# Rent Extraction with Information Acquisition

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**Abstract**

This paper revisits the classic mechanism design question of when buyers with private information in an auction setting can expect to receive economic rents. It is well known that under standard assumptions, the seller can fully extract rent for generic prior distributions over the valuations of the buyers. However, a crucial assumption underlying this result is that the buyers are not able to acquire any additional information about each other. This assumption can be seen as a special case of a general model where buyers have access to some information acquisition technology. We present a general model of information acquisition, and we provide necessary and sufficient conditions on the information acquisition technology for the seller to be able to guarantee full rent extraction. Unlike in the standard model, these conditions are not generically satisfied. Indeed, if buyers can choose from a continuum of information acquisition actions and we restrict attention to information acquisition technologies where the joint distribution is continuous in the choices of the buyers, the set of information acquisition technologies that allow full rent extraction is nowhere dense in the topology of uniform convergence.

*Keywords:* Rent extraction, information acquisition, auctions

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## 1 Introduction

Agents with private information who interact in strategic situations often have incentives to acquire information about each other before making their decisions, and in many relevant environments, it is also plausible that they have the opportunity to do so. This paper explores the implications of this observation for a seller in an auction setting who wishes to implement some allocation rule and fully extract rent from the buyers. Information acquisition is modelled in a general way by introducing a set of additional signals and a function—the information acquisition technology—that determines the joint distribution of signals for each information acquisition decision of the buyers. This approach represents the acquisition of higher order information in a simple and tractable manner and avoids the usual infinite regress problem that arises in a model with learning about the learning of others. We fully characterise the set of information acquisition technologies that allow the seller to fully extract rent, and we show that under natural assumptions this set is nowhere dense in the topology of uniform convergence.

In standard models of auctions with independent private values, buyers earn positive rents. In the optimal auction for the seller, the winning bidder pays the expectation of the second highest valuation, an amount that is lower than her own (the highest) valuation. The economic intuition is buyers must be incentivised to reveal their private information, and are rewarded for doing so with positive rents. However, this intuition seemingly depends on the assumption that the private information is independently distributed. Myerson (1981) showed by way of an example that when valuations are correlated, it may be possible for the seller to fully extract rent for any allocation rule. Intuitively when valuations are correlated, each buyer has information not only about her own valuation, but also about the valuations of the other buyers, and the seller can exploit this information to incentivise the buyers to reveal their valuations.

Indeed, it is well known that in the absence of information acquisition, full rent extraction is possible for generic prior distributions of the buyers' valuations (Crémer and McLean, 1985, 1988). For example, consider a mechanism where the payment of each buyer consists of two terms. First, each buyer pays the product of her valuation for the object and the probability of winning for each profile of her opponents' types (so that without the second term, she always receives exactly zero payoff ex post). The second term is a 'side bet' with the seller about the types of the other buyers. As long the valuations are correlated, it is possible to design these bets to have zero expected value under the belief of the reported type, and strictly positive expected value under the belief of every other type. As the bets become large, the incentive constraints for each type of buyer will be satisfied, since any type that pretends to be another type will have to pay the side bet, which can be made arbitrarily large in expectation.

Under this 'side bet' mechanism, each buyer's payoff will depend on the types of the other buyers, and the payment can be very large for some types and very small for others. This implies that buyers may have a strong incentive to learn about the types of their opponents, and having acquired this information, misreport their own type or drop out of the mechanism altogether. Thus, if buyers are able to acquire information, then this 'side bet' mechanism may fail to fully extract rent. However, this does not necessarily mean that the seller cannot fully extract rent using some other mechanism that exploits the buyers' information. For example, if each buyer can perfectly learn the other buyers' types, the seller can fully extract rent using a mechanism in which each buyer reports the types of all the other buyers. With this information acquisition technology, the seller can even fully extract rent when the underlying valuations are independent. On the other hand, if the information acquisition technology is such that each buyer independently receives a signal with probability $p < 1$, which perfectly reveals the types of the other buyers including whether they have received a signal, then the seller cannot always fully extract rent. For example, if the seller wishes to implement the efficient allocation rule, then a high type who learns that her opponents are low type and have no signal must earn positive rents.

Thus, the ability to acquire information may help or hurt the buyers in the presence of a seller who wishes to fully extract rent, depending on the information acquisition technology that is available, and so it is natural to ask which types of information acquisition technology can guarantee full rent extraction for the seller. In this paper, we will give an exact answer to this question by characterising the necessary and sufficient conditions on the information acquisition technology such that the seller is able to guarantee full rent extraction. Moreover, we show that under natural assumptions, the set of information acquisition technologies that allow full rent extraction is small in a topological sense. In particular, if buyers have infinitely many information acquisition actions to choose from, and if the information acquisition technology is a continuous function from the space of actions and valuations to the space of full support distributions over additional signals, then the closure of the set information acquisition technologies that allow full rent extraction has empty interior in the topology of uniform convergence. Thus, the possibility of information acquisition completely reverses the standard result that full rent extraction is generically possible and rescues the intuition that possessors of private information earn positive rents.

In our model, buyers observe not only their private valuations but also some payoff irrelevant signals that may be correlated with their opponents' types (where a type is both the valuation and the additional signal). After observing their valuations, each buyer covertly chooses an information acquisition action. The profile of the chosen information acquisition actions and the realised valuations then determine the joint distribution of the payoff irrelevant signals according to the information acquisition technology, which is represented by a function $\Sigma : V \times A \mapsto \Delta S$, where $V$ is the set of valuations, $A$ is the set of information acquisition actions, and $S$ is the set of payoff irrelevant signals. $V$ and $S$ are assumed to be finite, but we make no restrictions on $A$. Note that the distribution of signals depends on the choices of buyers—that is, information acquisition is endogenous.

This specification is broad enough to capture many relevant aspects of learning. For example, since the distribution of signals depends on the actions of all the buyers, it can be used to model a situation where agents take actions that prevent others from learning about them. Since the signals can be correlated conditional on the profile of valuations, buyers can acquire information about each other's signals as well as their valuations. Moreover, signals are also informative about the actions that have been chosen. Since all the learning is captured by a distribution over the set of additional signals, the infinite hierarchies of beliefs for each buyer that arises as a result of information acquisition is represented in a parsimonious way.

We address the question of when the seller can fully extract rent using a mechanism design framework. However, a nonstandard feature of our setup is that the distribution over the type space is not exogenously given, but determined as the result of buyers' choices. In particular, the buyers choose their information optimally, given the seller's choice of mechanism. Furthermore, the buyers are able to deviate not only by misreporting their types in the mechanism, but also by choosing alternative information acquisition actions. Thus, in designing the mechanism, the seller must take into account the optimal information choices of the buyers.

Following the optimal choice of information, the revelation principle implies that we can restrict attention to incentive compatible direct mechanisms. However, when buyers deviate to other information acquisition actions they will not necessarily report truthfully. We define a game between the buyers at the ex ante stage where the payoffs to each information acquisition strategy is the expectation of the maximum payoff from the mechanism, given the profile of information acquisition strategies, and we require that the choice of information is a Nash equilibrium of this game. A seller who wishes to implement a particular allocation rule can fully extract rent if there exists a mechanism that implements that allocation rule such that following the optimal choice of information, the buyers report truthfully and receive zero payoffs.

Without information acquisition, Crémer and McLean (1988) show that a necessary and sufficient condition for the seller to be able to fully extract rent for any allocation rule is that for each buyer $i$ with valuation $v_i$, the belief of type $v_i$ about $v_{-i}$ does not lie in the convex hull of the beliefs of any type $v_i'$, such that $v_i' \neq v_i$. When this condition holds, there exists a hyperplane that separates each type's belief from the beliefs of all other types. This is equivalent to the existence of lotteries, one for each type, with zero expected value for that type and strictly positive expected value for every other type. As we discussed previously, this implies that the condition is sufficient for full rent extraction.

Note that the condition depends only on the beliefs, and not on the valuations associated with each belief. This makes necessity less obvious. For example, if the only type whose belief lies in the convex hull of the beliefs of the other types also happens to have the highest valuation, then perhaps the seller can still fully extract rent even though the condition fails. Even though there does not exist a hyperplane separating the belief of the highest type from the beliefs of all other types, the seller does not need such a lottery for the highest type. In any mechanism where the highest type gets zero ex post payoff for every profile of her opponents' types, no type will deviate by pretending to be the highest type. Intuitively, 'side-bets' are required to prevent high types from deviating down, not low types deviating up.

In Crémer and McLean (1988), type $v_i$'s valuation for the object is given by a function $w_i : V_i \mapsto R_+$, and the condition is necessary and sufficient to guarantee full rent extraction for every possible $(w_i)_{i \in I}$, which is the reason why it does not depend on the valuation associated with each belief. However, in our model, since the seller can induce a different information acquisition strategy to fully extract rent for each specification of the payoffs, the conditions on beliefs will retain some dependence on the type that holds each belief.

In order to simplify the exposition, assume for now that the distribution over signals has full support (we relax this assumption in the body of the paper). Then a necessary and sufficient condition on the information acquisition technology for the seller to be able to guarantee full rent extraction is that for every $(w_i)_{i \in I}$, there exists an information acquisition strategy $\alpha : V \mapsto A$ such that for each type $(v_i, s_i)$ that does not have the highest valuation under $w_i$:

1. $(v_i, s_i)$'s belief is not in the relative interior of $C(\alpha_{-i})$, the convex hull of the beliefs of all types that could arise from any unilateral deviation to an alternative information acquisition strategy, given that the other players are following $\alpha_{-i}$.

2. The smallest exposed face of the closure of $C(\alpha_{-i})$ containing $(v_i, s_i)$'s belief does not intersect with the closure of the set of beliefs of all types $(v_i', s_i')$ with $w_i(v_i') > w_i(v_i)$ that could arise from any unilateral deviation to an alternative information acquisition strategy, given that the other players are following $\alpha_{-i}$.

The first condition ensures that for each type, there exists a lottery with zero expected value for that type, and weakly positive expected value for every type that could arise from any information acquisition strategy. The second condition ensures that the lottery for type $(v_i, s_i)$ can be chosen to have strictly positive expected value that is bounded away from zero for any type $(v_i', s_i')$ with $w_i(v_i') > w_i(v_i)$. Note that our conditions involve the beliefs of all types that could arise from any information acquisition strategy. This ensures that there is no profitable deviation to another information acquisition strategy and is a much stronger requirement than in the standard model, since there can be (infinitely) many beliefs that arise from any information acquisition strategy. If this set of beliefs is open (for example, if it always possible to acquire more information about each type of every opponent) then the seller will not be able to fully extract rent. Indeed, we show that these conditions fail "generically" when the information acquisition technology is sufficiently rich.

The first condition implies that full rent extraction may fail if the belief of some type $(v_i, s_i)$ is in the relative interior of $C(\alpha_{-i})$. Any lottery that the seller offers to type $(v_i, s_i)$ that can prevent some type $(v_i^*, s_i)$ from pretending to be $(v_i, s_i)$, where $w_i(v_i^*) > w_i(v_i)$ must have strictly positive expected value for type $(v_i^*, s_i)$. But since $(v_i, s_i)$ is in the relative interior of $C(\alpha_{-i})$, this lottery must then have strictly negative expected value for some other type $(v_i', s_i')$ following some information acquisition strategy $\alpha_i'$. In other words, any lottery that punishes $(v_i^*, s_i)$ will must reward some other type $(v_i', s_i')$. Suppose that $w_i(v_i^*)$ is large—then in any mechanism that fully extracts rent, any lottery that prevents $(v_i^*, s_i)$ from deviating must also be large. But in that case buyer $i$ can deviate when her valuation is $v_i'$ by choosing $\alpha_i'(v_i')$ and reporting $(v_i, s_i)$ after signal $s_i'$, where her payment will be large and negative in expectation.

The second condition implies that a reason why full rent extraction may fail is that with information acquisition, types with beliefs that lie on the same exposed face of the closure of $C(\alpha_{-i})$ may have different valuations. Exposed faces are relevant because (for an appropriately chosen $w_i$) any incentive compatible mechanism that fully extracts rent must offer a lottery to type $(v_i, s_i)$ that has the same expected value for all beliefs on the smallest exposed face of the closure of $C(\alpha_{-i})$ containing $(v_i, s_i)$'s belief. This is because if the seller makes a side bet with type $(v_i, s_i)$ such that the expected value is not zero for all beliefs on this exposed face, then some other type $(v_i', s_i')$ will strictly prefer to take that bet. As a result, every type with a belief on this exposed face will evaluate $(v_i, s_i)$'s expected payment to the seller in the same way as $(v_i, s_i)$. Thus, if any of these types have a strictly higher valuation than $(v_i, s_i)$, that type can report $(v_i, s_i)$ and earn positive rents from this deviation. An extreme example is where types with different valuations have the same belief—this is a failure of the *beliefs determine preferences* property introduced by Neeman (2004). Our second condition can thus be interpreted as a generalisation of this property.

## 2 Related Literature

The first example of a side bet mechanism that allows the seller to fully extract surplus appears in Myerson (1981). A series of papers, starting with Crémer and McLean (1985) argue that full surplus extraction is generically possible in various mechanism design settings. Crémer and McLean (1988) provide necessary and sufficient conditions for full surplus extraction to be Bayesian incentive compatible in an auction setting, and argue that these conditions are generically satisfied. McAfee, McMillan and Reny (1989) show that when buyers have continuous valuations, almost full surplus extraction is possible in a common value auction, and McAfee and Reny (1992) extend their result to general mechanism design problems.

Many papers have considered the limits of these results. For example, Robert (1991) introduces limited liability and risk aversion, and shows that full surplus extraction fails. Laffont and Martimort (2000) consider Collusion Proofness, and Lopomo, Rigotti and Shannon (2014) and Renou (2015) show that ambiguity aversion of the buyers prevents full surplus extraction from being a generic property. Our paper provides another natural setting in which full surplus extraction may fail.

Several papers have considered mechanism design with information acquisition. For example, Bikhchandani (2011) provides necessary and sufficient conditions for the existence of full extraction lotteries that are robust to the possibility of information acquisition. However, in Bikhchandani (2011), buyers can only acquire signals that are independent conditional on the profile of valuations. Our model differs in that the information acquisition stage is modelled as game, and we allow buyers to acquire information not just about each other's valuations, but also about each other's information. This is important because if buyers are able to acquire information only about each other's valuations, the seller can fully extract rent by offering side bets on the information instead. However, in such mechanisms, there is then a strong incentive to acquire information about each other's information.

A closely related paper is Obara (2008), which extends the Crémer and McLean (1988) result to a setting where agents take some hidden action that determines the distribution of their payoff relevant types. Obara (2008) provides necessary and sufficient conditions for full surplus extraction, and argues that the conditions are not necessarily satisfied when there are many actions to which the agents can deviate. In our setting, the distribution of the payoff relevant type is fixed, but the buyers can determine the distribution of payoff irrelevant signals after learning their payoff relevant type, and we show that when the action space is a continuum, the conditions for full rent extraction fail generically. Bikhchandani and Obara (2017) provide sufficient conditions for efficient implementation and full surplus extraction when buyers are able to acquire information about an unknown payoff relevant state of nature. On a technical level, unlike the papers by Obara (2008), Bikhchandani (2011), and Bikhchandani and Obara (2017), we do not restrict the action space to be finite, which is important for our genericity result.

Yamashita (2018) considers a related question about the highest revenue the seller can guarantee when the buyers have access to additional information. The seller considers the worst case information structure for each choice of mechanism, and chooses the mechanism to maximise the worst case revenue. One interpretation is that the buyers commit to some information structure, and then the seller offers the mechanism. A key difference in our approach is that the seller first commits to a mechanism, and then the buyers choose their information optimally. Thus, the seller is better off.

Finally Heifetz and Neeman (2006) argue that the result that full surplus extraction is possible for generic type spaces of a fixed and finite size that admit a common prior hinges on the nonconvexity of the set of priors, and generic priors on the universal type space do not allow for full surplus extraction. In our model, the type space is endogenous and depends on the information acquisition decisions of the buyers.

The rest of the paper proceeds as follows. In Section 3 we introduce the model of information acquisition, and motivate our definition of full rent extraction which is justified by the revelation principle. Section 4 contains several examples that illustrate the flexibility of our framework. Our main results are presented in Sections 5 and 6. Section 5 provides a complete characterisation of the set of information structures that guarantee full rent extraction for the seller. Section 6 shows that under natural assumptions, the set of information structures that guarantee full rent extraction is nowhere dense. All proofs are contained in the Appendix.

## 3 Model

There is a finite set $I$ of $n$ buyers who may buy a single indivisible object from a seller. Each $i \in I$ has a payoff relevant valuation $v_i \in V_i$ and a payoff irrelevant signal $s_i \in S_i$, where $V_i$ and $S_i$ are finite. Let $\theta_i = (v_i, s_i) \in V_i \times S_i \equiv \Theta_i$ be buyer $i$'s type. The profile of valuations $v \in V \equiv V_1 \times \ldots \times V_n$ is distributed according to a full support distribution function $\Pi$,[1] and buyer $i$'s valuation for the object is given by a function $w_i : V_i \mapsto R_+$. Let $\pi(v)$ denote the probability of $v$ according to the distribution $\Pi$. We assume that utility is quasilinear in transfers. After observing $v_i$, each buyer covertly chooses an information acquisition action $a_i \in A_i$, where $A_i$ can be an arbitrary set. Let $S \equiv S_1 \times \ldots \times S_n$ and $A \equiv A_1 \times \ldots \times A_n$. A function $\Sigma : A \times V \mapsto \Delta S$ determines a distribution over signals for each action profile $a \in A$ and each profile of valuations $v \in V$. That is, $\Sigma(a, v)$ is distribution over $S$. Let $\sigma(s|a, v)$ denote the probability of $s$ according to this distribution. Note that the distribution depends on the entire action profile $a$.

Let $\alpha_i : V_i \mapsto A_i$ be an information acquisition (pure) strategy for buyer $i$, and define $\Gamma : A^{|V|} \mapsto \Delta(V \times S)$ as $\gamma(v, s|\alpha) = \pi(v)\sigma(s|\alpha(v), v)$. That is, for any profile of information

[1] For all $v \in V$, $\pi(v) > 0$.

acquisition strategies $\alpha$, the resulting distribution over $\Theta = V \times S$ is given by $\Gamma(\alpha)$. Let $\Gamma_{\Theta_i}(\alpha)$ denote the marginal distribution on $\Theta_i$ given $\alpha$:

$$\begin{aligned}\gamma_{\Theta_i}(\theta_i|\alpha_i, \alpha_{-i}) &\equiv \sum_{\theta_{-i}} \gamma(\theta_i, \theta_{-i}|\alpha_i, \alpha_{-i}) \\ &= \sum_{v_{-i}} \sum_{s_{-i}} \pi(v_i, v_{-i}) \sigma(s_i, s_{-i}|\alpha_i(v_i), \alpha_{-i}(v_{-i}), v_i, v_{-i}).\end{aligned}$$

Finally, let $\Theta_i(\alpha) = \{\theta_i : \gamma_{\Theta_i}(\theta_i|\alpha) > 0\}$ be the set of $\theta_i$ that arises with positive probability under $\alpha$.

*Remark* 1. Our framework is general enough to incorporate mixed strategies over a finite set of actions as follows. Fix a finite set of actions $A^f$, a finite set of signals $S^f$, and an information acquisition function $\Sigma^f : A^f \times V \mapsto \Delta S^f$. Now suppose that each $i$ may randomise among actions in $A_i^f$. This is equivalent to each $i$ choosing a pure strategy from $A_i = \Delta A_i^f$, and receiving a signal from $S_i = S_i^f \times A_i^f$. Then $\Sigma : A \times V \mapsto \Delta S$ is given by $\sigma(s|a, v) = Pr(a^f|a)\sigma^f(s^f|a^f, v)$.

A direct mechanism $(x, t)$ is an allocation rule $x : V \times S \mapsto \Delta I$ and a transfer rule $t : V \times S \mapsto R^n$. The timing is as follows:

1. Seller commits to a mechanism
2. Each buyer observes the mechanism and $v_i$
3. Each buyer chooses $a_i$
4. Each buyer observes $s_i$ and reports $\theta_i = (v_i, s_i)$ to seller
5. Seller implements the mechanism and payoffs are realised.

Note that the seller commits to a mechanism first, and then the buyers choose their information acquisition actions. This is unlike the approach in the robust mechanism design literature where the seller considers the worst case information structure, which can be interpreted as the result of an optimal choice by buyers who commit to an information structure before the seller chooses the mechanism.

For $\theta_i \in \Theta_i(\alpha)$ and $\theta_i' \in \Theta_i$, define $U_i(\theta_i', \theta_i; \alpha)$ as $i$'s expected utility in the mechanism when Player $i$ reports $\theta_i'$, Player $i$ is type $\theta_i$, and the players are following the information acquisition strategy $\alpha$. That is:

$$U_i(\theta_i', \theta_i; \alpha) = \sum_{\theta_{-i}} \gamma(\theta_{-i}|\theta_i, \alpha) x_i(\theta_i', \theta_{-i}) w_i(v_i) - \sum_{\theta_{-i}} \gamma(\theta_{-i}|\theta_i, \alpha) t_i(\theta_i', \theta_{-i}).$$

Note that $\gamma(\theta_{-i}|\theta_i, \alpha) = \frac{\pi(v_{-i}|v_i)\sigma(s_i, s_{-i}|\alpha(v))}{\sum_{v'_{-i}, s'_{-i}} \pi(v'_{-i}|v_i)\sigma(s_i, s'_{-i}|\alpha_i(v_i), \alpha_{-i}(v'_{-i})))}$. Thus, $U_i(\theta_i', \theta_i; \alpha)$ depends only on $\alpha_i(v_i)$ and $\alpha_{-i}$, but to simplify notation we let $U_i(\theta_i', \theta_i; \alpha)$ depend on the entire strategy $\alpha$. Also note that given $\alpha$, $U_i(\theta_i', \theta_i; \alpha)$ is defined only for $\theta_i$ such that $\gamma_{\Theta_i}(\theta_i|\alpha) > 0$; hence our restriction to $\theta_i \in \Theta_i(\alpha)$. Finally, define $U_i^*(\theta_i; \alpha) = \max_{\theta_i'} U_i(\theta_i', \theta_i; \alpha)$. Note that the maximum exists since $\Theta_i$ is finite.

Given a mechanism $(x, t)$, *buyers optimally acquire information* $\alpha$ if $\alpha$ is a Nash equilibrium of the game given by payoffs:

$$\tilde{u}_i(\alpha_i, \alpha_{-i}) = \sum_{\theta_i \in \Theta_i(\alpha_i, \alpha_{-i})} \gamma_{\Theta_i}(\theta_i|\alpha_i, \alpha_{-i}) U_i^*(\theta_i; \alpha_i, \alpha_{-i}).$$

A mechanism is *incentive compatible* given $\alpha$ if for all $\theta_i \in \Theta_i(\alpha)$ and for all $\theta_i' \in \Theta_i$ :

$$U_i(\theta_i, \theta_i; \alpha) \geq U_i(\theta_i', \theta_i; \alpha).$$

A mechanism is *individually rational* given $\alpha$ if for all $\theta_i \in \Theta_i(\alpha)$:

$$U_i(\theta_i, \theta_i; \alpha) \geq 0.$$

A mechanism *fully extracts rents* given $\alpha$ if for all $\theta_i \in \Theta_i(\alpha)$:

$$U_i(\theta_i, \theta_i; \alpha) = 0.$$

An information structure is $(\Theta, \Pi, (A, \Sigma))$, and for each valuation function $w = (w_1, \ldots, w_n)$, $(\Theta, \Pi, (A, \Sigma), w)$ is the associated allocation problem. Let $\mathcal{W}$ be the set of all valuation functions. A *social choice function* $f : V \mapsto \Delta I$ maps a profile of payoff relevant types to a probability distribution over the set of players.

**Definition 1.** The information structure $(\Theta, \Pi, (A, \Sigma))$ *guarantees full rent extraction* for the social choice function $f : V \mapsto \Delta I$ if for all $w \in \mathcal{W}$, in the allocation problem $(\Theta, \Pi, (A, \Sigma)), w)$, there exists a mechanism $(x, t)$ and an information acquisition strategy $\alpha$ such that:

- $(x,t)$ is incentive compatible, individually rational, and fully extracts rent given $\alpha$
- Buyers optimally acquire information $\alpha$ given $(x,t)$
- $x(v,s) = f(v)$ for all $v, s$ such that $\gamma(v,s|\alpha) > 0$.

We will say that a mechanism $(x,t)$ is *consistent* with $f$ following $\alpha$ if $x(v,s) = f(v)$ for all $v, s$ such that $\gamma(v,s|\alpha) > 0$.

Definition 1 of full rent extraction requires that for a given social choice function $f : V \mapsto \Delta I$, the seller is able to fully extract rent for all possible valuation functions $w$. This approach is standard in the literature (for example, see Crémer and McLean (1988)) and simplifies the conditions on beliefs. Definition 1 is justified by the revelation principle, which we now prove for our environment.

Let $(x', t')$ be an arbitrary mechanism, where $x' : M \mapsto \Delta I$ and $t' : M \mapsto R^n$, and for each $i$, let $\mu_i : V_i \times S_i \times A_i \mapsto M_i$ be a strategy in the mechanism. Following the information acquisition strategy $\alpha$, for $\theta_i \in \Theta_i(\alpha)$, let $U_i'(m_i', \theta_i; \alpha, \mu_{-i})$ be $i$'s payoff when Player $i$ reports $m_i'$, Player $i$ is type $\theta_i$, and the other players are following the strategy $\mu_{-i}$ in the mechanism:

$$\begin{aligned} U_i'(m_i', \theta_i; \alpha, \mu_{-i}) = & \sum_{\theta_{-i}} \gamma(\theta_{-i}|\theta_i, \alpha) x_i(m_i', \mu_{-i}(\theta_{-i}, \alpha_{-i}(v_{-i}))) w_i(v_i) \\ & - \sum_{\theta_{-i}} \gamma(\theta_{-i}|\theta_i, \alpha) t_i(m_i', \mu_{-i}(\theta_{-i}, \alpha_{-i}(v_{-i}))). \end{aligned}$$

Define $m^{**}(\theta_i, \alpha, \mu_{-i}) = \arg\max_{m_i'} U_i'(m_i', \theta_i; \alpha, \mu_{-i})$, and $U_i^{**}(\theta_i; \alpha, \mu_{-i}) = U_i'(m_i, \theta_i; \alpha, \mu_{-i})$ for $m_i \in m^{**}(\theta_i, \alpha, \mu_{-i})$.

**Proposition 1.** *Suppose that there exists an information acquisition strategy $\alpha$, a mechanism $(x', t')$, and for each player $i$, a strategy $\mu_i$ in the mechanism such that for all $\alpha_i' \in A_i^{|V_i|}$:*

1. $\mu_i(\theta_i, \alpha_i(v_i)) \in m^{**}(\theta_i, \alpha, \mu_{-i})$ *for all* $\theta_i \in \Theta_i(\alpha)$
2. $\sum_{\theta_i \in \Theta_i(\alpha)} \gamma_{\Theta_i}(\theta_i|\alpha_i, \alpha_{-i}) U_i^{**}(\theta_i; \alpha, \mu_{-i}) \geq \sum_{\theta_i \in \Theta_i(\alpha_i', \alpha_{-i})} \gamma_{\Theta_i}(\theta_i|\alpha_i', \alpha_{-i}) U_i^{**}(\theta_i; \alpha_i', \alpha_{-i}, \mu_{-i})$.

*Then there exists a direct mechanism $(x,t)$ such that $(x,t)$ is incentive compatible given $\alpha$, buyers optimally acquire information $\alpha$ given $(x,t)$, and $(x(\theta), t(\theta)) = (x'(\mu(\theta, \alpha(v))), t'(\mu(\theta, \alpha(v))))$ for all $\theta \in \Theta$.*

*Proof.* Appendix. □

*Remark* 2. Conditions 1 and 2 in Proposition 1 are equivalent to $(\alpha, \mu)$ being a Nash equilibrium of the game $G' = (A^{|V|} \times M^{|\Theta \times A|}, u')$, where

$$u'(\alpha_i, \mu_i; \alpha_{-i}, \mu_{-i}) = \sum_{\theta_i \in \Theta_i(\alpha_i, \alpha_{-i})} \gamma_{\Theta_i}(\theta_i | \alpha_i, \alpha_{-i}) U_i'(\mu(\theta_i, \alpha_i(v_i)), \theta_i; \alpha_i, \alpha_{-i}, \mu_{-i}).$$

## 4 Examples

In the next section, we will establish necessary and sufficient conditions for the information structure to guarantee full rent extraction. First, we provide some examples.

**Example 1** (Crémer-McLean)**.** Suppose $S = \varnothing$, so there is no possibility of information acquisition. Then $((V, \varnothing), \Pi, (A, \Sigma))$ guarantees full rent extraction for all social choice functions $f : V \mapsto \Delta I$ if and only if for all $v_i$:

$$\pi(V_{-i}|v_i) \notin co\{\pi(V_{-i}|v_i') : v_i' \neq v_i\}.$$ [2]

With information acquisition, $\Pi$ exhibiting correlation is not sufficient for full rent extraction, as the following example shows:

**Example 2.** Let $n = 2$, $V_i = \{v^L, v^H\}$, $S_i = \{\varnothing, (v^L, \varnothing), (v^L, s), (v^H, \varnothing), (v^H, s)\}$, $w_i(v^L) = \underline{v} > 0$, $w_i(v^H) = 1$, $\Pi > 0$ ($\Pi$ can be any full support distribution), $A_i = \{N, Y\}$. If $a_i = N$, then player $i$ receives the null signal. If $a_i = Y$, then with probability $p < 1$ player $i$ receives a signal that perfectly reveals the valuation of her opponent and whether her opponent has received a null signal. Players receive signals independently. Note that for player $i$, types $(v^L, (v^L, \varnothing))$ and $(v^H, (v^L, \varnothing))$ have the same beliefs over $\Theta_{-i}$, namely that player $-i$ has valuation $v^L$ and the null signal. Thus, there exists an allocation rule where type $(v^H, (v^L, \varnothing))$ of player $i$ must receive positive rents. Let $x_i((v^L, (v^L, \varnothing)), (v^L, \varnothing)) \equiv x > 0$. Since type $(v^L, (v^L, \varnothing))$ of player $i$ knows the type of player $-i$ for sure, individual rationality implies $t_i((v^L, (v^L, \varnothing)), (v^L, \varnothing)) \leq x\underline{v}$. Now a possible deviation is for player $i$ to choose $\alpha_i(v^H) = Y$, drop out for all types other than $(v^H, (v^L, \varnothing))$, and for type $(v^H, (v^L, \varnothing))$, report $(v^L, (v^L, \varnothing))$. The utility of this deviation is at least $\pi(v^H)\pi(v^L|v^H)p(1-p)x(1-\underline{v}) > 0$.

[2] $\pi(V_{-i}|v_i) = (\pi(v_{-i}^1|v_i), \dots, \pi(v_{-i}^{|V_{-i}|}|v_i))$.

On the other hand, with information acquisition, sometimes full rent extraction is possible even when $\Pi$ is independent:

**Example 3.** Let $n = 2$, $V_i = \{v^L, v^H\}$, $S_i = \{v^L, v^H\}$, $\Pi$ uniform, $A_i = \{N, Y\}$. If $a_i = N$, then player $i$'s signal is uninformative. If $a_i = Y$, player $i$'s signal perfectly reveals the valuation of her opponent. That is, let $\sigma(s_i = v_{-i}, s_{-i} = v_i | v_i, v_{-i}, (Y, Y)) = 1$, $\sigma(s_i = v_{-i}, s_{-i} = v_i | v_i, v_{-i}, (N, Y)) = \frac{1}{2}$, $\sigma(s_i \neq v_{-i}, s_{-i} = v_i | v_i, v_{-i}, (N, Y)) = \frac{1}{2}$. For any $x$ and for any $w$, the following mechanism is incentive compatible, individually rational, and fully extracts rent given $\alpha_i(v_i) = Y$ for all $v_i$, and $\alpha_i(v_i) = Y$ for all $v_i$ is optimal for both players:

- $t_i((v_i, s_i), (v_{-i}, s_{-i})) = x_i((v_i, s_i), (v_{-i}, s_{-i}))w_i(v_i)$ if $v_i = s_{-i}$ and $s_i = v_{-i}$
- $t_i((v_i, s_i), (v_{-i}, s_{-i})) = \infty$ otherwise.

# 5 Main Result

In this section, we will characterise the necessary and sufficient conditions for the information structure $(\Theta, \Pi, (A, \Sigma))$ to guarantee full rent extraction.

Let $\gamma(\Theta_{-i}|\theta_i, \alpha) \equiv \left(\gamma(\theta^1_{-i}|\theta_i, \alpha), \gamma(\theta^2_{-i}|\theta_i, \alpha), \ldots, \gamma(\theta^{|\Theta_{-i}|}_{-i}|\theta_i, \alpha)\right)$ be the belief of type $\theta_i$ about $\Theta_{-i}$, given the profile of information acquisition strategies $\alpha$. Given $\alpha$ and $f$, for each $\theta_i$ we can define the subset of $\Theta_{-i}$ that arises with positive probability following $\alpha$, and the subset of $\Theta_{-i}$ that arises with positive probability following $\alpha$ where $\theta_i$ is allocated the object according to $f$. These sets are important because they specify the states of the world where $\theta_i$ must receive the object following $\alpha$ in any mechanism that is consistent with $f$. Define:

$$M_i(\theta_i, \alpha) = \{\theta_{-i} : \gamma(\theta|\alpha) > 0\}$$
$$M_i^f(\theta_i, \alpha) = \{\theta_{-i} : \gamma(\theta|\alpha) f_i(v_i, v_{-i}) > 0\}.$$

That is, $M_i(\theta_i, \alpha)$ is the subset of $\Theta_{-i}$ that type $\theta_i$ believes occurs with positive probability following $\alpha$, and $M_i^f(\theta_i, \alpha)$ is the subset of $M_i(\theta_i, \alpha)$ for which $\theta_i$ receives the object with positive probability according to $f$. Note that $\theta_i \in \Theta_i(\alpha)$ if and only if $M_i(\theta_i, \alpha) \neq \varnothing$. Let:

$$\Theta_i^f(\alpha) = \left\{\theta_i : M_i^f(\theta_i, \alpha) \neq \varnothing\right\}.$$

That is, $\Theta_i^f(\alpha)$ is the subset of $\Theta_i(\alpha)$ such that buyer $i$ that receives the object with positive probability according to $f$ following $\alpha$. Note that following $\alpha$, any mechanism such that $x(\theta) = f(v)$ for all $\theta$ such that $\gamma(\theta|\alpha) > 0$ (i.e. any mechanism that is consistent with $f$) must have $x_i(\theta_i, \theta_{-i}) > 0$ whenever $\theta_{-i} \in M_i^f(\theta_i, \alpha)$.

Let $\succeq_i$ be a complete and transitive binary relation over $V_i$, and let $\mathcal{R}_i$ be the set of all complete and transitive binary relations over $V_i$. Let $\succeq = (\succeq_1, \ldots, \succeq_n)$, and $\mathcal{R} = \mathcal{R}_1 \times \ldots \times \mathcal{R}_n$. Define:

- $C(\alpha_{-i}) = co\{\gamma(\Theta_{-i}|\theta_i', \alpha_i', \alpha_{-i}) : \alpha_i' \in A_i^{|V_i|}, \theta_i' \in \Theta_i(\alpha_i', \alpha_{-i})\}$
- $D(\theta_i, \alpha_{-i}, \succeq_i, M) = \left\{ \gamma(\Theta_{-i}|\theta_i', \alpha_i', \alpha_{-i}) : \begin{array}{c} \alpha_i' \in A_i^{|V_i|}, \theta_i' \in \Theta_i(\alpha_i', \alpha_{-i}), v_i' \succ_i v_i, \\ \sum_{\theta_{-i} \in M} \gamma(\theta_{-i}|\theta_i', \alpha_i', \alpha_{-i}) > 0 \end{array} \right\}$
- $v_i^*(\succeq_i) = \{v_i : v_i \succeq_i v_i' \text{ for all } v_i' \in V_i\}$

$C(\alpha_{-i})$ is the convex hull of the beliefs of every type of buyer $i$ that could arise from any information acquisition strategy of buyer $i$, fixing the information acquisition strategy of the other buyers at $\alpha_{-i}$. $D(\theta_i, \alpha_{-i}, \succeq_i, M)$ is the set of beliefs of every type of buyer $i$ with a valuation strictly greater than $v_i$ under $\succeq_i$ and according to which the probability that $\theta_{-i} \in M$ is strictly positive, arising from any information acquisition strategy of buyer $i$, fixing the information acquisition strategy of the other buyers at $\alpha_{-i}$. Note that $D(\theta_i, \alpha_{-i}, \succeq_i, M) \subset C(\alpha_{-i})$. If $A_i$ is finite, then $C(\alpha_{-i})$ is a polytope. If $A_i$ is compact and $\Sigma$ is continuous and has full support, then $C(\alpha_{-i})$ and $D(\theta_i, \alpha_{-i}, \succeq_i, M)$ are compact. $v_i^*(\succeq_i)$ is the set containing the greatest elements of $V_i$ under the relation $\succeq_i$.

For any $X \in R^n$, let $\overline{X}$ denote the closure of $X$, and let $\text{ri}(X)$ denote the relative interior of $X$. Recall that for any $X \subset R^n$ and $\bar{x} \in X$, $p \neq 0$ *supports* $X$ at $\bar{x}$ if $p \cdot y \geq p \cdot \bar{x}$ for all $y \in X$. The hyperplane $\{y \in R^n : p \cdot y = p \cdot x\}$ is a *supporting hyperplane* for $X$ at $\bar{x}$, and the support is *proper* if $p \cdot y > p \cdot \bar{x}$ for some $y \in X$.

**Definition 2.** For any convex set $K \subset R^n$, $F$ is an *exposed face* of $K$ if $F$ satisfies any of the following:

1. There exists a hyperplane $H$ supporting $K$ at some $\bar{x} \in K$ and $F = K \cap H$
2. $F = K$

3. $F = \varnothing$.

If $F \neq K \neq \varnothing$, then $F$ is a *proper exposed face* of $K$.

**Definition 3.** For any convex set $K \subset R^n$ and $\bar{x} \in K$, let $F_K(\bar{x})$ be the intersection of all exposed faces of $K$ containing $\bar{x}$.

**Proposition 2.** *The information structure $(\Theta, \Pi, (A, \Sigma))$ guarantees full rent extraction for the social choice function $f : V \mapsto \Delta I$ if and only if for every $\succeq \in \mathcal{R}$, there exists an $\alpha$ such that for all $i$ and for all $\theta_i \in \Theta_i^f(\alpha) \setminus (v_i^*(\succeq_i) \times S_i)$:*

1. $\gamma(\Theta_{-i}|\theta_i, \alpha) \cap \mathrm{ri}(C(\alpha_{-i})) = \varnothing$
2. $F_{\overline{C}(\alpha_{-i})}(\gamma(\Theta_{-i}|\theta_i, \alpha)) \cap \overline{D}(\theta_i, \alpha_{-i}, \succeq_i, M_i^f(\theta_i, \alpha)) = \varnothing$.[3]

*Proof.* Appendix. □

Condition 1 requires that $\theta_i$'s belief does not lie in the relative interior of $C(\alpha_{-i})$, which is the convex hull of the beliefs all of types of buyer $i$ that could arise from any information acquisition strategy of buyer $i$, fixing the information acquisition strategy of the other buyers at $\alpha_{-i}$. This is equivalent to the existence of a hyperplane properly supporting $C(\alpha_{-i})$ at $\gamma(\Theta_{-i}|\theta_i, \alpha)$. This requirement is stronger than the analogous condition in the standard model to the extent that there can be (possibly infinitely) many beliefs that arise from any information acquisition strategy, even when there are a small number of types that arise in equilibrium.

Condition 2 requires that the smallest exposed face of the closure of $C(\alpha_{-i})$ that contains $\gamma(\Theta_{-i}|\theta_i, \alpha)$ does not intersect with the closure of the set of beliefs of all types of buyer $i$ that could arise from any information acquisition strategy (fixing $\alpha_{-i}$) such that buyer $i$'s valuation is strictly greater than $v_i$ and buyer $i$ believes $M_i^f(\theta_i, \alpha)$ occurs with positive probability. By definition, there is a hyperplane that supports $C(\alpha_{-i})$ at $\gamma(\Theta_{-i}|\theta_i, \alpha)$ and intersects $C(\alpha_{-i})$ at exactly this exposed face. Condition 2 is equivalent to this hyperplane strongly separating $\gamma(\Theta_{-i}|\theta_i, \alpha)$ from any belief in $C(\alpha_{-i})$ with a valuation strictly greater than $v_i$ under $\succeq_i$ and according to which $M_i^f(\theta_i, \alpha)$ occurs with positive probability. This ensures that such a type can be prevented from pretending to be $\theta_i$. Example 2 illustrates a situation where Condition 2 fails.

[3] If we assume that $A_i$ is compact and $\sigma$ is continuous, then $C(\alpha_{-i})$ and $D(\theta_i, \alpha_{-i}, \succeq_i, M_i^f(\theta_i, \alpha))$ are closed, so we can drop the closure from the conditions. When Condition 1 is satisfied, there is always a hyperplane properly supporting $\overline{C}(\alpha_{-i})$ at $\gamma(\Theta_{-i}|\theta_i, \alpha)$, so $F_{\overline{C}(\alpha_{-i})}(\gamma(\Theta_{-i}|\theta_i, \alpha))$ is a proper exposed face of $\overline{C}(\alpha_{-i})$. On the other hand when Condition 1 fails, $F_{\overline{C}(\alpha_{-i})}(\gamma(\Theta_{-i}|\theta_i, \alpha)) = \overline{C}(\alpha_{-i})$, and Condition 2 cannot hold.

In that example, types $(l,(l,\varnothing))$ and $(h,(l,\varnothing))$ have the same beliefs; in particular, $(h,(l,\varnothing))$'s belief is contained in the smallest exposed face of the closure of $C(\alpha_{-i})$ containing $(l,(l,\varnothing))$'s belief.

For each $\succeq$, we require the existence of an $\alpha$ such that the conditions hold for every type $\theta_i$ such that $v_i$ is not the greatest element of $V_i$ under $\succeq_i$, and $\theta_i$ receives the object with positive probability in any mechanism that is consistent with $f$ following $\alpha$. If we want to guarantee full rent extraction for every $f$, then we can drop the requirement that $\theta_i$ receives the object with positive probability according to a particular $f$. Corollary 1 follows from the observation that $\Theta_i(\alpha) \supset \Theta_i^f(\alpha)$ and $M_i(\theta_i,\alpha) \supset M_i^f(\theta_i,\alpha)$, with equality when $f_i(v_i,v_{-i}) > 0$ for all $v_i, v_{-i}$.

**Corollary 1.** *The information structure $(\Theta,\Pi,(A,\Sigma))$ guarantees full rent extraction for every social choice function $f: V \mapsto \Delta I$ if and only if for every $\succeq \in \mathcal{R}$ there exists an $\alpha$ such that for all $i$ and for all $\theta_i \in \Theta_i(\alpha) \setminus (v_i^*(\succeq_i) \times S_i)$:*

1. $\gamma(\Theta_{-i}|\theta_i,\alpha) \cap \mathrm{ri}(C(\alpha_{-i})) = \varnothing$
2. $F_{\overline{C}(\alpha_{-i})}(\gamma(\Theta_{-i}|\theta_i,\alpha)) \cap \overline{D}(\theta_i,\alpha_{-i},\succeq_i,M_i(\theta_i,\alpha)) = \varnothing.$

*Remark* 3. In the standard case without information acquisition $S=\varnothing$, $\gamma(\Theta_{-i}|\theta_i,\alpha) = \pi(V_{-i}|v_i)$, and hence Corollary 1 requires:

1. For all $\succeq \in \mathcal{R}$ and for all $v_i \in V_i \setminus v_i^*(\succeq_i)$, $\pi(V_{-i}|v_i) \cap \mathrm{ri}(co\{\pi(V_{-i}|v_i') : v_i' \in V_i\}) = \varnothing$
2. For all $\succeq \in \mathcal{R}$ and for all $v_i \in V_i \setminus v_i^*(\succeq_i)$, $F_{co\{\pi(V_{-i}|v_i'):v_i' \in V_i\}}(\pi(V_{-i}|v_i)) \cap \{\pi(V_{-i}|v_i') : v_i' \succ_i v_i\} = \varnothing$.

Note that this is equivalent to $\pi(V_{-i}|v_i) \cap \mathrm{ri}(co\{\pi(V_{-i}|v_i') : v_i' \in V_i\}) = \varnothing$ for all $v_i \in V_i$ and $F_{co\{\pi(V_{-i}|v_i'):v_i' \in V_i\}}(\pi(V_{-i}|v_i)) \cap \{\pi(V_{-i}|v_i') : v_i' \neq v_i\} = \varnothing$ for all $v_i \in V_i$. Together they are equivalent to $\pi(V_{-i}|v_i) \cap co\{\pi(V_{-i}|v_i') : v_i' \neq v_i\} = \varnothing$, as we have stated in Example 1.

*Remark* 4. In Definition 1, the requirement that full rent extraction is possible for all $w$ simplifies the conditions on beliefs in the standard model. In particular, as the last remark shows, the conditions on beliefs do not depend on the valuation of the type holding each belief. However, this reduction is not possible in our setting, since for each permutation of the valuations, the seller can induce a different information acquisition strategy to fully extract rent; hence for each information acquisition strategy, the conditions on beliefs do not need to hold for every permutation of the valuations.

### 5.1 Proof Intuition

The intuition behind sufficiency is as follows. Condition 1 implies that for each $\theta_i \in \Theta_i^f(\alpha) \setminus (v_i^*(\succeq_i) \times S_i)$, there is a hyperplane $\{\gamma : \tau \cdot \gamma = \tau \cdot \gamma(\Theta_{-i}|\theta_i, \alpha)\}$ supporting $C(\alpha_{-i})$ at $\gamma(\Theta_{-i}|\theta_i, \alpha)$. That is, there exists a lottery $\tau(\theta_i)$ that has weakly positive expected value for all types $\theta_i'$ arising from any information acquisition strategy. Condition 2 implies that $\tau(\theta_i)$ can be chosen so that for all $\theta_i'$ with $w(v_i') > w(v_i)$ and $\sum_{\theta_{-i} \in M_i^f(\theta_i,\alpha)} \gamma(\theta_{-i}|\theta_i', \alpha_i', \alpha_{-i}) > 0$, there exists an $\varepsilon > 0$ such that the expected value of $\tau(\theta_i)$ for type $\theta_i'$ is strictly larger than $\varepsilon$—that is, the hyperplane $\tau(\theta_i)$ strongly separates the belief of $\theta_i$ from the beliefs of types $\theta_i'$ such that $w(v_i') > w(v_i)$ and according to which $M_i^f(\theta_i, \alpha)$ occurs with positive probability.

For each $\theta \in \Theta$ such that $\gamma(\theta|\alpha) > 0$, let $x_i(\theta_i, \theta_{-i}) = f_i(v_i, v_{-i})$; otherwise let $x_i(\theta_i, \theta_{-i}) = 0$. Note that $x_i(\theta_i, \theta_{-i}) > 0$ if and only if $\theta_{-i} \in M_i^f(\theta_i, \alpha)$. Now for each $\theta_i \notin \Theta_i^f(\alpha)$, we can let $t_i(\theta_i, \Theta_{-i}) = 0$,[4] in which case $U_i(\theta_i, \theta_i', \alpha_i', \alpha_{-i}) = 0$ for all $\alpha_i' \in A_i^{|V_i|}$, $\theta_i' \in \Theta_i(\alpha_i', \alpha_{-i})$. This is saying that no type has an incentive to deviate by pretending to be a type that does not receive the object. For $\theta_i \in (v_i^*(\succeq_i) \times S_i) \cap \Theta_i^f(\alpha)$, we can let $t_i(\theta_i, \Theta_{-i}) = w_i(v_i)x_i(\theta_i, \Theta_{-i})$.[5] Intuitively, if types with the highest valuation pay their valuation for the object whenever they receive the object, they receive no rents, and no other type would want to pretend to have the highest valuation as these other types value the object even less.

For types that do receive the object in equilibrium and do not have the highest valuation, i.e. for $\theta_i \in \Theta_i^f(\alpha) \setminus (v_i^*(\succeq_i) \times S_i)$, we can add a scaled up version of the lottery $\tau(\theta_i)$ to the payment. So let $t_i(\theta_i, \Theta_{-i}) = w_i(v_i)x_i(\theta_i, \Theta_{-i}) + K\tau(\theta_i)$ for sufficiently large $K$. Now following $\alpha_i' \in A_i^{|V_i|}$, any type $\theta_i'$ such that $w_i(v_i') > w_i(v_i)$ and $\sum_{\theta_{-i} \in M_i^f(\theta_i,\alpha)} \gamma(\theta_{-i}|\theta_i', \alpha_i', \alpha_{-i}) > 0$ will not report $\theta_i$, since the expected payment includes $K\tau(\theta_i) \cdot \gamma(\Theta_{-i}|\theta_i', \alpha_i', \alpha_{-i})$, which can be made arbitrarily large. For types $\theta_i'$ such that $w_i(v_i') \leq w_i(v_i)$ or $\sum_{\theta_{-i} \in M_i^f(\theta_i,\alpha)} \gamma(\theta_{-i}|\theta_i', \alpha_i', \alpha_{-i}) = 0$, it is possible that $\tau(\theta_i) \cdot \gamma(\Theta_{-i}|\theta_i', \alpha_i', \alpha_{-i}) = 0$, but in that case $U_i(\theta_i, \theta_i', \alpha_i', \alpha_{-i}) \leq 0$, since:

$$U_i(\theta_i, \theta_i', \alpha', \alpha_{-i}) = \underbrace{(w_i(v_i') - w_i(v_i))x_i(\theta_i, \Theta_{-i}) \cdot \gamma(\Theta_{-i}|\theta_i', \alpha', \alpha_{-i})}_{\leq 0} - \underbrace{K\tau(\theta_i) \cdot \gamma(\Theta_{-i}|\theta_i', \alpha', \alpha_{-i})}_{=0}.$$

Note that $\sum_{\theta_{-i} \in M_i^f(\theta_i,\alpha)} \gamma(\theta_{-i}|\theta_i', \alpha_i', \alpha_{-i}) = 0$ implies $x_i(\theta_i, \Theta_{-i}) \cdot \gamma(\theta_{-i}|\theta_i', \alpha_i', \alpha_{-i}) = 0$.

For necessity, first note that for any $\theta_i$, it is without loss of generality to let $t_i(\theta_i, \Theta_{-i}) =$

[4] $t_i(\theta_i, \Theta_{-i}) = (t_i(\theta_i, \theta_{-i}^1), \ldots, t_i(\theta_i, \theta_{-i}^{|\Theta_{-i}|}))$.

[5] $x_i(\theta_i, \Theta_{-i}) = (x_i(\theta_i, \theta_{-i}^1), \ldots, x_i(\theta_i, \theta_{-i}^{|\Theta_{-i}|}))$.

$w_i(v_i)x_i(\theta_i, \Theta_{-i}) + K\tau(\theta_i)$ for some $\tau(\theta_i)$. For the mechanism to fully extract rent, we need $\tau(\theta_i) \cdot \gamma(\Theta_{-i}|\theta_i, \alpha) = 0$. If Condition 1 fails, then there exists a type $\theta_i \in \Theta_i^f(\alpha) \setminus (v_i^*(\succeq_i) \times S_i)$ such that for any $\tau(\theta_i)$, either $\gamma(\Theta_{-i}|\theta_i', \alpha_i', \alpha_{-i}) \cdot \tau(\theta_i) = 0$ for all $\alpha_i' \in A_i^{|V_i|}$, $\theta_i' \in \Theta_i(\alpha_i', \alpha_{-i})$, or there exists $\alpha_i' \in A_i^{|V_i|}$, $\theta_i' \in \Theta_i(\alpha_i', \alpha_{-i})$ such that $\gamma(\Theta_{-i}|\theta_i', \alpha_i', \alpha_{-i}) \cdot \tau(\theta_i) < 0$. But incentive compatibility requires that $\gamma(\Theta_{-i}|\theta_i^*, \alpha) \cdot \tau(\theta_i) > 0$ for some $\theta_i^* \in v_i^*(\succeq_i) \times S_i$, and for $w_i$ such that $\max_{v_i'} w_i(v_i')$ is sufficiently large, $K$ must also be large. This means that there must exist $\alpha_i' \in A_i^{|V_i|}$, $\theta_i' \in \Theta_i(\alpha_i', \alpha_{-i})$ such that $\gamma(\Theta_{-i}|\theta_i', \alpha_i', \alpha_{-i}) \cdot \tau(\theta_i) < 0$. Then the utility from the following deviation is strictly positive: choose $\alpha_i'$, drop out for all types other than $\theta_i'$, and report $\theta_i$ when type $\theta_i'$, since the expected payment from reporting $\theta_i$ includes the term $K\tau(\theta_i) \cdot \gamma(\Theta_{-i}|\theta_i', \alpha_i', \alpha_{-i})$ which is large and negative for large $K$.

If Condition 2 fails, then there exists a type $\theta_i \in \Theta_i^f(\alpha) \setminus (v_i^*(\succeq_i) \times S_i)$ such that for any $\tau(\theta_i)$ that supports $C_i(\alpha_{-i})$ at $\gamma(\Theta_{-i}|\theta_i, \alpha)$, for every $\varepsilon > 0$, there exists an information acquisition strategy $\alpha_i' \in A_i^{|V_i|}$ such that for some type $\theta_i' \in \Theta_i(\alpha_i', \alpha_{-i})$ with $w(v_i') > w_i(v_i)$ and $\sum_{\theta_{-i} \in M_i^f(\theta_i, \alpha)} \gamma(\theta_{-i}|\theta_i' \alpha_i', \alpha_{-i}) > 0$, the expected value of $\tau(\theta_i)$ after having chosen $\alpha_i'$ is less than $\varepsilon$. Thus, $U_i(\theta_i, \theta_i', \alpha', \alpha_{-i}) > 0$, since:

$$U_i(\theta_i, \theta_i', \alpha', \alpha_{-i}) = \underbrace{(w_i(v_i') - w_i(v_i))x_i(\theta_i, \Theta_{-i}) \cdot \gamma(\Theta_{-i}|\theta_i', \alpha', \alpha_{-i})}_{>0} - \underbrace{K\tau(\theta_i) \cdot \gamma(\Theta_{-i}|\theta_i', \alpha', \alpha_{-i})}_{\leq \varepsilon}.$$

Then a profitable deviation is to choose $\alpha_i'$, drop out for all types other than $\theta_i'$, and report $\theta_i$ when type $\theta_i'$, since the expected payment is very close to the product of the probability of winning the object and the valuation of type $\theta_i$, which is strictly less than the valuation of type $\theta_i'$.

### 5.2 Simple Conditions

We now give simpler sufficient conditions and necessary conditions that depend only on the beliefs, and not the valuations of the types holding each belief. Define:

- $D^*(\theta_i, \alpha_{-i}, M) = \cup_{\succeq_i \in \mathcal{R}_i} D(\theta_i, \alpha_{-i}, \succeq_i, M)$
- $B(v_i, \alpha_{-i}, M) = \left\{ \gamma(\Theta_{-i}|\theta_i', \alpha_i', \alpha_{-i}) : \begin{array}{c} \alpha_i' \in A_i^{|V_i|}, \theta_i' \in \Theta_i(\alpha_i', \alpha_{-i}), v_i' = v_i, \\ \sum_{\theta_{-i} \in M} \gamma(\theta_{-i}|\theta_i', \alpha_i', \alpha_{-i}) > 0 \end{array} \right\}.$

$D^*(\theta_i, \alpha_{-i}, M)$ is the set of beliefs of every type of buyer $i$ with a valuation different from $v_i$ and according to which $M$ occurs with positive probability, arising from any information acquisition strategy of buyer $i$, fixing the information acquisition strategy of the other buyers at $\alpha_{-i}$. $B(v_i, \alpha_{-i}, M)$ is the set of beliefs of every types of buyer $i$ with valuation $v_i$ according to which $M$ occurs with positive probability, arising from any information acquisition strategy of buyer $i$, fixing the information acquisition strategy of the other buyers at $\alpha_{-i}$.

**Corollary 2** (Sufficient Condition)**.** *The information structure* $(\Theta, \Pi, (A, \Sigma))$ *guarantees full rent extraction for every social choice function* $f : V \mapsto \Delta I$ *if there exists an* $\alpha$ *such that for all* $i$*:*

1. *For all* $\theta_i \in \Theta_i(\alpha)$*,* $\gamma(\Theta_{-i}|\theta_i, \alpha) \cap \mathrm{ri}(C(\alpha_{-i})) = \varnothing$
2. *For all* $\theta_i \in \Theta_i(\alpha)$*,* $F_{\overline{C}(\alpha_{-i})}(\gamma(\Theta_{-i}|\theta_i, \alpha)) \cap \overline{D^*}(\theta_i, \alpha_{-i}, M_i(\theta_i, \alpha)) = \varnothing$.

**Corollary 3** (Necessary Condition)**.** *The information structure* $(\Theta, \Pi, (A, \Sigma))$ *guarantees full rent extraction for every social choice function* $f : V \mapsto \Delta I$ *only if there exists an* $\alpha$ *such that for all* $i$ *and for some* $v_i'$*:*

1. *For all* $\theta_i \in \Theta_i(\alpha) \setminus (\{v_i'\} \times S_i)$*,* $\gamma(\Theta_{-i}|\theta_i, \alpha) \cap \mathrm{ri}(C(\alpha_{-i})) = \varnothing$
2. *For all* $\theta_i \in \Theta_i(\alpha) \setminus (\{v_i'\} \times S_i)$*,* $F_{\overline{C}(\alpha_{-i})}(\gamma(\Theta_{-i}|\theta_i, \alpha)) \cap \overline{B}(v_i', \alpha_{-i}, M_i(\theta_i, \alpha)) = \varnothing$.

## 6 Generic Impossibility of Full Rent Extraction

In this section, we will argue that the set of information structures where the seller can fully extract rent is small in a topological sense. We make the following assumptions:

**A1**. $A_i = [0, 1]$

**A2**. $\Sigma : A \times V \mapsto \Delta S$ is continuous

**A3**. For all $v \in V$, $s \in S$ and $a \in A$, $\sigma(s|a, v) > 0$

**A4**. There exists $i \in I$ with $|V_i| \geq 2$ and $|S_i| > 2$

Fix $\Theta$, $\Pi$, and $A$. Let $\mathcal{T}$ be the set of all continuous $\Sigma : A \times V \mapsto \Delta S$ such that $\sigma(s|a, v) > 0$ for all $a \in A$, $v \in V$, $s \in S$, and equip $\mathcal{T}$ with the topology defined by the metric $d(f, g) = \sup_{a,v} ||f(a, v) - g(a, v)||$ (i.e. the topology of uniform convergence). Let $\mathcal{T}^*$ be the subset of $\mathcal{T}$ such that $\Sigma \in \mathcal{T}^*$ if and only if $(\Theta, \Pi, (A, \Sigma))$ guarantees full rent extraction for every social choice function.

**Proposition 3.** *Assume* **A1–A4**. *Then* $\overline{\mathcal{T}^*}$ *has empty interior (and hence* $\mathcal{T}^*$ *is nowhere dense) in the topology of uniform convergence.*

*Proof.* Appendix. $\square$

Proposition 3 is equivalent to the statement that under **A1–A4**, the set of information structures where the seller cannot fully extract rent is an open and dense subset of the set of all information structures. In this sense, full rent extraction fails "generically" in a model with information acquisition. **A2** and **A3** are important for the result. We conjecture that the proposition holds for any connected and compact topological action space.

If $|V_i| = 1$ for all $i \in I$, then the valuations are common knowledge and the seller can fully extract rent. If $|S_i| = 1$ for all $i \in I$, then signals are uninformative and we are in the standard model, where full rent extraction is generically possible. When $|S_i| > 2$ for some $i \in I$, Proposition 3 implies that full rent extraction is generically impossible; we conjecture that Proposition 3 remains true if we relax this requirement to $|S_i| > 1$ for some $i \in I$.

### 6.1 Proof Intuition

The first step of the proof is to show that if $\Sigma \in \overline{\mathcal{T}^*}$, then for any $\succeq$ there must exist an information strategy $\alpha$ such that for all $i$ and for all $\theta_i \in \Theta_i \setminus (v_i^*(\succeq_i) \times S_i)$, $\gamma(\Theta_{-i}|\theta_i, \alpha) \notin \text{int}(C(\alpha_{-i}))$. Note that this is weaker than than Condition 1 of Corollary 1, which under the full support assumption requires that for all $\theta_i \in \Theta_i \setminus (v_i^*(\succeq_i) \times S_i)$, $\gamma(\Theta_{-i}|\theta_i, \alpha) \notin \text{ri}(C(\alpha_{-i}))$. Then for any $\Sigma$ and for any $\varepsilon > 0$, we construct a $\hat{\Sigma}$ such that $d(\hat{\Sigma}, \Sigma) \leq \varepsilon$ and $\hat{\Sigma} \in \mathcal{T} \setminus \overline{\mathcal{T}^*}$ by defining $\hat{\Sigma}$ so that for some $\succeq$, for every information strategy $\alpha$, there exists $\theta_i \in \Theta_i \setminus (v_i^*(\succeq_i) \times S_i)$ such that $\hat{\gamma}(\Theta_{-i}|\theta_i, \alpha) \in \text{int}(\hat{C}(\alpha_{-i}))$.[6] Thus, the closure of the set of information structures that allow full rent extraction has empty interior.

The intuition for the construction of $\hat{\Sigma}$ is as follows. Without loss of generality, let $|V_1| \geq 2$ and $|S_1| > 2$. Let $\succeq_1$ be such that $v_1^1 \notin v_1^*(\succeq_1)$. $\hat{\Sigma}(a, v)$ will differ from $\Sigma(a, v)$ only when $v_1 = v_1^1$. For $v_1^1$, first we fix the value of $\hat{\Sigma}(a, (v_1^1, v_{-1}))$ to be equal to $\Sigma(a, (v_1^1, v_{-1}))$ whenever $a_1 \in \{a_1^1, \ldots, a_1^K\}$, where $a_1^1 = 0$, $a_1^K = 1$, $a_1^{k+1} - a_1^k = \delta$, and $\delta$ is chosen so that $|a_1 - a_1'| \leq \delta \implies \sup_{s, a_{-1}, v_{-1}} |\sigma(s|a_1, a_{-1}, v_1^1, v_{-1}) - \sigma(s|a_1', a_{-1}, v_1^1, v_{-1})| \leq \frac{\varepsilon}{2|\Theta_{-1}|^2}$. Such a $\delta$ exists since $\Sigma$ is continuous and $A$ is a compact set; thus, $\Sigma$ is uniformly continuous on $A$.

[6] $\hat{\gamma}(\Theta_{-i}|\theta_i, \alpha)$ and $\hat{C}(\alpha_{-i})$ denote the beliefs under $\hat{\Sigma}$.

Define $y_1(a_1, \alpha_{-1})$ and $y_2(a_1, \alpha_{-1})$ as the vector of joint probabilities for signals $s_1^1$ and $s_2^2$ respectively according to $\hat{\Sigma}$, when $v_1 = v_1^1$:

$$y_1(a_1, \alpha_{-1}) \equiv \Big( \hat{\sigma}(s_1^1, s_{-1}^1 | a_1, \alpha_{-1}(v_{-1}^1), v_1^1, v_{-1}^1), \ldots, \hat{\sigma}(s_1^1, s_{-1}^{|S_{-1}|} | a_1, \alpha_{-1}(v_{-1}^1), v_1^1, v_{-1}^1), \ldots$$

$$\ldots, \hat{\sigma}(s_1^1, s_{-1}^1 | a_1, \alpha_{-1}(v_{-1}^{|V_{-1}|}), v_1^1, v_{-1}^{|V_{-1}|}), \ldots, \hat{\sigma}(s_1^1, s_{-1}^{|S_{-1}|} | a_1, \alpha_{-1}(v_{-1}^{|V_{-1}|}), v_1^1, v_{-1}^{|V_{-1}|}) \Big)$$

$$y_2(a_1, \alpha_{-1}) \equiv \Big( \hat{\sigma}(s_1^2, s_{-1}^1 | a_1, \alpha_{-1}(v_{-1}^1), v_1^1, v_{-1}^1), \ldots, \hat{\sigma}(s_1^2, s_{-1}^{|S_{-1}|} | a_1, \alpha_{-1}(v_{-1}^1), v_1^1, v_{-1}^1), \ldots$$

$$\ldots, \hat{\sigma}(s_1^2, s_{-1}^1 | a_1, \alpha_{-1}(v_{-1}^{|V_{-1}|}), v_1^1, v_{-1}^{|V_{-1}|}), \ldots, \hat{\sigma}(s_1^2, s_{-1}^{|S_{-1}|} | a_1, \alpha_{-1}(v_{-1}^{|V_{-1}|}), v_1^1, v_{-1}^{|V_{-1}|}) \Big).$$

Note that each element of $y_j(a_1, \alpha_{-1})$ is the joint probability of $(s_1^j, s_{-1})$ for some $s_{-1} \in S_{-1}$, when the profile of valuations is $(v_1^1, v_{-1})$ for some $v_{-1} \in V_{-1}$, and the action profile is $(a_1, \alpha_{-1}(v_{-1}))$.

Recall that an $n$-dimensional *cross polytope* centred around $x \in R^n$ with circumradius $\frac{\varepsilon}{2}$ is the convex hull of the points $\{x \pm (\frac{\varepsilon}{2}, 0, \ldots), x \pm (0, \frac{\varepsilon}{2}, \ldots), \ldots\}$. We will define $\hat{\Sigma}$ so that for each $k \in \{1, \ldots, K-1\}$, and for all $\alpha_{-1}$, the convex hull of the image of $y_1(a_1, \alpha_{-1}) \in R^{|\Theta_{-1}|}$ over $a_1 \in [a_1^k, a_1^{k+1}]$ is a cross polytope $\mathcal{P}_1^k$ centred around $y_1(a_1^k, \alpha_{-1})$ with vertices:

- $y_1(a_1^k, \alpha_{-1}) + (\frac{\varepsilon}{2}, 0, \ldots)$
- $y_1(a_1^k, \alpha_{-1}) - (\frac{\varepsilon}{2}, 0, \ldots)$
- $y_1(a_1^k, \alpha_{-1}) + (0, \frac{\varepsilon}{2}, \ldots)$
- $y_1(a_1^k, \alpha_{-1}) - (0, \frac{\varepsilon}{2}, \ldots)$, etc.

Moreover, the image of $y_1(a_1, \alpha_{-1})$ over $a_1 \in [a_1^k, a_1^{k+1}]$ will consist of straight lines from $y_1(a_1^k, \alpha_{-1})$ to each of the vertices and a straight line from $y_1(a_1^k, \alpha_{-1})$ to $y_1(a_1^{k+1}, \alpha_{-1})$.

Similarly, the convex hull of the image of $y_2(a_1, \alpha_{-1})$ over $a_1 \in [a_1^k, a_1^{k+1}]$ will be a cross polytope $\mathcal{P}_2^k$ centred around $y_2(a_1^k, \alpha_{-1})$ with vertices:

- $y_2(a_1^k, \alpha_{-1}) + (\frac{\varepsilon}{2}, 0, \ldots)$
- $y_2(a_1^k, \alpha_{-1}) - (\frac{\varepsilon}{2}, 0, \ldots)$
- $y_2(a_1^k, \alpha_{-1}) + (0, \frac{\varepsilon}{2}, \ldots)$
- $y_2(a_1^k, \alpha_{-1}) - (0, \frac{\varepsilon}{2}, \ldots)$, etc.

Moreover, the image of $y_2(a_1, \alpha_{-1})$ over $a_1 \in [a_1^k, a_1^{k+1}]$ will consist of straight lines from $y_2(a_1^k, \alpha_{-1})$ to each of the vertices and a straight line from $y_2(a_1^k, \alpha_{-1})$ to $y_2(a_1^{k+1}, \alpha_{-1})$.

However, the key difference will be that the vertices of $\mathcal{P}_1^k$ and $\mathcal{P}_2^k$ occur at different values of $a_1$. That is, for any $a_1 \in [a_1^k, a_1^{k+1}]$ such that $y_1(a_1, \alpha_{-1})$ is a vertex of $\mathcal{P}_1^k$, $y_2(a_1, \alpha_{-1})$ is in the interior of $\mathcal{P}_2^k$. This will imply that for any $a_1 \in [a_1^k, a_1^{k+1}]$, and for any $\alpha_1$ such that $\alpha_1(v_1^1) = a_1$, either $\hat{\gamma}(\Theta_{-1}|(v_1^1, s_1^1), \alpha_1, \alpha_{-1}) \in \text{int}(\hat{C}(\alpha_{-1}))$ or $\hat{\gamma}(\Theta_{-1}|(v_1^1, s_1^2), \alpha_1, \alpha_{-1}) \in \text{int}(\hat{C}(\alpha_{-1}))$.

Enumerate $\Theta_{-1}$, $V_{-1}$ and $S_{-1}$ as follows:

$$\Theta_{-1} = \{\theta_{-1}^1, \dots, \theta_{-1}^{|\Theta_{-1}|}\}$$
$$V_{-1} = \{v_{-1}^1, \dots, v_{-1}^{|V_{-1}|}\}$$
$$S_{-1} = \{s_{-1}^1, \dots, s_{-1}^{|S_{-1}|}\}$$

Define a bijective function $\phi : \{1, \dots, |V_{-1}|\} \times \{1, \dots, |S_{-1}|\} \mapsto \{1, \dots, |\Theta_{-1}|\}$ so that $(v_{-1}^l, s_{-1}^m) = \theta_{-1}^{\phi_{l,m}}$. That is, for each $l$ and $m$ referring to an element of $V_{-1}$ and an element of $S_{-1}$ respectively, $\phi_{l,m}$ refers the corresponding element of $\Theta_{-1}$, and vice versa.

We split each interval $[a_1^k, a_1^{k+1}]$ into $|\Theta_{-1}| + 1$ parts, each with length $\kappa$. First we define $\hat{\Sigma}(a, (v_1^1, v_{-1}))$ for $a_1 \in [a_1^k, a_1^k + |\Theta_{-1}|\kappa]$. For every $a_{-1}$:

1. When $a_1$ is in the $\phi_{l,m}$-th part of the interval $[a_1^k, a_1^{k+1}]$,[7] let:

$$\hat{\sigma}(s_1^1, s_{-1}|a_1, a_{-1}, v_1^1, v_{-1}) = \begin{cases} \sigma(s_1^1, s_{-1}|a_1^k, a_{-1}, v_1^1, v_{-1}) + g(a_1 - a_1^k - (\phi_{l,m} - 1)\kappa) & \text{if } (v_{-1}, s_{-1}) = (v_{-1}^l, s_{-1}^m) \\ \sigma(s_1^1, s_{-1}|a_1^k, a_{-1}, v_1^1, v_{-1}) & \text{if } (v_{-1}, s_{-1}) \neq (v_{-1}^l, s_{-1}^m) \end{cases}$$

where $g : [0, \kappa] \mapsto [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ is piecewise linear, reaching a maximum at $\frac{\kappa}{5}$ and a minimum at $\frac{3\kappa}{5}$. The graph of $g$ is shown in Figure 1 (formal definitions are given in the Appendix).

[7] Since $\phi_{l,m} \leq |\Theta_{-1}|$, this implies that $a_1 \in [a_1^k, a_1^k + |\Theta_{-1}|\kappa]$.

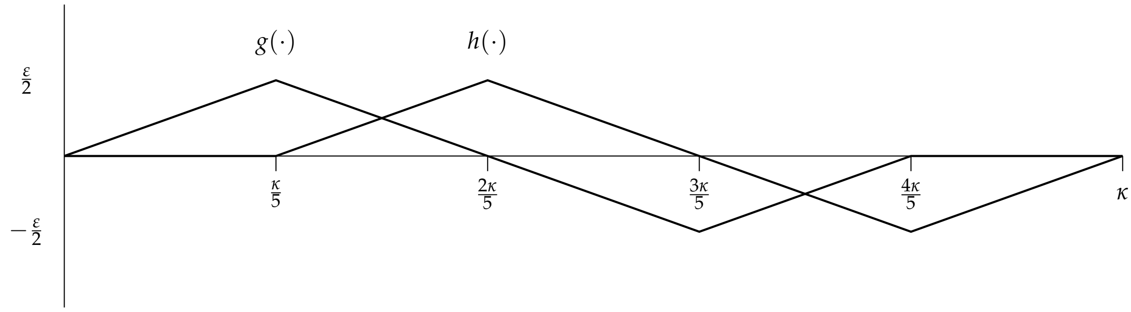


Figure 1: The functions $g$ and $h$

2. When $a_1$ is in the $\phi_{l,m}$-th part of the interval $[a_1^k, a_1^{k+1}]$, let:

$$\hat{\sigma}(s_1^2, s_{-1}|a_1, a_{-1}, v_1^1, v_{-1}) = \begin{cases} \sigma(s_1^2, s_{-1}|a_1^k, a_{-1}, v_1^1, v_{-1}) + h(a_1 - a_1^k - (\phi_{l,m} - 1)\kappa) & \text{if } (v_{-1}, s_{-1}) = (v_{-1}^l, s_{-1}^m) \\ \sigma(s_1^2, s_{-1}|a_1^k, a_{-1}, v_1^1, v_{-1}) & \text{if } (v_{-1}, s_{-1}) \neq (v_{-1}^l, s_{-1}^m) \end{cases}$$

   where $h : [0, \kappa] \mapsto [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ is piecewise linear, reaching a maximum at $\frac{2\kappa}{5}$ and a minimum at $\frac{4\kappa}{5}$. The graph of $h$ is shown in Figure 1.

3. For $a_1 \in [a_1^k, a_1^{k+1} + |\Theta_{-1}|\kappa]$, let:

$$\begin{aligned} \hat{\sigma}(s_1^3, s_{-1}|a_1, a_{-1}, v_1^1, v_{-1}) = \sigma(s_1^3, s_{-1}|a_1^k, a_{-1}, v_1^1, v_{-1}) \\ - (\hat{\sigma}(s_1^1, s_{-1}|a_1, a_{-1}, v_1^1, v_{-1}) - \sigma(s_1^1, s_{-1}|a_1^k, a_{-1}, v_1^1, v_{-1})) \\ - (\hat{\sigma}(s_1^2, s_{-1}|a_1, a_{-1}, v_1^1, v_{-1}) - \sigma(s_1^2, s_{-1}|a_1^k, a_{-1}, v_1^1, v_{-1})) \end{aligned}$$

$$\hat{\sigma}(s_1^j, s_{-1}|a_1, a_{-1}, v_1^1, v_{-1}) = \sigma(s_1^j, s_{-1}|a_1^k, a_{-1}, v_1^1, v_{-1}) \qquad \text{for } j > 3$$

   Note that this ensures that the signals sum to one.

Finally, when $a_1 \in [a_1^k + \kappa|\Theta_{-1}|, a_1^{k+1}]$, for every $a_{-1}$, we let $\hat{\sigma}(s_1, s_{-1}|a_1, a_{-1}, v_1^1, v_{-1})$ be a convex combination of $\sigma(s_1, s_{-1}|a_1^k, a_{-1}, v_1^1, v_{-1})$ and $\sigma(s_1, s_{-1}|a_1^{k+1}, a_{-1}, v_1^1, v_{-1})$, with the weight on $\sigma(s_1, s_{-1}|a_1^{k+1}, a_{-1}, v_1^1, v_{-1})$ changing continuously from zero at $a_1^k + \kappa|\Theta_{-1}|$ to one at $a_1^{k+1}$.

Our construction implies that for each $\alpha_{-1}$, the convex hull of the image of $y_1(a_1, \alpha_{-1})$ over $a_1 \in [a_1^k, a_1^k + |\Theta_{-1}|\kappa]$ is a cross polytope centred around $y_1(a_1^k, \alpha_{-1})$ with circumradius $\frac{\varepsilon}{2}$ and extreme points $y_1(a_1, \alpha_{-1})$ for $a_1 \in A_1^{k^*}$, where:

$$A_1^{k^*} \equiv \left\{ a_1^k + \tfrac{\kappa}{5}, a_1^k + \tfrac{3\kappa}{5}, \ldots, a_1^k + (|\Theta_{-1}| - 1)\kappa + \tfrac{\kappa}{5}, a_1^k + (|\Theta_{-1}| - 1)\kappa + \tfrac{3\kappa}{5} \right\}.$$

Moreover, for $a_1 \in [a_1^k, a_1^{k+1}] \setminus A_1^{k^*}$, $y_1(a_1, \alpha_{-1})$ is in the interior of this cross polytope.[8] Also, the convex hull of the image of $y_2(a_1, \alpha_{-1})$ over $a_1 \in [a_1^k, a_1^k + |\Theta_{-1}|\kappa]$ is a cross polytope centred around $y_2(a_1^k, \alpha_{-1})$ with circumradius $\frac{\varepsilon}{2}$ and extreme points occurring at $a_1 \in A_1^{k^{**}}$, where:

$$A_1^{k^{**}} \equiv \left\{ a_1^k + \tfrac{2\kappa}{5}, a_1^k + \tfrac{4\kappa}{5}, \ldots, a_1^k + (|\Theta_{-1}| - 1)\kappa + \tfrac{2\kappa}{5}, a_1^k + (|\Theta_{-1}| - 1)\kappa + \tfrac{4\kappa}{5} \right\}.$$

Moreover, for every $a_1 \in [a_1^k, a_1^{k+1}] \setminus A_1^{k^*}$, $y_1(a_1, \alpha_{-1})$ is in the interior of this cross polytope.

This means that for all $a_1 \in [a_1^k, a_1^{k+1}] \setminus A_1^{k^*}$, $\alpha_1(v_1^1) = a_1$ implies that $\hat{\gamma}(\Theta_{-1}(v_1^1, s_1^1), \alpha_1, \alpha_{-1}) \in \text{int}(\hat{C}(\alpha_{-1}))$, and for all $a_1 \in [a_1^k, a_1^{k+1}] \setminus A_1^{k^{**}}$, $\alpha_1(v_1^1) = a_1$ implies that $\hat{\gamma}(\Theta_{-1}(v_1^1, s_1^2), \alpha_1, \alpha_{-1}) \in \text{int}(\hat{C}(\alpha_{-1}))$.[9] Note that $A_1^{k^*} \cap A_1^{k^{**}} = \varnothing$, and so for every $a_1 \in [a_1^k, a_1^{k+1}]$, and for $\alpha_1$ such that $\alpha_1(v_1^1) = a_1$, either $\hat{\gamma}(\Theta_{-1}|(v_1^1, s_1^1), \alpha_1, \alpha_{-1}) \in \text{int}(\hat{C}(\alpha_{-1}))$ or $\hat{\gamma}(\Theta_{-1}|(v_1^1, s_1^2), \alpha_1, \alpha_{-1}) \in \text{int}(\hat{C}(\alpha_{-1}))$. Since this is true for all $\alpha_{-1}$, it follows that for every $\alpha$, there exists $\theta_1$ such that $\hat{\gamma}(\Theta_{-1}|\theta_1, \alpha) \in \text{int}(\hat{C}(\alpha_{-1}))$.

To conclude the proof, we show that $d(\hat{\Sigma}, \Sigma) \leq \varepsilon$, which is true as $\delta$ is chosen to be sufficiently small, $\hat{\Sigma}$ is continuous, which is true because it is the sum of continuous functions, and $\hat{\Sigma}$ has full support, which follows from the full support of $\Sigma$ when $\varepsilon$ is sufficiently small.

## 7 Conclusion

In this paper, we have given a full characterisation of information structures that guarantee full rent extraction when the buyers are able to acquire additional information about each other. The standard result in mechanism design with correlated information is that the seller can implement any allocation rule and fully rent from the buyers. In other words, buyers do not earn any rents from their private information. With information acquisition, buyers may earn positive rents. For example, a buyer who can always learn more about each type of her oppo-

[8]Note that $y_1(a_1^{k+1}, \alpha_{-1})$ is in the interior of this cross polytope since a cross polytope with circumradius $\frac{\varepsilon}{2}$ contains a ball of size $\frac{\varepsilon}{2\sqrt{|\Theta_{-1}|}}$, and $\delta$ was chosen to ensure that $||y_1(a_1^{k+1}, \alpha_{-1}) - y_1(a_1^k, \alpha_{-1})|| \leq \frac{\varepsilon}{2|\Theta_{-1}|} < \frac{\varepsilon}{2\sqrt{|\Theta_{-1}|}}$.

[9]This follows because for $j \in \{1, 2\}$, $y_j(a_1, \alpha_{-1}) = \lambda y_j(a_1', \alpha_{-1}) + (1 - \lambda) y_j(a_1'', \alpha_{-1})$ for $\lambda \in (0, 1)$ implies that $\hat{\gamma}(\Theta_{-1}|(v_1^1, s_1^j), \alpha_1', \alpha_{-1}) = \lambda' \hat{\gamma}(\Theta_{-1}|(v_1^1, s_1^j), \alpha_1'', \alpha_{-1}) + (1 - \lambda') \hat{\gamma}(\Theta_{-1}|(v_1^1, s_1), \alpha_1'', \alpha_{-1})$ for $\lambda' \in (0, 1)$, $\alpha_1'(v_1^1) = a_1'$, and $\alpha_1''(v_1^1) = a_1''$.

nents must earn positive rents whenever the seller wishes to implement any allocation rule in which she receives the object with positive probability. Moreover, if the action space is a continuum and the information acquisition technology is continuous, then full rent extraction fails "generically"—i.e. for an open and dense subset of the set of all information structures.

# A Omitted Proofs

*Proof of Proposition 1.* Assume that Condition 1 and Condition 2 of Proposition 1 are satisfied for $\alpha$, $(x', t')$ and $\mu$. Now define the direct mechanism $(x, t)$ as $x(\theta) = x'(\mu(\theta, \alpha(v)))$ and $t(\theta) = t'(\mu(\theta, \alpha(v)))$. First, note that $(x, t)$ is incentive compatible given $\alpha$ since:

$$
\begin{aligned}
U_i(\theta_i', \theta_i; \alpha) &= \sum_{\theta_{-i}} \gamma(\theta_{-i}|\theta_i, \alpha) x_i(\theta_i', \theta_{-i}) w_i(v_i) - \sum_{\theta_{-i}} \gamma(\theta_{-i}|\theta_i, \alpha) t_i(\theta_i', \theta_{-i}) \\
&= \sum_{\theta_{-i}} \gamma(\theta_{-i}|\theta_i, \alpha) x_i'(\mu_i(\theta_i', \alpha_i(v_i'))), \mu_{-i}(\theta_{-i}, \alpha_{-i}(v_{-i}))) w_i(v_i) \\
&\qquad - \sum_{\theta_{-i}} \gamma(\theta_{-i}|\theta_i, \alpha) t_i'(\mu_i(\theta_i', \alpha_i(v_i')), \mu_{-i}(\theta_{-i}, \alpha_{-i}(v_{-i}))) \\
&= U_i'(\mu_i(\theta_i', \alpha_i(v_i')), \theta_i; \alpha, \mu_{-i}) \\
&\leq U_i'(\mu_i(\theta_i, \alpha_i(v_i)), \theta_i; \alpha, \mu_{-i}) \\
&= \sum_{\theta_{-i}} \gamma(\theta_{-i}|\theta_i, \alpha) x_i'(\mu_i(\theta_i, \alpha_i(v_i)), \mu_{-i}(\theta_{-i}, \alpha_{-i}(v_{-i}))) w_i(v_i) \\
&\qquad - \sum_{\theta_{-i}} \gamma(\theta_{-i}|\theta_i, \alpha) t_i'(\mu_i(\theta_i, \alpha_i(v_i)), \mu_{-i}(\theta_{-i}, \alpha_{-i}(v_{-i}))) \\
&= \sum_{\theta_{-i}} \gamma(\theta_{-i}|\theta_i, \alpha) x_i(\theta_i, \theta_{-i}) w_i(v_i) - \sum_{\theta_{-i}} \gamma(\theta_{-i}|\theta_i, \alpha) t_i(\theta_i, \theta_{-i}) \\
&= U_i(\theta_i, \theta_i; \alpha).
\end{aligned}
$$

To see that buyers optimally acquire information $\alpha$ given $(x, t)$, note that for $\theta_i \in \Theta_i(\alpha_i', \alpha_{-i})$:

$$
\begin{aligned}
U_i^*(\theta_i;\alpha_i',\alpha_{-i}) &= \max_{\theta_i'} U_i(\theta_i',\theta_i;\alpha_i',\alpha_{-i}) \\
&\leq \max_{m_i'} U_i'(m_i',\theta_i;\alpha_i',\alpha_{-i},\mu_{-i}) \\
&= U_i^{**}(\theta_i;\alpha_i',\alpha_{-i},\mu_{-i})
\end{aligned}
$$

where the inequality follows because for all $\theta_i'$, $U_i(\theta_i',\theta_i;\alpha_i',\alpha_{-i}) = U_i'(\mu_i(\theta_i',\alpha_i(v_i')),\theta_i;\alpha_i',\alpha_{-i},\mu_{-i})$. Also note that for $\theta_i \in \Theta_i(\alpha)$:

$$
\begin{aligned}
U_i^*(\theta_i;\alpha) &= U_i(\theta_i,\theta_i;\alpha) \\
&= U_i'(\mu_i(\theta_i,\alpha_i(v_i)),\theta_i;\alpha,\mu_{-i}) \\
&= U_i^{**}(\theta_i;\alpha,\mu_{-i}).
\end{aligned}
$$

Thus:

$$
\begin{aligned}
\tilde{u}_i(\alpha_i,\alpha_{-i}) &= \sum_{\theta_i\in\Theta_i(\alpha)} \gamma_{\Theta_i}(\theta_i|\alpha_i,\alpha_{-i})U_i^*(\theta_i;\alpha_i,\alpha_{-i}) \\
&= \sum_{\theta_i\in\Theta_i(\alpha)} \gamma_{\Theta_i}(\theta_i|\alpha_i,\alpha_{-i})U_i^{**}(\theta_i;\alpha,\mu_{-i}) \\
&\geq \sum_{\theta_i\in\Theta_i(\alpha_i',\alpha_{-i})} \gamma_{\Theta_i}(\theta_i|\alpha_i',\alpha_{-i})U_i^{**}(\theta_i;\alpha_i',\alpha_{-i},\mu_{-i}) \\
&\geq \sum_{\theta_i\in\Theta_i(\alpha_i',\alpha_{-i})} \gamma_{\Theta_i}(\theta_i|\alpha_i',\alpha_{-i})U_i^*(\theta_i;\alpha_i',\alpha_{-i}) \\
&= \tilde{u}_i(\alpha_i',\alpha_{-i}).
\end{aligned}
$$

□

*Proof of Proposition 2.* For sufficiency, take $f : V \mapsto [0,1]^N$, and for any $w$, take $\succeq$ such that for all $i$, $v_i \succeq_i v_i'$ if and only if $w_i(v_i) \geq w_i(v_i')$. Then let information acquisition strategy $\alpha$ be such that:

1. For all $\theta_i \in \Theta_i^f(\alpha) \setminus (v_i^*(\succeq_i) \times S_i)$, $\gamma(\Theta_{-i}|\theta_i,\alpha) \cap \mathrm{ri}(C(\alpha_{-i})) = \varnothing$
2. For all $\theta_i \in \Theta_i^f(\alpha) \setminus (v_i^*(\succeq_i) \times S_i)$, $F_{\overline{C}(\alpha_{-i})}(\gamma(\Theta_{-i}|\theta_i,\alpha)) \cap \overline{D}(\theta_i,\alpha_{-i},\succeq_i,M_i^f(\theta_i,\alpha)) = \varnothing$.

Let $x(\theta) = f(v)$ for any $\theta$ that arises with positive probability under $\alpha$, and let $x(\theta) = 0$ otherwise. Note that $x_i(\theta_i,\theta_{-i}) > 0$ if and only if $\theta_{-i} \in M_i^f(\theta_i,\alpha)$. Now we argue that there exists transfers such that $(x, t)$ is incentive compatible, individually rational, and fully extracts rent given $\alpha$, and $\alpha$ is optimally chosen given $(x, t)$.

For each $\theta_i \in \Theta_i^f(\alpha) \setminus (v_i^*(\succeq_i) \times S_i)$, Condition 1 implies that there is a hyperplane:

$$\{\gamma : \tau \cdot \gamma = \tau \cdot \gamma(\Theta_{-i}|\theta_i, \alpha)\}$$

which properly supports $\overline{C}(\alpha_{-i})$ at $\gamma(\Theta_{-i}|\theta_i, \alpha)$. Thus, $F_{\overline{C}(\alpha_{-i})}(\gamma(\Theta_{-i}|\theta_i, \alpha))$ is a proper exposed face of $\overline{C}(\alpha_{-i})$. Let $\tau(\theta_i)$ be a lottery such that $F_{\overline{C}(\alpha_{-i})}(\gamma(\Theta_{-i}|\theta_i, \alpha)) = \{\gamma : \tau(\theta_i) \cdot \gamma = 0\} \cap \overline{C}(\alpha_{-i})$. This implies that $\tau(\theta_i) \cdot \gamma = 0$ for all $\gamma \in F_{\overline{C}(\alpha_{-i})}(\gamma(\Theta_{-i}|\theta_i, \alpha))$. Without loss of generality, let $\tau(\theta_i) \cdot \gamma > 0$ for all $\gamma \in \overline{C}(\alpha_{-i}) \setminus F_{\overline{C}(\alpha_{-i})}(\gamma(\Theta_{-i}|\theta_i, \alpha))$.

For $\theta_i \in \Theta_i^f(\alpha) \setminus (v_i^*(\succeq_i) \times S_i)$ let $t_i(\theta_i, \Theta_{-i}) = w_i(v_i)x_i(\theta_i, \Theta_{-i}) + K\tau(\theta_i)$,[10] where $K$ is sufficiently large. Note that $\gamma(\Theta_{-i}|\theta_i, \alpha) \cdot \tau(\theta_i) = 0$. For $\theta_i \notin \Theta_i^f(\alpha)$, let $t_i(\theta_i, \Theta_{-i}) = 0$. For $\theta_i \in (v_i^*(\succeq_i) \times S_i) \cap \Theta_i^f(\alpha)$, let $t_i(\theta_i, \Theta_{-i}) = w_i(v_i)x_i(\theta_i, \Theta_{-i})$. By construction, $(x, t)$ fully extracts rent given $\alpha$. We now show that $(x, t)$ is incentive compatible given $\alpha$, and $\alpha$ is chosen optimally.

First we show that for any $\theta_i \in \Theta_i(\alpha)$, $\theta_i' \in \Theta_i$, $U_i(\theta_i', \theta_i; \alpha) \leq U_i(\theta_i, \theta_i; \alpha)$. We will consider the cases where $\theta_i' \in \Theta_i^f(\alpha) \setminus (v_i^*(\succeq_i) \times S_i)$, $\theta_i' \notin \Theta_i^f(\alpha)$, and $\theta_i' \in (v_i^*(\succeq_i) \times S_i) \cap \Theta_i^f(\alpha)$ separately. Note that for $\theta_i' \in \Theta_i^f(\alpha) \setminus (v_i^*(\succeq_i) \times S_i)$:

$$\begin{aligned}
U_i(\theta_i', \theta_i; \alpha) &= w_i(v_i)\gamma(\Theta_{-i}|\theta_i, \alpha) \cdot x_i(\theta_i', \Theta_{-i}) - \gamma(\Theta_{-i}|\theta_i, \alpha) \cdot (w_i(v_i')x_i(\theta_i', \Theta_{-i}) + K\tau(\theta_i')) \\
&= (w_i(v_i) - w_i(v_i'))\gamma(\Theta_{-i}|\theta_i, \alpha) \cdot x_i(\theta_i', \Theta_{-i}) - K\gamma(\Theta_{-i}|\theta_i, \alpha) \cdot \tau(\theta_i') \\
&\leq 0.
\end{aligned}$$

The last inequality follows because $\gamma(\Theta_{-i}|\theta_i, \alpha) \in C(\alpha_i)$, so $\gamma(\Theta_{-i}|\theta_i, \alpha) \cdot \tau(\theta_i') \geq 0$, and when $\gamma(\Theta_{-i}|\theta_i, \alpha) \cdot \tau(\theta_i') = 0$, either $w_i(v_i) \leq w_i(v_i')$ or $\gamma(\Theta_{-i}|\theta_i, \alpha) \cdot x_i(\theta_i', \Theta_{-i}) = 0$ by Condition 2. To see this, first note that if $\gamma(\Theta_{-i}|\theta_i, \alpha) \cdot \tau(\theta_i') = 0$, then $\gamma(\Theta_{-i}|\theta_i, \alpha) \in F_{\overline{C}(\alpha_{-i})}(\gamma(\Theta_{-i}|\theta_i', \alpha)$, since $F_{\overline{C}(\alpha_{-i})}(\gamma(\Theta_{-i}|\theta_i', \alpha)) = \overline{C}(\alpha_{-i}) \cap \{\gamma : \tau(\theta_i') \cdot \gamma = 0\}$. But then Condition 2 requires that $\gamma(\Theta_{-i}|\theta_i, \alpha) \notin D(\theta_i', \alpha_{-i}, \succeq_i, M_i^f(\theta_i', \alpha))$. Since $\theta_i \in \Theta_i(\alpha)$, this implies that $\theta_i$ must be such that $w_i(v_i) \leq w_i(v_i')$ or $\gamma(\Theta_{-i}|\theta_i, \alpha) \cdot x_i(\theta_i', \Theta_{-i}) = 0$.

For $\theta_i' \notin \Theta_i^f(\alpha)$, $U_i(\theta_i', \theta_i; \alpha) = 0$ since $x_i(\theta_i', \Theta_{-i}) = t_i(\theta_i', \Theta_{-i}) = 0$. For $\theta_i' \in v_i^*(\succeq_i) \times S_i$,

[10] Recall that $t_i(\theta_i, \Theta_{-i}) = \left(t_i(\theta_i, \theta_{-i}^1), \ldots, t_i(\theta_i, \theta_{-i}^{|\Theta_{-i}|})\right)$ and $x_i(\theta_i, \Theta_{-i}) = \left(x_i(\theta_i, \theta_{-i}^1), \ldots, x_i(\theta_i, \theta_{-i}^{|\Theta_{-i}|})\right)$.

note that $w_i(v_i') \geq w_i(v_i)$. Then:

$$\begin{aligned}
U_i(\theta_i', \theta_i; \alpha) &= w(v_i)\gamma(\Theta_{-i}|\theta_i, \alpha) \cdot x_i(\theta_i', \Theta_{-i}) - \gamma(\Theta_{-i}|\theta_i, \alpha) \cdot (w_i(v_i')x_i(\theta_i', \Theta_{-i})) \\
&= (w_i(v_i) - w_i(v_i'))\gamma(\Theta_{-i}|\theta_i, \alpha) \cdot x_i(\theta_i', \Theta_{-i}) \\
&\leq 0.
\end{aligned}$$

Now we show that $\alpha$ is chosen optimally. That is, for any $\alpha_i' \in A_i^{|V_i|}$:

$$\sum_{\theta_i \in \Theta_i(\alpha_i', \alpha_{-i})} \gamma_{\Theta_i}(\theta_i|\alpha_i', \alpha_{-i}) U_i^*(\theta_i; \alpha_i', \alpha_{-i}) \leq \sum_{\theta_i \in \Theta_i(\alpha)} \gamma_{\Theta_i}(\theta_i|\alpha) U_i^*(\theta_i; \alpha).$$

Note that for all $\theta_i \in \Theta_i(\alpha_i', \alpha_{-i})$:

$$\begin{aligned}
U_i^*(\theta_i; \alpha_i', \alpha_{-i}) &= \max_{\theta_i'} U_i(\theta_i', \theta_i; \alpha_i', \alpha_{-i}) \\
&= \max_{\theta_i'} w_i(v_i)x(\theta_i', \Theta_{-i}) \cdot \gamma(\Theta_{-i}|\theta_i, \alpha_i', \alpha_{-i}) - t(\theta_i', \Theta_{-i}) \cdot \gamma(\Theta_{-i}|\theta_i, \alpha_i', \alpha_{-i}) \\
&= \max \Bigg\{ \max_{\theta_i' \in \Theta_i^f(\alpha) \setminus (v_i^*(\succeq_i) \times S_i)} w_i(v_i)x(\theta_i', \Theta_{-i}) \cdot \gamma(\Theta_{-i}|\theta_i, \alpha_i', \alpha_{-i}) \\
&\qquad - (w_i(v_i')x_i(\theta_i', \Theta_{-i}) + K\tau(\theta_i')) \cdot \gamma(\Theta_{-i}|\theta_i, \alpha_i', \alpha_{-i}), 0 \Bigg\} \\
&= \max \Bigg\{ \max_{\theta_i' \in \Theta_i^+(a,f) \setminus (v_i^*(\succeq_i) \times S_i)} (w_i(v_i) - w_i(v_i'))x(\theta_i', \Theta_{-i}) \cdot \gamma(\Theta_{-i}|\theta_i, \alpha_i', \alpha_{-i}) \\
&\qquad - K\tau(\theta_i') \cdot \gamma(\Theta_{-i}|\theta_i, \alpha_i', \alpha_{-i}), 0 \Bigg\} \\
&\leq 0.
\end{aligned}$$

The last inequality holds because for any $\alpha_i' \in A_i^{|V_i|}$, $\theta_i \in \Theta_i(\alpha_i', \alpha_{-i})$, and $\theta_i' \in \Theta_i^f(a) \setminus (v_i^*(\succeq_i) \times S_i)$, either $\gamma(\Theta_{-i}|\theta_i, \alpha_i', \alpha_{-i}) \cdot \tau(\theta_i') = 0$ and $w(v_i) \leq w(v_i')$ or $\gamma(\Theta_{-i}|\theta_i, \alpha) \cdot x_i(\theta_i', \Theta_{-i}) = 0$, or $\gamma(\Theta_{-i}|\theta_i, \alpha_i', \alpha_{-i}) \cdot \tau(\theta_i') > \varepsilon$ for some $\varepsilon > 0$ (i.e. $\varepsilon$ is a uniform lower bound across all $\alpha_i'$).[11] The former was shown when we established incentive compatibility. To see the latter, suppose that for every $\varepsilon > 0$, there exists $\alpha_i' \in A_i^{|V_i|}$, $\theta_i \in \Theta_i(\alpha_i', \alpha_{-i})$ where $w_i(v_i) > w_i(v_i')$ and $\gamma(\Theta_{-i}|\theta_i, \alpha) \cdot x_i(\theta_i', \Theta_{-i}) \neq 0$ such that $\gamma(\Theta_{-i}|\theta_i, \alpha_i', \alpha_{-i}) \cdot \tau(\theta_i') < \varepsilon$. Let $\gamma^\varepsilon \in \overline{D}(\theta_i', \alpha_{-i}, \succeq_i, M_i^f(\theta_i', \alpha))$ denote the sequence of such beliefs. Since $\overline{D}(\theta_i', \alpha_{-i}, \succeq_i, M_i^f(\theta_i', \alpha))$ is compact, as

[11] When we established incentive compatibility, we did not need this step because the set of $\theta_i$ is finite. However, since there can be infinitely many $\alpha_i'$, it is not enough that $\gamma(\Theta_{-i}|\theta_i, \alpha_i', \alpha_{-i}) \cdot \tau(\theta_i') > 0$ for all $\alpha_i'$, because now there is the possibility that $\gamma(\Theta_{-i}|\theta_i, \alpha_i', \alpha_{-i}) \cdot \tau(\theta_i')$ can be made arbitrarily close to 0.

$\varepsilon \to 0$, there is a convergent subsequence which converges to $\gamma \in \overline{D}(\theta_i', \alpha_{-i}, \succeq_i, M_i^f(\theta_i', \alpha))$, with $\gamma \cdot \tau(\theta_i') = 0$. Note that $\gamma^\varepsilon \in \overline{C}(\alpha_{-i})$, so $\gamma \in \overline{C}(\alpha_{-i})$, and $\gamma \in \{\gamma : \gamma \cdot \tau(\theta_i') = 0\}$. Since $F_{\overline{C}(\alpha_{-i})}(\gamma(\Theta_{-i}|\theta_i', \alpha) = \{\gamma : \gamma \cdot \tau(\theta_i') = 0\} \cap \overline{C}(\alpha_{-i})$, $\gamma \in F_{\overline{C}(\alpha_{-i})}(\gamma(\Theta_{-i}|\theta_i', \alpha)$ contradicting Condition 2. Thus, for sufficiently large $K$, the inequality holds.

Now we prove necessity. Suppose that there exists $\succeq$ such that Condition 1 is not satisfied. Define $w$ such that for all $i$, $w_i(v_i) \geq w_i(v_i')$ if and only if $v_i \succeq_i v_i'$, and for all $i$, for $v_i \in v_i^*(\succeq_i)$ let $w_i(v_i) = L$, where $L$ is sufficiently large. Take an arbitrary $\alpha$ and let $\hat{\theta}_i \in \Theta_i^f(\alpha) \setminus (v_i^*(\succeq_i) \times S_i)$ such that $\gamma(\Theta_{-i}|\hat{\theta}_i, \alpha) \cap \mathrm{ri}(C(\alpha_{-i})) \neq \varnothing$. Suppose for a contradiction that $(x, t)$ is incentive compatible, individually rational, and fully extracts rent given $\alpha$, $\alpha$ is chosen optimally, and $x_i(\theta_i, \theta_{-i}) = f_i(v_i, v_{-i})$ for all $\theta$ such that $\gamma(\theta|\alpha) > 0$. Without loss of generality, let $t_i(\hat{\theta}_i, \Theta_{-i}) = w_i(\hat{v}_i)x_i(\hat{\theta}_i, \Theta_{-i}) + K\tau(\hat{\theta}_i)$ for some $\tau(\hat{\theta}_i)$. For full rent extraction, we need $\tau(\hat{\theta}_i) \cdot \gamma(\Theta_{-i}|\hat{\theta}_i, \alpha) = 0$. Note that there must exist $\theta_i^* \in v_i^*(\succeq_i) \times S_i$ such that $\gamma(\Theta_{-i}|\theta_i, \alpha) \cdot x_i(\hat{\theta}_i, \Theta_{-i}) > 0$, otherwise Condition 1 must hold. Suppose that $\gamma(\Theta_{-i}|\theta_i, \alpha) \cdot x_i(\hat{\theta}_i, \Theta_{-i}) = 0$ for all $\theta_i$ such that $v_i \in v_i^*(\succeq_i)$. Then any $\tau(\hat{\theta}_i)$ that is strictly positive for and only for elements corresponding to $\theta_{-i}'$ with $f_i(\hat{v}_i, v_{-i}') > 0$ and $\gamma(\theta_{-i}'|\hat{\theta}_i, \alpha) = 0$ properly supports $C(\alpha_{-i})$ at $\gamma(\Theta_{-i}|\hat{\theta}_i, \alpha)$,[12] which implies that $\gamma(\Theta_{-i}|\hat{\theta}_i, \alpha) \in \mathrm{ri}(C(\alpha_{-i}))$, a contradiction. Incentive compatibility then requires $\gamma(\Theta_{-i}|\theta_i^*, \alpha) \cdot ((L - w(\hat{v}_i))x_i(\hat{\theta}_i, \Theta_{-i}) - K\tau(\hat{\theta}_i)) \leq 0$, which implies:

$$\gamma(\Theta_{-i}|\theta_i^*, \alpha) \cdot K\tau(\hat{\theta}_i) \geq \gamma(\Theta_{-i}|\theta_i^*, \alpha) \cdot (L - w(\hat{v}_i))x_i(\hat{\theta}_i, \Theta_{-i}) > 0.$$

Since Condition 1 is not satisfied, there must exist $\gamma \in C(\alpha_{-i})$ such that $\gamma \cdot \tau(\hat{\theta}_i) < 0$. Thus, there must exists $\alpha_i' \in A_i^{|V_i|}$, $\theta_i' \in \Theta_i(\alpha_i', \alpha_{-i})$ such that $\gamma(\Theta_{-i}|\theta_i', \alpha_i', \alpha_{-i}) \cdot \tau(\hat{\theta}_i) < 0$. The previous

[12] Since there must exist some $\theta_i$ such that $v_i \in v_i^*(\succeq_i)$ where $\sum_{s_{-i}} \gamma(v_{-i}', s_{-i}|\theta_i, \alpha) > 0$, and since $\gamma(\Theta_{-i}|\theta_i, \alpha) \cdot x_i(\hat{\theta}_i, \Theta_{-i}) = 0$, $\gamma(\theta_{-i}'|\theta_i, \alpha)$ must be positive for some $\theta_{-i}'$ with $f_i(\hat{v}_i, v_{-i}') > 0$ and $\gamma(\theta_{-i}'|\hat{\theta}_i, \alpha) = 0$

displayed inequality implies that for large $L$, $K$ must also be large. Then:

$$
\begin{aligned}
\sum_{\theta_i \in \Theta_i(\alpha'_i, \alpha_{-i})} & \gamma_{\Theta_i}(\theta_i | \alpha'_i, \alpha_{-i}) U_i^*(\theta_i; \alpha'_i, \alpha_{-i}) \\
& \geq \gamma_{\Theta_i}(\theta'_i | \alpha'_i, \alpha_{-i}) U_i^*(\theta'_i; \alpha'_i, \alpha_{-i}) \\
& \geq \gamma_{\Theta_i}(\theta'_i | \alpha'_i, \alpha_{-i}) U_i(\hat{\theta}_i, \theta'_i; \alpha'_i, \alpha_{-i}) \\
& = \gamma_{\Theta_i}(\theta'_i | \alpha'_i, \alpha_{-i}) \left( w_i(v'_i) x_i(\hat{\theta}_i, \Theta_{-i}) - t(\hat{\theta}_i, \Theta_{-i}) \right) \cdot \gamma(\Theta_{-i} | \theta'_i, \alpha'_i, \alpha_{-i}) \\
& = \gamma_{\Theta_i}(\theta'_i | \alpha'_i, \alpha_{-i}) \left( (w_i(v'_i) - w_i(\hat{v}_i)) x_i(\hat{\theta}_i, \Theta_{-i}) - K\tau(\hat{\theta}_i) \right) \cdot \gamma(\Theta_{-i} | \theta'_i, \alpha'_i, \alpha_{-i}) \\
& > 0.
\end{aligned}
$$

Suppose that there exists $\succeq$ such that Condition 2 is not satisfied. Then define $w$ as before, take an arbitrary $\alpha$, and let $\hat{\theta}_i \in \Theta_i^f(\alpha) \setminus (v_i^*(\succeq_i) \times S_i)$ be such that $F_{\overline{C}(\alpha_{-i})}(\gamma(\Theta_{-i}|\hat{\theta}_i, \alpha)) \cap \overline{D}(\hat{\theta}_i, \alpha_{-i}, \succeq_i, M_i^f(\theta_i, \alpha)) \neq \varnothing$. Suppose for a contradiction that $(x, t)$ is incentive compatible, individually rational, and fully extracts rent given $\alpha$, $\alpha$ is chosen optimally, and $x_i(\theta_i, \theta_{-i}) = f_i(v_i, v_{-i})$ for all $\theta$ such that $\gamma(\theta|\alpha) > 0$. Without loss of generality, let $t_i(\hat{\theta}_i, \Theta_{-i}) = w_i(\hat{v}_i) x_i(\hat{\theta}_i, \Theta_{-i}) + K\tau(\hat{\theta}_i)$ for some $\tau(\hat{\theta}_i)$. For full rent extraction, we need $\tau(\hat{\theta}_i) \cdot \gamma(\Theta_{-i}|\hat{\theta}_i, \alpha) = 0$. Assume that $\tau(\hat{\theta}_i) \cdot \gamma(\Theta_{-i}|\theta'_i, \alpha'_i, \alpha_{-i}) \geq 0$ for all $\gamma(\Theta_{-i}|\theta'_i, \alpha'_i, \alpha_{-i})$, otherwise by the same argument as in the previous paragraph, the strategy $\alpha'_i$ must yield strictly positive rents. Thus, $\tau(\hat{\theta}_i)$ is a supporting hyperplane for $\overline{C}(\alpha_{-i})$, which implies that $F_{\overline{C}(\alpha_{-i})}(\gamma(\Theta_{-i}|\hat{\theta}_i, \alpha)) \subset \{\gamma : \gamma \cdot \tau(\hat{\theta}_i) = 0\}$.

Since Condition 2 fails, there exists $\gamma \in F_{\overline{C}(\alpha_{-i})}(\gamma(\Theta_{-i}|\hat{\theta}_i, \alpha)) \cap \overline{D}(\hat{\theta}_i, \alpha_{-i}, \succeq_i, M_i^f(\hat{\theta}_i, \alpha))$. We show that for every $\varepsilon > 0$, we can find $\gamma(\Theta_{-i}|\theta'_i, \alpha'_i, \alpha_{-i})$, $\theta'_i \in \Theta_i(\alpha'_i, \alpha_{-i})$, such that $w_i(v'_i) > w_i(\hat{v}_i)$ and $\gamma(\Theta_{-i}|\theta'_i, \alpha'_i, \alpha_{-i}) \cdot x_i(\hat{\theta}_i, \Theta_{-i}) > 0$, and $\tau(\hat{\theta}_i) \cdot \gamma(\Theta_{-i}|\theta'_i, \alpha'_i, \alpha_{-i}) < \varepsilon$. To see this, take any sequence $\gamma^k \in D(\hat{\theta}_i, \alpha_{-i}, \succeq_i, M_i^f(\hat{\theta}_i, \alpha))$ converging to $\gamma$. As $k \to \infty$, $\tau(\hat{\theta}_i) \cdot \gamma^k \to \tau(\hat{\theta}_i) \cdot \gamma = 0$. Since $\gamma^k \in D(\hat{\theta}_i, \alpha_{-i}, \succeq_i, M_i^f(\hat{\theta}_i, \alpha))$, for each $k$, $\gamma^k = \gamma(\Theta_{-i}|\theta'_i, \alpha'_i, \alpha_{-i})$ such that $w_i(v'_i) > w_i(\hat{v}_i)$, $\theta'_i \in \Theta_i(\alpha'_i, \alpha_{-i})$, and $\gamma(\Theta_{-i}|\theta'_i, \alpha'_i, \alpha_{-i}) \cdot x_i(\hat{\theta}_i, \Theta_{-i}) > 0$. Take a sufficiently large $k$ and we are done. Then:

$$\sum_{\theta_i \in \Theta_i(\alpha_i', \alpha_{-i})} \gamma_{\Theta_i}(\theta_i|\alpha_i', \alpha_{-i}) U_i^*(\theta_i; \alpha_i', \alpha_{-i})$$
$$\geq \gamma_{\Theta_i}(\theta_i'|\alpha_i', \alpha_{-i}) U_i^*(\theta_i'; \alpha_i', \alpha_{-i})$$
$$\geq \gamma_{\Theta_i}(\theta_i'|\alpha_i', \alpha_{-i}) U_i(\hat{\theta}_i, \theta_i'; \alpha_i', \alpha_{-i})$$
$$= \gamma_{\Theta_i}(\theta_i'|\alpha_i', \alpha_{-i}) \left(w_i(v_i') x_i(\hat{\theta}_i, \Theta_{-i}) - t(\hat{\theta}_i, \Theta_{-i})\right) \cdot \gamma(\Theta_{-i}|\theta_i', \alpha_i', \alpha_{-i})$$
$$= \gamma_{\Theta_i}(\theta_i'|\alpha_i', \alpha_{-i}) \left((w_i(v_i') - w_i(\hat{v}_i)) x_i(\hat{\theta}_i, \Theta_{-i}) - K\tau(\hat{\theta}_i)\right) \cdot \gamma(\Theta_{-i}|\theta_i', \alpha_i', \alpha_{-i})$$
$$> 0. \qquad \square$$

*Proof of Proposition 3.* Without loss of generality, let $|V_1| \geq 2$ and $|S_1| > 2$. Let $V_1 = \{v_1^1, \ldots, v_1^{|V_i|}\}$, $S_1 = \{s_1^1, \ldots, s_1^{|S_i|}\}$, where $|V_1| \geq 2$ and $|S_1| > 3$. In what follows, we assume that $|\Theta_{-1}| > 1$, since for $|\Theta_{-1}| = 1$, trivially $\mathcal{T}^* = \varnothing$ and hence $T^*$ is nowhere dense. First, we show that any $\Sigma \in \overline{\mathcal{T}^*}$ is such that there exists an $\alpha$ such that for all $\theta_1$ with $v_1 = v_1^1$, $\gamma(\Theta_{-1}|\theta_1, \alpha) \notin \text{int}(C(\alpha_{-1}))$. To see this, let $\Sigma^n \to \Sigma$ so that $\Sigma^n \in \mathcal{T}^*$ for all $n$, and let $\alpha^n$ be such that for all $\theta_1$ with $v_1 = v_1^1$, $\gamma^n(\Theta_{-1}|\theta_1, \alpha^n) \cap \text{ri}(C^n(\alpha_{-1}^n)) = \varnothing$.[13] Such an $\alpha^n$ must exist by Corollary 3. Since $A^{|V|}$ is compact, we can assume that $\alpha^n$ converges to $\alpha$. Suppose for a contradiction that for some $\theta_1$ with $v_1 = v_1^1$, $\gamma(\Theta_{-1}|\theta_1, \alpha) \in \text{int}(C(\alpha_{-1}))$. Then for large enough $n$, $\gamma(\Theta_{-1}|\theta_1, \alpha) \in \text{int}(C^n(\alpha_{-1}^n))$ since $C^n(\alpha_{-1}^n)$ and $C(\alpha_{-1})$ are close (under the Hausdorff metric), and since $\gamma(\Theta_{-1}|\theta_1, \alpha)$ and $\gamma^n(\Theta_{-1}|\theta_1, \alpha^n)$ are close, $\gamma^n(\Theta_{-1}|\theta_1, \alpha^n) \in \text{int}(C^n(\alpha_{-1}^n))$, and hence $\gamma^n(\Theta_{-1}|\theta_1, \alpha^n) \in \text{ri}(C^n(\alpha_{-1}^n))$, a contradiction. Thus for each $\theta_1$ with $v_1 = v_1^1$, $\gamma(\Theta_{-1}|\theta_1, \alpha) \notin \text{int}(C(\alpha_{-1}))$. Note that we do not have the stronger conclusion that $\gamma(\Theta_{-1}|\theta_1, \alpha) \notin \text{ri}(C(\alpha_{-1}))$ for all $\theta_1$ with $v_1 = v_1^1$, since the relative boundary of the convex hull is not necessarily continuous.[14]

Thus, to show that $\overline{\mathcal{T}^*}$ has empty interior, it suffices to show that for any continuous $\Sigma : A \times V \mapsto \Delta S$ such that $\sigma(s|a, v) > 0$ for all $s \in S$, $a \in A$, $v \in V$, and for any $\varepsilon > 0$, there exists a continuous $\hat{\Sigma} : A \times V \mapsto \Delta S$ such that such that $\sigma(s|a, v) > 0$ for all $s \in S$, $a \in A$, $v \in V$, and for $v_1^1$, for every $\alpha$, there exists a $\theta_1 = (v_1^1, s_1)$ with $\hat{\gamma}(\Theta_{-1}|\theta_1, \alpha) \in \text{int}(\hat{C}(\alpha_{-1}))$,[15] where $d(\Sigma, \hat{\Sigma}) \equiv \sup_{s,a,v} |\hat{\sigma}(s|a, v) - \sigma(s|a, v)| \leq \varepsilon$ (i.e. for every $\Sigma$ and every $\varepsilon > 0$, we can find

[13] $\gamma^n(\Theta_{-1}|\theta_1, \alpha^n)$ and $C^n(\alpha_{-1}^n)$ denote the beliefs under $\Sigma^n$.
[14] For example, consider a rectangle that converges to a line.
[15] $\hat{\gamma}(\Theta_{-1}|\theta_1, \alpha)$ denotes the beliefs under $\hat{\Sigma}$, and $\hat{C}(\alpha_{-1})$ denotes the convex hull of the beliefs under $\hat{\Sigma}$.

$\hat{\Sigma} \in \mathcal{T} \setminus \overline{\mathcal{T}^*}$ such that $d(\Sigma, \hat{\Sigma}) \leq \varepsilon$).

As mentioned in the text, $\gamma(\Theta_{-1}|\theta_1, \alpha_1, \alpha_{-1})$ is a function of $\alpha_{-1}$ and $\alpha_1(v_1)$, which for notational simplicity we wrote as a function of $\alpha_{-1}$ and $\alpha_1(v_1)$. In this proof, it will be more convenient to explicitly restrict the argument to $\alpha_1(v_1)$, so with abuse of notation, we will write $\gamma(\Theta_{-1}|\theta_1, a_1, \alpha_{-1})$.

Since $\Sigma$ is continuous and $A$ is compact, $\Sigma$ is uniformly continuous on $A$. For each $\varepsilon > 0$, let $\delta(\varepsilon)$ be such $||a - a'|| \leq \delta \implies \sup_{s,v} |\sigma(s|a, v) - \sigma(s|a', v)| \leq \varepsilon$. Let:

$$A_1^* = \left\{ a_1^1, \ldots, a_1^K \right\},$$

where $a_1^1 = 0$, $a_1^K = 1$, and for all $k \in \{1, \ldots, K-1\}$, $a_1^{k+1} - a_1^k = \frac{1}{K} = \delta(\frac{\varepsilon}{2|\Theta_{-1}|^2}) \equiv \delta$. Let:

$$V_{-1} = \left\{ v_{-1}^1, \ldots v_{-1}^{|V_{-1}|} \right\}$$
$$S_{-1} = \left\{ s_{-1}^1, \ldots s_{-1}^{|S_{-1}|} \right\}.$$

For $1 \leq l \leq |V_{-1}|$ and $1 \leq m \leq |S_{-1}|$ , let $\phi_{l,m} \equiv (l-1)|S_{-1}| + m$. That is:

$$\Theta_{-1} = \left\{ (\theta_{-1}^{\phi_{1,1}}, \ldots, \theta_{-1}^{\phi_{1,|S_{-1}|}}, \ldots, \theta_{-1}^{\phi_{|V_{-1}|,1}}, \ldots, \theta_{-1}^{\phi_{|V_{-1}|,|S_{-1}|}} \right\},$$

where $\theta_{-1}^{\phi_{l,m}} = (v_{-1}^l, s_{-1}^m)$. We will split each interval $[a_1^k, a_1^{k+1}]$ into $|\Theta_{-1}| + 1$ parts, each with length $\kappa$, where:

$$\kappa \equiv \frac{\delta}{|\Theta_{-1}| + 1}.$$

Let $g : [0, \kappa] \mapsto [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ and $h : [0, \kappa] \mapsto [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ be defined as follows.

$$g(x) = \begin{cases} \frac{5}{\kappa}\frac{\varepsilon}{2}x & x \leq \frac{\kappa}{5} \\ \frac{\varepsilon}{2} - (x - \frac{\kappa}{5})\frac{5}{2\kappa}\varepsilon & x \in (\frac{\kappa}{5}, \frac{3\kappa}{5}] \\ -\frac{\varepsilon}{2} + (x - \frac{3\kappa}{5})\frac{5}{\kappa}\frac{\varepsilon}{2} & x \in (\frac{3\kappa}{5}, \frac{4\kappa}{5}] \\ 0 & x \in (\frac{4\kappa}{5}, \kappa] \end{cases}$$

$$h(x)=\begin{cases}0 & x\leq\frac{\kappa}{5}\\ \frac{5}{\kappa}\frac{\varepsilon}{2}(x-\frac{\kappa}{5}) & x\in(\frac{\kappa}{5},\frac{2\kappa}{5}]\\ \frac{\varepsilon}{2}-(x-\frac{2\kappa}{5})\frac{5}{2\kappa}\varepsilon & x\in(\frac{2\kappa}{5},\frac{4\kappa}{5}]\\ -\frac{\varepsilon}{2}+(x-\frac{4\kappa}{5})\frac{5}{\kappa}\frac{\varepsilon}{2} & x\in(\frac{4\kappa}{5},\kappa]\end{cases}$$

Let $\mathcal{I}(\cdot)$ denote the indicator function. We define $\hat{\Sigma}$ as follows.

1. $\hat{\Sigma}(a',v')=\Sigma(a,v)$ for $(a',v')$ such that $v'_1\neq v_1^1$

2. For $(v_1^1,s_1^1)$, $(v^l_{-1},s^m_{-1})$, for each $a_1\in[a_1^k,a_1^{k+1}]$, and for all $a_{-1}$:

$$\begin{aligned}\hat{\sigma}(s_1^1,s^m_{-1}|a_1,a_{-1},v_1^1,v^l_{-1}) &= \sigma(s_1^1,s^m_{-1}|a_1^k,a_{-1},v_1^1,v^l_{-1})\\ &+\mathcal{I}(a_1\in\left[a_1^k+(\phi_{l,m}-1)\kappa,a_1^k+\phi_{l,m}\kappa\right])g(a_1-a_1^k-(\phi_{l,m}-1)\kappa)\\ &+\mathcal{I}(a_1\in\left[a_1^k+|\Theta_{-1}|\kappa,a_1^{k+1}\right])\frac{1}{\kappa}\left(a_1-a_1^k-|\Theta_{-1}|\kappa\right)\\ &\quad\times\left(\sigma(s_1^1,s^m_{-1}|a_1^{k+1},a_{-1},v_1^1,v^l_{-1})-\sigma(s_1^1,s^m_{-1}|a_1^k,a_{-1},v_1^1,v^l_{-1})\right)\end{aligned}$$

3. For $(v_1^1,s_1^2)$, $(v^l_{-1},s^m_{-1})$, for each $a_1\in[a_1^k,a_1^{k+1}]$, and for all $a_{-1}$:

$$\begin{aligned}\hat{\sigma}(s_1^2,s^m_{-1}|a_1,a_{-1},v_1^1,v^l_{-1}) &= \sigma(s_1^2,s^m_{-1}|a_1^k,a_{-1},v_1^1,v^l_{-1})\\ &+\mathcal{I}(a_1\in\left[a_1^k+(\phi_{l,m}-1)\kappa,a_1^k+\phi_{l,m}\kappa\right])h(a_1-a_1^k-(\phi_{l,m}-1)\kappa)\\ &+\mathcal{I}(a_1\in\left[a_1^k+|\Theta_{-1}|\kappa,a_1^{k+1}\right])\frac{1}{\kappa}\left(a_1-a_1^k-|\Theta_{-1}|\kappa\right)\\ &\quad\times\left(\sigma(s_1^2,s^m_{-1}|a_1^{k+1},a_{-1},v_1^1,v^l_{-1})-\sigma(s_1^2,s^m_{-1}|a_1^k,a_{-1},v_1^1,v^l_{-1})\right)\end{aligned}$$

4. For $(v_1^1,s_1^3)$, $(v^l_{-1},s^m_{-1})$, for each $a_1\in[a_1^k,a_1^{k+1}]$, and for all $a_{-1}$:

$$\begin{aligned}\hat{\sigma}(s_1^3,s^m_{-1}|a_1,a_{-1},v_1^1,v^l_{-1}) &= \sigma(s_1^3,s^m_{-1}|a_1^k,a_{-1},v_1^1,v^l_{-1})\\ &-\mathcal{I}(a_1\in\left[a_1^k+(\phi_{l,m}-1)\kappa,a_1^k+\phi_{l,m}\kappa)\right]\\ &\quad\times\left(g(a_1-a_1^k-(\phi_{l,m}-1)\kappa)+h(a_1-a_1^k-(\phi_{l,m}-1)\kappa)\right)\\ &+\mathcal{I}(a_1\in\left[a_1^k+|\Theta_{-1}|\kappa,a_1^{k+1}\right])\frac{1}{\kappa}\left(a_1-a_1^k-|\Theta_{-1}|\kappa\right)\\ &\quad\times\left(\sigma(s_1^3,s^m_{-1}|a_1^{k+1},a_{-1},v_1^1,v^l_{-1})-\sigma(s_1^3,s^m_{-1}|a_1^k,a_{-1},v_1^1,v^l_{-1})\right)\end{aligned}$$

5. For $(v_1^1, s_1^j)$, $j \in \{3, \ldots, |S_1|\}$, $(v_{-1}^l, s_{-1}^m)$, for each $a_1 \in [a_1^k, a_1^{k+1}]$, and for all $a_{-1}$:

$$\begin{aligned}\hat{\sigma}(s_1^j, s_{-1}^m | a_1, a_{-1}, v_1^1, v_{-1}^l) = &\ \sigma(s_1^j, s_{-1}^m | a_1^k, a_{-1}, v_1^1, v_{-1}^l) \\ &+ \mathcal{I}(a_1 \in \left[a_1^k + |\Theta_{-1}|\kappa, a_1^{k+1}\right]) \frac{1}{\kappa} \left(a_1 - a_1^k - |\Theta_{-1}|\kappa\right) \\ &\times \left(\sigma(s_1^j, s_{-1}^m | a_1^{k+1}, a_{-1}, v_1^1, v_{-1}^l) - \sigma(s_1^j, s_{-1}^m | a_1^k, a_{-1}, v_1^1, v_{-1}^l)\right).\end{aligned}$$

Define:

$$\begin{aligned} y_1(a_1, \alpha_{-1}) \equiv \Big( &\hat{\sigma}(s_1^1, s_{-1}^1 | a_1, \alpha_{-1}(v_{-1}^1), v_1^1, v_{-1}^1), \ldots, \hat{\sigma}(s_1^1, s_{-1}^{|S_{-1}|} | a_1, \alpha_{-1}(v_{-1}^1), v_1^1, v_{-1}^1), \ldots \\ &\ldots, \hat{\sigma}(s_1^1, s_{-1}^1 | a_1, \alpha_{-1}(v_{-1}^{|V_{-1}|}), v_1^1, v_{-1}^{|V_{-1}|}), \ldots, \hat{\sigma}(s_1^1, s_{-1}^{|S_{-1}|} | a_1, \alpha_{-1}(v_{-1}^{|V_{-1}|}), v_1^1, v_{-1}^{|V_{-1}|}) \Big) \\ y_2(a_1, \alpha_{-1}) \equiv \Big( &\hat{\sigma}(s_1^2, s_{-1}^1 | a_1, \alpha_{-1}(v_{-1}^1), v_1^1, v_{-1}^1), \ldots, \hat{\sigma}(s_1^2, s_{-1}^{|S_{-1}|} | a_1, \alpha_{-1}(v_{-1}^1), v_1^1, v_{-1}^1), \ldots \\ &\ldots, \hat{\sigma}(s_1^2, s_{-1}^1 | a_1, \alpha_{-1}(v_{-1}^{|V_{-1}|}), v_1^1, v_{-1}^{|V_{-1}|}), \ldots, \hat{\sigma}(s_1^2, s_{-1}^{|S_{-1}|} | a_1, \alpha_{-1}(v_{-1}^{|V_{-1}|}), v_1^1, v_{-1}^{|V_{-1}|}) \Big). \end{aligned}$$

Note that $\hat{\Sigma}$ is defined so that the convex hull of the image of $y_1(a_1, \alpha_{-1})$ on $a_1 \in [a_1^k, a_1^k + |\Theta_{-1}|\kappa]$ is a cross polytope with centre $y_1(a_1^k, \alpha_{-1})$ and circumradius $\frac{\varepsilon}{2}$,[16] as $y_1(a_1, \alpha_{-1})$ is piecewise linear in $a_1$ with:

- $y_1(\alpha_1, \alpha_{-1}) = y_1(\alpha_1^k, \alpha_{-1})$ at $a_1 = a_1^k$
- $y_1(a_1, \alpha_{-1}) = y_1(a_1^k, \alpha_{-1}) + (\frac{\varepsilon}{2}, 0, 0, 0, \ldots)$ at $a_1 = a_1^k + \frac{\kappa}{5}$
- $y_1(a_1, \alpha_{-1}) = y_1(a_1^k, \alpha_{-1}) - (\frac{\varepsilon}{2}, 0, 0, 0, \ldots)$ at $a_1 = a_1^k + \frac{3\kappa}{5}$
- $y_1(a_1, \alpha_{-1}) = y_1(a_1^k, \alpha_{-1})$ at $a_1 = a_1^k + \frac{4\kappa}{5}$
- $y_1(a_1, \alpha_{-1}) = y_1(a_1^k, \alpha_{-1})$ at $a_1 = a_1^k + \kappa$
- $y_1(a_1, \alpha_{-1}) = y_1(a_1^k, \alpha_{-1}) + (0, \frac{\varepsilon}{2}, 0, 0, \ldots)$ at $a_1 = a_1^k + \kappa + \frac{\kappa}{5}$, etc.

Also $y_2(a_1, \alpha_{-1})$ is piecewise linear in $a_1$ with:

- $y_2(a_1, \alpha_{-1}) = y_2(a_1^k, \alpha_{-1})$ at $a_1 = a_1^k$
- $y_2(a_1, \alpha_{-1}) = y_1(a_1^k, \alpha_{-1})$ at $a_1 = a_1^k + \frac{\kappa}{5}$

[16] That is, the convex hull of the image is an $\frac{\varepsilon}{2}$ ball in the $l_1$ norm centred around $y_1(a_1^k, \alpha_{-1})$.

- $y_2(a_1, \alpha_{-1}) = y_2(a_1^k, \alpha_{-1}) + (\frac{\varepsilon}{2}, 0, 0, 0, \ldots)$ at $a_1 = a_1^k + \frac{2\kappa}{5}$

- $y_2(a_1, \alpha_{-1}) = y_2(a_1^k, \alpha_{-1}) - (\frac{\varepsilon}{2}, 0, 0, 0, \ldots)$ at $a_1 = a_1^k + \frac{4\kappa}{5}$

- $y_2(a_1, \alpha_{-1}) = y_2(a_1^k, \alpha_{-1})$ at $a_1 = a_1^k + \kappa$

- $y_2(a_1, \alpha_{-1}) = y_2(a_1^k, \alpha_{-1})$ at $a_1 = a_1^k + \kappa + \frac{\kappa}{5}$

- $y_2(a_1, \alpha_{-1}) = y_2(a_1^k, \alpha_{-1}) + (0, \frac{\varepsilon}{2}, 0, 0, \ldots)$ at $a_1 = a_1^k + \kappa + \frac{2\kappa}{5}$, etc.

**Lemma 1.** *Let $M \subset A_1$ be finite.*

1. *For any $a_1$ such that $y_1(a_1, \alpha_{-1}) \in \text{int}(co\{y_1(a_1', \alpha_{-1}) : a_1' \in M\})$, $\hat{\gamma}(\Theta_{-1}|(v_1^1, s_1^1), a_1, \alpha_{-1}) \in \text{int}(\hat{C}(\alpha_{-1}))$.*

2. *For any $a_1$ such that $y_2(a_1, \alpha_{-1}) \in \text{int}(co\{y_2(a_1', \alpha_{-1}) : a_1' \in M\})$, $\hat{\gamma}(\Theta_{-1}|(v_1^1, s_1^2), a_1, \alpha_{-1}) \in \text{int}(\hat{C}(\alpha_{-1}))$.*

*Proof.* Note that if $y_1(a_1, \alpha_{-1}) = \sum_{a_1' \in M} \lambda(a_1') y_1(a_1', \alpha_{-1})$, where $\lambda(a_1') \in (0,1)$ for each $a_1' \in M$ and $\sum_{a_i' \in M} \lambda(a_1') = 1$, then $\hat{\gamma}(\Theta_{-1}|(v_1^1, s_1^1), a_1, \alpha_{-1}) = \sum_{a_1'} \lambda'(a_1') \hat{\gamma}(\Theta_{-1}|(v_1^1, s_1^1), a_1', \alpha_{-1})$ for $\lambda'(a_1') \in (0,1)$ for each $a_1' \in M$ and $\sum_{a_i' \in M} \lambda'(a_1') = 1$. To see this, note that:

$$\begin{aligned}
&\hat{\gamma}(\theta_{-1}|(v_1^1, s_1^1), a_1, \alpha_{-1}) \\
&= \frac{\sum_{a_1' \in M} \lambda(a_1') \pi(v_{-1}|v_1^1) \sigma(s_1^1, s_{-1}|a_1', \alpha_{-1}(v_{-1}) v_1^1, v_{-1})}{\sum_{a_1' \in M} \lambda(a_1') \left( \sum_{v_{-1}', s_{-1}'} \pi(v_{-1}'|v_1^1) \sigma(s_1^1, s_{-1}'|a_1', \alpha_{-1}(v_{-1}'), v_1^1, v_{-1}') \right)} \\
&= \sum_{a_1' \in M} \lambda'(a_1') \hat{\gamma}(\theta_{-1}|(v_1^1, s_1^1), a_1', \alpha_{-1})
\end{aligned}$$

where:

$$\lambda'(a_1') = \frac{\lambda(a_1') \sum_{v_{-1}', s_{-1}'} \pi(v_{-1}'|v_1^1) \sigma(s_1^1, s_{-1}'|a_1', \alpha_{-1}(v_{-1}'), v_1^1, v_{-1}')}{\sum_{a_1' \in M} \lambda(a_1') \sum_{v_{-1}', s_{-1}'} \pi(v_{-1}'|v_1^1) \sigma(s_1^1, s_{-1}'|a_1', \alpha_{-1}(v_{-1}'), v_1^1, v_{-1}')}$$

Now, if $y_1(a_1, \alpha_{-1}) \in \text{int}(co\{y_1(a_1', \alpha_{-1}) : a_1' \in M\})$, then $y_1(a_1, \alpha_{-1})$ is a strict convex combination of $y_1(a_1', \alpha_{-1})$ for $a_1' \in M$. This implies that $\hat{\gamma}(\Theta_{-1}|(v_1^1, s_1^1), a_1, \alpha_{-1})$ is a strict convex combination of $\hat{\gamma}(\Theta_{-1}|(v_1^1, s_1^1), a_1', \alpha_{-1})$ for $a_1' \in M$, and hence $\hat{\gamma}(\Theta_{-1}|(v_1^1, s_1^1), a_1, \alpha_{-1}) \in \text{int}(co\{\hat{\gamma}(\Theta_{-1}|(v_1^1, s_1^1), a_1', \alpha_{-1}) : a_1' \in M\}) \subset \text{int}(\hat{C}(\alpha_{-1}))$. An analogous argument establishes the second part of the Lemma. □

Define:

$$A_1^{k^*} = \left\{a_1^k + \frac{\kappa}{5}, a_1^k + \frac{3\kappa}{5}, a_1^k + \kappa + \frac{\kappa}{5}, a_1^k + \kappa + \frac{3\kappa}{5}, \ldots, \right.$$
$$\left. a_1^k + (|\Theta_{-1}| - 1)\kappa + \frac{\kappa}{5}, a_1^k + (|\Theta_{-1}| - 1)\kappa + \frac{3\kappa}{5}\right\}$$
$$A_1^{k^{**}} = \left\{a_1^k + \frac{2\kappa}{5}, a_1^k + \frac{4\kappa}{5}, a_1^k + \kappa + \frac{2\kappa}{5}, a_1^k + \kappa + \frac{3\kappa}{5}, \ldots, \right.$$
$$\left. a_1^k + (|\Theta_{-1}| - 1)\kappa + \frac{2\kappa}{5}, a_1^k + (|\Theta_{-1}| - 1)\kappa + \frac{4\kappa}{5}\right\}$$

**Lemma 2.** *For any $k \in \{1, \ldots, K-1\}$, and for all $\alpha_{-1}$:*

1. *For all $a_1 \in [a_1^k, a_1^{k+1}] \setminus A_1^{k^*}$, $y_1(a_1, \alpha_{-1}) \in \text{int}(co\{y_1(a_1, \alpha_{-1}) : a_1 \in A_1^{k^*}\})$.*
2. *For all $a_1 \in [a_1^k, a_1^{k+1}] \setminus A_1^{k^{**}}$, $y_2(a_1, \alpha_{-1}) \in \text{int}(co\{y_2(a_1, \alpha_{-1}) : a_1 \in A_1^{k^{**}}\})$.*

*Proof.* Note that the convex hull of the image of $y_1(a_1, \alpha_{-1})$ on $a_1 \in [a_1^k, a_1^k + |\Theta_{-1}|\kappa]$ is a $\frac{\varepsilon}{2}$ cross polytope with centre $y_1(a_1^k, \alpha_{-1})$ that is equal to $co\{y_1(a_1, \alpha_{-1}) : a_1 \in A_1^{k^*}\}$. We will show that for $a_1 \in [a_1^k, a_1^{k+1}] \setminus A_1^{k^*}$, $y_1(a_1, \alpha_{-1}) \in \text{int}(co\{y_1(a_1, \alpha_{-1}) : a_1 \in A_1^{k^*}\})$.

Take any $k \in \{1, \ldots, K-1\}$, $\phi \le |\Theta_{-1}|$. For $a_1 \in [a_1^k + (\phi - 1)\kappa, a_1^k + (\phi - 1)\kappa + \frac{\kappa}{5})$, note that $y_1(a_1, \alpha_{-1})$ is a convex combination of $y_1(a_1^k, \alpha_{-1})$ and $y_1(a_1^k + (\phi - 1)\kappa + \frac{\kappa}{5}, \alpha_{-1})$, where the weight on the latter term is strictly less than one; thus, $y_1(a_1, \alpha_{-1}) \in \text{int}(co\{y_1(a_1, \alpha_{-1}) : a_1 \in A_1^{k^*}\})$. Similar arguments establish that for $a_1 \in [a_1^k, a_1^k + |\Theta_{-1}|\kappa] \setminus A_1^{k^*}$, $y_1(a_1, \alpha_{-1}) \in \text{int}(co\{y_1(a_1, \alpha_{-1}) : a_1 \in A_1^{k^*}\})$, and for $a_1 \in [a_1^k, a_1^k + |\Theta_{-1}|\kappa] \setminus A_1^{k^{**}}$, $y_2(a_1, \alpha_{-1}) \in \text{int}(co\{y_2(a_1, \alpha_{-1}) : a_1 \in A_1^{k^{**}}\})$.

For $a_1 \in [a_1^k + |\Theta_{-1}|\kappa, a_1^{k+1}]$, note that the convex hull of the image of $y_1(a_1, \alpha_{-1})$ on $a_1 \in [a_1^k, a_1^k + |\Theta_{-1}|\kappa]$ is a cross polytope that contains a hypersphere with radius $\frac{\varepsilon}{2\sqrt{|\Theta_{-1}|}}$ centred around $y_1(a_1^k, \alpha_{-1})$. Since $|\sigma(s|a_1^k, a_{-1}, v) - \sigma(s|a_1^{k+1}, a_{-1}, v)| \le \frac{\varepsilon}{2|\Theta_{-1}|^2}$, $y_1(a_1, \alpha_{-1})$ and $y_2(a_1, \alpha_{-1})$ are in the interior of this cross polytope for $a_1 \in [a_1^k + |\Theta_{-1}|\kappa, a_1^{k+1}]$. This is because $||y_1(a_1^k, \alpha_{-1}) - y_1(a_1^{k+1}, \alpha_{-1})|| \le \frac{\varepsilon}{2|\Theta_{-1}|} < \frac{\varepsilon}{2\sqrt{|\Theta_{-1}|}}$, and hence $y_1(a_1^{k+1}, \alpha_{-1}) \in \text{int}(co\{y_1(a_1, \alpha_{-1}) : a_1 \in A_1^{k^*}\})$. Since for $a_1 \in [a_1^k + |\Theta_{-1}|\kappa, a_1^{k+1}]$, $y_1(a_1, \alpha_{-1})$ is a convex combination of $y_1(a_1^k, \alpha_{-1})$ and $y_1(a_1^{k+1}, \alpha_{-1})$, it follows that $y_1(a_1, \alpha_{-1}) \in \text{int}(co\{y(a_1, \alpha_{-1}) : a_1 \in A_1^{k^*}\})$. An analogous argument establishes that for $a_1 \in [a_1^k + |\Theta_{-1}|\kappa, a_1^{k+1}]$, $y_2(a_1, \alpha_{-1}) \in \text{int}(co\{y_2(a_1, \alpha_{-1}) : a_1 \in A_1^{k^{**}}\})$. □

**Lemma 3.** *$\hat{\Sigma}$ is continuous, has full support, and $d(\hat{\Sigma}, \Sigma) \le \varepsilon$.*

*Proof.* To see that $\hat{\Sigma}$ is continuous, note that for each $s, v$, $\hat{\sigma}(s|a_1, a_{-1}, v)$ is the sum of $\sigma(s|a_1, a_{-1}, v)$ and a function that is continuous in $a_1$ and constant (and thus continuous) in $a_{-1}$. Note that for $a_1 \in [a_1^k, a_1^{k+1}]$, for all $a_{-1}, s, v$, $|\hat{\sigma}(s|a_1, a_{-1}, v) - \sigma(s|a_1, a_{-1}, v)| \leq |\hat{\sigma}(s|a_1, a_{-1}, v) - \sigma(s|a_1^k, a_{-1}, v)| + |\sigma(s|a_1, a_{-1}, v) - \sigma(s|a_1^k, a_{-1}, v)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2|\Theta_{-1}|^2} \leq \varepsilon$. Thus, $d(\Sigma, \hat{\Sigma}) \leq \varepsilon$. Finally, for sufficiently small $\varepsilon$, full support of $\Sigma$ implies the full support of $\hat{\Sigma}$. □

Lemma 1 and Lemma 2 imply that for every $a_1 \in [a_1^k, a_1^{k+1}]$, either $\hat{\gamma}(\Theta_{-1}|(v_1^1, s_1^1), a_1, \alpha_{-1}) \in \text{int}(\hat{C}(\alpha_{-1}))$ or $\hat{\gamma}(\Theta_{-1}|(v_1^1, s_1^2), a_1, \alpha_{-1}) \in \text{int}(\hat{C}(\alpha_{-1}))$. Since this is true for all $\alpha_{-1}$, it follows that for every $\alpha$, there exists $\theta_1$ such that $\hat{\gamma}(\Theta_{-1}|\theta_1, \alpha) \in \text{int}(\hat{C}(\alpha_{-1}))$. Combined with Lemma 3, this implies that $\hat{\Sigma} \in \mathcal{T} \setminus \overline{\mathcal{T}^*}$ and $d(\Sigma, \hat{\Sigma}) \leq \varepsilon$. Since $\Sigma$ and $\varepsilon$ were arbitrary, this implies that $\overline{\mathcal{T}^*}$ has empty interior. □

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