Large Strategy-Proof Mechanisms*

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Abstract

We introduce a distributional approach to mechanism design that proves to be useful for the analysis of large anonymous mechanisms in a private values setting. We use this setting to relate the classic notions of strategy-proofness and envy-freeness for anonymous mechanisms to approximate versions of these notions. We show that, in a generic sense, there is no difference between the exact and approximate versions of these notions.

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1 Introduction

There are many examples where an anonymous mechanism is used to allocate goods or services to a large number of individuals. For instance, consider the allocation of school seats for children living in a city, the election of a parliament in a country, and so on. The presence of a large number of individuals often facilitates the analysis of anonymous mechanisms and, more importantly, implies that the mechanism has desirable incentive properties. This has been shown by Córdoba and Hammond (1998), Jackson and Kremer (2007), McLean and Postlewaite (2015), Hashimoto (2018) and Azevedo and Budish (2019) among others, who constructed mechanisms in which individuals have an incentive to truthfully reveal their private information and do not envy the allocation of others in an approximate sense (i.e. are approximately strategy-proof and approximately envy-free) when the number of participants is large. Specifically, Azevedo and Budish (2019) argue that the mechanisms that are successful in practice are the ones that are strategy-proof in the large, i.e. satisfy a particular notion of approximate strategy-proofness.

A motivation for considering approximate versions of strategy-proofness and envy-freeness comes from several impossibility results, each stating that a certain desired property cannot hold in strategy-proof or envy-free mechanisms. In fact, as shown in the above mentioned papers, these impossibility results do not hold for mechanisms that are merely approximately strategy-proof and approximately envy-free.

This paper casts the relationship between the exact and approximate versions of strategy-proofness and envy-freeness in a different light. Indeed, we show that, for generic preferences with private values and for anonymous mechanisms satisfying a continuity property, approximate versions of strategy-proofness and envy-freeness can be strengthened at virtually no cost to exact ones when the number of participants is large. Thus, any desirable property that holds in a sufficiently large mechanism which is approximately strategy-proof (resp. approximately envy-free) will hold approximately in a strategy-proof (resp. envy-free) mechanism. Furthermore, when a sequence of approximately strategy-proof mechanisms is generic in a sense to be made precise, then each of these mechanisms is strategy-proof and envy-free when the number of participants is large. These results thus suggest that the difference between exact and approximate version of strategy-proofness and envy-freeness is small and, therefore, that strategy-proofness and envy-freeness are not restrictive properties after all.

The above results imply that, in the generic case, one only needs to check that an approximate version of strategy-proofness (e.g. strategy-proofness in the large) holds for a sequence of anonymous mechanisms to be strategy-proof when the number of participants is large. Since the requirements of the former are easier to demonstrate than the latter, this makes any such approximate notion of strategy-proofness a useful concept even when the focus is on strategy-proof mechanisms. This also identifies a way in which a sequence of anonymous mechanisms may fail to be strategy-proof, namely that the sequence may fail to be generic. In this case, a small perturbation of the sequence of mechanisms is enough to restore strategy-proofness. Similar considerations apply regarding envy-freeness.

The important concept for our results is the notion of a reduced mechanism which is a function that specifies the distribution of outcomes that results from the action of a player given the distribution of actions of the other players. Thus, it represents how a player can impact the distribution of outcomes through the choice of his own action given an action distribution of the other players. Formally, this notion corresponds to that of a large-market limit mechanism of Azevedo and Budish (2019) and defines a semi-anonymous game as in Kalai (2004); the key observation in our approach is that, in each finite-player anonymous mechanism, there is a reduced mechanism which equals the marginal distribution of outcomes for each player. That one such reduced mechanism is enough as opposed to one for each player is a consequence of anonymity, which also implies that only the distribution of actions of the other players matters for such marginal distribution of outcomes.

The observation that, for each finite-player anonymous mechanism, there is a reduced mechanism that equals the marginal distribution of outcomes for each player implies that finite-player anonymous mechanisms can be represented in the same space where limit mechanisms lie. This common space will be detailed below, the important point here being that it allows us to use a reduced mechanism as a reference point from which properties of finite-player mechanisms are derived. This is in contrast to many asymptotic results (such as those in Azevedo and Budish (2019)) which are obtained by focusing exclusively on the case of finitely many individuals.

This approach is conceptually the same as in Hashimoto (2018) who "constructs finite-market mechanisms from an infinite-market one." While our setting does not feature information aggregation,[1] in contrast to Hashimoto (2018), we present general results in terms of the underlying economic problem and anonymous mechanism used, i.e. they are not restricted to his generalized random priority mechanisms and his specific allocation problem.[2]

The space in which mechanisms are represented is the space of reduced mechanisms. Our results are for the continuous case and, thus, the space of reduced mechanisms is formally the space of continuous functions that specify a distribution of outcomes for each action and distribution of actions. Finite-player anonymous mechanisms are represented by a reduced mechanism which is only defined for the finitely many distributions of actions that can arise from the choices of the other players; in contrast, in the limit as the number of players increases to infinity, reduced mechanisms are defined for each distribution of actions. Thus, to compare both cases, we identify each reduced mechanism with its graph and endow the resulting space with the Hausdorff metric topology. We consider sequences of anonymous mechanisms and we focus on such sequences that converge to a reduced mechanism. We then show that:

1. When there are at least three outcomes and for generic preferences (e.g. in an open and dense subset of preferences), each mechanism in a sequence of finite-player anonymous mechanisms converging to a generic strategy-proof reduced

[1] Information aggregation plays an important role in his setting due to the attempt of the players to learn the state of nature by aggregating their information. In contrast, states of nature are absent from our setting.

[2] See Section 5.2 for a more detailed discussion of Hashimoto's (2018) approach.

mechanism (i.e. in an open and dense subset of such mechanisms) is, eventually, strictly strategy-proof and strictly envy-free.

2. When there are at least three outcomes and for generic preferences, each sequence of anonymous mechanisms converging to a (not necessarily generic) strategy-proof reduced mechanism can be slightly changed in an ex-ante sense to obtain a sequence of anonymous mechanisms whose elements are, eventually, strictly strategy-proof and strictly envy-free.

These results apply to, for example, sequences of anonymous mechanisms that are strategy-proof in the large since, as we show, the limits of such sequences are strategy-proof reduced mechanisms.

The requirement that a sequence of anonymous mechanisms converges is not demanding; it simply requires that the mechanisms change little when the number of its participants exceed a certain (large) bound. The substantive restriction in our setting comes from the requirement that the limit reduced mechanism is continuous. While we aim to extend our results in future work to cover some discontinuous mechanisms, we stress here that the point of this paper is to argue that the difference between exact and approximate versions of strategy-proofness and envy-freeness may amount to nothing, and this has been established by the above results in the continuous case. The analytical convenience of the continuous case allows us to make this point in a (relatively) simple and transparent way. Several examples discuss these issues.

The paper is organized as follows. Section 2 contains a brief literature review. Our framework is in Section 3 and includes several examples. Our main results are in Section 4. We provide a discussion of their relationship to the recent work of Hashimoto (2018) and Azevedo and Budish (2019) in Section 5, where we also discuss our continuity assumption. The proofs of our results are in the appendix, Section A.

2 Literature review

A large literature has demonstrated that strategy-proofness is a desirable but restrictive objective of market design. Strategy-proof mechanisms satisfy an important robustness property: truthful revelation is optimal regardless of participants' beliefs.[3] Nevertheless, strategy-proofness can be restrictive in markets with finitely many players. For example, several papers have demonstrated that strategy-proofness can be incompatible with efficiency in a variety of settings.[4] On the other hand, strategy-proofness may appear to become less restrictive in large markets. To take a specific example, even though there does not exist a strategy-proof mechanism that implements competitive equilibrium outcomes in a finite-player exchange economy, the Walrasian mechanism is exactly strategy-proof when there is a continuum of agents and approximately so when the number of players is large but finite.[5]

Given the above, many papers have attempted to construct specific finite-player mechanisms that approximate limit mechanisms with desirable properties, the goal being that the finite-player mechanisms will also satisfy the properties in an approximate sense. For example, in the context of an exchange economy, Córdoba and Hammond (1998) and Kovalenkov (2002) construct strategy-proof mechanisms whose outcomes converge to those of a competitive equilibrium, and Hashimoto (2018) (discussed further in Section 5.2) introduces a strategy-proof mechanism that can approximate many limit mechanisms of interest. In contrast to these papers, we are less concerned with properties of specific mechanisms, but rather provide general results about sequences of approximating mechanisms. In particular, our Theorem 1 implies

[3]The argument that this property is desirable originates from Wilson (1987); Bergemann and Morris (2005) show that only strategy-proof mechanisms satisfy this property. As well as being robust, Azevedo and Budish (2019) note that strategy-proof mechanisms are strategically simple (Fudenberg and Tirole (1991) and Roth (2008)), fair (Friedman (1991), Pathak and Sönmez (2008) and Abdulkadiroğlu, Pathak, Roth, and Sönmez (2006)) and generate useful information about the true preferences of its participants (Roth (2008) and Abdulkadiroğlu, Agarwal, and Pathak (2017)).

[4]See Hurwicz (1972), Abdulkadiroğlu, Pathak, and Roth (2009), Papai (2001), Ehlers and Klaus (2003), Hatfield (2009), Zhou (1990) and Bogomolnaia and Moulin (2001).

[5]See, respectively, Hammond (1979) and Roberts and Postlewaite (1976).

that, in the generic case, any mechanism in a sequence of anonymous mechanisms that converges to a strategy-proof mechanism is itself eventually strategy-proof.

An alternative approach is to replace strategy-proofness with a weaker criterion. For example, Azevedo and Budish (2019) introduce the notion of strategy-proofness in the large, which is an approximate and asymptotic form of strategy-proofness.[6] They consider a number of existing mechanisms and argue that the mechanisms that perform well in practice are precisely those that are strategy-proof in the large. We provide an alternative perspective by showing that mechanisms that are strategy-proof in the large, since they converge to a limit mechanism that is strategy proof (by Theorem 3), can in fact be made strategy-proof at virtually no cost (by Theorem 2).

On a technical level, Azevedo and Budish (2019) focus on the case of finitely many individuals and establish their results using asymptotic methods such as the Dvoretzky, Kiefer, and Wolfowitz's (1956) inequality. In contrast, our approach consists in, first, representing finite-player anonymous mechanisms in a space, the space of reduced mechanisms, that also contains their limits and, second, using the properties of such limits to obtain properties of large finite-player mechanisms. We have used this approach in the context of normal-form games in Carmona and Podczeck (2020), Carmona and Podczeck (2021) and Carmona and Podczeck (2022).[7] As in here, the contribution of these papers consists in dropping the "approximate" qualifier generically from results in Rashid (1983), Khan and Sun (1999), Kalai (2004), Carmona and Podczeck (2009), Carmona and Podczeck (2012), and Deb and Kalai (2015) showing that sufficiently large finite-player games have pure strategy approximate equilibria.[8]

Besides strategy-proofness, another desirable robustness property of a mechanism

[6]For a survey of other notions of approximate strategy-proofness that have been used in the literature, see footnote 12 in Azevedo and Budish (2019).

[7]See also Greinecker and Kah (2021), who used this approach in the context of one-to-one matching.

[8]Here "pure strategy approximate equilibrium" means a strategy profile such that for some numbers $\varepsilon > 0$ and $0 \leq \eta < 1$, players which make up a fraction of at least $1 - \eta$ cannot deviate so that payoffs increase more than ε (in Kalai (2004) and some of the results of Carmona and Podczeck (2009) the number η is zero), and "sufficiently large" means that these numbers can be made arbitrarily small if one takes the number of players to be large enough.

is full implementation, i.e. the requirement that the intended outcome is implemented by every equilibrium. Sinander and Escudé (2020) point out that strictly strategy-proof mechanisms satisfy both robustness criteria and show that, in the canonical auction environment, every strategy-proof mechanism can be made strict by an arbitrarily small modification. In our setting, Theorem 2 establishes an analogous result for sequences of anonymous mechanisms that converge to a strategy-proof mechanism and thus shows that a designer who wishes to implement the outcome of a mechanism from the tail of such a sequence can achieve full implementation at a small cost.

3 Model

3.1 Preferences

Individuals have preferences that depend on their type and on which outcome occurs. Specifically, there is a common finite type space T and a finite set of outcomes X_0. There is a payoff function $u : T \times X_0 \to \mathbb{R}$ that is common to all individuals, with $u(t, x_0)$ being each individual's payoff when he is of type t and the outcome is x_0. Thus, preferences are private values since an individual's payoff depends only on his type and outcome. Let $X = M(X_0)$ be the set of probability distributions over outcomes and we extend the common payoff function from $T \times X_0$ to $T \times X$ by setting $u(t, x) = \sum_{x_0 \in X_0} x(x_0)u(t, x_0)$ for each $t \in T$ and $x \in X$.[9]

The common payoff function $u : T \times X_0 \to \mathbb{R}$ is identified with an element of $\mathbb{R}^{|T||X_0|}$. We say that a subset U of $\mathbb{R}^{|T||X_0|}$ is *generic* if the closure of its complement has Lebesgue measure zero in $\mathbb{R}^{|T||X_0|}$.

3.2 Anonymous mechanisms

An *anonymous mechanism* is defined by a finite set $I = \{1, \ldots, n\}$ of players, a set $Y_n \subseteq X_0^n$ of feasible outcomes, a finite action set A and a function $\Phi_n : A^n \to M(Y_n)$

[9]More generally, if Z is a finite set, then $M(Z)$ denotes the set of probability distributions over Z. For each $\sigma \in M(Z)$, $\text{supp}(\sigma) = \{z \in Z : \sigma(z) > 0\}$ denotes the support of σ.

such that

$$\Phi_n(a_1, \dots, a_n)(x_1, \dots, x_n) = \Phi_n(a_{k(1)}, \dots, a_{k(n)})(x_{k(1)}, \dots, x_{k(n)})$$

for each $(a_1, \dots, a_n) \in A^n$, $(x_1, \dots, x_n) \in X_0^n$ and bijection k mapping $\{1, \dots, n\}$ onto itself. The interpretation is that Φ_n maps action profiles to probability distributions over feasible outcomes. An anonymous mechanism is *direct* if $A = T$. This concludes the definition of an anonymous mechanism and, thus, we write (I, Y_n, A, Φ_n) for the anonymous mechanism just defined; in the case of a direct mechanism, we simply write (I, Y_n, Φ_n).

In the above notation, n stands for the cardinality of the set I of players. We often consider sequences $\langle (I_n, Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ of anonymous direct mechanisms where $I_n = \{1, \dots, n\}$ so that the sequence is indexed by the number of players. In this case, we simply omit I_n, thus writing such sequence as $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$.

The following is a key lemma in the analysis of anonymous mechanisms. Let $M_{n-1}(A) = \{\pi \in M(A) : \pi = \sum_{j=1}^{n-1} 1_{a_j}/(n-1) \text{ for some } a_1, \dots, a_{n-1} \in A\}$ be the set of action distributions that $n-1$ players can induce and Φ_i^n be the marginal of Φ_n on the ith coordinate of X_0^n, where $i \in \{1, \dots, n\}$.

Lemma 1. *If Φ_n is anonymous, then there exists $\gamma_n : A \times M_{n-1}(A) \to X$ such that*

$$\Phi_i^n(a_1, \dots, a_n) = \gamma_n(a_i, \sum_{j \neq i} 1_{a_j}/(n-1))$$

for each $i \in \{1, \dots, n\}$ and $(a_1, \dots, a_n) \in A^n$.

Lemma 1 allows us to express each player's payoff function as a function U of his type, his choice and the distribution of the choices of all other players. The function $U : T \times A \times M_{n-1}(A) \to \mathbb{R}$ is defined by setting, for each $t \in T$, $a \in A$ and $\pi \in M_{n-1}(A)$,

$$U(t, a, \pi) = u(t, \gamma_n(a, \pi)).$$

The focus is on direct mechanisms and on strategy-proofness. Formally, the direct mechanism (I, Y_n, Φ_n) is *strategy-proof* if $U(t, t, \pi) \geq U(t, t', \pi)$ for each $t, t' \in T$ and $\pi \in M_{n-1}(T)$; it is *strictly strategy-proof* if all these inequalities hold strictly.

We also consider envy-freeness. A direct mechanism (I, Y_n, Φ_n) is *envy-free* if, for each $i, j \in I$ and $t \in T^n$, $u(t_i, \Phi_i^n(t)) \geq u(t_i, \Phi_j^n(t))$. Using Lemma 1, this inequality can be written as

$$u(t_i, \gamma_n(t_i, \sum_{k \neq i} 1_{t_k}/(n-1))) \geq u(t_i, \gamma_n(t_j, \sum_{k \neq j} 1_{t_k}/(n-1))).$$

Since $\sum_{k \neq j} 1_{t_k}/(n-1) = \sum_{k \neq i} 1_{t_k}/(n-1) + (1_{t_i} - 1_{t_j})/(n-1)$, the envy-freeness of (I, Y_n, Φ_n) can be stated as follows: for each $t, t' \in T$ and $\pi \in M_{n-1}(T)$ with $\pi(t') > 0$,

$$u(t, \gamma_n(t, \pi)) \geq u(t, \gamma_n(t', \pi + (1_t - 1_{t'})/(n-1))).$$

We say that (I, Y_n, Φ_n) is *strictly envy-free* if all these inequalities hold strictly.

3.3 Space of reduced mechanisms

For fixed preferences, described by the function u, the relevant feature of an anonymous mechanism is the function γ_n, which describes the marginal distribution over outcomes of each individual. This provides a reduced-form description of the mechanism, which proves to be useful in the analysis of large mechanisms.

Our results require some assumptions on reduced mechanisms and a convergence notion. In this paper, we focus on a convenient case from a technical viewpoint, namely on reduced mechanisms that are continuous, and on a uniform convergence notion. We focus on direct mechanisms, i.e. set $A = T$ for the remainder of this paper.

Let $\mathcal{M}$ denote the space of all continuous functions $\gamma : T \times C \to X$ where C is a nonempty closed subset of $M(T)$. This is the space of mechanisms that we consider. A *direct reduced mechanism* is $\gamma \in \mathcal{M}$.

For example, the function γ_n in Lemma 1 with $A = T$ belongs to $\mathcal{M}$ with $C = M_{n-1}(T)$. The case where $C = M(T)$ arises in the limit when the number of players goes to infinity. The reason is that the relevant set of distributions induced by players other than some fixed player is then $M(T)$. Accordingly, we let $\mathcal{L} \subseteq \mathcal{M}$ be the space of all continuous functions $\gamma : T \times M(T) \to X$.

A direct reduced mechanism $\gamma \in \mathcal{L}$ is *strategy-proof* if $u(t, \gamma(t, \pi)) \geq u(t, \gamma(t', \pi))$ for each $t, t' \in T$ and $\pi \in M(T)$. Let $\mathcal{S}$ denote the set of $\gamma \in \mathcal{L}$ that are strategy-proof. The set $\mathcal{S}$ is nonempty since it contains constant reduced mechanisms (i.e. $\gamma \in \mathcal{S}$ whenever $\gamma(t, \pi) = \gamma(t', \pi')$ for each $t, t' \in T$ and $\pi, \pi' \in M(T)$).

Our results are for sequences of anonymous direct mechanisms that converge according to the following convergence notion. Each $\gamma \in \mathcal{M}$ is identified with its graph, graph(γ), which is a nonempty and compact subset of $T \times M(T) \times X$. We endow the space of nonempty and compact subsets of $T \times M(T) \times X$ with the Hausdorff metric topology and obtain in this way a topology on $\mathcal{M}$; $\mathcal{L}$ is then endowed with the resulting relative topology. In particular, a sequence $\langle \gamma_n \rangle_n$ in $\mathcal{M}$ *converges* to $\gamma \in \mathcal{M}$, which we write as $\gamma_n \to \gamma$, if graph(γ_n) $\to$ graph(γ) in the Hausdorff metric topology of the space of nonempty and compact subsets of $T \times M(T) \times X$.[10] Furthermore, we say that a sequence $\langle (I_n, Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ of anonymous direct mechanisms *converges* to $\gamma \in \mathcal{M}$ if $\gamma_n \to \gamma$ where, for each $n \in \mathbb{N}$, γ_n corresponds to (I_n, Y_n, Φ_n) via Lemma 1.

One of our main results considers sequences of anonymous direct mechanisms that converge to a generic strategy-proof direct reduced mechanism. We say that a subset S of $\mathcal{S}$ is *generic* if S is open and dense in $\mathcal{S}$.

3.4 Examples

The following examples illustrate the notion of a reduced mechanism and the convergence notion we use. We consider the Boston mechanism with a single round as in Azevedo and Budish (2019), the random priority mechanism as in Hashimoto (2018) and an auction. The latter is an example where the continuity of the reduced mechanism fails, but we show that there is a small perturbation of the auction that yields a continuous reduced mechanism.

3.4.1 Boston mechanism with a single round

This example is based on Azevedo and Budish (2019, Section E.1). Consider a set of students who apply to a finite set S of schools. Each school can accommodate a proportion $q_s \in (0, 1)$ of the market. Specifically, if there are n students, then school s

[10] We often let $d(\gamma, \gamma')$ stand for the Hausdoff distance between graph(γ) and graph(γ') whenever $\gamma, \gamma' \in \mathcal{M}$.

can accommodate $\lfloor q_s n \rfloor$ students, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. It is possible that some students are not allocated to any school, an outcome denoted by $\emptyset$. Thus, the set of outcomes is $X_0 = S \cup \{\emptyset\}$. Each student chooses a school, thus, $A = S$. The set of types is also the set of schools: $T = S$. Preferences are such that $u(s,s) > u(s,s') > 0 = u(s,\emptyset)$ for each $s, s' \in S$ such that $s \neq s'$; thus, each student's type indicates his favorite school and not being allocated to a school is the worst outcome.

Let L be the set of orders $\prec$ on $\{1,\ldots,n\}$ (thus, $|L| = n!$) with $j \prec i$ meaning that j comes before (or has priority over) i according to $\prec$. For each possible order $\prec \in L$ and school s, the first $n_s = \lfloor q_s n \rfloor$ students according to $\prec$ amongst those who choose s are allocated to school s, while the remaining ones who choose s are allocated to $\emptyset$. For each $(s_1,\ldots,s_n) \in S^n$, $(x_1,\ldots,x_n) \in X_0^n$ and $s \in S$, let

$$\begin{aligned} I(s) &= \{i \in I : s_i = s\}, \\ I(s,s) &= \{i \in I : s_i = s \text{ and } x_i = s\} \text{ and} \\ I(s,\emptyset) &= \{i \in I : s_i = s \text{ and } x_i = \emptyset\}. \end{aligned}$$

Then $(x_1,\ldots,x_n) \in \operatorname{supp}(\Phi_n(s_1,\ldots,s_n))$ if and only if, for each $s \in S$,

$$\begin{aligned} &|I(s)| = |I(s,s)| + |I(s,\emptyset)|, \\ &|I(s,s)| \leq n_s, \text{ and} \\ &|I(s,\emptyset)| > 0 \text{ only if } |I(s,s)| = n_s. \end{aligned}$$

Furthermore, for each $(x_1,\ldots,x_n) \in \operatorname{supp}(\Phi_n(s_1,\ldots,s_n))$,

$$\begin{aligned} \Phi_n(s_1,\ldots,s_n)(x_1,\ldots,x_n) = \frac{1}{n!}|\{\prec \in L : &\text{ for each } s \in S, \\ &j \prec i \text{ for each } j \in I(s,s) \text{ and } i \in I(s,\emptyset)\}|. \end{aligned} \tag{1}$$

The marginal distribution γ_n of Φ_n is then as follows: Let $s \in S$ and $\pi \in M_{n-1}(T)$, with the interpretation that s is the school choice of a given student – student 1 for concreteness – and π is the distribution of the choices of the other students. Then $(n-1)\pi(s)$ is the number of other students who choose s and $1 + (n-1)\pi(s)$ is the total number of students who choose s. Thus, if $1 + (n-1)\pi(s) \leq n_s$, then student 1 is allocated to school s. If, instead, $1 + (n-1)\pi(s) > n_s$, then student 1 is allocated to school s if and only if he is one of the first n_s students according to $\prec$ amongst those $1 + (n-1)\pi(s)$ students who choose s. The number of orders where student 1 is the lth student amongst those $k = 1 + (n-1)\pi(s)$ students who choose s is

$$\frac{n!}{k!(n-k)!}(k-1)!(n-k)! = \frac{n!}{k} = \frac{n!}{1+(n-1)\pi(s)}$$

since there are $C_k^n = \frac{n!}{k!(n-k)!}$ ways of taking k elements out of n, i.e. of ordering $I(s)$ and $I \setminus I(s)$ by determining whether each of the n positions is occupied by someone from $I(s)$ or from $I \setminus I(s)$, and, for each such ordering, there are $(k-1)!$ ways of ordering those in $I(s)$ such that player 1 is the lth student amongst them and $(n-k)!$ ways of ordering those in $I \setminus I(s)$. Thus, the probability of student 1 being the lth student amongst those in $I(s)$ is

$$\frac{n!}{1+(n-1)\pi(s)}\frac{1}{n!} = \frac{1}{1+(n-1)\pi(s)}$$

and the probability of him being one of the n_s first students in $I(s)$ is

$$\frac{n_s}{1+(n-1)\pi(s)}.$$

Thus, for each $s \in S$ and $\pi \in M_{n-1}(S)$,

$$\gamma_n(s,\pi) = 1_s \min\left\{\frac{\lfloor q_s n\rfloor}{1+(n-1)\pi(s)}, 1\right\} + 1_\emptyset\left(1 - \min\left\{\frac{\lfloor q_s n\rfloor}{1+(n-1)\pi(s)}, 1\right\}\right).$$

Define for each $s \in S$ and $\pi \in M(S)$,

$$\gamma(s,\pi) = 1_s \min\left\{\frac{q_s}{\pi(s)}, 1\right\} + 1_\emptyset\left(1 - \min\left\{\frac{q_s}{\pi(s)}, 1\right\}\right). \tag{2}$$

Then γ is continuous and, thus, γ is a reduced mechanism. Furthermore, $\gamma_n \to \gamma$; this can easily be shown using

$$\gamma_n(s,\pi) = 1_s \min\left\{\frac{\lfloor q_s n\rfloor/n}{1/n+(1-1/n)\pi(s)}, 1\right\} + 1_\emptyset\left(1 - \min\left\{\frac{\lfloor q_s n\rfloor/n}{1/n+(1-1/n)\pi(s)}, 1\right\}\right)$$

and $\lfloor q_s n\rfloor/n \to q_s$.

Furthermore, γ is strategy-proof provided that

$$\frac{u(s,s)}{u(s,s')} \geq \frac{\min\left\{\frac{q_{s'}}{\pi(s')}, 1\right\}}{\min\left\{\frac{q_s}{\pi(s)}, 1\right\}}$$

for each $s, s' \in S$ such that $s \neq s'$ and $\pi \in M(S)$. Since the right-hand side of this inequality is maximized when π is such that $\pi(s) = 1$ (hence, $\pi(s') = 0$), it follows that γ is strategy-proof if and only if $q_s u(s, s) \geq u(s, s')$ for each $s, s' \in S$ such that $s \neq s'$. Furthermore, γ is strictly strategy-proof if and only if $q_s u(s, s) > u(s, s')$ for each $s, s' \in S$ such that $s \neq s'$.

3.4.2 A random priority mechanism

The Boston mechanism considered in the previous example is an example of a random priority mechanism in the sense that the allocation that each individual gets depends on an ordering of the set of individuals which is randomly selected. Here we consider the random priority mechanism in a private values version of the illustrating example in Hashimoto (2018).

There are $\lfloor qn \rfloor$ identical objects to be allocated to n individuals, with $q \in (0, 1)$. Each individual is allocated at most one object and, if he receives one of them, then he makes a payment $0 < p < 1$. The set of outcomes is then $X_0 = \{0, 1\} \times \{0, p\}$. The set of T types is contained in $[0, 1]$ and $t \in T$ represents the valuation of the good. Thus, preferences are described by $u(t, y, z) = ty - z$ for each $t \in T$ and $(y, z) \in X_0$.

Let L be the set of orders $\prec$ on $\{1, \ldots, n\}$ with $j \prec i$ meaning that j has priority over i according to $\prec$. For each possible order $\prec \in L$, the first $\lfloor qn \rfloor$ individuals according to $\prec$ amongst those who reported $t > p$ are allocated one unit and pay p, while the remaining ones are allocated zero units and pay 0. For each $(t_1, \ldots, t_n) \in T^n$ and $(x_1, \ldots, x_n) \in X_0^n$, let

$$
\begin{aligned}
&I^+ = \{i \in I : t_i > p\}, \\
&I^- = I \setminus I^+, \\
&I^+(1) = \{i \in I^+ : x_i = (1, p)\} \text{ and} \\
&I^+(0) = \{i \in I^+ : x_i = (0, 0)\}.
\end{aligned}
$$

Then $(x_1, \ldots, x_n) \in \text{supp}(\Phi_n(t_1, \ldots, t_n))$ if and only if

$$\begin{aligned} &x_i = (0,0) \text{ for each } i \in I^-, \\ &|I^+| = |I^+(1)| + |I^+(0)|, \\ &|I^+(1)| \leq \lfloor qn \rfloor, \text{ and} \\ &|I^+(0)| > 0 \text{ only if } |I^+(1)| = \lfloor qn \rfloor. \end{aligned}$$

Furthermore, for each $(x_1, \ldots, x_n) \in \text{supp}(\Phi_n(t_1, \ldots, t_n))$,

$$\Phi_n(t_1, \ldots, t_n)(x_1, \ldots, x_n) = \frac{1}{n!}|\{\prec \in L : j \prec i \text{ for each } j \in I^+(1) \text{ and } i \in I^+(0)\}|.$$

The marginal distribution γ_n of Φ_n is then obtained in an analogous way to the Boston mechanism. Thus, for each $t \in T$ and $\pi \in M_{n-1}(T)$,

$$\gamma_n(t, \pi) = \begin{cases} 1_{(0,0)} & \text{if } t \leq p, \\ 1_{(1,p)} \min\left\{\frac{\lfloor qn \rfloor}{1+(n-1)\sum_{s>p}\pi(s)}, 1\right\} & \\ +1_{(0,0)}\left(1 - \min\left\{\frac{\lfloor qn \rfloor}{1+(n-1)\sum_{s>p}\pi(s)}, 1\right\}\right) & \text{otherwise.} \end{cases}$$

Define for each $t \in T$ and $\pi \in M(T)$,

$$\gamma(t, \pi) = 1_{(1,p)} \min\left\{\frac{q}{\sum_{s>p}\pi(s)}, 1\right\} + 1_{(0,0)}\left(1 - \min\left\{\frac{q}{\sum_{s>p}\pi(s)}, 1\right\}\right).$$

Then γ is continuous and, thus, γ is a reduced mechanism. Furthermore, $\gamma_n \to \gamma$.

We have that γ is strategy-proof. Indeed, for each $t, t' \in T$ and $\pi \in M(T)$ such that $t \neq t'$,

$$u(t, \gamma(t, \pi)) - u(t, \gamma(t', \pi)) = \begin{cases} 0 & \text{if } t \leq p \text{ and } t' \leq p, \\ -\min\left\{\frac{q}{\sum_{s>p}\pi(s)}, 1\right\}(t-p) & \text{if } t \leq p \text{ and } t' > p, \\ 0 & \text{if } t > p \text{ and } t' > p, \\ \min\left\{\frac{q}{\sum_{s>p}\pi(s)}, 1\right\}(t-p) & \text{if } t > p \text{ and } t' \leq p. \end{cases}$$

3.4.3 An auction

We consider in this section a version of the gth-price auction when there are g units of an object to be allocated. The main distinction to the usual gth-price auction is that instead of setting the price of the good deterministically to gth highest bid and rationing stochastically among agents who are indifferent between buying at that price or not, we propose randomizing the price each potential winner pays between the gth highest bid and the (gth highest bid $+1$) such that as the excess demand at the gth highest bid goes to zero, the probability that the price is equal to the gth highest bid goes to one.[11]

There are $n \in \mathbb{N}$ single-unit-demand bidders and $\lfloor qn \rfloor$ goods, where $q \in (0,1)$. The set of agent types is finite and ordered, $T = \{1, \ldots, m\}$. The utility of an agent depends on whether they get the good $y \in \{0,1\}$, their type $t \in T$, and how much they pay $p \in T \cup \{0\}$: $u(t,y,p) = ty - p$. The set of outcomes is then $X_0 = \{0,1\} \times (T \cup \{0\})$.

Consider the following auction $\Phi_{n,g}$ to allocate g units of the good to n individuals; we will then focus on the case where $g = \lfloor qn \rfloor$ but this slightly more general description will be useful in Section 3.4.4 below. For each $n \in \mathbb{N}$ and $\pi \in M(T)$, let

$$p_{n,g}(\pi) = \min\left\{ t \in T : \frac{g}{n} \geq \sum_{t' \geq t} \pi(t') \right\} \text{ and}$$
$$\alpha_{n,g}(\pi) = \frac{\frac{g}{n} - \sum_{t \geq p_{n,g}(\pi)} \pi(t)}{\pi(p_{n,g}(\pi) - 1)}.$$

Note that the definition of $p_{n,g}$ implies that $\pi(p_{n,g}(\pi) - 1) > \frac{g}{n} - \sum_{t \geq p_{n,g}(\pi)} \pi(t)$. For each $\hat{t} \in T^n$, let

$$\pi = \frac{1}{n} \sum_{j \in I} 1_{\hat{t}_j},$$
$$I^+ = \{i \in I : \hat{t}_i \geq p_{n,g}(\pi)\},$$
$$I^- = \{i \in I : \hat{t}_i = p_{n,g}(\pi) - 1\}, \text{ and}$$
$$I^0 = I \setminus (I^+ \cup I^-).$$

If $\frac{g}{n} = \sum_{t \geq p_{n,g}(\pi)} \pi(t)$, then each $i \in I^+$ gets outcome $(1, p_{n,g}(\pi))$ whereas each $i \notin I \setminus I^+$ gets outcome $(0,0)$. In this case, for each $i \in I$, let $x_i = (1, p_{n,g}(\pi))$ if $i \in I^+$

[11] When there is excess demand at the gth highest bid, the gth highest bid corresponds to $p_{n,g}(\pi) - 1$ as defined in this section and excess demand is proportional to $1 - \alpha_{n,g}(\pi)$, which will be the probability weight on $p_{n,g}(\pi)$. When the gth highest bid exactly clears the market, the gth highest bid is equal to $p_{n,g}(\pi)$ and in this case $\alpha_{n,g}(\pi) = 0$.

and $x_i = (0,0)$ if $i \in I \setminus I^+$, and

$$\Phi_{n,g}(\hat{t}_1, \dots, \hat{t}_n)(x_1, \dots, x_n) = 1.$$

If $\frac{g}{n} > \sum_{t \geq p_{n,g}(\pi)} \pi(t)$, then those in I^+ get the good with probability one but pay $p_{n,g}(\pi)$ with probability $(1 - \alpha_{n,g}(\pi))$ and $p_{n,g}(\pi) - 1$ with probability $\alpha_{n,g}(\pi)$. Moreover, the first $g - n \sum_{t \geq p_{n,g}(\pi)} \pi(t)$ individuals in I^- according to $\prec$ get outcome $(1, p_{n,g}(\pi) - 1)$. The remaining individuals get outcome $(0,0)$. In this case, let

$$\begin{aligned}
&I^+(0) = \{i \in I^+ : x_i = (1, p_{n,g}(\pi))\}, \\
&I^+(-1) = \{i \in I^+ : x_i = (1, p_{n,g}(\pi) - 1)\}, \\
&I^-(1) = \{i \in I^- : x_i = (1, p_{n,g}(\pi) - 1)\} \text{ and} \\
&I^-(0) = \{i \in I^- : x_i = (0,0)\}.
\end{aligned}$$

Then $(x_1, \dots, x_n) \in \text{supp}(\Phi_{n,g}(\hat{t}_1, \dots, \hat{t}_n))$ if and only if

$$\begin{aligned}
&x_i = (0,0) \text{ for each } i \in I^0, \\
&|I^-(1)| = g - n \sum_{t \geq p_n(\pi)} \pi(t), \\
&|I^-(0)| = |I^-| - |I^-(1)|, \text{ and} \\
&|I^+| = |I^+(0)| + |I^+(-1)|.
\end{aligned}$$

Furthermore, let L be the set of orders $\prec$ on I^- with $j \prec i$ meaning that j has priority over i according to $\prec$. Let $\nu \in M(\{-1, 0\})$ be such that $\nu(-1) = \alpha_n(\pi)$ and

$$\nu^+ = \underbrace{\nu \otimes \cdots \otimes \nu}_{|I^+| \text{ times}}.$$

Then let, for each $(x_1, \dots, x_n) \in \text{supp}(\Phi_{n,g}(\hat{t}_1, \dots, \hat{t}_n))$,

$$\begin{aligned}
\Phi_{n,g}(\hat{t}_1, \dots, \hat{t}_n)(x_1, \dots, x_n) = &\frac{1}{|I^-|!} |\{\prec \in L : j \prec i \text{ for each } j \in I^-(1) \text{ and } i \in I^-(0)\}| \\
&\times \nu^+(\{z \in \{-1, 0\}^{|I^+|} : z_i = -1 \text{ for each } i \in I^+(-1) \text{ and} \\
&z_i = 0 \text{ for each } i \in I^+(0)\}).
\end{aligned}$$

The marginal distribution $\gamma_{n,g}$ of $\Phi_{n,g}$ is then as follows. Let $t \in T$ and $\pi \in M_{n-1}(T)$. The distribution of reports is then $\hat{\pi} = \frac{1}{n}1_t + \left(1 - \frac{1}{n}\right)\pi$ and

$$\gamma_{n,g}(t,\pi) = \begin{cases} (1-\alpha_{n,g}(\hat{\pi}))1_{(1,p_{n,g}(\hat{\pi}))} + \alpha_{n,g}(\hat{\pi})1_{(1,p_{n,g}(\hat{\pi})-1)} & \text{if } t \geq p_{n,g}(\hat{\pi}), \\ (1-\alpha_{n,g}(\hat{\pi}))1_{(0,0)} + \alpha_{n,g}(\hat{\pi})1_{(1,p_{n,g}(\hat{\pi})-1)} & \text{if } t = p_{n,g}(\hat{\pi}) - 1, \\ 1_{(0,0)} & \text{otherwise.} \end{cases}$$

We focus on the case where $g = \lfloor qn \rfloor$ and let $\Phi_n = \Phi_{n,\lfloor qn \rfloor}$ and $\gamma_n = \gamma_{n,\lfloor qn \rfloor}$. Define, for each $t \in T$ and $\pi \in M(T)$,

$$\gamma(t,\pi) = \begin{cases} (1-\alpha(\pi))1_{(1,p(\pi))} + \alpha(\pi)1_{(1,p(\pi)-1)} & \text{if } t \geq p(\pi), \\ (1-\alpha(\pi))1_{(0,0)} + \alpha(\pi)1_{(1,p(\pi)-1)} & \text{if } t = p(\pi) - 1, \\ 1_{(0,0)} & \text{otherwise,} \end{cases}$$

where

$$p(\pi) = \min\left\{ t \in T : q \geq \sum_{t' \geq t} \pi(t') \right\} \text{ and}$$
$$\alpha(\pi) = \frac{q - \sum_{t \geq p(\pi)} \pi(t)}{\pi(p(\pi)-1)}.$$

This is an example of a reduced mechanism that fails to be continuous. Indeed, if $q = 1/2$, $T = \{1,2,3,4\}$, $\pi = (1/2, 0, 1/4, 1/4)$ and $\langle \pi_k \rangle_{k \in \mathbb{N}}$ is such that $\pi_k = (1/2 - 2/k, 1/k, 1/4 + 1/k, 1/4)$ for each $k \in \mathbb{N}$, it follows that $p(\pi_k) = 4$ for each $k \in \mathbb{N}$ and $p(\pi) = 2$. Furthermore, $\alpha(\pi_k) = \frac{1/2 - 1/4}{1/4 + 1/k} = \frac{1}{1 + 4/k}$ for each $k \in \mathbb{N}$ and $\alpha(\pi) = 0$. Then $\gamma(4, \pi_k) = (1 - \alpha(\pi_k))1_{(1,4)} + \alpha(\pi_k)1_{(1,3)} \to 1_{(1,3)}$ but $\gamma(4,\pi) = 1_{(1,2)}$.

Finally, we have that γ is strategy-proof. Indeed, for each $t, t' \in T$ and $\pi \in M(T)$ such that $t \neq t'$,

$$u(t,\gamma(t,\pi)) - u(t,\gamma(t',\pi)) = \begin{cases} 0 & \text{if } t \geq p(\pi) \text{ and } t' \geq p(\pi), \\ (1 - \alpha(\pi)(t - p(\pi)) & \text{if } t \geq p(\pi) \text{ and } t' = p(\pi) - 1, \\ t - p(\pi) + \alpha(\pi) & \text{if } t \geq p(\pi) \text{ and } t' < p(\pi) - 1, \\ -(t - p(\pi) + \alpha(\pi)) & \text{if } t < p(\pi) \text{ and } t' \geq p(\pi), \\ -\alpha(\pi)(t - p(\pi) + 1) & \text{if } t < p(\pi) \text{ and } t' = p(\pi) - 1, \\ 0 & \text{if } t < p(\pi) \text{ and } t' < p(\pi) - 1 \end{cases}$$

and, whenever $t < p(\pi)$, we have that $t \leq p(\pi) - 1$, hence, $t - p(\pi) + \alpha(\pi) \leq t - p(\pi) + 1 \leq 0$.

3.4.4 A continuous auction

The auction in the previous section can be made continuous via a slight perturbation. Indeed, its reduced mechanism γ fails to be continuous only at distributions π in the boundary of $M(T)$, i.e. when $\pi(t) = 0$ for some $t \in T$. One way of avoiding this case is by adding, for each $t \in T$, "fake" individuals to the mechanism who always report t as we detail in what follows.

We consider the allocation of $\lfloor qn \rfloor$ goods between n people and the mechanism we will use consists in adding $m\lfloor \delta n \rfloor$ fake individuals, where $\delta > 0$, to the auction of Section 3.4.3; these individuals are fake in the sense that units of the good allocated to them are effectively not allocated (e.g. remain with the auctioneer). In addition, for each $t \in T$, there are $\lfloor \delta n \rfloor$ fake individuals reporting t. Thus, the perturbed auction Φ_n we consider is defined as follows. Let $n \in \mathbb{N}$, $(\hat{t}_1, \dots, \hat{t}_n) \in T^n$, $(x_1, \dots, x_n) \in X_0^n$ and $\tilde{t} = (\overbrace{1, \dots, 1}^{\lfloor \delta n \rfloor \text{ times}}, \dots, \overbrace{m, \dots, m}^{\lfloor \delta n \rfloor \text{ times}})$. Then

$$\Phi_n(\hat{t}_1, \dots, \hat{t}_n)(x_1, \dots, x_n) = \sum_{\tilde{x} \in X_0^{m\lfloor \delta n \rfloor}} \Phi_{n+m\lfloor \delta n \rfloor, \lfloor qn \rfloor}(\hat{t}_1, \dots, \hat{t}_n, \tilde{t})(x_1, \dots, x_n, \tilde{x}),$$

where $\Phi_{n+m\lfloor \delta n \rfloor, \lfloor qn \rfloor}$ is as in Section 3.4.3. Note that from reports $(\hat{t}_1, \dots, \hat{t}_n)$ by n individuals, we obtain $n + m\lfloor \delta n \rfloor$ reports $(\hat{t}_1, \dots, \hat{t}_n, \tilde{t})$. Hence, the distribution of all reports, including those from fake individuals, is

$$\frac{1}{n + m\lfloor \delta n \rfloor} \left(\sum_{i=1}^{n} 1_{\hat{t}_i} + \lfloor \delta n \rfloor \sum_{t \in T} 1_t \right) = \frac{n}{n + m\lfloor \delta n \rfloor} \frac{1}{n} \sum_{i=1}^{n} 1_{\hat{t}_i} + \frac{m\lfloor \delta n \rfloor}{n + m\lfloor \delta n \rfloor} \chi,$$

where $\chi = (1/m, \dots, 1/m)$ is the uniform distribution on T.

The advantage of this perturbation is that we can obtain each of its marginal distributions from $\gamma_{n,g}$. Let $t \in T$ and $\pi \in M_{n-1}(T)$. The distribution of all reports of the non-fake individuals is then $\hat{\pi} = \frac{1}{n} 1_t + \left(1 - \frac{1}{n}\right) \pi$. The distribution of all reports, including the fake individuals is then

$$\bar{\pi}_n = \frac{n}{n + m\lfloor \delta n \rfloor} \left(\frac{1}{n} 1_t + \left(1 - \frac{1}{n}\right) \pi \right) + \frac{m\lfloor \delta n \rfloor}{n + m\lfloor \delta n \rfloor} \chi$$

Writing $p_n(t,\pi)$ instead of $p_{n+m\lfloor \delta n\rfloor,\lfloor qn\rfloor}(\bar{\pi}_n)$, $\alpha_n(t,\pi)$ instead of $\alpha_{n+m\lfloor \delta n\rfloor,\lfloor qn\rfloor}(\bar{\pi}_n)$ and $q_n = \lfloor qn\rfloor/n$, we have that

$$p_n(t,\pi) = \min\left\{\hat{t}\in T : q_n \geq \sum_{t'\geq \hat{t}} \bar{\pi}_n(t')\right\} \text{ and}$$
$$\alpha_n(t,\pi) = \frac{q_n - \sum_{\hat{t}\geq p_n(t,\pi)} \bar{\pi}_n(\hat{t})}{\bar{\pi}_n(p_n(t,\pi)-1)}.$$

Then

$$\gamma_n(t,\pi) = \begin{cases} (1-\alpha_n(t,\pi))1_{(1,p_n(t,\pi))} + \alpha_n(t,\pi)1_{(1,p_n(t,\pi)-1)} & \text{if } t \geq p_n(t,\pi),\\ (1-\alpha_n(t,\pi))1_{(0,0)} + \alpha_n(t,\pi)1_{(1,p_n(t,\pi)-1)} & \text{if } t = p_n(t,\pi)-1,\\ 1_{(0,0)} & \text{otherwise.}\end{cases}$$

For each $\pi \in M(T)$, let

$$\bar{\pi} = \frac{1}{1+m\delta}\pi + \frac{m\delta}{1+m\delta}\chi,$$
$$p(\pi) = \min\left\{t\in T : q \geq \sum_{t'\geq t}\bar{\pi}(t')\right\}, \text{ and}$$
$$\alpha(\pi) = \frac{q - \sum_{t\geq p(\pi)}\bar{\pi}(t)}{\bar{\pi}(p(\pi)-1)}.$$

Finally, define γ by setting, for each $(t,\pi)\in T\times M(T)$,

$$\gamma(t,\pi) = \begin{cases} (1-\alpha(\pi))1_{(1,p(\pi))} + \alpha(\pi)1_{(1,p(\pi)-1)} & \text{if } t\geq p(\pi),\\ (1-\alpha(\pi))1_{(0,0)} + \alpha(\pi)1_{(1,p(\pi)-1)} & \text{if } t = p(\pi)-1,\\ 1_{(0,0)} & \text{otherwise.}\end{cases}$$

We have that γ is strategy-proof by the same argument as in Section 3.4.3.

In Section A.2, we show that γ is continuous (Claim 1) and that $\gamma_n \to \gamma$ (Claim 2) using the fact that, for each $\pi \in M(T)$ and $t\in T$, $\bar{\pi}(t) > 0$. To see how this property is used to establish the continuity of γ, note first that one difficulty with the auctions we have defined is that we only have $\lim_k p(\pi_k) \geq p(\pi)$ whenever $\pi_k \to \pi \in M(T)$. More importantly, we may have $\lim_k p(\pi_k) - p(\pi) > 1$ in the case of the auction in Section 3.4.3. This cannot happen in the auction of this section since, as we show, $\lim_k p(\bar{\pi}_k) - p(\bar{\pi}) \leq 1$. This property together with the definition of γ will then allow us to establish the continuity of γ and the convergence of $\langle\gamma_n\rangle_{n\in\mathbb{N}}$ to γ.

4 Results

4.1 Generic asymptotic strategy-proofness

As we have discussed, several authors have pointed out that several known mechanisms are, asymptotically and approximately, strategy-proof. It is often the case that the notion of approximate strategy-proofness is such that the limit point of a sequence of anonymous direct mechanisms that satisfy it is strategy-proof — see e.g. Theorem 3 below showing that this conclusion holds when the notion of approximate strategy-proofness is Azevedo and Budish's (2019) strategy-proof in the large.

We complement the above result by showing that when the limit point of a sequence of anonymous direct mechanisms is generic (i.e. it belongs to an open and dense subset of the set of strategy-proof mechanisms), then any mechanism in the sequence is strategy-proof provided that the number of its participants is sufficiently large. Thus, in the generic case and with a large number of participants, any approximate notion of strategy-proofness can be strengthened to strategy-proofness at no cost.

The above conclusion requires that there are at least three outcomes and that preferences are generic. The latter is needed to guarantee that there is a reduced mechanism which is independent of the type distribution and is strictly strategy-proof, as the following lemma states.

Lemma 2. *If $|X_0| \geq 3$, then there is a generic subset $\mathcal{U}$ of $\mathbb{R}^{|T||X_0|}$ such that, for each $u \in \mathcal{U}$ and each distinct elements x_1, x_2, x_3 of X_0, there exists $\sigma : T \to X$ such that $u(t, \sigma(t)) > u(t, \sigma(t'))$ and $\operatorname{supp}(\sigma(t)) \subseteq \{x_1, x_2, x_3\}$ for each $t, t' \in T$.*

For the remainder of the paper, let $\mathcal{U}$ be the generic subset of $\mathbb{R}^{|T||X_0|}$ such that the conclusion of Lemma 2 holds.

Theorem 1. *Suppose that $|X_0| \geq 3$. Then, for each $u \in \mathcal{U}$, there is a generic subset $\mathcal{S}^*$ of $\mathcal{S}$ such that if $\gamma \in \mathcal{S}^*$ and $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ is a sequence of anonymous direct mechanisms converging to γ, then there is $N \in \mathbb{N}$ such that (Y_n, Φ_n) is strictly strategy-proof and strictly envy-free whenever $n \geq N$.*

Our proof of Theorem 1 uses for $\mathcal{S}^*$ the set of strictly strategy-proof reduced mechanisms. Lemma 2 shows that $\mathcal{S}^*$ is nonempty and then we show that this set is generic in $\mathcal{S}$. Moreover, we show that mechanisms in a sequence of direct anonymous mechanisms that converges to a strictly strategy-proof reduced mechanism must eventually be strictly strategy-proof and strictly envy-free.

While our proof of Theorem 1 lets $\mathcal{S}^*$ be the set of strictly strategy-proof reduced mechanisms, we remark that this is just a convenient set for our argument. Indeed, let $\mathcal{S}^{**}$ be the set of $\gamma \in \mathcal{S}$ such that the conclusion of Theorem 1 holds, i.e. for each sequence $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ of anonymous direct mechanisms converging to γ, there is $N \in \mathbb{N}$ such that (Y_n, Φ_n) is strictly strategy-proof and strictly envy-free whenever $n \geq N$. Since $\mathcal{S}^*$ is open and dense and the conclusion of Theorem 1 holds for any $\gamma \in \mathcal{S}^*$, it follows that $\mathcal{S}^* \subseteq \mathcal{S}^{**}$, $\mathcal{S}^* = \text{int}(\mathcal{S}^*) \subseteq \text{int}(\mathcal{S}^{**})$ and, hence, $\text{int}(\mathcal{S}^{**})$ is open and dense. Thus, $\text{int}(\mathcal{S}^{**})$ is the largest open and dense subset of $\mathcal{S}$ such that the conclusion of Theorem 1 holds. Furthermore, $\mathcal{S}^{**}$ is, by definition, the largest subset of $\mathcal{S}$ such that the conclusion of Theorem 1 holds and contains a generic set.

Theorem 1 is strong because its conclusion applies to *every* sequence of anonymous direct mechanisms (converging to some element of $\mathcal{S}^*$). It is also strong since its conclusion holds for *both* strategy-proofness and envy-freeness and their *strict* versions. The former, i.e. the fact that the conclusion of Theorem 1 applies to every sequence of anonymous direct mechanisms converging to some element of $\mathcal{S}^*$, is important when the goal is to argue that a specific sequence $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ of anonymous direct mechanisms is strategy-proof simply by studying its limit γ.

Alternatively, one may want to focus on the limit case of an infinite population and focus on elements of $\mathcal{S}$. In this case, it may be enough to ask whether a direct reduced mechanism $\gamma \in \mathcal{S}$ can be justified by a sequence $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ of anonymous direct mechanisms with finitely many players. Indeed, if $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ converges to γ and (Y_n, Φ_n) is strategy-proof for each $n \geq N$, then γ is also the limit of strategy-proof anonymous direct mechanisms with finitely many players and, thus, not an artifact of an infinite population. Clearly, Theorem 1 implies that, if $|X_0| \geq 3$ and $u \in \mathcal{U}$, there is a generic subset of $\mathcal{S}$ such that this conclusion holds.[12]

The notion of genericity in Theorem 1 can be strengthened in some specific cases. We illustrate this in the context of the Boston mechanism of Section 3.4.1. Let u be given as in Section 3.4.1. Each $q = (q_s)_{s \in S}$ defines a reduced Boston mechanism γ_q by (2) and $\gamma_q \in \mathcal{S}$ if $q \in \mathcal{Q} = \{q \in \mathbb{R}^{|S|} : q_s \geq \max_{s' \in S \setminus \{s\}} \frac{u(s,s')}{u(s,s)}$ for each $s \in S\}$. The set $\mathcal{Q}^* = \{q \in \mathbb{R}^{|S|} : q_s > \max_{s' \in S \setminus \{s\}} \frac{u(s,s')}{u(s,s)}$ for each $s \in S\}$ is generic in $\mathcal{Q}$ in the sense that the closure of $\mathcal{Q} \setminus \mathcal{Q}^*$ has Lebesgue measure zero in $\mathbb{R}^{|S|}$. Furthermore, $\gamma_q \in \mathcal{S}^*$ if $q \in \mathcal{Q}^*$, where, as above, $\mathcal{S}^*$ is the set of strictly strategy-proof reduced mechanisms.[13] Thus, these arguments together with Theorem 1 imply the following: for each u as in Section 3.4.1, there is a generic subset $\mathcal{Q}^*$ of $\mathcal{Q}$ such that if $q \in \mathcal{Q}^*$ and $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ is the sequence of Boston mechanisms defined by q via (1) (thus, converging to γ_q), then there is $N \in \mathbb{N}$ such that (Y_n, Φ_n) is strictly strategy-proof and strictly envy-free whenever $n \geq N$.

The notion of genericity in Theorem 1 is already strong enough to imply that any strategy-proof direct reduced mechanism can be arbitrarily well approximated by $\gamma \in \mathcal{S}^*$ (since $\mathcal{S}^*$ is dense). This suggests that any sequence $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ of anonymous direct mechanisms converging to a strategy-proof direct reduced mechanism can be arbitrarily well approximated by a sequence of anonymous direct mechanisms whose elements are eventually strictly strategy-proof and strictly envy-free. We will indeed establish a result along these lines below.

Lemma 2 is key to the existence of arbitrarily close approximations in the above sense and is also the harder and more novel part of the proof of Theorem 1. In its proof, we consider for each distinct elements x_1, x_2, x_3 of X_0 the set $\mathcal{U}_{x_1,x_2,x_3}$ of $u \in \mathbb{R}^{|T||X_0|}$ such that

(a) $u(t,x) \neq u(t,x')$ for each $t \in T$ and $x, x' \in \{x_1, x_2, x_3\}$ with $x \neq x'$, and

[12]Indeed, let $\mathcal{S}^{***}$ be the set of $\gamma \in \mathcal{S}$ such that there is a sequence $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ of anonymous direct mechanisms converging to γ and $N \in \mathbb{N}$ such that (Y_n, Φ_n) is strategy-proof whenever $n \geq N$. Then $\mathcal{S}^* \subseteq \mathcal{S}^{***}$, $\mathcal{S}^* = \text{int}(\mathcal{S}^*) \subseteq \text{int}(\mathcal{S}^{***})$ and, hence, $\text{int}(\mathcal{S}^{***})$ is open and dense.

[13]Note that $\mathcal{Q}^* \neq \emptyset$ and, hence, $\mathcal{S}^* \neq \emptyset$ in this application. Thus, there is no need to require $|X_0| \geq 3$ or that $u \in \mathcal{U}$ since these conditions are needed only to guarantee that $\mathcal{S}^* \neq \emptyset$ in the general context of Theorem 1.

(b) $\frac{u(t,x')-u(t,x)}{u(t,x)-u(t,\hat{x})} \neq \frac{u(t',x')-u(t',x)}{u(t',x)-u(t',\hat{x})}$ for each $t, t' \in T$ with $t \neq t'$ and $x, x', \hat{x} \in \{x_1, x_2, x_3\}$ with $x \neq x'$, $x \neq \hat{x}$ and $x' \neq \hat{x}$.[14]

As a concrete example to illustrate the definition of $\mathcal{U}_{x_1,x_2,x_3}$, consider the case where $X_0 = \{0,1\} \times F$ and F is a finite subset of $[0,1]$, with the interpretation that the first coordinate of $x_0 = (y,z) \in X_0$ indicates an individual's consumption of a good and the second consists of his payment. Let $\theta, \tau \in F$ be such that $\theta \neq \tau$, $\theta \notin T$ and $\tau \notin T$, i.e. assume that F is such that θ and τ exist, as well as $0 \in F$. If payoffs are such that $u(t,x_0) = ty - z$, then u belongs to $\mathcal{U}_{x_1,x_2,x_3}$ for $x_1 = (1,\theta)$, $x_2 = (1,\tau)$ and $x_3 = (0,0)$.[15]

We let $\mathcal{U} = \cap_{(x_1,x_2,x_3)\in Z} \mathcal{U}_{x_1,x_2,x_3}$, where Z is the set of distinct elements x_1, x_2, x_3 of X_0. The proof of Lemma 2 then shows that the complement of $\mathcal{U}$ is closed and has Lebesgue measure zero. It also constructs σ satisfying the properties in the statement of the lemma.[16]

4.2 Approximation by strategy-proof mechanisms

Theorem 1 describes the case where a sequence of anonymous direct mechanisms converges to a generic strategy-proof reduced mechanism. For instance, this happens when the limit mechanism is strictly strategy-proof (whose general existence is guaranteed by Lemma 2) but, as we have pointed out, there are other examples.

Nevertheless, one may be interested in some particular sequence of anonymous direct mechanisms that e.g. are typically used in practice and whose limit may not (be easily shown to) be generic. Theorem 1 and Lemma 2 are still useful in this case as they allow us to obtain another sequence of anonymous direct mechanisms, arbitrarily close to the original one, and such that all of the mechanisms in it are eventually strictly strategy-proof and strictly envy-free. Thus, for sequences of di-

[14] In Section A.5 we illustrate how Lemma 2 can fail when there are only two outcomes.

[15] Indeed, $u(t,x_1) - u(t,x_2) = \theta - \tau$, $u(t,x_1) - u(t,x_3) = t - \theta$, $u(t,x_2) - u(t,x_3) = t - \tau$, $\frac{u(t,x_2)-u(t,x_1)}{u(t,x_1)-u(t,x_3)} = \frac{\tau-\theta}{t-\theta}$, $\frac{u(t,x_3)-u(t,x_1)}{u(t,x_1)-u(t,x_2)} = \frac{t-\theta}{\tau-\theta}$, $\frac{u(t,x_1)-u(t,x_2)}{u(t,x_2)-u(t,x_3)} = \frac{\theta-\tau}{t-\tau}$, $\frac{u(t,x_3)-u(t,x_2)}{u(t,x_2)-u(t,x_1)} = \frac{t-\tau}{\theta-\tau}$, $\frac{u(t,x_1)-u(t,x_3)}{u(t,x_3)-u(t,x_2)} = \frac{t-\theta}{\tau-t}$ and $\frac{u(t,x_2)-u(t,x_3)}{u(t,x_3)-u(t,x_1)} = \frac{\tau-t}{t-\theta}$.

[16] See Section A.3 for a brief outline of the proof of Lemma 2.

rect anonymous mechanisms converging to a (not necessarily generic) strategy-proof reduced mechanism, strategy-proofness can be obtained at a small cost.

Recall that $\mathcal{U}$ is the generic subset of $\mathbb{R}^{|T||X_0|}$ such that the conclusion of Lemma 2 holds.

Theorem 2. *Suppose that $|X_0| \geq 3$. Then, for each $u \in \mathcal{U}$, $\varepsilon > 0$, $\gamma \in \mathcal{S}$ and sequence $\langle Y_n, \Phi_n \rangle_{n \in \mathbb{N}}$ of anonymous direct mechanisms converging to γ such that, for each $n \in \mathbb{N}$, there is $\bar{x}_0 \in X_0$ such that*

(a) $(\bar{x}_0, \ldots, \bar{x}_0) \in Y_n$ and

(b) there are at least two elements $x_0 \in X_0 \setminus \{\bar{x}_0\}$ such that, for each $i \in \{1, \ldots, n\}$, $y \in Y_n$ if $y_i = x_0$ and $y_j = \bar{x}_0$ for each $j \neq i$,

there is $N \in \mathbb{N}$ and $\langle \Phi'_n \rangle_{n=N}^{\infty}$ such that $\Phi'_n : T^n \to M(Y_n)$, $||\Phi_n - \Phi'_n|| < \varepsilon$ and Φ'_n is strictly strategy-proof and strictly envy-free for each $n \geq N$.

Conditions (a) and (b) on Y_n are satisfied in several examples. The easiest one is when $Y_n = X_0^n$, which could arise when $X_0 = \{0, 1\} \times F$, where F is a finite subset of $\mathbb{R}_+$, with the interpretation that if $x_0 = (y, z)$, then y denotes access to a public good and z a payment to the provider of such public good. Another similar example with m units of a private good would have $X_0 = \{0, \ldots, m\} \times F$ with $0, 1, 2 \in F$ and Y_n such that $((0, 0), \ldots, (0, 0)) \in Y_n$ and, say, for each $i \in \{1, \ldots, n\}$, $y \in Y_n$ if $y_i \in \{(1, 1), (1, 2)\}$ and $y_j = (0, 0)$ for each $j \neq i$. In this example, it is then feasible that no one gets any unit of the good and no one pays. Furthermore, it is also feasible that only one individuals gets one unit of the good by either paying a price of 1 or a price of 2.

We illustrate Theorem 2 using the continuous auction of Section 3.4.4. In this example, $|X_0| > 3$ and u belongs to $\mathcal{U}$.[17] The sequence $\langle Y_n, \Phi_n \rangle_{n \in \mathbb{N}}$ with

$$Y_n = \{(x_1, \ldots, x_n) \in X_0^n : |\{i \in I : x_i \in \{1\} \times (T \cup \{0\})\}| \leq \lfloor qn \rfloor\}$$

[17] The latter can be seen by letting $x_1 = (0, 0)$, $x_2 = (0, p)$ and $x_3 = (1, 0)$ with $p \in T$ (thus, $p > 0$). Indeed, $u(t, x_1) - u(t, x_2) = p$, $u(t, x_1) - u(t, x_3) = -t$, $u(t, x_2) - u(t, x_3) = t + p$, $\frac{u(t,x_2)-u(t,x_1)}{u(t,x_1)-u(t,x_3)} = \frac{p}{t}$, $\frac{u(t,x_3)-u(t,x_1)}{u(t,x_1)-u(t,x_2)} = \frac{t}{p}$, $\frac{u(t,x_1)-u(t,x_2)}{u(t,x_2)-u(t,x_3)} = -\frac{p}{t+p}$, $\frac{u(t,x_3)-u(t,x_2)}{u(t,x_2)-u(t,x_1)} = -\frac{t+p}{p}$, $\frac{u(t,x_1)-u(t,x_3)}{u(t,x_3)-u(t,x_2)} = -\frac{t}{t+p}$ and $\frac{u(t,x_2)-u(t,x_3)}{u(t,x_3)-u(t,x_1)} = -\frac{t+p}{t}$.

converges to γ and $\gamma \in \mathcal{S}$. The final assumption of Theorem 2 is also satisfied, for instance, by letting n be such that $\lfloor qn \rfloor \geq 1$, $\bar{x}_0 = (0,0)$ and noting that $y \in Y_n$ if $p \in T$, $y_i \in \{(1,0),(1,p)\}$ and $y_j = 0$ for each $i,j \in \{1,\ldots,n\}$ with $i \neq j$. Thus, Theorem 2 applies and, for each $\varepsilon > 0$, yields a sequence $\langle Y_n, \Phi'_n \rangle_{n=N}$ of strictly strategy-proof and strictly envy-free anonymous direct mechanisms such that $||\Phi'_n - \Phi_n|| < \varepsilon$ for each $n \geq N$. Thus, a small modification of Φ_n yields an anonymous direct mechanism in which each player strictly prefers to truthfully reveal his type and strictly prefers his allocation to that of any of the other players.

The conclusion of Theorem 2 holds for a variation of the auction of Section 3.4.3 even though its reduced mechanism is discontinuous; we discuss this in Section 5.3 below. We also illustrate Theorem 2 using the random priority mechanism of Section 3.4.2 in Section 5.2 below, where we discuss Hashimoto's (2018) work.

5 Discussion

In this section we establish formal relationships between our setting and results with those of Azevedo and Budish (2019) and Hashimoto (2018). We also comment on the continuity assumption we make on reduced mechanisms.

5.1 Relationship with Azevedo and Budish (2019)

Our results apply to any sequence of anonymous direct mechanisms whose marginal distributions converge to a strategy-proof reduced mechanism and, as we show, this happens when such sequence satisfies the asymptotic notion of strategy-proofness in Azevedo and Budish (2019).

Such asymptotic notion of strategy-proofness is as follows. Let (Y_n, Φ_n) be an anonymous mechanism and let γ_n correspond to (Y_n, Φ_n) via Lemma 1. Define $e_{n-1} : A^{n-1} \to M_{n-1}(A)$ by setting, for each $a^{n-1} \in A^{n-1}$,

$$e_{n-1}(a^{n-1}) = \sum_{j=1}^{n-1} 1_{a_j}/(n-1);$$

$e_{n-1}(a^{n-1})$ is the empirical distribution of a profile of actions $a^{n-1} = (a_1, \ldots, a_{n-1})$. Let $M^0(A) = \{\pi \in M(A) : \pi(a) > 0 \text{ for all } a \in A\}$ be the set of distributions on A with full support. Define $\phi_n : A \times M(A) \to X$ by setting, for each $a \in A$ and $m \in M(A)$,

$$\phi_n(a, m) = \sum_{a^{n-1} \in A^{n-1}} m^{n-1}(a^{n-1}) \gamma_n(a, e_{n-1}(a^{n-1})),$$

where $m^{n-1}(a^{n-1}) = \prod_{j=1}^{n-1} m(a_j)$ and $a^{n-1} = (a_1, \ldots, a_{n-1})$.

A sequence $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ of anonymous direct mechanisms is *strategy-proof in the large* if, for each $m \in M^0(T)$ and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$u(t, \phi_n(t, m)) \geq u(t, \phi_n(t', m)) - \varepsilon$$

for each $t, t' \in T$ and $n \geq N$.

Theorem 3 shows that whenever a sequence of anonymous direct mechanisms that is strategy-proof in the large converges to a reduced mechanism, then this limit reduced mechanism is strategy-proof.

Theorem 3. *If $\gamma \in \mathcal{L}$ and $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ is a sequence of anonymous direct mechanisms that is strategy-proof in the large and converges to γ, then γ is strategy-proof.*

This result uses some ideas that are familiar to the literature on large games (e.g. Kalai (2004)) and others already used in Azevedo and Budish (2019). Nevertheless, it does not use the notion of a *large-market limit* ϕ^∞ of $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ that the latter consider and which satisfied, by definition, $\phi^\infty(t, m) = \lim_n \phi_n(t, m)$ for each $t \in T$ and $m \in M(T)$. In contrast, we focus on the sequence of marginal distributions $\langle \gamma_n \rangle_{n \in \mathbb{N}}$ and conclude from $\gamma_n \to \gamma$ and $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ being strategy-proof in the large that $u(t, \gamma(t, \pi)) \geq u(t, \gamma(t', \pi))$ for each $t, t' \in T$ and $\pi \in M(T)$. Part of the argument consists in showing that ϕ_n in the definition of strategy-proof in the large can be replaced by γ_n provided that $\gamma_n(t, \pi)$ is close to $\gamma_n(t, \pi')$ whenever n is large and π is close to π'; more precisely, under the latter condition, we show that $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ is envy-free in the large using a result on the concentration of probability measures, which is a standard tool in the analysis of large games since at least Kalai (2004).[18]

[18]See Section A.8 or Azevedo and Budish (2019) for a definition of envy-free in the large.

Since the requirement that $\gamma_n(t, \pi)$ be close to $\gamma_n(t, \pi')$ whenever n is large and π is close to π' is satisfied when $\gamma_n \to \gamma$ and γ is continuous, the remainder of the argument consists in showing that $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ being envy-free in the large implies that γ is strategy-proof.

5.2 Relationship with Hashimoto (2018)

An important difference between our setting and that of Hashimoto (2018) concerns information aggregation, which is absent in ours and is the source of common values in his. In the case where there is no state of nature to learn from individual types or signals, then Hashimoto's (2018) setting can be accommodated in ours (modulo some technical assumptions) and we use his motivating example to illustrate Theorem 2.

The setting in Hashimoto (2018) is described by a set T of types or signals, a set Θ of states of nature, a vector $q \in \mathbb{N}^L$ describing the total supply of goods, with $L \in \mathbb{N}$, a finite consumption set C and a function v such that individuals' payoffs are

$$v(t, y, \theta) - z$$

where $y \in C$ is the consumption bundle, $z \in \mathbb{R}_+$ is the payment, t is the type or signal and θ is the state of nature. There are two additional elements, namely, the density $(t, \theta) \mapsto f(t, \theta)$ of $t \in T$ given θ and the density $\theta \mapsto f(\theta)$ of θ.

The presence of states of nature is, in Hashimoto (2018), the source of common values since individuals use the (truthful) profile of reported types to infer the state of nature, through the probability distribution over states of nature conditional on the type profile. In contrast, our framework does not feature states of nature; once Θ is absent from Hashimoto's (2018) model, one obtains a private values setting which can be accommodated in our framework as follows. Let T be the set of types, $X_0 = C \times F$, where $F \subseteq \mathbb{R}_+$, and

$$u(t, y, z) = v(t, y) - z.$$

Given the absence of Θ, there is only one relevant density, namely $t \mapsto f(t)$ which plays no role in our framework due to the focus on ex-post properties.

We use the example in Section 3.4.2 to illustrate Theorem 2 in a particular instance of Hashimoto's (2018) setting. The assumptions of Theorem 2 hold in this example for the same reasons as in the continuous auction (see Section 4.2). Thus, Theorem 2 applies and, for each $\varepsilon > 0$, yields a sequence $\langle Y_n, \Phi'_n \rangle_{n=N}$ of strictly strategy-proof and strictly envy-free anonymous direct mechanisms such that $||\Phi'_n - \Phi_n|| < \varepsilon$ for each $n \geq N$.

The random priority mechanism (Y_n, Φ_n) in Section 3.4.2 is strategy-proof and envy-free, hence the gain from the above application of Theorem 2 is that both properties hold strictly. In general, Hashimoto's (2018) results yield envy-freeness only asymptotically (i.e. the amount of envy converges to zero) whereas Theorem 2 implies that it eventually holds in a strict way.

5.3 Continuity

We have focused on reduced mechanisms that are continuous and this allowed us to establish our results in a relatively simple and transparent way. Nevertheless, it may be possible to extend our results to a more general class of mechanisms.

Extending Theorem 1 requires finding conditions weaker than continuity that are nevertheless sufficient for its conclusions. The literature on discontinuous games, surveyed in Carmona (2013), has been successful in extending results from continuous to discontinuous payoff functions and the same may happen here. The problem considered in this paper is nevertheless different from the ones considered in the discontinuous games literature and, hence, such an extension is not merely the application of known results from the latter.

It is still possible to extend Theorem 2 without extending Theorem 1 but this requires analysing the case of finitely many participants. This is illustrated by the following variation of the auction of Section 3.4.3.

Let Φ_n be like the auction of Section 3.4.3 except that those in I^+ pay $p_n(\pi) - 1$, where $\pi = \frac{1}{n} \sum_{i=1}^{n} 1_{\hat{t}_i}$. Hence, its marginal distribution γ_n is as follows. For each $t \in T$

and $\pi \in M_{n-1}(T)$, let $\hat{\pi} = \frac{1}{n} 1_t + \left(1 - \frac{1}{n}\right) \pi$ and

$$\gamma_n(t, \pi) = \begin{cases} 1_{(1, p_n(\hat{\pi})-1)} & \text{if } t \geq p_n(\hat{\pi}), \\ (1 - \alpha_n(\hat{\pi})) 1_{(0,0)} + \alpha_n(\hat{\pi}) 1_{(1, p_n(\hat{\pi})-1)} & \text{if } t = p_n(\hat{\pi}) - 1, \\ 1_{(0,0)} & \text{otherwise .} \end{cases}$$

It follows that Φ_n is strategy-proof (see Section A.10). Thus, the argument of the proof of Theorem 2 applies; namely, by combining Φ_n with a strictly strategy-proof mechanism, whose reduced mechanism is provided by Lemma 2, yields the following conclusion: There is $N \in \mathbb{N}$ and $\langle \Phi'_n \rangle_{n=N}^{\infty}$ such that $\Phi'_n : T^n \to M(Y_n)$, $||\Phi_n - \Phi'_n|| < \varepsilon$ and Φ'_n is strictly strategy-proof for each $n \geq N$.[19]

The lack of continuity of γ prevents us from applying Theorem 2 to obtain the above conclusion. While the argument of its proof still holds, the strictly strategy-proofness of Φ'_n is no longer a consequence of the strictly strategy-proofness of its reduced mechanism. Instead, it is now the consequence of the strategy-proofness of Φ_n. Thus, the analysis of discontinuous mechanisms may require the explicit analysis of the case with a large finite number of participants; in contrast, continuous mechanisms require only the analysis of, the often much simpler, limit case. This also accounts for the technical convenience of focusing on continuous mechanisms.

A Appendix

A.1 Proof of Lemma 1

Let $n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$ be given. Let $(a_1, \ldots, a_n), (a'_1, \ldots, a'_n) \in A^n$ be such that $a_i = a'_i$ and $\sum_{j \neq i} 1_{a_j}/(n-1) = \sum_{j \neq i} 1_{a'_j}/(n-1)$. Then there exists a bijection $k : \{1, \ldots, n\} \setminus \{i\} \to \{1, \ldots, n\} \setminus \{i\}$ such that

$$(a'_1, \ldots, a'_{i-1}, a'_i, a'_{i+1}, \ldots, a'_n) = (a_{k(1)}, \ldots, a_{k(i-1)}, a_i, a_{k(i+1)}, \ldots, a_{k(n)}).$$

[19] Recall that $Y_n = \{(x_1, \ldots, x_n) \in X_0^n : |\{i \in I : x_i \in \{1\} \times (T \cup \{0\})\}| \leq \lfloor qn \rfloor\}$ and that $u \in \mathcal{U}$.

Let $k(i) = i$. Thus, for each $(x'_1, \dots, x'_n)$, defining $(x_1, \dots, x_n)$ such that $x_i = x'_i$ and $x_{k(j)} = x'_j$ for each $j \neq i$, it follows that

$$\Phi_n(a'_1, \dots, a'_n)(x'_1, \dots, x'_n) = \Phi_n(a_{k(1)}, \dots, a_{k(n)})(x_{k(1)}, \dots, x_{k(n)})$$
$$= \Phi_n(a_1, \dots, a_n)(x_1, \dots, x_n) = \Phi_n(a_1, \dots, a_n)(x'_{k^{-1}(1)}, \dots, x'_{k^{-1}(n)}).$$

The function $x'_{-i} \mapsto x_{-i}$ mapping X_0^{n-1} into itself and defined (as above) by $x_{k(j)} = x'_j$ for each $j \neq i$ is a bijection. Hence, for each $x'_i \in X_0$,

$$\Phi_i^n(a'_1, \dots, a'_n)(x'_i) = \sum_{x'_{-i} \in X_0^{n-1}} \Phi_n(a'_1, \dots, a'_n)(x'_1, \dots, x'_n)$$
$$= \sum_{x_{-i} \in X_0^{n-1}} \Phi_n(a_1, \dots, a_n)(x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n) = \Phi_i^n(a_1, \dots, a_n)(x'_i).$$

Thus, there exists $\gamma_i^n : A \times M_{n-1}(A) \to X$ such that

$$\Phi_i^n(a_1, \dots, a_n) = \gamma_i^n(a_i, \sum_{j \neq i} 1_{a_j}/(n-1))$$

for each $i \in \{1, \dots, n\}$ and $(a_1, \dots, a_n) \in A^n$.

In addition, by considering k such that $k(i) = 1$, $k(1) = i$ and $k(j) = j$ for each $j \notin \{1, i\}$, it follows that $\gamma_i^n(a, \pi) = \gamma_1^n(a, \pi)$ for each $i \in \{1, \dots, n\}$, $a \in A$ and $\pi \in M_{n-1}(A)$. Indeed, for each $i \neq 1$, $a \in A$ and $\pi \in M_{n-1}(A)$, let $a_i = a$ and $a_{-i} \in X_0^{n-1}$ be such that $\sum_{j \neq i} 1_{a_j}/(n-1) = \pi$. Then, for each $x_i \in X_0$,

$$\gamma_i^n(a, \pi)(x_i) = \sum_{x_{-i} \in X_0^{n-1}} \Phi_n(a_1, \dots, a_n)(x_1, \dots, x_n)$$
$$= \sum_{x_{-i} \in X_0^{n-1}} \Phi_n(a_i, a_2, \dots, a_{i-1}, a_1, a_{i+1}, \dots, a_n)(x_i, x_2, \dots, x_{i-1}, x_1, x_{i+1}, \dots, x_n)$$
$$= \gamma_1^n(a, \pi)(x_i).$$

Thus, let $\gamma_n = \gamma_1^n$.

A.2 Details for the continuous auction example

For each $n \in \mathbb{N}$, let Φ_n, γ_n and γ be as in Section 3.4.4.

Claim 1. γ *is continuous.*

Proof. Let $t \in T$, $\pi \in M(T)$ and π^k be a sequence converging to π. Then

$$\bar{\pi}^k := \frac{1}{1+m\delta}\pi^k + \frac{m\delta}{1+m\delta}\chi \to \bar{\pi}.$$

Case I: $\sum_{t \geq p(\pi)} \bar{\pi}(t) < q < \sum_{t \geq p(\pi)-1} \bar{\pi}(t)$.

As $\bar{\pi}^k \to \bar{\pi}$, there exists $K \in \mathbb{N}$ such that $\sum_{t \geq p(\pi^k)} \bar{\pi}^k(t) < q < \sum_{t \geq p(\pi^k)-1} \bar{\pi}^k(t)$ for all $k \geq K$. Hence, $p(\pi^k) = p(\pi)$ and $\alpha(\pi^k) = \alpha(\pi)$. Therefore, for all $k \geq K$, we have that

$$\gamma(t, \pi^k) = \gamma(t, \pi) = \begin{cases} (1-\alpha(\pi))1_{(1,p(\pi))} + \alpha(\pi)1_{(1,p(\pi)-1)} & \text{if } t \geq p(\pi), \\ (1-\alpha(\pi))1_{(0,0)} + \alpha(\pi)1_{(1,p(\pi)-1)} & \text{if } t = p(\pi)-1, \\ 1_{(0,0)} & \text{otherwise.} \end{cases}$$

Case II: $q = \sum_{t \geq p(\pi)} \bar{\pi}(t) < \sum_{t \geq p(\pi)-1} \bar{\pi}(t)$.

We divide the sequence $\{\pi^k\}_{k=1}^{\infty}$ in three subsequences: (i) where $q = \sum_{t \geq p(\pi)} \bar{\pi}^{k^0}(t)$; (ii) where $q > \sum_{t \geq p(\pi)} \bar{\pi}^{k^+}(t)$; and (iii) where $\sum_{t \geq p(\pi)} \bar{\pi}^{k^-}(t) < q$. We then show that in all such subsequences $\gamma(t, \pi^{k^l}) \to \gamma(t, \pi)$, $l \in \{0, +, -\}$.

Consider first the subsequence k^0 such that $q = \sum_{t \geq p(\pi)} \bar{\pi}^{k^0}(t)$. Then, $p(\pi^{k^0}) = p(\pi)$. Hence,

$$\gamma(t, \pi^{k^0}) = \gamma(t, \pi) = \begin{cases} 1_{(1,p(\pi))} & \text{if } t \geq p(\pi), \\ 1_{(0,0)} & \text{otherwise} \end{cases}$$

for all k^0.

Consider next the subsequence k^+ such that $q > \sum_{t \geq p(\pi)} \bar{\pi}^{k^+}(t)$. There exists $K^+ \in \mathbb{N}$ such that $p(\pi^{k^+}) = p(\pi)$ for all $k^+ \geq K^+$. Therefore, for all $k^+ \geq K^+$,

$$\alpha(\pi^{k^+}) = \frac{q - \sum_{t \geq p(\pi)} \bar{\pi}^{k^+}(t)}{\bar{\pi}^{k^+}(p(\pi)-1)} \text{ and}$$

$$\gamma(t, \pi^{k^+}) = \begin{cases} (1-\alpha(\pi^{k^+}))1_{(1,p(\pi))} + \alpha(\pi^{k^+})1_{(1,p(\pi)-1)} & \text{if } t \geq p(\pi), \\ (1-\alpha(\pi^{k^+}))1_{(0,0)} + \alpha(\pi^{k^+})1_{(1,p(\pi)-1)} & \text{if } t = p(\pi)-1, \\ 1_{(0,0)} & \text{otherwise.} \end{cases}$$

Since $\alpha(\pi^{k^+}) \to 0$ due to $\sum_{t \geq p(\pi)} \bar{\pi}^{k^+}(t) \to \sum_{t \geq p(\pi)} \bar{\pi}(t) = q$ and $\bar{\pi}^{k^+}(p(\pi)-1) \to \bar{\pi}(p(\pi)-1) > 0$, it follows that

$$\gamma(t, \pi^{k^+}) \to \gamma(t, \pi) = \begin{cases} 1_{(1,p(\pi))} & \text{if } t \geq p(\pi), \\ 1_{(0,0)} & \text{otherwise.} \end{cases}$$

Finally, consider the subsequence k^- such that $q < \sum_{t \geq p(\pi)} \bar{\pi}^{k^-}(t)$. There exists $K^- \in \mathbb{N}$ such that $p(\pi^{k^-}) = p(\pi)+1$ for all $k^- \geq K^-$. This follows because $\bar{\pi}(p(\pi)) > 0$, hence

$$\sum_{t \geq p(\pi)+1} \bar{\pi}(t) = \sum_{t \geq p(\pi)} \bar{\pi}(t) - \bar{\pi}(p(\pi)) = q - \bar{\pi}(p(\pi)) < q;$$

thus, $\sum_{t \geq p(\pi)+1} \bar{\pi}^{k^-}(t) < q$ for all k^- sufficiently large. Therefore, for all $k^- \geq K^-$,

$$\alpha(\pi^{k^-}) = \frac{q - \sum_{t \geq p(\pi)+1} \bar{\pi}^{k^-}(t)}{\bar{\pi}^{k^-}(p(\pi))} \text{ and}$$

$$\gamma(t, \pi^{k^-}) = \begin{cases} (1-\alpha(\pi^{k^-}))1_{(1,p(\pi)+1)} + \alpha(\pi^{k^-})1_{(1,p(\pi))} & \text{if } t \geq p(\pi)+1, \\ (1-\alpha(\pi^{k^-}))1_{(0,0)} + \alpha(\pi^{k^-})1_{(1,p(\pi))} & \text{if } t = p(\pi), \\ 1_{(0,0)} & \text{otherwise.} \end{cases}$$

Since $\alpha(t, \pi^{k^-}) \to 1$ as $\sum_{t \geq p(\pi)+1} \bar{\pi}^{k^-}(t) \to \sum_{t \geq p(\pi)+1} \bar{\pi}(t) = q - \bar{\pi}(p(\pi))$ and $\bar{\pi}^{k^-}(p(\pi)) \to \bar{\pi}(p(\pi)) > 0$, it follows that

$$\gamma(t, \pi^{k^-}) \to \gamma(t, \pi) = \begin{cases} 1_{(1,p(\pi))} & \text{if } t \geq p(\pi), \\ 1_{(0,0)} & \text{otherwise.} \end{cases}$$

□

Claim 2. *$\gamma_n \to \gamma$.*

Proof. Note that it suffices to show that $\text{Ls}(\text{graph}(\gamma_n)) \subseteq \text{graph}(\gamma) \subseteq \text{Li}(\text{graph}(\gamma_n))$ by Hildenbrand (1974, Theorem 1, p. 17) or Aliprantis and Border (2006, Theorem 3.93, p. 121). Recall that $(t, \pi, g) \in \text{Li}(\text{graph}(\gamma_n))$ if and only if there is $\bar{n} \in \mathbb{N}$ and $\{\pi_n\}_{n=1}^{\infty}$ such that $\pi_n \in M_{n-1}(T)$ for each $n \geq \bar{n}$ and $\lim_n(\pi_n, \gamma_n(t, \pi_n)) = (\pi, g)$; furthermore, $(t, \pi, g) \in \text{Ls}(\text{graph}(\gamma_n))$ if and only if there is a subsequence

$\{\text{graph}(\gamma_{n_k})\}_{k=1}^{\infty}$ of $\{\text{graph}(\gamma_n)\}_{n=1}^{\infty}$ and $\pi_{n_k} \in M_{n_k-1}(T)$ for each $k \in \mathbb{N}$ such that $\lim_k(\pi_{n_k}, \gamma_{n_k}(t, \pi_{n_k})) = (\pi, g)$ (see Hildenbrand (1974, p. 15)).

We start by showing that $\text{Ls}(\text{graph}(\gamma_n)) \subseteq \text{graph}(\gamma)$. Let $(t, \pi, g) \in \text{Ls}(\text{graph}(\gamma_n))$, $\{\text{graph}(\gamma_{n_k})\}_{k=1}^{\infty}$ be a subsequence of $\{\text{graph}(\gamma_n)\}_{n=1}^{\infty}$ and $\pi_{n_k} \in M_{n_k-1}(T)$ for each $k \in \mathbb{N}$ be such that $\lim_k(\pi_{n_k}, \gamma_{n_k}(t, \pi_{n_k})) = (\pi, g)$. We will show that $\gamma_{n_k}(t, \pi_{n_k}) \to \gamma(t, \pi)$, from which it follows that $g = \gamma(t, \pi)$, i.e. $(t, \pi, g) \in \text{graph}(\gamma)$.

By the definition of $p(\pi)$, we have that $\sum_{t \geq p(\pi)} \bar{\pi}(t) \leq q < \sum_{t \geq (p(\pi)-1)} \bar{\pi}(t)$. We then divide the proof in two cases, analogously to the proof of Claim 1.

Case I: $\sum_{t \geq p(\pi)} \bar{\pi}(t) < q < \sum_{t \geq p(\pi)-1} \bar{\pi}(t)$.

Then there exists $K \in \mathbb{N}$ such that, for each $k \geq K$,

$$\sum_{t \geq p(\pi)} \bar{\pi}_{n_k}(t) < q_{n_k} < \sum_{t \geq p(\pi)-1} \bar{\pi}_{n_k}(t).$$

Hence, for all $k \geq K$, $p_{n_k}(t, \pi_{n_k}) = p(\pi)$ and

$$\alpha_{n_k}(t, \pi_{n_k}) = \frac{q_n - \sum_{\hat{t} \geq p(\pi)} \bar{\pi}_{n_k}(\hat{t})}{\bar{\pi}_{n_k}(p(\pi) - 1)} \to \frac{q - \sum_{\hat{t} \geq p(\pi)} \bar{\pi}(\hat{t})}{\bar{\pi}(p(\pi) - 1)} = \alpha(\pi).$$

Therefore, $\gamma_{n_k}(t, \pi_{n_k}) \to \gamma(t, \pi)$.

Case II: $\sum_{t \geq p(\pi)} \bar{\pi}(t) = q < \sum_{t \geq p(\pi)-1} \bar{\pi}(t)$.

Then there exists $K \in \mathbb{N}$ such that, for each $k \geq K$,

$$q_{n_k} < \sum_{t \geq p(\pi)-1} \bar{\pi}_{n_k}(t).$$

We divide the sequence $\{\pi_{n_k}\}_{k=K}^{\infty}$ in three subsequences: (i) where $q_{n_k^0} = \sum_{t \geq p(\pi)} \bar{\pi}_{n_k^0}(t)$; (ii) where $q_{n_k^+} > \sum_{t \geq p(\pi)} \bar{\pi}_{n_k^+}(t)$; and (iii) where $\sum_{t \geq p(\pi)} \bar{\pi}_{n_k^-}(t) < q_{n_k^-}$. We then show that in all such subsequences $\gamma_{n_k^l}(t, \pi_{n_k^l}) \to \gamma(t, \pi)$, $l \in \{0, +, -\}$.

Consider first the subsequence n_k^0 such that $q_{n_k^0} = \sum_{t \geq p(\pi)} \bar{\pi}_{n_k^0}(t)$. Then $p_{n_k^0}(t, \pi_{n_k^0}) = p(\pi)$ and

$$\gamma_{n_k^0}(t, \pi_{n_k^0}) = \gamma(t, \pi) = \begin{cases} 1_{(1, p(\pi))} & \text{if } t \geq p(\pi), \\ 1_{(0,0)} & \text{otherwise} \end{cases}$$

for each n_k^0. Hence, $\gamma_{n_k^0}(t, \pi_{n_k^0}) \to \gamma(t, \pi)$.

Consider next the subsequence n_k^+ such that $q_{n_k^+} > \sum_{t \geq p(\pi)} \bar{\pi}_{n_{k+}}(t)$. Hence, there exists $N_k^+ \in \mathbb{N}$ such that $p_{n_k^+}(t, \pi) = p(\pi)$ for all $n_k^+ \geq N_k^+$. Therefore, for all $n_k^+ \geq N_k^+$,

$$\alpha_{n_k^+}(t, \pi_{n_k^+}) = \frac{q_{n_k^+} - \sum_{\hat{t} \geq p(\pi)} \bar{\pi}_{n_k^+}(\hat{t})}{\bar{\pi}_{n_k^+}(p(\pi) - 1)} \text{ and}$$

$$\gamma_{n_k^+}(t, \pi_{n_k^+}) = \begin{cases} (1 - \alpha_{n_k^+}(t, \pi_{n_k^+}))1_{(1,p(\pi))} + \alpha_{n_k^+}(t, \pi_{n_k^+})1_{(1,p(\pi)-1)} & \text{if } t \geq p(\pi), \\ (1 - \alpha_{n_k^+}(t, \pi_{n_k^+}))1_{(0,0)} + \alpha_{n_k^+}(t, \pi_{n_k^+})1_{(1,p(\pi)-1)} & \text{if } t = p(\pi) - 1, \\ 1_{(0,0)} & \text{otherwise.} \end{cases}$$

Note that $q_{n_k^+} \to q$ (since $q_n \to q$), $\alpha_{n_k^+}(t, \pi_{n_k^+}) \to 0$ and

$$\gamma_{n_k^+}(t, \pi_{n_k^+}) \to \gamma(t, \pi) = \begin{cases} 1_{(1,p(\pi))} & \text{if } t \geq p(\pi), \\ 1_{(0,0)} & \text{otherwise.} \end{cases}$$

Finally, consider the subsequence n_{k^-} such that $q_{n_k^-} < \sum_{t \geq p(\pi)} \bar{\pi}_{n_k^-}(t)$. Hence, there exists $N_k^- \in \mathbb{N}$ such that $p_{n_k^-}(t, \pi) = p(\pi) + 1$ for all $n_k^- \geq N_k^-$.[20] Therefore, for all $n_k^- \geq N_k^-$,

$$\alpha_{n_k^-}(t, \pi_{n_k^-}) = \frac{q_{n_k^-} - \sum_{\hat{t} \geq p(\pi)+1} \bar{\pi}_{n_k^-}(\hat{t})}{\bar{\pi}_{n_k^-}(p(\pi))} \text{ and}$$

$$\gamma_{n_k^-}(t, \pi_{n_k^-}) = \begin{cases} (1 - \alpha_{n_k^-}(t, \pi_{n_k^-}))1_{(1,p(\pi)+1)} + \alpha_{n_k^-}(t, \pi_{n_k^-})1_{(1,p(\pi))} & \text{if } t \geq p(\pi) + 1, \\ (1 - \alpha_{n_k^-}(t, \pi_{n_k^-}))1_{(0,0)} + \alpha_{n_k^-}(t, \pi_{n_k^-})1_{(1,p(\pi))} & \text{if } t = p(\pi), \\ 1_{(0,0)} & \text{otherwise.} \end{cases}$$

Note that $q_{n_k^-} \to q$ (since $q_n \to q$), $\alpha_{n_k^-}(t, \pi_{n_k^-}) \to 1$ and

$$\gamma_{n_k^-}(t, \pi_{n_k^-}) \to \gamma(t, \pi) = \begin{cases} 1_{(1,p(\pi))} & \text{if } t \geq p(\pi), \\ 1_{(0,0)} & \text{otherwise.} \end{cases}$$

We next show that $\text{graph}(\gamma) \subseteq \text{Li}(\text{graph}(\gamma))$. Let $(t, \pi) \in T \times M(T)$ and let $\{\pi_n\}_{n=1}^{\infty}$ be such that $\pi_n \in M_{n-1}(T)$ for each $n \in \mathbb{N}$ and $\pi_n \to \pi$; simply set, for each

[20] This follows as in the proof of Claim 1. We have that $\sum_{t \geq p(\pi)+1} \bar{\pi}(t) = \sum_{t \geq p(\pi)} \bar{\pi}(t) - \bar{\pi}(p(\pi)) = q - \bar{\pi}(p(\pi)) < q$ since because $\bar{\pi}(p(\pi)) > 0$. Thus, $\sum_{t \geq p(\pi)+1} \bar{\pi}_{n_k}(t) < q$ for all k sufficiently large.

$n \in \mathbb{N}$ and $t \in T \setminus \{m\}$, $\pi_n(t) = \lfloor (n-1)\pi(t) \rfloor / (n-1)$ and $\pi_n(m) = 1 - \sum_{t<m} \pi_n(t)$. The same argument used above shows that $\gamma_n(t, \pi_n) \to \gamma(t, \pi)$, from which it follows that $(t, \pi, \gamma(t, \pi)) \in \mathrm{Li}(\mathrm{graph}(\gamma))$. $\square$

A.3 Outline of the proof of Lemma 2

The proof of Lemma 2 considers the set $\mathcal{U}_{x_1,x_2,x_3}$ as defined in the body of the paper, where x_1, x_2 and x_3 are three distinct elements of X_0. It shows that shows that the complement of $\mathcal{U}_{x_1,x_2,x_3}$ is closed and has a dimension lower than $|T||X_0|$, hence, has Lebesgue measure zero. We then let $\mathcal{U} = \cap_{(x_1,x_2,x_3) \in Z} \mathcal{U}_{x_1,x_2,x_3}$, where Z is the set of distinct elements x_1, x_2, x_3 of X_0.

It then constructs σ as follows. Part (a) of the definition of $\mathcal{U}_{x_1,x_2,x_3}$ implies that $T = \cup_{i,j:i \neq j} T^{ij}$, where, for each $i, j \in \{1, 2, 3\}$ with $i \neq j$, $T^{ij} = \{t \in T : u(t, x_i) > u(t, x_j) > u(t, x_l), l \in \{1, 2, 3\} \setminus \{i, j\}\}$ is the set of types that rank x_i above x_j and x_j above x_l. We start by considering $\sigma' : T \to X$ such that $\sigma'(t) = (1 - \varepsilon)1_{x_i} + \varepsilon 1_{x_j}$ if $t \in T^{ij}$ for some $i, j \in \{1, 2, 3\}$ with $i \neq j$ and where $\varepsilon > 0$. Thus, type $t \in T^{ij}$ strictly prefers $\sigma'(t)$ to $\sigma'(t')$ whenever $t' \notin T^{ij}$. The problem is that $\sigma'(t) = \sigma'(t')$ if $t, t' \in T^{ij}$. To deal with this problem, we define σ by slightly changing σ' as follows.

Let $\eta > 0$ and, for each $i, j \in \{1, 2, 3\}$ with $i \neq j$ and $\alpha, \beta \in [0, \eta]$, define

$$\sigma_{ij}(\alpha, \beta) = (1 - \varepsilon + \alpha)1_{x_i} + (\varepsilon - \alpha - \beta)1_{x_j} + \beta 1_{x_l},$$

where $l \in \{1, 2, 3\} \setminus \{i, j\}$. We then choose $\eta > 0$ and $(\alpha_t, \beta_t) \in [0, \eta]^2$ for each $t \in T$ and set $\sigma(t) = \sigma_{ij}(\alpha_t, \beta_t)$ whenever $t \in T_{ij}$.

We choose $\eta > 0$ sufficiently small such that $u(t, \sigma_{ij}(\alpha, \beta)) > u(t, \sigma_{km}(\alpha', \beta'))$ whenever $t \in T^{ij}$, $k \neq i$, $m \neq k$ and $\alpha, \beta, \alpha', \beta' \in [0, \eta]$; this is possible since $\sigma_{ij}(0, 0) = \sigma'(t)$. Thus, no matter how we choose (α_t, β_t), a type $t \in T^{ij}$ strictly prefers $\sigma(t)$ to $\sigma(t')$ whenever $t' \notin T^{ij}$. The key argument is to show we can choose $\langle(\alpha_t, \beta_t)\rangle_{t \in T}$ in such a way that a type $t \in T^{ij}$ strictly prefers $\sigma(t)$ to $\sigma(t')$ even when $t' \in T^{ij}$. This is where part (b) of the definition of $\mathcal{U}_{x_1,x_2,x_3}$ is used, as we explore the differences across types in T^{ij} in their willingness to substitute between the three outcomes x_1, x_2 and x_3.

A.4 Proof of Lemma 2

For each distinct elements x_1, x_2, x_3 of X_0, let $\mathcal{U}_{x_1,x_2,x_3}$ be the set of $u \in \mathbb{R}^{|T||X_0|}$ such that

(a) $u(t,x) \neq u(t,x')$ for each $t \in T$ and $x, x' \in \{x_1, x_2, x_3\}$ with $x \neq x'$, and

(b) $\frac{u(t,x')-u(t,x)}{u(t,x)-u(t,\hat{x})} \neq \frac{u(t',x')-u(t',x)}{u(t',x)-u(t',\hat{x})}$ for each $t, t' \in T$ with $t \neq t'$ and $x, x', \hat{x} \in \{x_1, x_2, x_3\}$ with $x \neq x'$, $x \neq \hat{x}$ and $x' \neq \hat{x}$.

We first note that $\mathcal{U}_{x_1,x_2,x_3}$ is generic. To see this, let $D = \{(t,t') \in T^2 : t \neq t'\}$, $D_2 = \{(x,x') \in \{x_1,x_2,x_3\}^2 : x \neq x'\}$, $D_3 = \{(x,x',\hat{x}) \in \{x_1,x_2,x_3\}^3 : x \neq x', x \neq \hat{x} \text{ and } x' \neq \hat{x}\}$, $V = \cup_{(t,x,x') \in T \times D_2} \{u \in \mathbb{R}^{|T||X_0|} : u(t,x) = u(t,x')\}$ and $W = \big(\cup_{(t,t',x,x',\hat{x}) \in D \times D_3} W(t,t',x,x',\hat{x})\big) \cap V^c$, where

$$W(t,t',x,x',\hat{x}) = \left\{u \in \mathbb{R}^{|T||X_0|} : u(t,x') = u(t,x) + \frac{u(t,x)-u(t,\hat{x})}{u(t',x)-u(t',\hat{x})}\big(u(t',x')-u(t',x)\big)\right\}.$$

Then $V \cup W$ is the complement of $\mathcal{U}_{x_1,x_2,x_3}$, is closed and has Lebesgue measure zero by Tonelli's Theorem since both V and W have dimension lower than $|T||X_0|$.

Let $Z = \{(x_1,x_2,x_3) \in X_0^3 : x_1 \neq x_2, x_2 \neq x_3 \text{ and } x_1 \neq x_3\}$. Then let $\mathcal{U} = \cap_{(x_1,x_2,x_3) \in Z} \mathcal{U}_{x_1,x_2,x_3}$ and note that $\mathcal{U}$ is generic since Z is finite, $\mathcal{U}$ is open (it is the intersection of finitely many open sets) and its complement has Lebesgue measure zero (it equals $\cup_{(x_1,x_2,x_3) \in Z} \mathcal{U}^c_{x_1,x_2,x_3}$, hence it is the union of null sets).

Let $u \in \mathcal{U}$ and distinct elements x_1, x_2, x_3 of X_0 be given. Then $u \in \mathcal{U}_{x_1,x_2,x_3}$. For each $i, j \in \{1,2,3\}$ with $i \neq j$, let $T^{ij} = \{t \in T : u(t,x_i) > u(t,x_j) > u(t,x_l), l \in \{1,2,3\} \setminus \{i,j\}\}$. Since $u \in \mathcal{U}_{x_1,x_2,x_3}$, $T = \cup_{i,j:i\neq j} T^{ij}$. Define $\sigma' : T \to X$ by $\sigma'(t) = (1-\varepsilon)1_{x_i} + \varepsilon 1_{x_j}$ if $t \in T^{ij}$ for some $i, j \in \{1,2,3\}$ with $i \neq j$ and where $\varepsilon > 0$. Note that if $0 < \varepsilon < 1/3$, then $u(t,\sigma'(t)) > u(t,\sigma'(t'))$ whenever $t \in T^{ij}$ and $t' \notin T^{ij}$. Indeed, if $t' \in T^{il}$, then

$$u(t,\sigma'(t)) - u(t,\sigma'(t')) = \varepsilon(u(t,x_j) - u(t,x_l)) > 0;$$

if $t' \in T^{km}$ with $k \in \{j, l\}$ and $m \neq k$, then

$$\begin{aligned} u(t, \sigma'(t)) - u(t, \sigma'(t')) &= (1-\varepsilon)(u(t, x_i) - u(t, x_k)) + \varepsilon(u(t, x_j) - u(t, x_m)) \\ &\geq (1-\varepsilon)(u(t, x_i) - u(t, x_j)) + \varepsilon(u(t, x_j) - u(t, x_i)) \\ &= (1-2\varepsilon)(u(t, x_i) - u(t, x_j)) > 0. \end{aligned}$$

Of course, $\sigma'(t) = \sigma'(t')$ if $t, t' \in T^{ij}$; we will now define σ by slightly changing σ'. Let $\eta > 0$ and, for each $i, j \in \{1, 2, 3\}$ with $i \neq j$ and $\alpha, \beta \in [0, \eta]$, define

$$\sigma_{ij}(\alpha, \beta) = (1 - \varepsilon + \alpha)1_{x_i} + (\varepsilon - \alpha - \beta)1_{x_j} + \beta 1_{x_l},$$

where $l \in \{1, 2, 3\}\backslash\{i, j\}$. Then there is $\eta > 0$ sufficiently small such that $u(t, \sigma_{ij}(\alpha, \beta)) > u(t, \sigma_{km}(\alpha', \beta'))$ whenever $t \in T^{ij}$, $k \neq i$, $m \neq k$ and $\alpha, \beta, \alpha', \beta' \in [0, \eta]$ since $\sigma_{ij}(0, 0) = \sigma'(t)$.

Fix $i, j \in \{1, 2, 3\}$ with $i \neq j$ and let $t \in T^{ij}$. For each $\alpha, \beta \in [0, \eta]$, let $I_t(\alpha, \beta)$ ($L_t(\alpha, \beta)$ and $U_t(\alpha, \beta)$ respectively) be the set of $(\alpha', \beta') \in (0, \eta)^2$ such that $u(t, \sigma_{ij}(\alpha', \beta')) = u(t, \sigma_{ij}(\alpha, \beta))$ ($u(t, \sigma_{ij}(\alpha', \beta')) < u(t, \sigma_{ij}(\alpha, \beta))$ and $u(t, \sigma_{ij}(\alpha', \beta')) > u(t, \sigma_{ij}(\alpha, \beta))$ respectively). Let also

$$\begin{aligned} s_t &= \frac{u(t, x_j) - u(t, x_l)}{u(t, x_i) - u(t, x_j)} \text{ and} \\ \theta_t(\alpha, \beta) &= \frac{u(t, \sigma_{ij}(\alpha, \beta)) - u(t, \sigma_{ij}(0, 0))}{u(t, x_i) - u(t, x_j)} = \alpha - \beta s_t. \end{aligned}$$

Then

$$\begin{aligned} I_t(\alpha, \beta) &= \left\{(\alpha', \beta') \in (0, \eta)^2 : \alpha' = \beta' s_t + \theta_t(\alpha, \beta)\right\} \\ L_t(\alpha, \beta) &= \left\{(\alpha', \beta') \in (0, \eta)^2 : \alpha' < \beta' s_t + \theta_t(\alpha, \beta)\right\} \\ U_t(\alpha, \beta) &= \left\{(\alpha', \beta') \in (0, \eta)^2 : \alpha' > \beta' s_t + \theta_t(\alpha, \beta)\right\}. \end{aligned}$$

Let $m = |T^{ij}|$ and order the elements of T^{ij} so that $T^{ij} = \{t_1, \ldots, t_m\}$ and

$$\frac{u(t_1, x_j) - u(t_1, x_l)}{u(t_1, x_i) - u(t_1, x_j)} < \cdots < \frac{u(t_m, x_j) - u(t_m, x_l)}{u(t_m, x_i) - u(t_m, x_j)};$$

this is possible because $u \in \mathcal{U}_{x_1, x_2, x_3}$. For convenience, let $s_k = s_{t_k}$ and $\theta_k = \theta_{t_k}$ for each $1 \leq k \leq m$. In addition, let $s_{m+1} > s_m$ and

$$I_{t_{m+1}}(0, 0) = \left\{(\alpha', \beta') \in (0, \eta)^2 : \alpha' = \beta' s_{m+1}\right\}.$$

Let $(\alpha_1, \beta_1) \in I_{t_2}(0,0)$. Assuming that $(\alpha_1, \beta_1), \dots, (\alpha_{k-1}, \beta_{k-1})$ have been defined, let

$$(\alpha_k, \beta_k) \in I_{t_{k+1}}(0,0) \cap L_{t_{k-1}}(\alpha_{k-1}, \beta_{k-1}).$$

That such (α_k, β_k) exists can be seen as follows. First, note that $(\alpha_{k-1}, \beta_{k-1}) \in I_{t_k}(0,0) \subseteq U_{t_{k-1}}(0,0)$ by (5); hence,

$$\begin{aligned}
&u(t_{k-1}, \sigma_{ij}(\alpha_{k-1}, \beta_{k-1})) > u(t_{k-1}, \sigma_{ij}(0,0)) \text{ and} \\
&\theta_{k-1}(\alpha_{k-1}, \beta_{k-1}) = \frac{u(t_{k-1}, \sigma_{ij}(\alpha_{k-1}, \beta_{k-1})) - u(t_{k-1}, \sigma_{ij}(0,0))}{u(t_{k-1}, x_i) - u(t_{k-1}, x_j)} > 0.
\end{aligned}$$

Next, let $(\bar{\alpha}, \bar{\beta})$ be such that $\{(\bar{\alpha}, \bar{\beta})\} = I_{t_{k+1}}(0,0) \cap I_{t_{k-1}}(\alpha_{k-1}, \beta_{k-1})$; thus,

$$\bar{\beta} s_{k+1} = \bar{\beta} s_{k-1} + \theta_{k-1}(\alpha_{k-1}, \beta_{k-1}) \Leftrightarrow \bar{\beta} = \frac{\theta_{k-1}(\alpha_{k-1}, \beta_{k-1})}{s_{k+1} - s_{k-1}} > 0.$$

Hence, each (α, β) such that $\beta \in (0, \bar{\beta})$ and $\alpha = \beta s_{k+1}$ belong to $I_{t_{k+1}}(0,0) \cap L_{t_{k-1}}(\alpha_{k-1}, \beta_{k-1})$. Indeed, it is clear that $(\alpha, \beta) \in I_{t_{k+1}}(0,0)$; in addition,

$$\begin{aligned}
\alpha &= \beta s_{k+1} = \bar{\beta} s_{k+1} - (\bar{\beta} - \beta) s_{k+1} = \bar{\beta} s_{k-1} + \theta_{k-1}(\alpha_{k-1}, \beta_{k-1}) - (\bar{\beta} - \beta) s_{k+1} \\
&< \bar{\beta} s_{k-1} + \theta_{k-1}(\alpha_{k-1}, \beta_{k-1}) - (\bar{\beta} - \beta) s_{k-1} = \beta s_{k-1} + \theta_{k-1}(\alpha_{k-1}, \beta_{k-1}),
\end{aligned}$$

implying that $(\alpha, \beta) \in L_{t_{k-1}}(\alpha_{k-1}, \beta_{k-1})$.

We then obtain that, for each $k, r \in \{1, \dots, m\}$ with $k \neq r$,

$$u(t_k, \sigma_{ij}(\alpha_k, \beta_k)) > u(t_k, \sigma_{ij}(\alpha_r, \beta_r)). \tag{3}$$

To see (3), consider first the case $r < k$. We have that, whenever $r < k$,

$$I_{t_r}(0,0) \subseteq L_{t_k}(0,0) \text{ and} \tag{4}$$

$$I_{t_k}(0,0) \subseteq U_{t_r}(0,0). \tag{5}$$

Indeed, if $(\alpha, \beta) \in I_{t_r}(0,0)$, then $\alpha = \beta s_r < \beta s_k$ and, hence, $(\alpha, \beta) \in L_{t_k}(0,0)$; this shows (4). If $(\alpha, \beta) \in I_{t_k}(0,0)$, then $\alpha = \beta s_k > \beta s_r$ and, hence, $(\alpha, \beta) \in U_{t_r}(0,0)$; this shows (5).

Then $(\alpha_k, \beta_k) \in I_{t_{k+1}}(0,0) \subseteq U_{t_k}(0,0)$, $(\alpha_{k-1}, \beta_{k-1}) \in I_{t_k}(0,0)$ and $(\alpha_h, \beta_h) \in I_{t_{h+1}}(0,0) \subseteq L_{t_k}(0,0)$ for each $h < k-1$ imply that

$$u(t_k, \sigma_{ij}(\alpha_k, \beta_k)) > u(t_k, \sigma_{ij}(0,0)) \geq u(t_k, \sigma_{ij}(\alpha_r, \beta_r)).$$

Next, consider the case $r > k$. To establish (3) in this case, we first show that $\beta_k < \beta_{k-1}$ for each $k = 2, \ldots, m$. Indeed, $\alpha_k = \beta_k s_{k+1}$, $\alpha_{k-1} = \beta_{k-1} s_k$ and, recalling that $\theta_{k-1}(\alpha_{k-1}, \beta_{k-1}) = \alpha_{k-1} - \beta_{k-1} s_{k-1}$, $\alpha_k < \beta_k s_{k-1} + \alpha_{k-1} - \beta_{k-1} s_{k-1}$. From this we obtain that

$$\beta_k s_{k+1} - \beta_k s_{k-1} < \beta_{k-1} s_k - \beta_{k-1} s_{k-1} \Leftrightarrow \beta_k < \frac{s_k - s_{k-1}}{s_{k+1} - s_{k-1}} \beta_{k-1}.$$

Since $\frac{s_k - s_{k-1}}{s_{k+1} - s_{k-1}} < 1$, it follows that $\beta_k < \beta_{k-1}$.

We now claim that $(\alpha_r, \beta_r) \in L_{t_k}(\alpha_k, \beta_k)$ for each $r > k$. This holds for $r = k+1$ since $(\alpha_{k+1}, \beta_{k+1}) \in L_{t_k}(\alpha_k, \beta_k)$ by construction. Suppose next that $(\alpha_r, \beta_r) \in L_{t_k}(\alpha_k, \beta_k)$, i.e. $\alpha_r < \beta_r s_k + \alpha_k - \beta_k s_k$; we will show that $(\alpha_{r+1}, \beta_{r+1}) \in L_{t_k}(\alpha_k, \beta_k)$. To see this, note first that since $\beta_{r+1} < \beta_r$,

$$\beta_{r+1} s_r + \beta_r s_k - \beta_r s_r < \beta_{r+1} s_k.$$

Since $(\alpha_{r+1}, \beta_{r+1}) \in L_{t_r}(\alpha_r, \beta_r)$, it follows that

$$\begin{aligned}
\alpha_{r+1} &< \beta_{r+1} s_r + \alpha_r - \beta_r s_r \\
&< \beta_{r+1} s_r + \beta_r s_k + \alpha_k - \beta_k s_k - \beta_r s_r \\
&< \beta_{r+1} s_k + \alpha_k - \beta_k s_k.
\end{aligned}$$

Hence, $(\alpha_{r+1}, \beta_{r+1}) \in L_{t_k}(\alpha_k, \beta_k)$ as claimed.

To conclude the proof, for each $t \in T^{ij}$, let $(\alpha_t, \beta_t) = (\alpha_k, \beta_k)$ if $t = t_k$. Then define σ by setting $\sigma(t) = \sigma_{ij}(\alpha_t, \beta_t)$ if $t \in T^{ij}$.

A.5 On the assumption $|X_0| \geq 3$

This section illustrates the difficulties with $|X_0| = 2$. Let $X_0 = \{x_0, x_1\}$, $T^0 = \{t \in T : u(t, x_0) > u(t, x_1)\}$, $T^1 = \{t \in T : u(t, x_1) > u(t, x_0)\}$ and assume that $T = T^0 \cup T^1$. If γ is strategy-proof, then $\gamma(t, \pi) = \gamma(t', \pi)$ and $\gamma(t, \pi)(x_i) \geq \gamma(\hat{t}, \pi)(x_j)$ for each $i \in \{0, 1\}$, $t, t' \in T^i$, $\hat{t} \in T^j$, $j \neq i$ and $\pi \in M(T)$. The case of $|X_0| = 2$ adds a strong requirement on strategy-proofness, namely that $\gamma(t, \pi) = \gamma(t', \pi)$ whenever $t, t' \in T^i$. For this reason, there is no strict strategy-proof mechanisms whenever $|T^i| > 1$ for some $i \in \{0, 1\}$.

Likewise, a sequence $\langle(Y_n, \Phi_n)\rangle_n$ of direct mechanisms that are strategy-proof in the large will, in general, fail to be such that (Y_n, Φ_n) is strategy-proof whenever n is sufficiently large. Indeed, this will hold provided that $\gamma_n(t, \pi) \neq \gamma_n(t', \pi)$ for some $\pi \in M_{n-1}(T)$, $t, t' \in T^i$ and $i \in \{0, 1\}$.

A.6 Proof of Theorem 1

(a) Let $\mathcal{S}^*$ be the subset of $\mathcal{S}$ consisting of those γ such that $u(t, \gamma(t, \pi)) > u(t, \gamma(t', \pi))$ for each $t, t' \in T$ and $\pi \in M(T)$.

(b) $\mathcal{S}^*$ is an open subset of $\mathcal{S}$. Let $\gamma \in \mathcal{S}^*$ and

$$\varepsilon(\gamma) = \min_{\pi \in M(T), t, t' \in T, t \neq t'} [u(t, \gamma(t, \pi)) - u(t, \gamma(t', \pi))].$$

Because $M(T) \times \{(t, t') \in T^2 : t \neq t'\}$ is compact and γ is continuous, the mininum is achieved; as $\gamma \in \mathcal{S}^*$, $\varepsilon(\gamma) > 0$.

We have that the mapping $\hat{\gamma} \mapsto \varepsilon(\hat{\gamma})$ is continuous. It then follows that, for some $\delta > 0$, the open ball of radius δ of γ is contained in $\mathcal{S}^*$.

(c) $\mathcal{S}^*$ is dense in $\mathcal{S}$. Let $\gamma \in \mathcal{S}$ and $\varepsilon > 0$. Moreover, by Lemma 2, let $\sigma : T \to M(X)$ be such that $u(t, \sigma(t)) > u(t, \sigma(t'))$ for all $t, t' \in T$. Define ψ by setting, for each $t \in T$ and $\pi \in M(T)$, $\psi(t, \pi) = (1 - \varepsilon)\gamma(t, \pi) + \varepsilon\sigma(t)$. Then ψ is continuous and, for each $t, t' \in T$ and $\pi \in M(T)$,

$$u(t, \psi(t, \pi)) - u(t, \psi(t', \pi)) =$$
$$(1 - \varepsilon)(u(t, \gamma(t, \pi)) - u(t, \gamma(t', \pi))) + \varepsilon(u(t, \sigma(t)) - u(t, \sigma(t'))) > 0.$$

Thus, $\psi \in \mathcal{S}^*$. By making ε small enough, ψ is as close to γ as desired.

(d) Let $\gamma \in \mathcal{S}^*$ and let $\langle(Y_n, \Phi_n)\rangle_{n \in \mathbb{N}}$ be a sequence of anonymous direct mechanisms such that $\gamma_n \to \gamma$. Let $\varepsilon > 0$ be such that $3\varepsilon < \varepsilon(\gamma)$, $\eta > 0$ be such that

$$\sup\{|u(t, x) - u(t, x')| : t \in T, x, x' \in X, ||x - x'|| \leq \eta\} < \varepsilon$$

and $0 < \delta < \eta$ be such that $||\gamma(t, \pi') - \gamma(t, \pi)|| < \eta$ whenever $t \in T$, $\pi, \pi' \in M(T)$ and $||\pi - \pi'|| < 3\delta$. Thus, whenever n is such that $d(\gamma_n, \gamma) < \delta$, we have that, for each

$t, t' \in T$ and $\tilde{\pi} \in M_{n-1}(T)$, there is $\pi, \pi' \in M(T)$ such that $||\gamma_n(t, \tilde{\pi}) - \gamma(t, \pi)|| < \delta$, $||\gamma_n(t', \tilde{\pi}) - \gamma(t', \pi')|| < \delta$, $||\pi - \tilde{\pi}|| < \delta$ and $||\pi' - \tilde{\pi}|| < \delta$. Thus,

$$|u(t, \gamma_n(t, \tilde{\pi})) - u(t, \gamma(t, \pi))| < \varepsilon,$$
$$|u(t, \gamma_n(t', \tilde{\pi})) - u(t, \gamma(t', \pi'))| < \varepsilon \text{ and}$$
$$|u(t, \gamma(t', \pi)) - u(t, \gamma(t', \pi'))| < \varepsilon.$$

Therefore,

$$u(t, \gamma_n(t, \tilde{\pi})) - u(t, \gamma_n(t', \tilde{\pi})) > u(t, \gamma(t, \pi)) - u(t, \gamma(t', \pi')) - 2\varepsilon$$
$$> u(t, \gamma(t, \pi)) - u(t, \gamma(t', \pi)) - 3\varepsilon \geq \varepsilon(\gamma) - 3\varepsilon > 0.$$

since $\gamma \in \mathcal{S}^*$. It follows that (Y_n, Φ_n) is strictly strategy-proof.

We next show that there is $N \in \mathbb{N}$ such that (Y_n, Φ_n) is strictly strategy-proof for each $n \geq N$. Let $N \in \mathbb{N}$ be such that, for each $n \geq N$, $d(\gamma_n, \gamma) < \delta$ and $1/(n-1) < \delta$. Thus, for each $t, t' \in T$ and $\tilde{\pi} \in M_{n-1}(T)$ with $\tilde{\pi}(t') > 0$, there is $\pi, \pi' \in M(T)$ such that $||\gamma_n(t, \tilde{\pi}) - \gamma(t, \pi)|| < \delta$, $||\gamma_n(t', \tilde{\pi} + (1_t - 1_{t'})/(n-1)) - \gamma(t', \pi')|| < \delta$, $||\pi - \tilde{\pi}|| < \delta$ and $||\pi' - (\tilde{\pi} + (1_t - 1_{t'})/(n-1))|| < \delta$. Thus,

$$||\pi - \pi'|| \leq ||\pi - \tilde{\pi}|| + ||\tilde{\pi} - (\tilde{\pi} + (1_t - 1_{t'})/(n-1))|| + ||\tilde{\pi} + (1_t - 1_{t'})/(n-1) - \pi'|| < 3\delta,$$
$$|u(t, \gamma_n(t, \tilde{\pi})) - u(t, \gamma(t, \pi))| < \varepsilon,$$
$$|u(t, \gamma_n(t', \tilde{\pi} + (1_t - 1_{t'})/(n-1))) - u(t, \gamma(t', \pi'))| < \varepsilon \text{ and}$$
$$|u(t, \gamma(t', \pi)) - u(t, \gamma(t', \pi'))| < \varepsilon.$$

Therefore,

$$u(t, \gamma_n(t, \tilde{\pi})) - u(t, \gamma_n(t', \tilde{\pi} + (1_t - 1_{t'})/(n-1))) > u(t, \gamma(t, \pi)) - u(t, \gamma(t', \pi')) - 2\varepsilon$$
$$> u(t, \gamma(t, \pi)) - u(t, \gamma(t', \pi)) - 3\varepsilon \geq \varepsilon(\gamma) - 3\varepsilon > 0.$$

since $\gamma \in \mathcal{S}^*$. It follows that (Y_n, Φ_n) is strictly envy-free.

A.7 Proof of Theorem 2

Let $\tilde{X}_0$ consist of $\bar{x}_0$ and the two elements of $X_0 \setminus \{\bar{x}_0\}$ given by condition (b). Writing $\tilde{X}_0 = \{\bar{x}_0, x_1, x_2\}$, then let σ be as in Lemma 2, i.e. σ is strictly strategy-proof and $\sigma(t)$ is supported on $\tilde{X}_0$ for each $t \in T$.

Claim 3. *There exists* $\sigma^* \in \mathcal{S}^*$ *such that* $\text{supp}(\sigma^*(t)) \subseteq \tilde{X}_0$ *for each* $t \in T$ *and, for each* $n \in \mathbb{N}$, $\Psi_n : T^n \to M(Y_n)$ *such that* Ψ_n *is anonymous and its marginal distributions equal* σ^*.

Proof. Let $M = \max_{\tau \in T^n} \sum_{i=1}^n \sum_{x_0 \neq \bar{x}_0} \sigma(\tau_i)(x_0)$. Then $M > 0$ since otherwise $\sigma(t)(\bar{x}_0) = 1$ for each $t \in T$ and, hence, σ is not strictly strategy-proof. For each $t \in T$, define $\sigma^*(t)$ by setting, for each $x_0 \in X_0$

$$\sigma^*(t)(x_0) = \begin{cases} \frac{\sigma(t)(x_0)}{M} & \text{if } x_0 \neq \bar{x}_0, \\ 1 - \sum_{x_0 \neq \bar{x}_0} \frac{\sigma(t)(x_0)}{M} & \text{if } x_0 = \bar{x}_0. \end{cases}$$

It follows that $\text{supp}(\sigma^*(t)) \subseteq \tilde{X}_0$ for each $t \in T$. Furthermore, $\sigma^* \in \mathcal{S}^*$ since, for each $t, t' \in T$,

$$\begin{aligned} u(t, \sigma^*(t)) - u(t, \sigma^*(t')) &= \frac{1}{M} \sum_{x_0 \neq \bar{x}_0} \big(u(t, x_0) - u(t, \bar{x}_0)\big)\big(\sigma(t)(x_0) - \sigma(t')(x_0)\big) \\ &= \frac{1}{M}\big(u(t, \sigma(t)) - u(t, \sigma(t'))\big) > 0. \end{aligned}$$

For each $n \in \mathbb{N}$, define Ψ_n by setting, for each $(t_1, \ldots, t_n) \in T^n$ and $(x_1, \ldots, x_n) \in X_0^n$,

$$\Psi_n(t_1, \ldots, t_n)(x_1, \ldots, x_n) = \begin{cases} 1 - \sum_{j=1}^n \sum_{x_0 \neq \bar{x}_0} \sigma^*(t_j)(x_0) & \text{if } x_i = \bar{x}_0 \text{ for all } i, \\ \sigma^*(t_i)(x_i) & \text{if } x_i \neq \bar{x}_0 \text{ and } x_j = \bar{x}_0 \text{ for all } j \neq i, \\ 0 & \text{otherwise.} \end{cases}$$

Then each marginal of Ψ_n equals σ^*. Indeed, for each $x_0 \neq \bar{x}_0$,

$$\sum_{y : y_1 = x_0} \Psi_n(t_1, \ldots, t_n)(y) = \Psi_n(t_1, \ldots, t_n)(x_0, \bar{x}_0, \ldots, \bar{x}_0) = \sigma^*(t_1)(x_0).$$

Furthermore, Ψ_n is feasible since, letting $y(i, x_0) \in Y_n$ be such that $y(i, x_0)_i = x_0$ and $y(i, x_0)_j = \bar{x}_0$ for each $j \neq i$,

$$\begin{aligned} &\sum_{y \in Y_n} \Psi_n(t_1, \ldots, t_n)(y) = \Psi(t_1, \ldots, t_n)(\bar{x}_0, \ldots, \bar{x}_0) + \sum_{j=1}^n \sum_{x_0 \neq \bar{x}_0} \Phi(t_1, \ldots, t_n)(y(i, x_0)) \\ &= 1 - \sum_{j=1}^n \sum_{x_0 \neq \bar{x}_0} \sigma^*(t_j)(x_0) + \sum_{j=1}^n \sum_{x_0 \neq \bar{x}_0} \sigma^*(t_j)(x_0) = 1. \end{aligned}$$

Finally, note that Ψ_n is anonymous. Indeed, let $k : \{1, \dots, n\} \to \{1, \dots, n\}$ be a bijection. If $(x_1, \dots, x_n) = (\bar{x}_0, \dots, \bar{x}_0)$, then $\Psi_n(t_{k(1)}, \dots, t_{k(n)})(x_{k(1)}, \dots, x_{k(n)}) = 1 - \sum_{j=1}^{n} \sum_{x_0 \neq \bar{x}_0} \sigma^*(t_j)(x_0) = \Psi_n(t_1, \dots, t_n)(x_1, \dots, x_n)$. If $(x_1, \dots, x_n)$ is such that $x_i \neq \bar{x}_0$ and $x_j = \bar{x}_0$ for all $j \neq i$, then, letting $\tau = (t_{k(1)}, \dots, t_{k(n)})$ and $y = (x_{k(1)}, \dots, x_{k(n)})$, we have that $y_{k^{-1}(i)} = x_i \neq \bar{x}_0$ and $t_{k^{-1}(i)} = t_i$. Thus,

$$\Psi_n(t_{k(1)}, \dots, t_{k(n)})(x_{k(1)}, \dots, x_{k(n)}) = \sigma^*(t_i)(x_i) = \Psi_n(t_1, \dots, t_n)(x_1, \dots, x_n).$$

□

For each $n \in \mathbb{N}$, define $\Phi'_n = (1 - \varepsilon)\Phi_n + \varepsilon\Psi_n$. Let, for each $n \in \mathbb{N}$, γ_n is the marginal of Φ_n described in Lemma 1 and define $\gamma'_n = (1 - \varepsilon)\gamma_n + \varepsilon\sigma$. Then the following claims hold: (a) Φ'_n are anonymous, (b) the marginal of Φ'_n is γ'_n and (c) $\lim_n \gamma'_n = (1 - \varepsilon)\gamma + \varepsilon\sigma$. Since $\gamma \in \mathcal{S}$, it follows $(1 - \varepsilon)\gamma + \varepsilon\sigma \in \mathcal{S}^*$. It then follows by Theorem 1 that Φ'_n is strictly strategy-proof and strictly envy-free for all n sufficiently large. Finally, we clearly have that $||\Phi_n - \Phi'_n|| < \varepsilon$.

A.8 A lemma

In this section we show that, under a continuity condition, strategy-proof in the large implies envy-free in the large. This result, which will be used in the proof of Theorem 3, has some independent interest as it provides a converse to Theorem 1 in Azevedo and Budish (2019).

A sequence $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ of direct mechanism is *envy-free in the large* if, for each $\varepsilon > 0$ and $m \in M^0(T)$, there exists $N \in \mathbb{N}$ and $\delta > 0$ such that

$$u(t, \gamma_n(t, \pi)) \geq u(t, \gamma_n(t', \pi + (1_t - 1_{t'})/(n - 1))) - \varepsilon.$$

for each $n \geq N$, $t, t' \in T$ and $\pi \in M_{n-1}(T)$ with $\pi(t') > 0$ and $||\pi - m|| \leq \delta$. The difference between envy-free and envy-free in the large is that, in the latter, (1) its requirement is only for sufficiently large n, (2) the utility from own allocation must exceeds the utility from the allocation of other agent minus ε and (3) the distribution of reports must be close to a distribution with full support.

For each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $||x|| = \max_{1 \leq i \leq n} |x_i|$. Define, for each $\varepsilon > 0$,

$$\tilde{\omega}_n(\varepsilon) = \sup\{||\gamma_n(t,\pi) - \gamma_n(t,\pi')|| : t \in T, \pi, \pi' \in M_{n-1}(T) \text{ and } ||\pi - \pi'|| \leq \varepsilon\} \text{ and}$$

$$\omega_n(\varepsilon) = \sup\{|u(t,x) - u(t,x')| : t \in T, x, x' \in X \text{ and } ||x - x'|| \leq \tilde{\omega}_n(\varepsilon)\}.$$

We consider the case where $\lim_{\varepsilon \to 0} \sup_n \tilde{\omega}_n(\varepsilon) = 0$; the interpretation of this condition is that the effect of an action of any single agent on the outcome of any agent is negligible. Since u is continuous, it follows that $\lim_{\varepsilon \to 0} \sup_n \tilde{\omega}_n(\varepsilon) = 0$ implies that $\lim_{\varepsilon \to 0} \sup_n \omega_n(\varepsilon) = 0$.

Lemma 3. *If $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ is strategy-proof in the large and $\lim_{\varepsilon \to 0} \sup_n \tilde{\omega}_n(\varepsilon) = 0$, then $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ is envy-free in the large.*

Proof. Let $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ be strategy-proof in the large and consider $\varepsilon > 0$ and $m \in M^0(T)$. Let $\alpha = 1/4$ and $\delta > 0$ be such that $\delta < \min_{t \in T} m(t)$, in which case $||\pi - m|| \leq \delta$ implies that $\pi(t) > 0$ for each $t \in T$, and

$$2 \sup_n \omega_n(2\delta) < \alpha\varepsilon.$$

Fix $t, t' \in T$. We have that

$$\begin{aligned}
&u(t, \phi^n(t', m)) - u(t, \phi^n(t, m)) = \\
&\sum_{\pi \in M_{n-1}(T) \setminus \overline{B}_\delta(m)} m^{n-1} \circ e_{n-1}^{-1}(\pi) \Big(u(t, \gamma_n(t', \pi)) - u(t, \gamma_n(t, \pi)) \Big) + \\
&\sum_{\pi \in M_{n-1}(T) \cap \overline{B}_\delta(m)} m^{n-1} \circ e_{n-1}^{-1}(\pi) \Big(u(t, \gamma_n(t', \pi + (1_t - 1_{t'})/(n-1))) - u(t, \gamma_n(t, \pi)) \Big) + \\
&\sum_{\pi \in M_{n-1}(T) \cap \overline{B}_\delta(m)} m^{n-1} \circ e_{n-1}^{-1}(\pi) \Big(u(t, \gamma_n(t', \pi)) - u(t, \gamma_n(t', \pi + (1_t - 1_{t'})/(n-1))) \Big).
\end{aligned}$$

Lemma 4 in Kalai (2004) implies that

$$m^{n-1} \circ e_{n-1}^{-1}(M_{n-1}(T) \setminus \overline{B}_\delta(m)) \leq 2|T| e^{-2\delta^2 n},$$

where $\overline{B}_\delta(m) = \{\pi \in M_{n-1}(T) : ||\pi - m|| \leq \delta\}$ is the closed ball of radius δ around m. Thus,

$$\begin{aligned}
&\sum_{\pi \in M_{n-1}(T) \cap \overline{B}_\delta(m)} m^{n-1} \circ e_{n-1}^{-1}(\pi) \Big(u(t, \gamma_n(t', \pi + (1_t - 1_{t'})/(n-1))) - u(t, \gamma_n(t, \pi)) \Big) < \\
&u(t, \phi^n(t', m)) - u(t, \phi^n(t, m)) + \omega_n(1/(n-1)) + 4 \max_{\tilde{t} \in T, x_0 \in X_0} |u(\tilde{t}, x_0)| |T| e^{-2\delta^2 n}.
\end{aligned}$$

Since $\langle (Y_n, \Phi_n)\rangle_{n\in\mathbb{N}}$ is strategy-proof in the large, $\omega_n(1/(n-1)) \to 0$ and $e^{-2\delta^2 n} \to 0$, there is $N_1 \in \mathbb{N}$ such that

$$\sum_{\pi \in M_{n-1}(T)\cap \overline{B}_\delta(m)} m^{n-1} \circ e_{n-1}^{-1}(\pi)\Big(u(t, \gamma_n(t', \pi + (1_t - 1_{t'})/(n-1))) - u(t, \gamma_n(t, \pi))\Big) < \alpha\varepsilon.$$

For each $n \geq N_1$, let $\pi_n \in \overline{B}_\delta(m)$ and

$$e_n = u(t, \gamma_n(t', \pi_n + (1_t - 1_{t'})/(n-1))) - u(t, \gamma_n(t, \pi_n)).$$

Then

$$\begin{aligned}
&e_n m^{n-1} \circ e_{n-1}^{-1}(\overline{B}_\delta(m)) = \\
&\sum_{\pi \in M_{n-1}(T)\cap \overline{B}_\delta(m)} m^{n-1} \circ e_{n-1}^{-1}(\pi)\Big(e_n - u(t, \gamma_n(t', \pi + (1_t - 1_{t'})/(n-1))) + u(t, \gamma_n(t, \pi))\Big) + \\
&\sum_{\pi \in M_{n-1}(T)\cap \overline{B}_\delta(m)} m^{n-1} \circ e_{n-1}^{-1}(\pi)\Big(u(t, \gamma_n(t', \pi + (1_t - 1_{t'})/(n-1))) - u(t, \gamma_n(t, \pi))\Big) < \\
&2\omega_n(2\delta) + \alpha\varepsilon < 2\alpha\varepsilon.
\end{aligned}$$

Hence,

$$e_n < \frac{2\alpha\varepsilon}{m^{n-1} \circ e_{n-1}^{-1}(\overline{B}_\varepsilon(m))} \leq \frac{2\alpha\varepsilon}{1 - 2|T|e^{-2\varepsilon^2 n}}.$$

Thus, there is $N_2 \in \mathbb{N}$ such that $N_2 > N_1$ and $e_n < 3\alpha\varepsilon$ for each $n \geq N_2$.

Finally, for each $n \geq N_2$ and $\pi \in M_{n-1}(T) \cap \overline{B}_\delta(m)$,

$$u(t, \gamma_n(t', \pi + (1_t - 1_{t'})/(n-1))) - u(t, \gamma_n(t, \pi)) \leq e_n + 2\omega_n(2\delta) < 4\alpha\varepsilon = \varepsilon.$$

Thus, $\langle (Y_n, \Phi_n)\rangle_{n\in\mathbb{N}}$ is envy-free in the large. □

A.9 Proof of Theorem 3

The proof of Theorem 3 requires the following lemma.

Lemma 4. *If $\gamma_n \to \gamma$ and $\gamma \in \mathcal{L}$ then* $\lim_{\varepsilon\to 0} \sup_n \tilde{\omega}_n(\varepsilon) = 0$.

Proof. Note first that it is enough to show that, for each $\eta > 0$, there exists $\bar{\varepsilon} > 0$ and $N \in \mathbb{N}$ such that $\sup_{n\geq N} \tilde{\omega}_n(\varepsilon) < \eta$ for each $0 < \varepsilon \leq \bar{\varepsilon}$. Indeed, let $\varepsilon^* = \min\{\bar{\varepsilon}, \frac{1}{N}\} > 0$. Thus, for each $\varepsilon \leq \varepsilon^*$ and $n < N$,

$$||\pi - \pi'|| \geq \frac{1}{n-1} > \frac{1}{N-1} > \varepsilon^* \geq \varepsilon$$

for each $\pi, \pi' \in M_{n-1}(T)$ with $\pi \neq \pi'$, implying that $\tilde{\omega}_n(\varepsilon) = 0$. Thus, for each $0 < \varepsilon \leq \varepsilon^*$, $\sup_n \tilde{\omega}_n(\varepsilon) = \sup_{n \geq N} \tilde{\omega}_n(\varepsilon) < \eta$.

Let $\eta > 0$ be given. Since γ is continuous, let $\varepsilon' > 0$ be such that $\varepsilon' < \eta$ and $||\gamma(t,\pi) - \gamma(t,\pi')|| < \eta/2$ whenever $t \in T$ and $\pi, \pi' \in M(T)$ are such that $||\pi - \pi'|| < \varepsilon'$. Let $\bar{\varepsilon} = \varepsilon'/2$, $N \in \mathbb{N}$ be such that $d(\text{graph}(\gamma_n), \text{graph}(\gamma)) < \varepsilon'/4$ for each $n \geq N$, $0 < \varepsilon \leq \bar{\varepsilon}$ and $n \geq N$.

Let $t \in T$ and $\pi, \pi' \in M_{n-1}(T)$ be such that $||\pi - \pi'|| \leq \varepsilon \leq \bar{\varepsilon} = \varepsilon'/2$. Then there are $\tilde{\pi}$ and $\tilde{\pi}'$, both in $M(T)$, such that $||\pi - \tilde{\pi}|| < \varepsilon'/4$, $||\pi' - \tilde{\pi}'|| < \varepsilon'/4$, $||\gamma_n(t,\pi) - \gamma(t,\tilde{\pi})|| < \varepsilon'/4$ and $||\gamma_n(t,\pi') - \gamma(t,\tilde{\pi}')|| < \varepsilon'/4$. Thus, $||\tilde{\pi} - \tilde{\pi}'|| < \varepsilon'/2 + 2\varepsilon'/4 = \varepsilon'$ and, hence, $||\gamma(t,\tilde{\pi}) - \gamma(t,\tilde{\pi}')|| < \eta/2$. This then implies that

$$||\gamma_n(t,\pi) - \gamma_n(t,\pi')|| < \frac{2\varepsilon'}{4} + \frac{\eta}{2}.$$

Thus, $\sup_{n \geq N} \tilde{\omega}_n(\varepsilon) \leq \frac{\varepsilon' + \eta}{2} < \eta$. □

We now turn to the proof of Theorem 3. Let γ, $\langle \gamma_n \rangle_{n \in \mathbb{N}}$ and $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ be as in the statement of the theorem and let $t, t' \in T$ and $\pi \in M^0(T)$ be given. By Lemmas 3 and 4, it follows that $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ is envy-free in the large. Fix $\varepsilon > 0$ and let $N \in \mathbb{N}$ and $\delta > 0$ be given as in the definition of envy-free in the large; we may assume that $\delta < 2\varepsilon$, $||\gamma(t',\tilde{\pi}) - \gamma(t',\pi)|| < \varepsilon$ whenever $||\pi - \tilde{\pi}|| < 3\delta$ and $\delta < \min_{t \in T} \pi(t)$, the latter implying that $\min_{t \in T} \pi'(t) > 0$ for each $\pi' \in M(T)$ with $||\pi' - \pi|| \leq \delta$.

Let $N' \in \mathbb{N}$ be such that $N' \geq N$, $1/(n-1) < \delta$ and $d(\text{graph}(\gamma_n), \text{graph}(\gamma)) < \delta$ for each $n \geq N'$. Thus, for each $n \geq N'$, there exists $\pi' \in M_{n-1}(T)$ and $\tilde{\pi} \in M(T)$ such that $||\pi - \pi'|| < \delta$, $||\pi' + \frac{1_t - 1_{t'}}{n-1} - \tilde{\pi}|| < \delta$, $||\gamma_n(t,\pi') - \gamma(t,\pi)|| < \delta$ and $||\gamma_n(t', \pi' + \frac{1_t - 1_{t'}}{n-1}) - \gamma(t',\tilde{\pi})|| < \delta$. Then $||\pi - \tilde{\pi}|| < 2\delta + \frac{1}{n-1} < 3\delta$ and, therefore, $||\gamma(t',\tilde{\pi}) - \gamma(t',\pi)|| < \varepsilon$. Thus,

$$\left\| \gamma_n\left(t', \pi' + \frac{1_t - 1_{t'}}{n-1}\right) - \gamma(t',\pi) \right\| < \varepsilon + \delta < 2\varepsilon$$

Letting $n \geq N'$ and, for each $\eta > 0$,

$$\omega(\eta) = \sup\{|u(\hat{t},m) - u(\hat{t},m')| : \hat{t} \in T, m, m' \in X \text{ such that } ||m - m'|| \leq \eta\},$$

it follows that,

$$u(t,\gamma(t,\pi)) \geq u(t,\gamma_n(t,\pi')) - \omega(\delta) \geq$$
$$u\left(t,\gamma_n\left(t',\pi' + \frac{1_t - 1_{t'}}{n-1}\right)\right) - \varepsilon - \omega(\delta) \geq u(t,\gamma(t',\pi)) - \varepsilon - 2\omega(2\varepsilon).$$

Letting $\varepsilon \to 0$, it follows that $u\left(t,\gamma\left(t,\pi\right)\right) \geq u\left(t,\gamma\left(t',\pi\right)\right)$.

Finally, for each $t,t' \in T$ and $\pi \in M(T)$, there exists $\pi_k \to \pi$ such that $\pi_k \in M^0(T)$ for each $k \in \mathbb{N}$. The above argument implies that $u\left(t,\gamma\left(t,\pi_k\right)\right) \geq u\left(t,\gamma\left(t',\pi_k\right)\right)$ for each $k \in \mathbb{N}$ and, hence, we obtain from the continuity of γ that $u\left(t,\gamma\left(t,\pi\right)\right) \geq u\left(t,\gamma\left(t',\pi\right)\right)$. Thus, γ is strategy-proof.

A.10 Details for Section 5.3

Let Γ_n be the auction in Section 5.3 and γ_n be its marginal. We show that Φ_n is strategy-proof.

Let $\hat{\pi} = \frac{1}{n}1_t + \left(1 - \frac{1}{n}\right)\pi$, $\tilde{\pi} = \frac{1}{n}1_{t'} + \left(1 - \frac{1}{n}\right)\pi$, $p = p_n(\hat{t})$, $\alpha = \alpha_n(\hat{\pi})$, $p' = p_n(\tilde{t})$ and $\alpha' = \alpha_n(\tilde{\pi})$. Note that $\tilde{\pi} = \hat{\pi} + \frac{1}{n}(1_{t'} - 1_t)$.

Suppose first that $t' > t$. If $t \geq p$, then $\sum_{\hat{t}\geq p}\tilde{\pi}(\hat{t}) = \sum_{\hat{t}\geq p}\hat{\pi}(\hat{t})$ and $\tilde{\pi}(p-1) = \hat{\pi}(p-1)$; hence $p = p'$ and $\alpha = \alpha'$. Thus, $u(t,\gamma_n(t,\pi)) = u(t,\gamma_n(t',\pi))$.

If $t < p$, then $u(t,\gamma_n(t,\pi)) = 0$, $p' \geq p$ and $u(t,\gamma_n(t',\pi)) \leq 0$.

Suppose next that $t' < t$. If $t < p$, then $u(t,\gamma_n(t,\pi)) = 0 = u(t,\gamma_n(t',\pi))$. If $t \geq p$ and $t' \geq p$, then $\sum_{\hat{t}\geq p}\tilde{\pi}(\hat{t}) = \sum_{\hat{t}\geq p}\hat{\pi}(\hat{t})$ and $\tilde{\pi}(p-1) = \hat{\pi}(p-1)$; hence $p = p'$ and $\alpha = \alpha'$. Thus, $u(t,\gamma_n(t,\pi)) = u(t,\gamma_n(t',\pi))$.

Thus, consider $t' < p \leq t$. For each $\theta \in T$ such that $t' < \theta < t$, we have that $q_n < \sum_{\hat{t}\geq\theta}\hat{\pi}(\hat{t})$ by the definition of p and $\sum_{\hat{t}\geq\theta}\tilde{\pi}(\hat{t}) = \sum_{\hat{t}\geq\theta}\hat{\pi}(\hat{t}) - \frac{1}{n}$ since $\tilde{\pi} = \hat{\pi} + \frac{1}{n}(1_{t'} - 1_t)$. Furthermore, $\sum_{\hat{t}\geq t'}\tilde{\pi}(\hat{t}) = \sum_{\hat{t}\geq t'}\hat{\pi}(\hat{t}) > q_n$. Thus, $t' < p' \leq p$.

Thus $u(t,\gamma_n(t',\pi)) = 0 \leq u(t,\gamma_n(t,\pi))$ unless $t' = p' - 1$ and $q_n > \sum_{\hat{t}\geq p'}\tilde{\pi}(\hat{t})$. Thus, assume that $t' = p' - 1$ and $q_n > \sum_{\hat{t}\geq p'}\tilde{\pi}(\hat{t})$.

If $p' < p$, then

$$q_n - \sum_{\hat{t}\geq p'}\tilde{\pi}(\hat{t}) = q_n - \sum_{\hat{t}\geq p'}\hat{\pi}(\hat{t}) + \frac{1}{n} < \frac{1}{n}.$$

This, together with $nq_n \in \mathbb{N}$ and $n \sum_{\hat{t} \geq p'} \tilde{\pi}(\hat{t}) \in \mathbb{N}$, implies that $q_n - \sum_{\hat{t} \geq p'} \tilde{\pi}(\hat{t}) = 0$, contradicting $q_n > \sum_{\hat{t} \geq p'} \tilde{\pi}(\hat{t})$. Thus, it follows that $p = p'$. Hence, $u(t, \gamma_n(t, \pi)) - u(t, \gamma_n(t', \pi)) = (1 - \alpha')(t - t') \geq 0$.

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