hat{m}_3) = (A, A, M)$, $\pi(m_1'', m_2'', \hat{m}_3) = (B, B, M)$ and $\pi(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3) = (A, A, L)$. Clearly, $|\text{supp}(\phi_i^*)| = 1$ for each $i \in N$. We now show that the conditions in Theorem 2 are satisfied. We have that $u_i(\pi(m)) = \max_{a \in A} u_i(a)$ for each $i \in N$ and $m \in \text{supp}(\phi_i^*)$. If $i \in \{1, 2\}$ and $m_i \in \cup_{j \neq i} \text{supp}(\phi_{j,M_i}^*)$, then $\pi_i(m_i) \in BR_i(\pi_{-i}(m_{-i}))$ where $m_{-i}$ is the unique $\bar{m}_{-i}$ such that $(m_i, \bar{m}_{-i}) \in \cup_{j \neq i} \text{supp}(\phi_j^*)$ and, for each $\alpha_i \in \Delta(A_i)$,

$$\sum_{\bar{m}_{-i}} \frac{\sum_{j \neq i} \beta_j \phi_j^*[m_i, \bar{m}_{-i}]}{\sum_{j \neq i} \beta_j \phi_{j,M_i}^*[m_i]} u_i(\alpha_i, \pi_{-i}(\bar{m}_{-i})) = u_i(\alpha_i, \pi_{-i}(m_{-i})).$$

Finally, if $i = 3$ and $m_3 \in \cup_{j \neq 3} \text{supp}(\phi_{j,M_3}^*)$, then $m_3 = \hat{m}_3$ and $\pi_3(\hat{m}_3) = M$ solves

$$\begin{aligned} &\max_{\alpha_3 \in \Delta(A_3)} \sum_{\bar{m}_{-3}} \frac{\sum_{j \neq 3} \beta_j \phi_j^*[\hat{m}_3, \bar{m}_{-i}]}{\sum_{j \neq 3} \beta_j \phi_{j,M_3}^*[\hat{m}_3]} u_3(\alpha_3, \pi_{-3}(\bar{m}_{-3})) \\ &= \max_{\alpha_3 \in \Delta(A_3)} \frac{\beta_1 u_3(\alpha_3, A, A) + \beta_2 u_3(\alpha_3, B, B)}{\beta_1 + \beta_2} \end{aligned}$$

since $\frac{\beta_2}{2} \leq \beta_1 \leq 2\beta_2$.

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