Large Strategy-Proof Mechanisms*

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Abstract

We introduce a distributional approach to large anonymous private-value mechanisms, representing finite mechanisms and their large-market limits in a common space of reduced-form mechanisms. This framework allows us to compare large-market approximate strategy-proofness and envy-freeness to exact finite-population properties. For generic preferences and sequences of mechanisms that are approximately strategy-proof and envy-free, we show that an arbitrarily small ex ante perturbation yields mechanisms that remain arbitrarily close to the originals and are strictly strategy-proof and strictly envy-free for all sufficiently large populations.

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1 Introduction

There are many settings in which anonymous mechanisms are used to allocate goods or services among a large number of individuals, or to choose collective outcomes for them. Examples include the assignment of courses to university students, the allocation of school seats to children, and the election of representatives to a parliament. The presence of a large number of individuals may facilitate the analysis for both economic and technical reasons. Economically, several papers have argued that in large markets, it may be justified to replace exact notions of strategy-proofness and envy-freeness with approximate notions that hold as the market grows.[1] Technically, in the presence of a large number of individuals, it is tempting to study mechanisms through their large-market limit, in which the population is (implicitly) modelled as a continuum.[2]

The relaxation of incentive and fairness requirements is motivated in part by impossibility results showing that exact strategy-proofness and exact envy-freeness are often incompatible with other desirable welfare properties.[3] However, only requiring that reporting truthfully is approximately optimal admits the possibility that all agents may optimally misreport.[4] Consequently, there is no guarantee that the resulting outcomes are close to the theoretical predictions. Studying the large-market limit also poses problems since a single individual's report has no effect on the aggregate distribution of reports; thus, each individual would be indifferent between all possible reports under the most naive definition of the limit mechanism.

In this paper, we define the appropriate notion of the limit for anonymous mechanisms and show that incentive properties of the limit carry over to large finite populations. Thus, it becomes possible to derive properties of mechanisms in large finite

[1] For example, Azevedo and Budish (2019) argue that, in large markets, mechanisms that are not strategy-proof can perform well in practice as long as they are strategy-proof in the large, that is, they satisfy a suitable large-market notion of approximate strategy-proofness.

[2] The continuum approach has been fruitful dating back to at least Aumann (1964).

[3] See, for example, Zhou (1990), Bogomolnaia and Moulin (2001), Klaus and Miyagawa (2002), Papai (2001), Ehlers and Klaus (2003), and Hatfield (2009).

[4] This point is made by Nguyen, Teytelboym, and Vardi (2025).

populations by directly studying the limit. In particular, our results imply an alternative way to circumvent the above impossibility results: instead of relaxing incentive and fairness requirements while preserving exact welfare properties, we show that exact (and strict) finite-population incentive and fairness requirements can be restored at small cost to welfare.

We first introduce *reduced-form* mechanisms as a way to represent anonymous mechanisms. A reduced-form mechanism describes the distribution of outcomes faced by an individual as a continuous function of the individual's own action and the distribution of others' actions. We show that each anonymous mechanism induces a unique reduced-form mechanism, which we call its reduced-form, and that this representation is rich enough to capture both finite anonymous mechanisms and their large-market limits.[5] To compare these objects, we identify each reduced-form mechanism with its graph, endow the resulting space with the Hausdorff metric topology, and study sequences of finite anonymous mechanisms whose reduced-forms converge in this topology as the number of players tends to infinity. The continuity requirement in the definition of reduced-form mechanisms is the main substantive restriction of our approach, though it is trivially satisfied by any finite mechanism; our focus is therefore on sequences of finite mechanisms that converge to a continuous limit.

The main results are the following, for which we assume there are at least three outcomes and preferences are generic:

1. There exists an open dense subset of the set of strategy-proof reduced-form mechanisms such that any sequence of finite anonymous mechanisms whose reduced-forms converge to an element of this subset is strictly strategy-proof and strictly envy-free for all sufficiently large populations.

2. Under a richness condition on the sequence of feasible outcomes, any sequence of finite anonymous mechanisms whose reduced-forms converge to a strategy-proof reduced-form mechanism can be perturbed ex ante so that the perturbed

[5] Our notion of a reduced-form mechanism corresponds to the large-market limit mechanism of Azevedo and Budish (2019) and to the payoff-relevant description of a semi-anonymous game in Kalai (2004).

mechanisms are arbitrarily close to the original mechanisms and are strictly strategy-proof and strictly envy-free for all sufficiently large populations.

The results described above impose both a convergence requirement on sequences of anonymous mechanisms and a richness condition on the sequence of feasible outcomes. The convergence requirement is a mild stability condition. It requires that, once the number of participants is sufficiently large, the induced reduced-form mechanisms vary little as the population grows. The richness condition on the sequence of feasible outcomes imposed in the second result is what we call *freeness in the large*. Informally, it requires that, in every sufficiently large economy, it is feasible to vary the individual outcomes of a positive fraction of participants while keeping the outcomes of the remaining participants fixed. The simplest sufficient case is unrestricted feasibility: if the feasible set is the full product of individual outcome sets, then each individual's outcome can be varied independently of the others. Freeness in the large, however, does not require the entire feasible set to have a unrestricted product structure. It only requires enough independent variation along a non-vanishing fraction of individual coordinates. Thus, it is compatible with aggregate feasibility constraints, such as proportional capacity constraints in school allocation or auction environments in which the supply of goods grows linearly with the population, as illustrated by the examples below.

The main substantive restriction of our approach is the continuity of the limiting reduced-form mechanism, which allows us to establish our results in a simple and transparent way. The continuous case already captures the main point of the paper: under the regularity conditions above, approximate large-market versions of strategy-proofness and envy-freeness can be strengthened to exact finite-population properties while remaining arbitrarily close to the original mechanisms. Although we aim to extend the analysis to some discontinuous mechanisms in future work, the examples below illustrate both the role and the limitations of this assumption.

We illustrate the framework and the results through several examples. The single-round Boston mechanism, following Azevedo and Budish (2019), illustrates the reduced-mechanism representation and the genericity condition in the first result. A random-priority mechanism, following Hashimoto (2018), illustrates the perturbation approach in the second result and its connection with large-market envy-freeness. An auction example shows that continuity of the limiting reduced-form mechanism should not be taken for granted. However, a continuous approximation of the same auction then illustrates how our insights may remain useful even in settings where the original limiting mechanism is discontinuous.

In summary, our contribution is to demonstrate that properties of mechanisms in large but finite populations can usefully be derived by directly studying the large-market limit. For continuous limit mechanisms, we obtain the strong conclusion that, in the generic case, any sequence of mechanisms converging to a strategy-proof limit (e.g. a sequence of mechanisms that is strategy-proof in the large) is, in fact, strictly strategy-proof and strictly envy-free for large enough populations.

The remainder of the paper is organized as follows. Section 2 reviews the related literature. Section 3 presents the framework and several examples. Section 4 contains the main results. Section 5 discusses the relationship between our results and the recent work of Hashimoto (2018) and Azevedo and Budish (2019), as well as the role of the continuity assumption. The proofs are collected in Appendix A.

2 Literature review

This paper contributes to the literature on strategy-proofness in large anonymous markets. Strategy-proofness is a central objective in market design because it makes truthful revelation optimal independently of participants' beliefs. This robustness motivation goes back to Wilson (1987), and Bergemann and Morris (2005) formalize the sense in which strategy-proof mechanisms are robust to agents' higher-order beliefs. Strategy-proof mechanisms are also attractive because they are strategically simple, facilitate preference elicitation, and are often connected with fairness considerations; see, among others, Fudenberg and Tirole (1991), Roth (2008), Friedman (1991), Pathak and Sönmez (2008), Abdulkadiroğlu, Pathak, Roth, and Sönmez (2006), and Abdulkadiroğlu, Agarwal, and Pathak (2017). At the same time, exact strategy-proofness is restrictive in finite markets. It can conflict with efficiency or other desirable requirements in exchange economies, school choice, assignment problems, and random allocation problems; see Hurwicz (1972), Abdulkadiroğlu, Pathak, and Roth (2009), Papai (2001), Ehlers and Klaus (2003), Hatfield (2009), Zhou (1990), and Bogomolnaia and Moulin (2001).

One response to these finite-market restrictions is to study how incentive constraints behave as markets become large. In exchange economies, Hammond (1979) shows that the Walrasian mechanism is strategy-proof in a continuum economy, while Roberts and Postlewaite (1976) show that it is approximately strategy-proof in large finite economies. Related work constructs finite-player mechanisms whose outcomes converge to desirable large-market benchmarks. For example, Córdoba and Hammond (1998) and Kovalenkov (2002) construct strategy-proof mechanisms whose outcomes converge to competitive equilibrium outcomes. Hashimoto (2018) provides a strategy-proof mechanism that can approximate many limiting mechanisms of interest. Our paper differs from these contributions in that we do not construct a particular approximating mechanism. Instead, we study arbitrary sequences of anonymous direct mechanisms and ask when properties of their large-market limits imply exact properties for large finite markets.

A complementary approach is to weaken exact strategy-proofness. Most closely related is Azevedo and Budish (2019), who introduce strategy-proofness in the large, an approximate and asymptotic notion of strategy-proofness, and argue that many mechanisms that perform well in practice satisfy it. We connect their notion to our reduced-form approach by showing that, whenever a sequence of anonymous direct mechanisms is strategy-proof in the large and its reduced-forms converge, the limiting reduced-form mechanism is strategy-proof. Thus, strategy-proofness in the large provides one route to the strategy-proof limits studied in this paper.

On a technical level, Azevedo and Budish (2019) establish their large-market results using asymptotic tools such as the Dvoretzky, Kiefer, and Wolfowitz's (1956) inequality. Our approach is instead to embed finite anonymous mechanisms and their limits in a common space of reduced-form mechanisms. A finite anonymous mechanism induces a reduced-form mechanism on the finite grid of empirical distributions of other agents' reports, while a limiting reduced-form mechanism is defined on the whole simplex. We compare these objects through graph convergence in the Hausdorff metric.

Finally, our use of strict strategy-proofness is related to Sinander and Escudé (2020). They emphasize that strictly strategy-proof mechanisms satisfy strong robustness properties, including full implementation, and show that in a canonical auction environment every strategy-proof mechanism can be made strict by an arbitrarily small modification. Our approximation result establishes an analogous conclusion for large anonymous mechanisms: when a sequence converges to a strategy-proof reduced-form mechanism and the feasible sets are sufficiently rich, a small perturbation yields mechanisms that are strictly strategy-proof and strictly envy-free for all sufficiently large populations.

3 Model

3.1 Preferences

Individuals have preferences that depend on their type and on which outcome occurs. Specifically, there is a common finite type space T and a finite set of outcomes X_0. The payoff function $u : T \times X_0 \to \mathbb{R}$ is common to all individuals, with $u(t, x_0)$ being each individual's payoff when she is of type t and the outcome is x_0. Thus, preferences are private values since an individual's payoff depends only on her type and outcome. Let $X = M(X_0)$ be the set of probability distributions over outcomes and we extend the common payoff function from $T \times X_0$ to $T \times X$ by setting $u(t, x) = \sum_{x_0 \in X_0} x(x_0)u(t, x_0)$ for each $t \in T$ and $x \in X$.[6]

The common payoff function $u : T \times X_0 \to \mathbb{R}$ is identified with an element of $\mathbb{R}^{|T||X_0|}$. We say that a subset U of $\mathbb{R}^{|T||X_0|}$ is *generic* if the closure of its complement has Lebesgue measure zero in $\mathbb{R}^{|T||X_0|}$.

[6]More generally, if Z is a finite set, then $M(Z)$ denotes the set of probability distributions over Z. For each $\sigma \in M(Z)$, $\mathrm{supp}(\sigma) = \{z \in Z : \sigma(z) > 0\}$ denotes the support of σ.

3.2 Mechanisms

An (n-player) *mechanism* consists of a set $Y_n \subseteq X_0^n$ of feasible outcome profiles, a finite action set A, and a function $\Phi_n : A^n \to M(Y_n)$. We identify $M(Y_n)$ with the set of probability distributions on X_0^n whose support is contained in Y_n; thus, if $p \in M(Y_n)$, then $p(x) = 0$ for every $x \in X_0^n \setminus Y_n$. For each action profile $a = (a_1, \dots, a_n) \in A^n$, the mechanism selects a probability distribution $\Phi_n(a)$ over feasible outcome profiles. Thus, if $x = (x_1, \dots, x_n) \in Y_n$, then $\Phi_n(a)(x)$ is the probability that the outcome profile x is chosen when the agents choose the action profile a. The coordinate x_i is the individual outcome assigned to agent i.

We impose no product structure on the feasible set Y_n. In particular, we do not assume that every agent can receive any individual outcome independently of the outcomes assigned to the other agents. The set Y_n can therefore capture feasibility constraints that link different agents' outcomes. For example, in a public decision problem, such as an election, all agents face the same implemented alternative. In that case, the feasible outcome profiles are diagonal:

$$Y_n \subseteq \{(s, \dots, s) \in X_0^n : s \in X_0\}.$$

By contrast, if $Y_n = X_0^n$, then there are no feasibility constraints across agents' individual outcomes.

A mechanism is *anonymous* if relabeling the agents has no effect on the induced lottery, provided that actions and outcomes are relabeled in the same way. To state this formally, let K_n be the set of bijections from $\{1, \dots, n\}$ onto itself. For each $k \in K_n$ and a profile $z = (z_1, \dots, z_n) \in Z^n$, where Z is a finite set, write

$$k(z) = (z_{k(1)}, \dots, z_{k(n)}).$$

Thus, the same notation applies to action profiles and to outcome profiles.

A mechanism (Y_n, A, Φ_n) is *anonymous* if for every $k \in K_n$,

$$\Phi_n(k(a))(k(x)) = \Phi_n(a)(x)$$

for every $a \in A^n$ and every $x \in \operatorname{supp}(\Phi_n(a))$.[7]

An anonymous mechanism is *direct* if $A = T$. We denote an anonymous mechanism by (Y_n, A, Φ_n). In the direct case, we suppress the action set and write (Y_n, Φ_n). We often consider sequences of anonymous direct mechanisms $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$.

3.2.1 A Reduced-Form Representation

The following lemma gives the reduced-form representation of anonymous mechanisms. Given a finite set Z and $m \in \mathbb{N}$, define $e_m : Z^m \to M(Z)$ by

$$e_m(z_1, \ldots, z_m) = \sum_{i=1}^{m} \frac{1_{z_i}}{m}.$$

Thus, $e_m(z_1, \ldots, z_m)$ is the empirical distribution of the profile $(z_1, \ldots, z_m)$.[8] Let $M_m(Z) = e_m(Z^m)$ denote the set of empirical distributions over Z that can be induced by m agents. For an anonymous mechanism (Y_n, A, Φ_n), let $\Phi_i^n(a)$ denote the marginal of $\Phi_n(a)$ on the ith coordinate of X_0^n.

Lemma 1. *If (Y_n, A, Φ_n) is anonymous, then there exists a unique map $\gamma_n : A \times M_{n-1}(A) \to X$ such that*

$$\Phi_i^n(a_1, \ldots, a_n) = \gamma_n\big(a_i, e_{n-1}(a_{-i})\big)$$

for every $i \in \{1, \ldots, n\}$ and every $(a_1, \ldots, a_n) \in A^n$.

We call the map γ_n in Lemma 1 the *reduced-form* of the anonymous mechanism. It records the marginal lottery assigned to an agent as a function of the agent's own action and the empirical distribution of the other agents' actions.

3.2.2 Properties

We will be interested in anonymous direct mechanisms satisfying the following properties, defined through the reduced-form.

[7] We do not assume that $k(Y_n) \subseteq Y_n$ for every relabeling $k \in K_n$; anonymity implies instead that $k(Y_n^\Phi) = Y_n^\Phi$ for $Y_n^\Phi := \bigcup_{a \in A^n} \operatorname{supp}(\Phi_n(a))$, so the outcome profiles used by the mechanism are preserved under relabelings.

[8] For each $z \in Z$, $1_z \in M(Z)$ denotes the probability distribution degenerate on z.

A direct mechanism (Y_n, Φ_n) with reduced-form γ_n is *strategy-proof* if, for every $t, t' \in T$ and every $\pi \in M_{n-1}(T)$,

$$u(t, \gamma_n(t, \pi)) \geq u(t, \gamma_n(t', \pi))$$

It is *strictly strategy-proof* if the inequality is strict whenever $t' \neq t$.

A direct mechanism (Y_n, Φ_n) with reduced-form γ_n is *envy-free* if, for every $t, t' \in T$ and every $\pi \in M_{n-1}(T)$ with $\pi(t') > 0$,

$$u(t, \gamma_n(t, \pi)) \geq u\left(t, \gamma_n\left(t', \pi + \frac{1_t - 1_{t'}}{n-1}\right)\right).$$[9]

It is *strictly envy-free* if this inequality is strict whenever $t' \neq t$.

3.3 Space of reduced-form mechanisms

We now view reduced-forms as objects in their own right and introduce the topology used to compare them. For the remainder of the paper, we focus on direct mechanisms.

Let $\mathcal{M}$ denote the collection of all continuous functions $\gamma : T \times C \to X$ for some nonempty closed subset C of $M(T)$. We call an element of $\mathcal{M}$ a *direct reduced mechanism*. When $\gamma : T \times C \to X$, we write $C_\gamma = C$ for the projection of its domain in $M(T)$.

For example, the reduced-form γ_n associated with an anonymous direct n-agent mechanism by Lemma 1 belongs to $\mathcal{M}$ with $C_{\gamma_n} = M_{n-1}(T)$. In the limit as the number of agents grows, the relevant domain is the full simplex $M(T)$. Accordingly, let $\mathcal{L} = \{\gamma \in \mathcal{M} : C_\gamma = M(T)\}$ denote the space of direct reduced-form mechanisms defined on all of $T \times M(T)$.

We define convergence through graph convergence. Each $\gamma \in \mathcal{M}$ is identified with its graph

$$\text{graph}(\gamma) = \{(t, \pi, \gamma(t, \pi)) \in T \times M(T) \times X : t \in T,\ \pi \in C_\gamma\}.$$

Since C_γ is closed and $M(T)$ is compact, C_γ is compact; therefore, by continuity of γ, graph(γ) is a nonempty compact subset of $T \times M(T) \times X$. We endow the space of

[9] See the supplemental material for an equivalent definition of envy-freeness.

nonempty compact subsets of $T \times M(T) \times X$ with the Hausdorff metric. This induces a topology on $\mathcal{M}$: a sequence $\langle \gamma_n \rangle_n$ in $\mathcal{M}$ converges to $\gamma \in \mathcal{M}$, written $\gamma_n \to \gamma$, if $\text{graph}(\gamma_n) \to \text{graph}(\gamma)$ in the Hausdorff metric. The space $\mathcal{L}$ is endowed with the relative topology inherited from $\mathcal{M}$.[10]

A sequence $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ of anonymous direct mechanisms converges to $\gamma \in \mathcal{M}$ if the associated reduced-forms γ_n, obtained from Lemma 1, satisfy $\gamma_n \to \gamma$.

We use the same reduced-form notion of strategy-proofness for elements of $\mathcal{L}$. A direct reduced-form mechanism $\gamma \in \mathcal{L}$ is *strategy-proof* if $u(t, \gamma(t, \pi)) \geq u(t, \gamma(t', \pi))$ for every $t, t' \in T$ and every $\pi \in M(T)$. Let $\mathcal{S}$ denote the set of strategy-proof mechanisms in $\mathcal{L}$. The set $\mathcal{S}$ is nonempty because it contains all constant reduced-form mechanisms.

One of our main results concerns sequences of anonymous direct mechanisms that converge to a generic strategy-proof direct reduced-form mechanism. A subset $G \subseteq \mathcal{S}$ is *generic* if it is open and dense in $\mathcal{S}$, where $\mathcal{S}$ is endowed with the relative topology inherited from $\mathcal{L}$.

3.4 Examples

This subsection illustrates both the reduced-form representation of anonymous mechanisms and the convergence notion introduced above. We consider two examples: the one-round Boston mechanism, following Azevedo and Budish (2019); and the random priority mechanism, following Hashimoto (2018).[11]

3.4.1 Boston mechanism with a single round

This example is based on Azevedo and Budish (2019, Section E.1). It serves three purposes. First, it shows how an anonymous mechanism induces a reduced-form γ_n. Second, it identifies the limiting reduced-form mechanism γ and verifies convergence

[10]We write $d(\gamma, \gamma')$ for the Hausdorff distance between $\text{graph}(\gamma)$ and $\text{graph}(\gamma')$ whenever $\gamma, \gamma' \in \mathcal{M}$.

[11]In this section, we directly describe each mechanism reduced-form. In Section 2 of the supplemental material, we provide the complete description of each mechanism in its original form.

in the graph topology. Third, it illustrates how strategy-proofness of the limit can be checked directly from the reduced-form.

Consider a finite set S of schools. Each school $s \in S$ can accommodate a proportion $q_s \in (0,1)$ of the market. Thus, when there are n students, school s has capacity $n_s = \lfloor q_s n \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Students may also remain unmatched, denoted by $\emptyset$, so $X_0 = S \cup \{\emptyset\}$. Each student applies to one school, so $A = S$. We also let $T = S$, with the interpretation that a student's type is her favorite school. Preferences satisfy $u(s,s) > u(s,s') > 0 = u(s,\emptyset)$ for all distinct $s, s' \in S$.

Given a report profile $a = (a_1, \ldots, a_n) \in S^n$, an order of the students is drawn uniformly at random. For each school s, the first n_s students in this order among those who report s are assigned to s, while the remaining students who report s are assigned to $\emptyset$. This defines an anonymous direct mechanism.

We now compute its reduced-form. Fix a student who reports school $s \in S$, and let $\pi \in M_{n-1}(S)$ be the empirical distribution of the reports of the other students. Then $1 + (n-1)\pi(s)$ students in total report school s. If this number is at most n_s, the student is assigned to s with probability one. Otherwise, she is assigned to s if and only if she is among the first n_s students, in the random order, among those who report s. Hence the probability of assignment to s is

$$\alpha_n(s,\pi) = \min\left\{\frac{n_s}{1 + (n-1)\pi(s)}, 1\right\}.$$

Therefore, for every $s \in S$ and $\pi \in M_{n-1}(S)$,

$$\gamma_n(s,\pi) = \alpha_n(s,\pi)1_s + \big(1 - \alpha_n(s,\pi)\big)1_\emptyset.$$

The limiting reduced-form is given by

$$\gamma(s,\pi) = \alpha(s,\pi)1_s + \big(1 - \alpha(s,\pi)\big)1_\emptyset,$$

where

$$\alpha(s,\pi) = \min\left\{\frac{q_s}{\pi(s)}, 1\right\},$$

with the convention that $q_s/\pi(s) = +\infty$ when $\pi(s) = 0$. Thus γ is continuous. Moreover, $\gamma_n \to \gamma$. To see this, write

$$\alpha_n(s,\pi) = \min\left\{\frac{n_s/n}{1/n + (1-1/n)\pi(s)}, 1\right\}.$$

Since $n_s/n \to q_s$, these assignment probabilities converge uniformly to $\alpha(s,\pi)$ on $M(S)$, and the grids $M_{n-1}(S)$ become dense in $M(S)$. Hence the corresponding graphs converge in the Hausdorff metric.

Finally, we characterize when the limiting reduced-form mechanism is strategy-proof. The mechanism γ is strategy-proof if and only if

$$q_s u(s,s) \geq u(s,s')$$

for every $s, s' \in S$ with $s' \neq s$, and it is strictly strategy-proof if and only if all these inequalities are strict. Indeed, truthful reporting gives a type-s student payoff $u(s,s)\alpha(s,\pi)$, while reporting $s' \neq s$ gives payoff $u(s,s')\alpha(s',\pi)$. Since $\alpha(s,\pi) \geq q_s$ and $\alpha(s',\pi) \leq 1$ for every π, the displayed condition is sufficient. It is necessary by taking $\pi(s) = 1$, in which case $\alpha(s,\pi) = q_s$ and $\alpha(s',\pi) = 1$.

3.4.2 A random priority mechanism

This example is a private-values version of the motivating example in Hashimoto (2018). It illustrates the same reduced-form and convergence ideas as the Boston example, but in a setting where agents decide whether to seek an object at a fixed price.

There are $\lfloor qn \rfloor$ identical objects to be allocated to n individuals, where $q \in (0,1)$. Each individual receives at most one object and pays a fixed price $p \in (0,1)$ if she receives one. Let $X_0 = \{0,1\} \times \{0,p\}$, where the first coordinate indicates whether the individual receives an object and the second coordinate is the payment. The finite type set T is contained in $(0,1)$, and $t \in T$ represents the individual's valuation for the object. Preferences are given by

$$u(t,y,z) = ty - z$$

for each $t \in T$ and $(y, z) \in X_0$.

The direct mechanism works as follows. Given a report profile $(t_1, \ldots, t_n) \in T^n$, an order of the individuals is drawn uniformly at random. Among those who report a type strictly above p, the first $\lfloor qn \rfloor$ individuals in the order receive one object and pay p; all remaining individuals receive no object and pay nothing. This defines an anonymous direct mechanism.

We now compute the reduced-form. Fix an individual with report $t \in T$, and let $\pi \in M_{n-1}(T)$ be the empirical distribution of the other individuals' reports. Let

$$H(\pi) = \sum_{s \in T : s > p} \pi(s)$$

be the fraction of other individuals who report a type above p. If $t \leq p$, the individual is not eligible and receives $(0, 0)$ with probability one. If $t > p$, then $[1 + (n-1)H(\pi)]$ individuals in total report a type above p. Hence the probability that the individual receives an object is

$$\alpha_n(\pi) = \min\left\{ \frac{\lfloor qn \rfloor}{1 + (n-1)H(\pi)}, 1 \right\}.$$

Therefore,

$$\gamma_n(t, \pi) = \begin{cases} 1_{(0,0)}, & \text{if } t \leq p, \\ \alpha_n(\pi) 1_{(1,p)} + \big(1 - \alpha_n(\pi)\big) 1_{(0,0)}, & \text{if } t > p. \end{cases}$$

The limiting reduced-form is given by

$$\gamma(t, \pi) = \begin{cases} 1_{(0,0)}, & \text{if } t \leq p, \\ \alpha(\pi) 1_{(1,p)} + \big(1 - \alpha(\pi)\big) 1_{(0,0)}, & \text{if } t > p, \end{cases}$$

where

$$\alpha(\pi) = \min\left\{ \frac{q}{H(\pi)}, 1 \right\},$$

with the convention that $q/H(\pi) = +\infty$ when $H(\pi) = 0$. The map γ is continuous. Moreover, $\gamma_n \to \gamma$. Indeed,

$$\alpha_n(\pi) = \min\left\{ \frac{\lfloor qn \rfloor / n}{1/n + (1 - 1/n)H(\pi)}, 1 \right\},$$

and these assignment probabilities converge uniformly to $\alpha(\pi)$ as $n \to \infty$.

Finally, γ is strategy-proof. If $t \leq p$, truthful reporting gives payoff zero, while any report above p gives payoff $\alpha(\pi)(t-p) \leq 0$. If $t > p$, any report above p gives the same payoff $\alpha(\pi)(t-p) \geq 0$, while any report at or below p gives payoff zero. Thus truthful reporting is optimal for every type. Note, however, that the limiting mechanism is not strictly strategy-proof, as an agent with true type $t < p$ would be indifferent between reporting any below p and any agent with true type $t > p$ would be indifferent between reporting any type above p.

4 Results

4.1 Generic Asymptotic Strategy-Proofness

The convergence notion introduced above allows us to ask when incentive properties of a limiting reduced-form mechanism are inherited by large finite anonymous mechanisms. Strategy-proofness of the limit alone is not enough to guarantee strategy-proofness of nearby finite mechanisms: because strategy-proofness is defined by weak inequalities, some incentive constraints may bind at the limit, and small finite-population perturbations may create profitable deviations.

This difficulty disappears for strictly strategy-proof limits. If all incentive constraints of the limiting reduced-form mechanism hold with a strict margin, then sufficiently close finite mechanisms inherit these incentive inequalities. The main question is therefore whether restricting attention to strictly strategy-proof reduced-form mechanisms reduces the set of outcomes that can be generated compared to what could be achieved with weakly strategy-proof mechanisms. The answer is generically no: under a genericity condition on preferences, strictly strategy-proof reduced-form mechanisms are dense in the set of strategy-proof reduced-form mechanisms.

The key step is the following lemma. It says that, for generic preferences and at least three outcomes, one can assign to each type a lottery in such a way that every type strictly prefers its own assigned lottery to the lotteries assigned to all other types. Importantly, this separating assignment is independent of the distribution of types.

Lemma 2. *Suppose that $|X_0| \geq 3$. There is a generic subset $\mathcal{U}$ of $\mathbb{R}^{|T||X_0|}$ such that, for each $u \in \mathcal{U}$, there exists a strictly strategy proof $\sigma : T \to X$.*

The lemma constructs a type-separating menu of lotteries using only three outcomes: each type strictly prefers the lottery assigned to itself to the lotteries assigned to all other types. To see the idea, fix three distinct outcomes $x_1, x_2, x_3 \in X_0$. The genericity condition used in the proof requires, first, that no type is indifferent between any two of these three outcomes and, second, that different types have different relative tradeoffs among them. Formally, for each distinct $x, x', \hat{x} \in \{x_1, x_2, x_3\}$ and each pair of distinct types $t, t' \in T$, the proof rules out equalities of the form

$$\frac{u(t, x') - u(t, x)}{u(t, x) - u(t, \hat{x})} = \frac{u(t', x') - u(t', x)}{u(t', x) - u(t', \hat{x})}.$$

These restrictions exclude only a finite union of lower-dimensional sets, and are therefore generic.

The construction of σ then proceeds in two steps. First, types are grouped according to their ranking of x_1, x_2, x_3. If a type ranks x_i first and x_j second, assign it a lottery close to

$$(1 - \varepsilon)1_{x_i} + \varepsilon 1_{x_j}.$$

For $\varepsilon > 0$ small, this separates types with different rankings of the three outcomes. The remaining problem is to separate types with the same ranking. This is where the distinct-tradeoff condition is used. Within each ranking class, the proof perturbs the assigned lotteries in two independent directions inside the simplex spanned by x_1, x_2, x_3. For example, if x_i is ranked first, x_j second, and x_ℓ third, the perturbed lotteries have the form

$$\sigma_{ij}(\alpha, \beta) = (1 - \varepsilon + \alpha)1_{x_i} + (\varepsilon - \alpha - \beta)1_{x_j} + \beta 1_{x_\ell},$$

for small $\alpha, \beta \geq 0$. The generic tradeoff condition ensures that these perturbations can be chosen so that each type strictly prefers its own lottery to the lotteries assigned to all other types in the same ranking class. Taking the perturbations sufficiently small preserves the strict comparisons across different ranking classes. This yields the function σ in Lemma 2.

Let $\mathcal{U}$ be a generic subset of $\mathbb{R}^{|T||X_0|}$ satisfying the conclusion of Lemma 2. For a fixed $u \in \mathcal{U}$, let $\mathcal{S}^*$ denote the set of strictly strategy-proof reduced-form mechanisms in $\mathcal{S}$.

Theorem 1. *Suppose that $|X_0| \geq 3$. Then, for each $u \in \mathcal{U}$, $\mathcal{S}^*$ is generic in $\mathcal{S}$. Moreover, if $\gamma \in \mathcal{S}^*$ and $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ is a sequence of anonymous direct mechanisms converging to γ, then there exists $N \in \mathbb{N}$ such that (Y_n, Φ_n) is strictly strategy-proof and strictly envy-free for every $n \geq N$.*

Theorem 1 has two components. The first is a genericity statement: strictly strategy-proof reduced-form mechanisms are open and dense in the set of strategy-proof reduced-form mechanisms. Openness follows from strict inequalities and compactness. Density uses Lemma 2. If $\gamma \in \mathcal{S}$ and σ is as in the lemma, then, for $\varepsilon \in (0,1)$, define

$$\gamma^\varepsilon(t, \pi) = (1 - \varepsilon)\gamma(t, \pi) + \varepsilon\sigma(t).$$

Since γ is strategy-proof, its incentive inequalities hold weakly. Since σ strictly separates types, the perturbation makes all incentive inequalities strict. Hence $\gamma^\varepsilon \in \mathcal{S}^*$, and $\gamma^\varepsilon \to \gamma$ as $\varepsilon \to 0$.

The second component is a finite-population implication. If $\gamma \in \mathcal{S}^*$, then its strict incentive constraints hold with a uniform margin. Any sequence of anonymous direct mechanisms converging to γ has reduced-forms that are eventually close enough to preserve this margin, yielding strict strategy-proofness. The same argument, together with the fact that one agent's report affects another agent's empirical distribution by only $1/(n-1)$, yields strict envy-freeness for all sufficiently large n.

One illustration of Theorem 1 is to apply it to the Boston mechanism example in Section 3.4. Recall that if we impose $q_s u(s, s) > u(s, s')$ for every pair of distinct schools $s, s' \in T$, then the limiting reduced-form mechanism is continuous and strictly strategy-proof. Hence, the limiting reduced-form mechanism belongs to $\mathcal{S}^*$, and the theorem implies that the corresponding finite mechanisms are eventually strictly strategy-proof and strictly envy-free.

Remark 1. The set $\mathcal{S}^*$ used in Theorem 1 is not meant to be maximal. Let $\mathcal{S}^{**}$ be the set of $\gamma \in \mathcal{S}$ such that, for every sequence $\langle (Y_n, \Phi_n) \rangle_{n\in\mathbb{N}}$ of anonymous direct mechanisms converging to γ, there exists $N \in \mathbb{N}$ such that (Y_n, Φ_n) is strictly strategy-proof and strictly envy-free for every $n \geq N$. By Theorem 1, $\mathcal{S}^* \subseteq \mathcal{S}^{**}$. Since $\mathcal{S}^*$ is open and dense in $\mathcal{S}$, it follows that $\mathcal{S}^* = \mathrm{int}_{\mathcal{S}}(\mathcal{S}^*) \subseteq \mathrm{int}_{\mathcal{S}}(\mathcal{S}^{**})$, and therefore $\mathrm{int}_{\mathcal{S}}(\mathcal{S}^{**})$ is also open and dense in $\mathcal{S}$. Thus $\mathcal{S}^*$ is only a convenient generic subset on which the conclusion of Theorem 1 holds; the conclusion may hold on a larger subset of $\mathcal{S}$.

4.2 Approximation by Strategy-Proof Mechanisms

Theorem 1 applies when the limit of a sequence of anonymous direct mechanisms belongs to a generic subset of the strategy-proof reduced-form mechanisms. In applications, however, one may be interested in a particular sequence of anonymous direct mechanisms whose limit is strategy-proof but is not generic. The purpose of this subsection is to show that, under a mild feasibility condition, such a sequence can be modified by an arbitrarily small amount so that the modified mechanisms are eventually strictly strategy-proof and strictly envy-free.

Thus, unlike Theorem 1, the result below does not require the original limit to be generic. Instead, it uses the density of strictly strategy-proof reduced-form mechanisms to perturb the original sequence. The challenge is that the perturbation must be implemented using outcome profiles that remain feasible. This motivates the following condition on the feasible sets.

We say that a sequence $\langle Y_n \rangle_{n\in\mathbb{N}}$ is *free in the large* if, for each $n \in \mathbb{N}$, there exist $m_n \in \{1, \ldots, n\}$ and $z_n \in X_0^{n-m_n}$ such that $\langle m_n/n \rangle_{n\in\mathbb{N}}$ converges, $\lim_{n\to\infty} \frac{m_n}{n} > 0$, and, for every bijection $k \in K_n$,

$$k\big(X_0^{m_n} \times \{z_n\}\big) \subseteq Y_n.$$

Here k acts on outcome profiles by $k(x_1, \ldots, x_n) = (x_{k(1)}, \ldots, x_{k(n)})$.

In words, freeness in the large requires that a non-vanishing fraction of agents can be assigned arbitrary individual outcomes, while the outcomes of the remaining agents are fixed at a background profile. Moreover, the free group of agents can be any group of that size, because the feasibility condition is required to hold after any relabeling of agents.

The condition is trivially satisfied in the least restrictive case $Y_n = X_0^n$, where every outcome profile is feasible. It is also satisfied in standard capacity-constrained environments whenever one can fix the outcomes of all but m_n agents in a way that leaves enough slack for the remaining m_n agents to receive arbitrary individual outcomes. For example, in a school-assignment problem with capacities proportional to the number of students, as in the Boston mechanism example, one can take m_n no larger than the smallest capacity and let the background profile assign the empty school for the remaining agents, thus leaving m_n available seats at every school. Freeness in the large is also satisfied in the setting of the random priority mechanism, which we return to after the statement of the theorem.

Recall that $\mathcal{U}$ is the generic subset of $\mathbb{R}^{|T||X_0|}$ satisfying the conclusion of Lemma 2, and let $\|\cdot\|$ denote the sup norm.

Theorem 2. *Suppose that $|X_0| \geq 3$ and let $u \in \mathcal{U}$, $\varepsilon > 0$, $\gamma \in \mathcal{S}$, and $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ be a sequence of anonymous direct mechanisms converging to γ. If $\langle Y_n \rangle_{n \in \mathbb{N}}$ is free-in-the-large, then there exist $N \in \mathbb{N}$ and a sequence $\langle \Phi'_n \rangle_{n \geq N}$ such that, for every $n \geq N$, (Y_n, Φ'_n) is an anonymous direct mechanism, strictly strategy-proof, strictly envy-free, and satisfies $\|\Phi_n - \Phi'_n\| < \varepsilon$.*

The proof clarifies the role of the freeness assumption. Let $\sigma : T \to X$ be the type-separating map from Lemma 2. Freeness in the large allows us to construct, for large n, an anonymous auxiliary mechanism $\Psi_n : T^n \to M(Y_n)$ that assigns outcomes according to σ to each agent in a randomly chosen block of m_n agents and assigns the fixed background profile z_n to the remaining agents. Since the free block is chosen symmetrically, the mechanism Ψ_n is anonymous; since every relabeling of $X_0^{m_n} \times \{z_n\}$ is contained in Y_n, it is feasible. The reduced-form of Ψ_n puts weight m_n/n on $\sigma(t)$

and the remaining weight on a lottery independent of the agent's report. Mixing a small amount of Ψ_n into the original mechanism therefore creates strict strategy-proof inequalities, while changing the original mechanism by an arbitrarily small amount.

This theorem applies to the random priority mechanism in Section 3.4. Recall that the limiting reduced-form mechanism is strategy-proof, but it is not strictly strategy-proof. Thus the finite-population implication in Theorem 1 cannot be invoked for the original sequence. Theorem 2 instead uses the slack in the feasible sets to construct nearby mechanisms that are eventually strictly strategy-proof and strictly envy-free. Note that the feasible sets are free in the large: take $m_n \leq \lfloor qn \rfloor$ and fix the remaining agents at a background profile in which no scarce objects are used. Then the m_n free agents can be assigned arbitrary individual outcomes without exceeding the supply of objects, and feasibility is preserved under relabeling. Theorem 2 then gives mechanisms arbitrarily close to the original ones that are eventually strictly strategy-proof and strictly envy-free.[12]

Variants of freeness in the large. Theorem 2 is stated using freedom over the whole set of individual outcomes X_0. However, this freedom is only needed for the outcomes of σ to be feasible and the proof of Lemma 2 constructs σ using only three outcomes. Thus the same conclusion holds if the feasible sets are free in the large relative to some subset $C \subseteq X_0$ with $|C| \geq 3$.

Formally, for $C \subseteq X_0$, we say that a sequence $\langle Y_n \rangle_{n\in\mathbb{N}}$ is *free in the large relative to C* if, for each $n \in \mathbb{N}$, there exist $m_n \in \{1, \ldots, n\}$ and $z_n \in X_0^{n-m_n}$ such that $\langle m_n/n \rangle_{n\in\mathbb{N}}$ converges, $\lim_{n\to\infty} m_n/n > 0$, and, for every bijection $k \in K_n$,

$$k\big(C^{m_n} \times \{z_n\}\big) \subseteq Y_n.$$

Let $\mathcal{U}_C$ denote the set of payoff functions on $T \times X_0$ whose restriction to $T \times C$

[12]In the setting of the random priority mechanism, we have $u \in \mathcal{U}$ as long as $p \notin T$ since all types have different relative tradeoffs among, e.g. $x_1 = (0,0)$, $x_2 = (0,p)$ and $x_3 = (1,0)$. Indeed, $u(t,x_1) - u(t,x_2) = p$, $u(t,x_1) - u(t,x_3) = -t$, $u(t,x_2) - u(t,x_3) = -(t+p)$, $\frac{u(t,x_2)-u(t,x_1)}{u(t,x_1)-u(t,x_3)} = \frac{p}{t}$, $\frac{u(t,x_3)-u(t,x_1)}{u(t,x_1)-u(t,x_2)} = \frac{t}{p}$, $\frac{u(t,x_1)-u(t,x_2)}{u(t,x_2)-u(t,x_3)} = -\frac{p}{t+p}$, $\frac{u(t,x_3)-u(t,x_2)}{u(t,x_2)-u(t,x_1)} = -\frac{t+p}{p}$, $\frac{u(t,x_1)-u(t,x_3)}{u(t,x_3)-u(t,x_2)} = -\frac{t}{t+p}$ and $\frac{u(t,x_2)-u(t,x_3)}{u(t,x_3)-u(t,x_1)} = -\frac{t+p}{t}$.

belongs to the generic set obtained from Lemma 2 with X_0 replaced by C. Then Theorem 2 remains valid with $\mathcal{U}$ replaced by $\mathcal{U}_C$ and with "free in the large" replaced by "free in the large relative to C."

If we only require strict strategy-proofness but not envy-freeness, the feasibility condition can be weakened further. The reason is that as long as the auxiliary perturbation used in the proof generates the separating lottery for a positive fraction of agents, it will be strictly strategy proof. Thus, the outcomes assigned to the remaining agents may depend on the realized outcomes of this free group, weakening the feasibility requirement. However, this dependence may destroy envy-freeness of the auxiliary mechanism.

We say that a sequence $\langle Y_n \rangle_{n\in\mathbb{N}}$ is *projection-free in the large relative to* $C \subseteq X_0$ if there exists a sequence $\langle m_n \rangle_{n\in\mathbb{N}}$ with $m_n \in \{1, \ldots, n\}$ and $\lim_{n\to\infty} \frac{m_n}{n} > 0$, such that, for every $n \in \mathbb{N}$ and every $c \in C^{m_n}$, there exists $w \in X_0^{n-m_n}$ satisfying $k(c, w) \in Y_n$ for every bijection $k \in K_n$.

Equivalently, after any relabeling of agents, the projection of Y_n on the first m_n coordinates contains C^{m_n}.

Theorem 3. *Let $C \subseteq X_0$ be such that $|C| \geq 3$. Then, for each $u \in \mathcal{U}_C$, $\varepsilon > 0$, $\gamma \in \mathcal{S}$, and sequence $\langle (Y_n, \Phi_n) \rangle_{n\in\mathbb{N}}$ of anonymous direct mechanisms converging to γ, if $\langle Y_n \rangle_{n\in\mathbb{N}}$ is projection-free in the large relative to C, then there exist $N \in \mathbb{N}$ and a sequence $\langle \Phi'_n \rangle_{n\geq N}$ such that, for every $n \geq N$, (Y_n, Φ'_n) is an anonymous direct mechanism, strictly strategy-proof and satisfies $\|\Phi_n - \Phi'_n\| < \varepsilon$.*

The difference between Theorem 2 and Theorem 3 is that projection-freeness allows the background outcomes to depend on the realized outcomes assigned to the free agents. This is enough for strict strategy-proofness, because the background part of the reduced-form does not depend on the agent's own report. It is not enough, in general, for strict envy-freeness, since the background part may depend on the empirical distribution of the other agents' reports.

5 Discussion

This section relates our framework and results to two strands of the literature. First, we compare our reduced-form convergence approach with the notion of strategy-proofness in the large introduced by Azevedo and Budish (2019). Second, we discuss the connection with Hashimoto (2018). We then return to the continuity assumption on reduced-form mechanisms and suggest directions for future work.

5.1 Relationship with Azevedo and Budish (2019)

Azevedo and Budish (2019) study an asymptotic notion of strategy-proofness in which an agent compares reports after averaging over the reports of the other agents, drawn independently from a fixed population distribution. We now relate their notion to our reduced-form convergence approach. The main point is that, if a sequence of anonymous direct mechanisms is strategy-proof in the large and its reduced-forms converge to a limit γ, then γ is strategy-proof.

Let (Y_n, Φ_n) be an anonymous direct mechanism, and let γ_n be its reduced-form, as given by Lemma 1. Let $M^0(T) = \{\pi \in M(T) : \pi(t) > 0 \text{ for all } t \in T\}$ denote the set of full-support distributions on T. For each $m \in M(T)$, let m^{n-1} denote the product distribution of m on T^{n-1}. Define $\phi_n : T \times M(T) \to X$ by

$$\phi_n(t, m) = \sum_{\tau \in T^{n-1}} m^{n-1}(\tau)\, \gamma_n\big(t, e_{n-1}(\tau)\big),$$

where $\tau = (\tau_1, \ldots, \tau_{n-1})$. Thus, $\phi_n(t, m)$ is the interim marginal lottery assigned to an agent who reports t when the other agents' reports are drawn independently according to m.

A sequence $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ of anonymous direct mechanisms is *strategy-proof in the large* if, for every $m \in M^0(T)$ and every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$u(t, \phi_n(t, m)) \geq u(t, \phi_n(t', m)) - \varepsilon$$

for every $t, t' \in T$ and every $n \geq N$.

The following result shows that strategy-proofness in the large is sufficient for strategy-proofness of any reduced-form limit.

Theorem 4. *If $\gamma \in \mathcal{L}$ and $\langle (Y_n, \Phi_n) \rangle_{n\in\mathbb{N}}$ is a sequence of anonymous direct mechanisms that is strategy-proof in the large and converges to γ, then γ is strategy-proof.*

Theorem 4 uses concentration ideas familiar from the literature on large games, going back at least to Kalai (2004), as well as arguments used by Azevedo and Budish (2019). However, the object of convergence is different. Azevedo and Budish (2019) work with the large-market limit of the interim functions ϕ_n, namely a function ϕ^∞ satisfying $\phi^\infty(t,m) = \lim_{n\to\infty} \phi_n(t,m)$ for each $t \in T$ and $m \in M(T)$, whenever this limit exists. In contrast, our primitive convergence notion concerns the reduced-forms γ_n.

The main step is to connect the interim object ϕ_n in the definition of strategy-proofness in the large with the reduced-form γ_n. The value $\phi_n(t,m)$ averages $\gamma_n\big(t, e_{n-1}(\tau)\big)$ over profiles $\tau \in T^{n-1}$ drawn independently according to m. When n is large, the empirical distribution $e_{n-1}(\tau)$ is close to m with high probability. Hence, if $\gamma_n(t,\pi)$ is close to $\gamma_n(t,\pi')$ whenever π and π' are close, then $\phi_n(t,m)$ is close to the value of γ_n at empirical distributions near m. This continuity property is implied by $\gamma_n \to \gamma$ together with the continuity of γ.

Hence, strategy-proofness in the large allows one to replace the averaged comparison involving ϕ_n by a comparison involving the marginal reduced-form γ_n. More precisely, the proof first shows that the sequence is envy-free in the large.[13] Intuitively, in a large anonymous mechanism, one agent's report changes the empirical distribution faced by another agent by only $1/(n-1)$, so the payoff comparison in the definition of strategy-proofness in the large can be translated into a comparison between the marginal lotteries assigned to different agents. The final step shows that envy-freeness in the large passes to the reduced-form limit and yields the strategy-proof inequalities for γ.

Thus, strategy-proofness in the large provides one route to a strategy-proof reduced-form limit. Once such a limit is obtained, the results of the previous sections can be applied. In particular, if the limit belongs to the generic set identified in Theorem 1, then every sufficiently large mechanism in the sequence is strictly strategy-proof

[13]See Section A.8 or Azevedo and Budish (2019) for the definition of envy-freeness in the large.

and strictly envy-free. If the limit is strategy-proof but not generic, Theorem 2 gives an arbitrarily close sequence that is eventually strictly strategy-proof and strictly envy-free, provided the feasible sets are free in the large.

5.2 Relationship with Hashimoto (2018)

We relate our framework to Hashimoto (2018) through the private-values case of his model. In the general formulation of Hashimoto (2018), agents' reports may also convey information about a payoff-relevant state. Since our analysis focuses on ex-post strategy-proofness and envy-freeness of anonymous mechanisms, we abstract from this informational role of reports. The common part of the two settings is obtained by taking the state to be absent, or fixed, so that payoffs depend only on an agent's type and individual outcome.

In the general model of Hashimoto (2018), there is a set T of types or signals, a set Θ of states of nature, a vector $q \in \mathbb{N}^L$ describing the aggregate supply of L goods, a finite consumption set C, and payoff functions of the form

$$v(t, y, \theta) - z,$$

where $y \in C$ is the consumption bundle, $z \in \mathbb{R}_+$ is the payment, $t \in T$ is the type or signal, and $\theta \in \Theta$ is the state of nature. The model also specifies the distribution of signals conditional on the state and the prior distribution over states. When the state is absent, or fixed, the inference problem disappears and the environment becomes a private-values environment. In that case, the primitives can be written in our notation by taking $X_0 = C \times F$, where $F \subseteq \mathbb{R}_+$ is a finite set of feasible payments, and by defining

$$u(t, y, z) = v(t, y) - z.$$

The distribution of types plays no role in our definitions because strategy-proofness and envy-freeness are imposed ex post.

The random priority mechanism in Section 3.4.2 is a simple instance of this private-values case. There are $\lfloor qn \rfloor$ identical objects, each agent has unit demand, and an agent who receives an object pays a fixed price p. Agents with reported values above

p are ordered uniformly at random, and the first $\lfloor qn \rfloor$ of them receive the object. As shown in Section 3.4.2, the sequence of reduced-forms converges to a strategy-proof reduced-form mechanism.

The original random priority mechanism is already strategy-proof and envy-free, but these properties need not hold strictly. Theorem 2 implies that, for every $\varepsilon > 0$, there exist $N \in \mathbb{N}$ and a sequence $\langle (Y_n, \Phi'_n) \rangle_{n \geq N}$ of anonymous direct mechanisms such that, for every $n \geq N$, $\|\Phi'_n - \Phi_n\| < \varepsilon$ and (Y_n, Φ'_n) is strictly strategy-proof and strictly envy-free.

5.3 Relaxing Continuity

The main results above are stated for sequences whose reduced-forms converge to a continuous limiting reduced-form mechanism. Continuity is useful because it allows us to study large finite mechanisms through their limit, but it is also a substantive restriction. The auction example below illustrates how continuity can fail and suggests two ways to proceed in such cases.

Consider a multi-unit auction with n single-unit-demand bidders and $\lfloor qn \rfloor$ units of an object, where $q \in (0, 1)$. The type set is finite and ordered, $T = \{1, \ldots, m\}$. If a bidder of type t receives $y \in \{0, 1\}$ units and pays $z \in T \cup \{0\}$, her utility is

$$u(t, y, z) = ty - z.$$

Thus the individual outcome space is $X_0 = \{0, 1\} \times (T \cup \{0\})$.

The auction is a version of the gth-price auction with g units. The main distinction from the usual gth-price auction is that, instead of setting the price deterministically equal to the gth highest bid and rationing stochastically among bidders at that bid, the mechanism randomizes the price paid by potential winners between the cutoff bid and the bid one increment higher. The probability of paying the cutoff bid goes to one as excess demand at the cutoff bid goes to zero.

To describe the reduced-form, for $\rho \in M(T)$ and $r \in \{1, \ldots, m + 1\}$, write the mass of reports above a certain type

$$F_r(\rho) = \sum_{s \in T: s \geq r} \rho(s),$$

with the convention that $F_{m+1}(\rho) = 0$. For $g < n$, define

$$p_{n,g}(\rho) = \min \left\{ r \in \{2, \dots, m+1\} : \frac{g}{n} \geq F_r(\rho) \right\}$$

and

$$\alpha_{n,g}(\rho) = \frac{\frac{g}{n} - F_{p_{n,g}(\rho)}(\rho)}{\rho(p_{n,g}(\rho) - 1)}.$$

The number $p_{n,g}(\rho) - 1$ is the cutoff bid, and $\alpha_{n,g}(\rho)$ is the fraction of cutoff bidders who can be served after all bidders above the cutoff have been served.

Given a report profile, all bidders with reports at least $p_{n,g}(\rho)$ receive the object. Each such bidder pays $p_{n,g}(\rho) - 1$ with probability $\alpha_{n,g}(\rho)$ and pays $p_{n,g}(\rho)$ with probability $1 - \alpha_{n,g}(\rho)$. Among bidders who report exactly $p_{n,g}(\rho) - 1$, the remaining units are assigned uniformly at random, and winners pay $p_{n,g}(\rho) - 1$. All other bidders receive no object and pay zero. This defines an anonymous direct mechanism.[14]

Let $\gamma_{n,g}$ be its reduced-form. If a bidder reports t and the empirical distribution of the other bidders' reports is $\pi \in M_{n-1}(T)$, set

$$\hat{\pi} = \frac{1}{n} 1_t + \left(1 - \frac{1}{n}\right) \pi.$$

Then

$$\gamma_{n,g}(t, \pi) = \begin{cases} (1 - \alpha_{n,g}(\hat{\pi})) 1_{(1, p_{n,g}(\hat{\pi}))} + \alpha_{n,g}(\hat{\pi}) 1_{(1, p_{n,g}(\hat{\pi}) - 1)} & \text{if } t \geq p_{n,g}(\hat{\pi}), \\ (1 - \alpha_{n,g}(\hat{\pi})) 1_{(0,0)} + \alpha_{n,g}(\hat{\pi}) 1_{(1, p_{n,g}(\hat{\pi}) - 1)} & \text{if } t = p_{n,g}(\hat{\pi}) - 1, \\ 1_{(0,0)} & \text{otherwise.} \end{cases}$$

Now take $g = \lfloor qn \rfloor$. The natural limiting reduced-form map is given by

$$\gamma(t, \pi) = \begin{cases} (1 - \alpha(\pi)) 1_{(1, p(\pi))} + \alpha(\pi) 1_{(1, p(\pi) - 1)} & \text{if } t \geq p(\pi), \\ (1 - \alpha(\pi)) 1_{(0,0)} + \alpha(\pi) 1_{(1, p(\pi) - 1)} & \text{if } t = p(\pi) - 1, \\ 1_{(0,0)} & \text{otherwise,} \end{cases}$$

[14]Section 2 of the supplemental material provides the complete formal description of the mechanism.

where

$$p(\pi) = \min\{r \in \{2, \ldots, m+1\} : q \geq F_r(\pi)\}$$

and

$$\alpha(\pi) = \frac{q - F_{p(\pi)}(\pi)}{\pi(p(\pi) - 1)}.$$

This limiting reduced-form map is strategy-proof by the standard cutoff argument. However, it need not be continuous.

For example, let $q = 1/2$, $T = \{1, 2, 3, 4\}$,

$$\pi = \left(\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4}\right),$$

and, for k large,

$$\pi_k = \left(\frac{1}{2} - \frac{2}{k}, \frac{1}{k}, \frac{1}{4} + \frac{1}{k}, \frac{1}{4}\right).$$

Then $\pi_k \to \pi$, but $p(\pi_k) = 4$ for every k, whereas $p(\pi) = 2$. Moreover,

$$\alpha(\pi_k) = \frac{1}{1 + 4/k} \quad \text{and} \quad \alpha(\pi) = 0.$$

Hence

$$\gamma(4, \pi_k) = (1 - \alpha(\pi_k))1_{(1,4)} + \alpha(\pi_k)1_{(1,3)} \to 1_{(1,3)},$$

while

$$\gamma(4, \pi) = 1_{(1,2)}.$$

Thus γ is discontinuous and therefore does not belong to $\mathcal{L}$.

There are two ways to deal with this kind of discontinuity. The first is to replace the auction by a nearby continuous perturbation. In Appendix A.10, we formally describe such a perturbation. The discontinuity in the limiting reduced-form above arises when the relevant empirical distribution is on the boundary of $M(T)$, that is, when $\pi(t) = 0$ for some $t \in T$. The perturbation keeps the distribution used to determine the cutoff away from this boundary by adding, for each type $t \in T$, a small number of fake bidders who always report t. Units allocated to fake bidders are not allocated to real bidders.

At the limit, this amounts to replacing π by

$$\bar{\pi} = \frac{1}{1 + m\delta}\pi + \frac{m\delta}{1 + m\delta}\chi,$$

where χ is the uniform distribution on T and $\delta > 0$ controls the size of the perturbation. Since every coordinate of $\bar{\pi}$ is strictly positive, the cutoff cannot jump by more than one increment; together with the price randomization described above, this yields a continuous limiting reduced-form mechanism. Hence the perturbed auction falls within the scope of Theorem 2.

The second approach is to work directly with the finite mechanisms. This does not extend Theorem 2 as stated, because that theorem uses continuity of the limiting reduced-form mechanism. But the finite-population argument behind the theorem can still apply in specific cases. To illustrate, consider a variation of the auction above in which all bidders above the cutoff pay the cutoff bid $p_{n,g}(\rho) - 1$. Its reduced-form is

$$\gamma_n(t, \pi) = \begin{cases} 1_{(1, p_n(\hat{\pi}) - 1)} & \text{if } t \geq p_n(\hat{\pi}), \\ (1 - \alpha_n(\hat{\pi})) 1_{(0,0)} + \alpha_n(\hat{\pi}) 1_{(1, p_n(\hat{\pi}) - 1)} & \text{if } t = p_n(\hat{\pi}) - 1, \\ 1_{(0,0)} & \text{otherwise,} \end{cases}$$

where

$$\hat{\pi} = \frac{1}{n} 1_t + \left(1 - \frac{1}{n}\right) \pi.$$

As shown in Appendix A.11, each finite mechanism in this sequence is strategy-proof. Therefore, one can combine it directly with the strictly strategy-proof perturbation provided by Lemma 2. Under the same feasibility condition used in Theorem 2, this gives, for every $\varepsilon > 0$, a sequence of strictly strategy-proof anonymous direct mechanisms for all sufficiently large n.

This second route shows why continuity is technically convenient but not conceptually indispensable in every application. When the limiting reduced-form mechanism is continuous, the analysis can be carried out at the limit and then transferred to all large finite mechanisms. When the limit is discontinuous, one may instead need to verify the relevant finite-population inequalities directly.

6 Conclusion

This paper develops a reduced-form approach to large anonymous private-value mechanisms. By representing finite mechanisms and their limits in a common space, we show how properties of limiting reduced-form mechanisms can deliver exact finite-population conclusions. For generic preferences, strictly strategy-proof reduced-form mechanisms are dense in the set of strategy-proof reduced-form mechanisms, and convergence to such limits implies that all sufficiently large finite mechanisms are strictly strategy-proof and strictly envy-free. More generally, under freeness in the large, any sequence converging to a strategy-proof continuous reduced-form mechanism can be approximated by mechanisms that are eventually strictly strategy-proof and strictly envy-free.

The analysis highlights both the usefulness and the limitations of reduced-form convergence. Continuity of the limiting reduced-form mechanism allows large finite mechanisms to be studied through their limits, but the assumption may be violated in some settings. The discussion of continuous perturbations and direct finite-population arguments suggests that the approach can be extended beyond the continuous case. A natural direction for future work is to identify broader classes of discontinuous reduced-form mechanisms for which the same exact finite-population conclusions continue to hold.

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A Appendix

A.1 Proof of Lemma 1

Let $n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$ be given. Let $(a_1, \ldots, a_n), (a'_1, \ldots, a'_n) \in A^n$ be such that $a_i = a'_i$ and $\sum_{j \neq i} 1_{a_j}/(n-1) = \sum_{j \neq i} 1_{a'_j}/(n-1)$. Then there exists a bijection $k : \{1, \ldots, n\} \setminus \{i\} \to \{1, \ldots, n\} \setminus \{i\}$ such that

$$(a'_1, \ldots, a'_{i-1}, a'_i, a'_{i+1}, \ldots, a'_n) = (a_{k(1)}, \ldots, a_{k(i-1)}, a_i, a_{k(i+1)}, \ldots, a_{k(n)}).$$

Let $k(i) = i$. Thus, for each $(x'_1, \ldots, x'_n)$, defining $(x_1, \ldots, x_n)$ such that $x_i = x'_i$ and $x_{k(j)} = x'_j$ for each $j \neq i$, it follows that

$$\begin{aligned}
&\Phi_n(a'_1, \ldots, a'_n)(x'_1, \ldots, x'_n) = \Phi_n(a_{k(1)}, \ldots, a_{k(n)})(x_{k(1)}, \ldots, x_{k(n)}) \\
&= \Phi_n(a_1, \ldots, a_n)(x_1, \ldots, x_n) = \Phi_n(a_1, \ldots, a_n)(x'_{k^{-1}(1)}, \ldots, x'_{k^{-1}(n)}).
\end{aligned}$$

The function $x'_{-i} \mapsto x_{-i}$ mapping X_0^{n-1} into itself and defined (as above) by $x_{k(j)} = x'_j$ for each $j \neq i$ is a bijection. Hence, for each $x'_i \in X_0$,

$$\begin{aligned}
&\Phi_i^n(a'_1, \ldots, a'_n)(x'_i) = \sum_{x'_{-i} \in X_0^{n-1}} \Phi_n(a'_1, \ldots, a'_n)(x'_1, \ldots, x'_n) \\
&= \sum_{x_{-i} \in X_0^{n-1}} \Phi_n(a_1, \ldots, a_n)(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n) = \Phi_i^n(a_1, \ldots, a_n)(x'_i).
\end{aligned}$$

Thus, there exists $\gamma_i^n : A \times M_{n-1}(A) \to X$ such that

$$\Phi_i^n(a_1, \ldots, a_n) = \gamma_i^n(a_i, \sum_{j \neq i} 1_{a_j}/(n-1))$$

for each $i \in \{1, \ldots, n\}$ and $(a_1, \ldots, a_n) \in A^n$.

In addition, by considering k such that $k(i) = 1$, $k(1) = i$ and $k(j) = j$ for each $j \notin \{1, i\}$, it follows that $\gamma_i^n(a, \pi) = \gamma_1^n(a, \pi)$ for each $i \in \{1, \ldots, n\}$, $a \in A$ and $\pi \in M_{n-1}(A)$. Indeed, for each $i \neq 1$, $a \in A$ and $\pi \in M_{n-1}(A)$, let $a_i = a$ and $a_{-i} \in A^{n-1}$ be such that $\sum_{j \neq i} 1_{a_j}/(n-1) = \pi$. Then, for each $x_i \in X_0$,

$$\begin{aligned}
&\gamma_i^n(a, \pi)(x_i) = \sum_{x_{-i} \in X_0^{n-1}} \Phi_n(a_1, \ldots, a_n)(x_1, \ldots, x_n) \\
&= \sum_{x_{-i} \in X_0^{n-1}} \Phi_n(a_i, a_2, \ldots, a_{i-1}, a_1, a_{i+1}, \ldots, a_n)(x_i, x_2, \ldots, x_{i-1}, x_1, x_{i+1}, \ldots, x_n) \\
&= \gamma_1^n(a, \pi)(x_i).
\end{aligned}$$

Thus, let $\gamma_n = \gamma_1^n$.

A.2 Proof of Lemma 2

Let x_1, x_2, x_3 be distinct elements of X_0 and $\mathcal{U} = \mathcal{U}_{x_1,x_2,x_3}$ be the set of $u \in \mathbb{R}^{|T||X_0|}$ such that

(a) $u(t,x) \neq u(t,x')$ for each $t \in T$ and $x, x' \in \{x_1, x_2, x_3\}$ with $x \neq x'$, and

(b) $\frac{u(t,x')-u(t,x)}{u(t,x)-u(t,\hat{x})} \neq \frac{u(t',x')-u(t',x)}{u(t',x)-u(t',\hat{x})}$ for each $t, t' \in T$ with $t \neq t'$ and $x, x', \hat{x} \in \{x_1, x_2, x_3\}$ with $x \neq x'$, $x \neq \hat{x}$ and $x' \neq \hat{x}$.

We first note that $\mathcal{U}_{x_1,x_2,x_3}$ is generic. To see this, let $D = \{(t,t') \in T^2 : t \neq t'\}$, $D_2 = \{(x,x') \in \{x_1,x_2,x_3\}^2 : x \neq x'\}$, $D_3 = \{(x,x',\hat{x}) \in \{x_1,x_2,x_3\}^3 : x \neq x', x \neq \hat{x} \text{ and } x' \neq \hat{x}\}$, $V = \cup_{(t,x,x') \in T \times D_2} \{u \in \mathbb{R}^{|T||X_0|} : u(t,x) = u(t,x')\}$ and $W = \left(\cup_{(t,t',x,x',\hat{x}) \in D \times D_3} W(t,t',x,x',\hat{x})\right) \cap V^c$, where

$$W(t,t',x,x',\hat{x}) = \left\{ u \in \mathbb{R}^{|T||X_0|} : u(t,x') = u(t,x) + \frac{u(t,x)-u(t,\hat{x})}{u(t',x)-u(t',\hat{x})}\big(u(t',x')-u(t',x)\big) \right\}.$$

Then $V \cup W$ is the complement of $\mathcal{U}_{x_1,x_2,x_3}$, is closed and has Lebesgue measure zero by Tonelli's Theorem since both V and W have dimension lower than $|T||X_0|$.

Let $u \in \mathcal{U}_{x_1,x_2,x_3}$. For each $i, j \in \{1,2,3\}$ with $i \neq j$, let $T^{ij} = \{t \in T : u(t,x_i) > u(t,x_j) > u(t,x_l), l \in \{1,2,3\} \setminus \{i,j\}\}$. Since $u \in \mathcal{U}_{x_1,x_2,x_3}$, $T = \cup_{i,j:i \neq j} T^{ij}$. Define $\sigma' : T \to X$ by $\sigma'(t) = (1-\varepsilon)1_{x_i} + \varepsilon 1_{x_j}$ if $t \in T^{ij}$ for some $i, j \in \{1,2,3\}$ with $i \neq j$ and where $\varepsilon > 0$. Note that if $0 < \varepsilon < 1/3$, then $u(t,\sigma'(t)) > u(t,\sigma'(t'))$ whenever $t \in T^{ij}$ and $t' \notin T^{ij}$. Indeed, if $t' \in T^{il}$, then

$$u(t,\sigma'(t)) - u(t,\sigma'(t')) = \varepsilon(u(t,x_j) - u(t,x_l)) > 0;$$

if $t' \in T^{km}$ with $k \in \{j,l\}$ and $m \neq k$, then

$$\begin{aligned} u(t,\sigma'(t)) - u(t,\sigma'(t')) &= (1-\varepsilon)(u(t,x_i) - u(t,x_k)) + \varepsilon(u(t,x_j) - u(t,x_m)) \\ &\geq (1-\varepsilon)(u(t,x_i) - u(t,x_j)) + \varepsilon(u(t,x_j) - u(t,x_i)) \\ &= (1-2\varepsilon)(u(t,x_i) - u(t,x_j)) > 0. \end{aligned}$$

Of course, $\sigma'(t) = \sigma'(t')$ if $t, t' \in T^{ij}$; we will now define σ by slightly changing σ'. Let $\eta \in (0, \varepsilon/2)$ and, for each $i, j \in \{1, 2, 3\}$ with $i \neq j$ and $\alpha, \beta \in [0, \eta]$, define

$$\sigma_{ij}(\alpha, \beta) = (1 - \varepsilon + \alpha)1_{x_i} + (\varepsilon - \alpha - \beta)1_{x_j} + \beta 1_{x_l},$$

where $l \in \{1, 2, 3\} \backslash \{i, j\}$. Then there is $\eta > 0$ sufficiently small such that $u(t, \sigma_{ij}(\alpha, \beta)) > u(t, \sigma_{km}(\alpha', \beta'))$ whenever $t \in T^{ij}$, $(k, m) \neq (i, j)$, $m \neq k$ and $\alpha, \beta, \alpha', \beta' \in [0, \eta]$ since $\sigma_{ij}(0, 0) = \sigma'(t)$.

Fix $i, j \in \{1, 2, 3\}$ with $i \neq j$ and let $t \in T^{ij}$. For each $\alpha, \beta \in [0, \eta]$, let $I_t(\alpha, \beta)$ ($L_t(\alpha, \beta)$ and $U_t(\alpha, \beta)$ respectively) be the set of $(\alpha', \beta') \in (0, \eta)^2$ such that $u(t, \sigma_{ij}(\alpha', \beta')) = u(t, \sigma_{ij}(\alpha, \beta))$ ($u(t, \sigma_{ij}(\alpha', \beta')) < u(t, \sigma_{ij}(\alpha, \beta))$ and $u(t, \sigma_{ij}(\alpha', \beta')) > u(t, \sigma_{ij}(\alpha, \beta))$ respectively). Let also

$$\begin{aligned} s_t &= \frac{u(t, x_j) - u(t, x_l)}{u(t, x_i) - u(t, x_j)} \text{ and} \\ \theta_t(\alpha, \beta) &= \frac{u(t, \sigma_{ij}(\alpha, \beta)) - u(t, \sigma_{ij}(0, 0))}{u(t, x_i) - u(t, x_j)} = \alpha - \beta s_t. \end{aligned}$$

Then

$$\begin{aligned} I_t(\alpha, \beta) &= \left\{(\alpha', \beta') \in (0, \eta)^2 : \alpha' = \beta' s_t + \theta_t(\alpha, \beta)\right\} \\ L_t(\alpha, \beta) &= \left\{(\alpha', \beta') \in (0, \eta)^2 : \alpha' < \beta' s_t + \theta_t(\alpha, \beta)\right\} \\ U_t(\alpha, \beta) &= \left\{(\alpha', \beta') \in (0, \eta)^2 : \alpha' > \beta' s_t + \theta_t(\alpha, \beta)\right\}. \end{aligned}$$

Let $m = |T^{ij}|$ and order the elements of T^{ij} so that $T^{ij} = \{t_1, \ldots, t_m\}$ and

$$\frac{u(t_1, x_j) - u(t_1, x_l)}{u(t_1, x_i) - u(t_1, x_j)} < \cdots < \frac{u(t_m, x_j) - u(t_m, x_l)}{u(t_m, x_i) - u(t_m, x_j)};$$

this is possible because $u \in \mathcal{U}_{x_1, x_2, x_3}$. For convenience, let $s_k = s_{t_k}$ and $\theta_k = \theta_{t_k}$ for each $1 \leq k \leq m$. In addition, let $s_{m+1} > s_m$ and

$$I_{t_{m+1}}(0, 0) = \left\{(\alpha', \beta') \in (0, \eta)^2 : \alpha' = \beta' s_{m+1}\right\}.$$

Let $(\alpha_1, \beta_1) \in I_{t_2}(0, 0)$. Assuming that $(\alpha_1, \beta_1), \ldots, (\alpha_{k-1}, \beta_{k-1})$ have been defined, let

$$(\alpha_k, \beta_k) \in I_{t_{k+1}}(0, 0) \cap L_{t_{k-1}}(\alpha_{k-1}, \beta_{k-1}).$$

That such (α_k, β_k) exists can be seen as follows. First, note that $(\alpha_{k-1}, \beta_{k-1}) \in I_{t_k}(0,0) \subseteq U_{t_{k-1}}(0,0)$ by (3); hence,

$$u(t_{k-1}, \sigma_{ij}(\alpha_{k-1}, \beta_{k-1})) > u(t_{k-1}, \sigma_{ij}(0,0)) \text{ and}$$
$$\theta_{k-1}(\alpha_{k-1}, \beta_{k-1}) = \frac{u(t_{k-1}, \sigma_{ij}(\alpha_{k-1}, \beta_{k-1})) - u(t_{k-1}, \sigma_{ij}(0,0))}{u(t_{k-1}, x_i) - u(t_{k-1}, x_j)} > 0.$$

Next, let $(\bar{\alpha}, \bar{\beta})$ be such that $\{(\bar{\alpha}, \bar{\beta})\} = I_{t_{k+1}}(0,0) \cap I_{t_{k-1}}(\alpha_{k-1}, \beta_{k-1})$; thus,

$$\bar{\beta} s_{k+1} = \bar{\beta} s_{k-1} + \theta_{k-1}(\alpha_{k-1}, \beta_{k-1}) \Leftrightarrow \bar{\beta} = \frac{\theta_{k-1}(\alpha_{k-1}, \beta_{k-1})}{s_{k+1} - s_{k-1}} > 0.$$

Hence, each (α, β) such that $\beta \in (0, \bar{\beta})$ and $\alpha = \beta s_{k+1}$ belong to $I_{t_{k+1}}(0,0) \cap L_{t_{k-1}}(\alpha_{k-1}, \beta_{k-1})$. Indeed, it is clear that $(\alpha, \beta) \in I_{t_{k+1}}(0,0)$; in addition,

$$\begin{aligned}\alpha = \beta s_{k+1} &= \bar{\beta} s_{k+1} - (\bar{\beta} - \beta) s_{k+1} = \bar{\beta} s_{k-1} + \theta_{k-1}(\alpha_{k-1}, \beta_{k-1}) - (\bar{\beta} - \beta) s_{k+1} \\ &< \bar{\beta} s_{k-1} + \theta_{k-1}(\alpha_{k-1}, \beta_{k-1}) - (\bar{\beta} - \beta) s_{k-1} = \beta s_{k-1} + \theta_{k-1}(\alpha_{k-1}, \beta_{k-1}),\end{aligned}$$

implying that $(\alpha, \beta) \in L_{t_{k-1}}(\alpha_{k-1}, \beta_{k-1})$.

We then obtain that, for each $k, r \in \{1, \dots, m\}$ with $k \neq r$,

$$u(t_k, \sigma_{ij}(\alpha_k, \beta_k)) > u(t_k, \sigma_{ij}(\alpha_r, \beta_r)). \tag{1}$$

To see (1), consider first the case $r < k$. We have that, whenever $r < k$,

$$I_{t_r}(0,0) \subseteq L_{t_k}(0,0) \text{ and} \tag{2}$$
$$I_{t_k}(0,0) \subseteq U_{t_r}(0,0). \tag{3}$$

Indeed, if $(\alpha, \beta) \in I_{t_r}(0,0)$, then $\alpha = \beta s_r < \beta s_k$ and, hence, $(\alpha, \beta) \in L_{t_k}(0,0)$; this shows (2). If $(\alpha, \beta) \in I_{t_k}(0,0)$, then $\alpha = \beta s_k > \beta s_r$ and, hence, $(\alpha, \beta) \in U_{t_r}(0,0)$; this shows (3).

Then $(\alpha_k, \beta_k) \in I_{t_{k+1}}(0,0) \subseteq U_{t_k}(0,0)$, $(\alpha_{k-1}, \beta_{k-1}) \in I_{t_k}(0,0)$ and $(\alpha_h, \beta_h) \in I_{t_{h+1}}(0,0) \subseteq L_{t_k}(0,0)$ for each $h < k-1$ imply that

$$u(t_k, \sigma_{ij}(\alpha_k, \beta_k)) > u(t_k, \sigma_{ij}(0,0)) \geq u(t_k, \sigma_{ij}(\alpha_r, \beta_r)).$$

Next, consider the case $r > k$. To establish (1) in this case, we first show that $\beta_k < \beta_{k-1}$ for each $k = 2, \dots, m$. Indeed, $\alpha_k = \beta_k s_{k+1}$, $\alpha_{k-1} = \beta_{k-1} s_k$ and, recalling that $\theta_{k-1}(\alpha_{k-1}, \beta_{k-1}) = \alpha_{k-1} - \beta_{k-1}s_{k-1}$, $\alpha_k < \beta_k s_{k-1} + \alpha_{k-1} - \beta_{k-1}s_{k-1}$. From this we obtain that

$$\beta_k s_{k+1} - \beta_k s_{k-1} < \beta_{k-1}s_k - \beta_{k-1}s_{k-1} \Leftrightarrow \beta_k < \frac{s_k - s_{k-1}}{s_{k+1} - s_{k-1}}\beta_{k-1}.$$

Since $\frac{s_k - s_{k-1}}{s_{k+1}-s_{k-1}} < 1$, it follows that $\beta_k < \beta_{k-1}$.

We now claim that $(\alpha_r, \beta_r) \in L_{t_k}(\alpha_k, \beta_k)$ for each $r > k$. This holds for $r = k+1$ since $(\alpha_{k+1}, \beta_{k+1}) \in L_{t_k}(\alpha_k, \beta_k)$ by construction. Suppose next that $(\alpha_r, \beta_r) \in L_{t_k}(\alpha_k, \beta_k)$, i.e. $\alpha_r < \beta_r s_k + \alpha_k - \beta_k s_k$; we will show that $(\alpha_{r+1}, \beta_{r+1}) \in L_{t_k}(\alpha_k, \beta_k)$. To see this, note first that since $\beta_{r+1} < \beta_r$,

$$\beta_{r+1}s_r + \beta_r s_k - \beta_r s_r < \beta_{r+1}s_k.$$

Since $(\alpha_{r+1}, \beta_{r+1}) \in L_{t_r}(\alpha_r, \beta_r)$, it follows that

$$\begin{aligned} \alpha_{r+1} &< \beta_{r+1}s_r + \alpha_r - \beta_r s_r \\ &< \beta_{r+1}s_r + \beta_r s_k + \alpha_k - \beta_k s_k - \beta_r s_r \\ &< \beta_{r+1}s_k + \alpha_k - \beta_k s_k. \end{aligned}$$

Hence, $(\alpha_{r+1}, \beta_{r+1}) \in L_{t_k}(\alpha_k, \beta_k)$ as claimed.

To conclude the proof, for each $t \in T^{ij}$, let $(\alpha_t, \beta_t) = (\alpha_k, \beta_k)$ if $t = t_k$. Then define σ by setting $\sigma(t) = \sigma_{ij}(\alpha_t, \beta_t)$ if $t \in T^{ij}$.

A.3 On the assumption $|X_0| \geq 3$

This section illustrates the difficulties with $|X_0| = 2$. Let $X_0 = \{x_0, x_1\}$, $T^0 = \{t \in T : u(t, x_0) > u(t, x_1)\}$, $T^1 = \{t \in T : u(t, x_1) > u(t, x_0)\}$ and assume that $T = T^0 \cup T^1$. If γ is strategy-proof, then $\gamma(t, \pi) = \gamma(t', \pi)$ and $\gamma(t, \pi)(x_i) \geq \gamma(\hat{t}, \pi)(x_i)$ for each $i \in \{0, 1\}$, $t, t' \in T^i$, $\hat{t} \in T^j$, $j \neq i$ and $\pi \in M(T)$. The case of $|X_0| = 2$ adds a strong requirement on strategy-proofness, namely that $\gamma(t, \pi) = \gamma(t', \pi)$ whenever $t, t' \in T^i$. For this reason, there is no strict strategy-proof mechanisms whenever $|T^i| > 1$ for some $i \in \{0, 1\}$.

Likewise, a sequence $\langle (Y_n, \Phi_n) \rangle_n$ of direct mechanisms that are strategy-proof in the large will, in general, fail to be such that (Y_n, Φ_n) is strategy-proof whenever n

is sufficiently large. Indeed, this will hold provided that $\gamma_n(t,\pi) \neq \gamma_n(t',\pi)$ for some $\pi \in M_{n-1}(T)$, $t, t' \in T^i$ and $i \in \{0,1\}$.

A.4 A lemma

The following lemma will be used in the proof of Theorems 1 and 2. Recall that $\mathcal{S}^*$ is the set of strictly strategy-proof reduced-form mechanisms.

Lemma 3. *Let $\gamma \in \mathcal{S}^*$, $\lambda \in (0,1]$ and $\langle (Y_n, \Phi_n), \gamma_n, \mu_n, \lambda_n \rangle_{n\in\mathbb{N}}$ be such that, for each $n \in \mathbb{N}$, $\lambda_n \in [0,1]$, (Y_n, Φ_n) is an anonymous direct mechanism whose marginal is $\lambda_n\gamma_n + (1-\lambda_n)\mu_n$, μ_n is strategy-proof and envy-free, $\gamma_n \to \gamma$ and $\lambda_n \to \lambda$. Then, there is $N \in \mathbb{N}$ such that (Y_n, Φ_n) is strictly strategy-proof and strictly envy-free whenever $n \geq N$.*

Proof. Let

$$\varepsilon(\gamma) = \min_{\pi \in M(T), t, t' \in T, t \neq t'} [u(t, \gamma(t,\pi)) - u(t, \gamma(t',\pi))].$$

Because $M(T) \times \{(t,t') \in T^2 : t \neq t'\}$ is compact and γ is continuous, the mininum is achieved; as $\gamma \in \mathcal{S}^*$, $\varepsilon(\gamma) > 0$.

Let $\varepsilon > 0$ be such that $3\varepsilon < \varepsilon(\gamma)$, $\eta > 0$ be such that

$$\sup\{|u(t,x) - u(t,x')| : t \in T, x, x' \in X, ||x - x'|| \leq \eta\} < \varepsilon$$

and $0 < \delta < \eta$ be such that $||\gamma(t,\pi') - \gamma(t,\pi)|| < \eta$ whenever $t \in T$, $\pi, \pi' \in M(T)$ and $||\pi - \pi'|| < 3\delta$. Let $N \in \mathbb{N}$ be such that $\lambda_n > 0$ and $d(\gamma_n, \gamma) < \delta$ for each $n \geq N$.

Thus, whenever $n \geq N$, we have that, for each $t, t' \in T$ with $t \neq t'$ and $\tilde{\pi} \in M_{n-1}(T)$, there is $\pi, \pi' \in M(T)$ such that $||\gamma_n(t,\tilde{\pi}) - \gamma(t,\pi)|| < \delta$, $||\gamma_n(t',\tilde{\pi}) - \gamma(t',\pi')|| < \delta$, $||\pi - \tilde{\pi}|| < \delta$ and $||\pi' - \tilde{\pi}|| < \delta$. Thus,

$$|u(t, \gamma_n(t,\tilde{\pi})) - u(t, \gamma(t,\pi))| < \varepsilon,$$
$$|u(t, \gamma_n(t',\tilde{\pi})) - u(t, \gamma(t',\pi'))| < \varepsilon \text{ and}$$
$$|u(t, \gamma(t',\pi)) - u(t, \gamma(t',\pi'))| < \varepsilon.$$

Therefore,

$$\begin{aligned}&u(t,\gamma_n(t,\tilde\pi)) - u(t,\gamma_n(t',\tilde\pi)) > u(t,\gamma(t,\pi)) - u(t,\gamma(t',\pi')) - 2\varepsilon\\ &> u(t,\gamma(t,\pi)) - u(t,\gamma(t',\pi)) - 3\varepsilon \geq \varepsilon(\gamma) - 3\varepsilon > 0.\end{aligned}$$

It follows that (Y_n, Φ_n) is strictly strategy-proof since

$$\begin{aligned}&u(t,\lambda_n\gamma_n(t,\tilde\pi) + (1-\lambda_n)\mu_n(t,\tilde\pi)) - u(t,\lambda_n\gamma_n(t',\tilde\pi) + (1-\lambda_n)\mu_n(t',\tilde\pi)) =\\ &\lambda_n\Big(u(t,\gamma_n(t,\tilde\pi)) - u(t,\gamma_n(t',\tilde\pi))\Big) + (1-\lambda_n)\Big(u(t,\mu_n(t,\tilde\pi)) - u(t,\mu_n(t',\tilde\pi))\Big) > 0.\end{aligned}$$

We next show that there is $N \in \mathbb{N}$ such that (Y_n, Φ_n) is strictly envy-free for each $n \geq N$. Let $N \in \mathbb{N}$ be such that, for each $n \geq N$, $\lambda_n > 0$, $d(\gamma_n, \gamma) < \delta$ and $1/(n-1) < \delta$. Thus, for each $t, t' \in T$ with $t \neq t'$ and $\tilde\pi \in M_{n-1}(T)$ with $\tilde\pi(t') > 0$, there is $\pi, \pi' \in M(T)$ such that $||\gamma_n(t,\tilde\pi) - \gamma(t,\pi)|| < \delta$, $||\gamma_n(t', \tilde\pi + (1_t - 1_{t'})/(n-1)) - \gamma(t',\pi')|| < \delta$, $||\pi - \tilde\pi|| < \delta$ and $||\pi' - (\tilde\pi + (1_t - 1_{t'})/(n-1))|| < \delta$. Thus,

$$\begin{aligned}&||\pi - \pi'|| \leq ||\pi - \tilde\pi|| + ||\tilde\pi - (\tilde\pi + (1_t - 1_{t'})/(n-1))||\\ &\quad + ||\tilde\pi + (1_t - 1_{t'})/(n-1) - \pi'|| < 3\delta,\\ &|u(t,\gamma_n(t,\tilde\pi)) - u(t,\gamma(t,\pi))| < \varepsilon,\\ &|u(t,\gamma_n(t',\tilde\pi + (1_t - 1_{t'})/(n-1))) - u(t,\gamma(t',\pi'))| < \varepsilon \text{ and}\\ &|u(t,\gamma(t',\pi)) - u(t,\gamma(t',\pi'))| < \varepsilon.\end{aligned}$$

Therefore,

$$\begin{aligned}&u(t,\gamma_n(t,\tilde\pi)) - u(t,\gamma_n(t',\tilde\pi + (1_t - 1_{t'})/(n-1))) > u(t,\gamma(t,\pi)) - u(t,\gamma(t',\pi')) - 2\varepsilon\\ &> u(t,\gamma(t,\pi)) - u(t,\gamma(t',\pi)) - 3\varepsilon \geq \varepsilon(\gamma) - 3\varepsilon > 0.\end{aligned}$$

It follows that (Y_n, Φ_n) is strictly envy-free since

$$\begin{aligned}&u(t,\lambda_n\gamma_n(t,\tilde\pi) + (1-\lambda_n)\mu_n(t,\tilde\pi))\\ &- u(t,\lambda_n\gamma_n(t',\tilde\pi + (1_t - 1_{t'})/(n-1)) + (1-\lambda_n)\mu_n(t',\tilde\pi + (1_t - 1_{t'})/(n-1))) =\\ &\lambda_n\Big(u(t,\gamma_n(t,\tilde\pi)) - u(t,\gamma_n(t',\tilde\pi + (1_t - 1_{t'})/(n-1)))\Big)\\ &+ (1-\lambda_n)\Big(u(t,\mu_n(t,\tilde\pi)) - u(t,\mu_n(t',\tilde\pi + (1_t - 1_{t'})/(n-1)))\Big) > 0.\end{aligned}$$

$\square$

A.5 Proof of Theorem 1

(a) Let $\mathcal{S}^*$ be the subset of $\mathcal{S}$ consisting of those γ such that $u(t, \gamma(t, \pi)) > u(t, \gamma(t', \pi))$ for each $t, t' \in T$ with $t \neq t'$ and $\pi \in M(T)$.

(b) $\mathcal{S}^*$ is an open subset of $\mathcal{S}$. Let $\gamma \in \mathcal{S}^*$ and

$$\varepsilon(\gamma) = \min_{\pi \in M(T), t, t' \in T, t \neq t'} [u(t, \gamma(t, \pi)) - u(t, \gamma(t', \pi))].$$

Because $M(T) \times \{(t, t') \in T^2 : t \neq t'\}$ is compact and γ is continuous, the mininum is achieved; as $\gamma \in \mathcal{S}^*$, $\varepsilon(\gamma) > 0$.

We have that the mapping $\hat{\gamma} \mapsto \varepsilon(\hat{\gamma})$ is continuous. It then follows that, for some $\delta > 0$, the open ball of radius δ of γ is contained in $\mathcal{S}^*$.

(c) $\mathcal{S}^*$ is dense in $\mathcal{S}$. Let $\gamma \in \mathcal{S}$ and $\varepsilon > 0$. Moreover, by Lemma 2, let $\sigma : T \to X$ be such that $u(t, \sigma(t)) > u(t, \sigma(t'))$ for all $t, t' \in T$. Define ψ by setting, for each $t \in T$ and $\pi \in M(T)$, $\psi(t, \pi) = (1 - \varepsilon)\gamma(t, \pi) + \varepsilon\sigma(t)$. Then ψ is continuous and, for each $t, t' \in T$ and $\pi \in M(T)$,

$$u(t, \psi(t, \pi)) - u(t, \psi(t', \pi)) =$$
$$(1 - \varepsilon)(u(t, \gamma(t, \pi)) - u(t, \gamma(t', \pi))) + \varepsilon(u(t, \sigma(t)) - u(t, \sigma(t'))) > 0.$$

Thus, $\psi \in \mathcal{S}^*$. By making ε small enough, ψ is as close to γ as desired.

(d) Let $\gamma \in \mathcal{S}^*$ and let $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ be a sequence of anonymous direct mechanisms whose sequence of marginal distributions $\langle \gamma_n \rangle_{n \in \mathbb{N}}$ converges to γ. It then follows by Lemma 3 that there $N \in \mathbb{N}$ such that, for each $n \geq N$, (Y_n, Φ_n) is strictly strategy-proof and strictly envy-free.

A.6 Proof of Theorem 2

Let u, ε, γ and $\langle Y_n, \Phi_n \rangle_{n \in \mathbb{N}}$ be as in the statement of the theorem. Since $\langle Y_n \rangle_{n \in \mathbb{N}}$ if free in the large, let m_n and z_n be as in the definition of freeness in the large. Let $\sigma : T \to X$ be as in Lemma 2, i.e. σ is strictly strategy-proof, $\alpha_n = m_n/n$ for each $n \in \mathbb{N}$ and $\alpha = \lim_n \alpha_n$.

Claim 1. *For each $n \in \mathbb{N}$, there exists $\Psi_n : T^n \to M(Y_n)$ and $\mu_n \in X$ such that Ψ_n is anonymous and its marginal distribution ψ_n satisfies $\psi_n = \alpha_n\sigma + (1 - \alpha_n)\mu_n$.*

Proof. For each $n \in \mathbb{N}$, define an anonymous direct mechanism $\Psi_n : T^n \to M(Y_n)$ as follows. Let K_n denote the set of all bijections on $\{1, \ldots, n\}$ and $|K_n| = n!$ be its cardinality. Then, for each $(t_1, \ldots, t_n) \in T^n$ and $(x_1, \ldots, x_n) \in X_0^n$, set

$$\Psi_n(t_1, \ldots, t_n)(x_1, \ldots, x_n) = \frac{1}{|K_n|} \sum_{\kappa \in K_n} \prod_{i=1}^{m_n} \sigma(t_{\kappa(i)})(x_{\kappa(i)}) 1_{z_n}(x_{\kappa(m_n+1)}, \ldots, x_{\kappa(n)}).$$

If $k \in K_n$ then $K_n = \{\kappa \circ k^{-1} : \kappa \in K_n\}$ since the inverse of a bijection is a bijection and so is the composition of two bijections. Hence, Ψ_n is anonymous since, for each $(t_1, \ldots, t_n) \in A^n$ and $(x_1, \ldots, x_n) \in X_0^n$,

$$\Psi_n(t_{k(1)}, \ldots, t_{k(n)})(x_{k(1)}, \ldots, x_{k(n)}) = \frac{1}{|K_n|} \sum_{\kappa \in K_n} \prod_{i=1}^{m_n} \sigma(t_{\kappa \circ k(i)})(x_{\kappa \circ k(i)}) 1_{z_n}(x_{\kappa \circ k(m_n+1)}, \ldots, x_{\kappa \circ k(n)})$$

$$= \frac{1}{|K_n|} \sum_{\hat{\kappa} \in K_n} \prod_{i=1}^{m_n} \sigma(t_{\hat{\kappa} \circ k^{-1} \circ k(i)})(x_{\hat{\kappa} \circ k^{-1} \circ k(i)}) 1_{z_n}(x_{\hat{\kappa} \circ k^{-1} \circ k(m_n+1)}, \ldots, x_{\hat{\kappa} \circ k^{-1} \circ k(n)})$$

$$= \frac{1}{|K_n|} \sum_{\hat{\kappa} \in K_n} \prod_{i=1}^{m_n} \sigma(t_{\hat{\kappa}(i)})(x_{\hat{\kappa}(i)}) 1_{z_n}(x_{\hat{\kappa}(m_n+1)}, \ldots, x_{\hat{\kappa}(n)}) = \Psi_n(t_1, \ldots, t_n)(x_1, \ldots, x_n).$$

If $(x_1, \ldots, x_n) \in \operatorname{supp}(\Psi_n(t_1, \ldots, t_n))$, then for some $\kappa \in K_n$, $(x_{\kappa(1)}, \ldots, x_{\kappa(n)}) \in X_0^{m_n} \times \{z_n\} \subseteq Y_n$. Hence, $(x_1, \ldots, x_n) = (x_{\kappa^{-1} \circ \kappa(1)}, \ldots, x_{\kappa^{-1} \circ \kappa(n)}) \in Y_n$ since $\kappa^{-1} \in K$. Thus, $\Psi_n(t_1, \ldots, t_n) \in M(Y_n)$.

Let $\mu_n = e_{n-m_n}(z_n) \in M(X_0)$. For each $(t, \pi) \in T \times M_{n-1}(T)$, let $t_1 = t$ and $(t_2, \ldots, t_n) \in T^n$ be such that $\pi = \sum_{j=2}^n 1_{t_j}/(n-1)$. Then, for each $x_1 \in X_0$,

$$\psi_n(t, \pi)(x_1) = \sum_{(x_2, \ldots, x_n) \in X_0^{n-1}} \Psi_n(t_1, \ldots, t_n)(x_1, \ldots, x_n)$$

$$= \frac{1}{|K_n|} \sum_{\kappa \in K_n} \sum_{(x_2, \ldots, x_n) \in X_0^{n-1}} \prod_{i=1}^{m_n} \sigma(t_{\kappa(i)})(x_{\kappa(i)}) 1_{z_n}(x_{\kappa(m_n+1)}, \ldots, x_{\kappa(n)})$$

$$= \frac{1}{|K_n|} \left(\sum_{\kappa \in K_n : \kappa^{-1}(1) \in \{1, \ldots, m_n\}} \sigma(t)(x_1) + \sum_{j=m_n+1}^{n} \sum_{\kappa \in K_n : \kappa^{-1}(1) = j} 1_{z_{n,j-m_n}}(x_1) \right)$$

$$= \frac{|\{\kappa \in K_n : \kappa^{-1}(1) \in \{1, \ldots, m_n\}\}|}{|K_n|} \sigma(t)(x_1) + \sum_{j=m_n+1}^{n} \frac{|\{\kappa \in K_n : \kappa^{-1}(1) = j\}|}{|K_n|} 1_{z_{n,j-m_n}}(x_1)$$

$$= \frac{m_n}{n} \sigma(t)(x_1) + \frac{1}{n} \sum_{j=m_n+1}^{n} 1_{z_{n,j-m_n}}(x_1)$$

$$= \frac{m_n}{n} \sigma(t)(x_1) + \left(1 - \frac{m_n}{n}\right) \mu_n(x_1)$$

since $|K_n| = n!$, $|\{\kappa \in K_n : \kappa^{-1}(1) \in \{1,\ldots,m_n\}\}| = m_n(n-1)!$ and $|\{\kappa \in K_n : \kappa^{-1}(1) = j\}| = (n-1)!$. Since $\alpha_n = \frac{m_n}{n}$, it follows that $\psi_n = \alpha_n\sigma + (1-\alpha_n)\mu_n$. $\square$

Let $\varepsilon' \in (0,\varepsilon)$ and, for each $n \in \mathbb{N}$, define $\Phi'_n = (1-\varepsilon')\Phi_n + \varepsilon'\Psi_n$. Let, for each $n \in \mathbb{N}$, γ_n be the marginal of Φ_n described in Lemma 1 and define $\gamma'_n = (1-\varepsilon')\gamma_n + \varepsilon'\psi_n$. Then Φ'_n is anonymous and the marginal of Φ'_n is γ'_n. Then

$$\gamma'_n = \lambda_n(\theta_n\sigma + (1-\theta_n)\gamma_n) + (1-\lambda_n)\mu_n$$

with

$$\lambda_n = 1 - \varepsilon' + \varepsilon'\alpha_n \text{ and}$$
$$\theta_n = \frac{\varepsilon'\alpha_n}{\lambda_n}.$$

Letting

$$\lambda = 1 - \varepsilon' + \varepsilon'\alpha > 0 \text{ and}$$
$$\theta = \frac{\varepsilon'\alpha}{1 - \varepsilon' + \varepsilon'\alpha} > 0,$$

it follows that $\lambda_n \to \lambda$, $\theta_n \to \theta$ and $\theta\sigma + (1-\theta)\gamma \in \mathcal{S}^*$ since $\theta > 0$, $\sigma \in \mathcal{S}^*$ and $\gamma \in \mathcal{S}$. It then follows by Lemma 3 that Φ'_n is strictly strategy-proof and strictly envy-free for all n sufficiently large. Finally, we clearly have that $||\Phi_n - \Phi'_n|| < \varepsilon$.

A.7 Proof of Theorem 3

Let C, u, ε, γ and $\langle Y_n, \Phi_n\rangle_{n\in\mathbb{N}}$ be as in the statement of the theorem. Since $\langle Y_n\rangle_{n\in\mathbb{N}}$ is projection-free in the large relative to C, for each $n \in \mathbb{N}$, let m_n be as in the definition and for each $x \in C^{m_n}$, let $z_n(x) \in X_0^{n-m_n}$ be such that $k(x, z_n(x)) \in Y_n$ for every bijection $k : \{1,\ldots,n\} \to \{1,\ldots,\}$. Let σ be as in Lemma 2 with X_0 replaced by C, i.e. $\sigma : T \to \Delta(C)$ is strictly strategy-proof, $\alpha_n = m_n/n$ for each $n \in \mathbb{N}$ and $\alpha = \lim_n \alpha_n$.

Claim 2. *For each $n \in \mathbb{N}$, there exists $\Psi_n : T^n \to M(Y_n)$ and $\mu_n : M_{n-1}(T) \to X$ such that Ψ_n is anonymous and its marginal distribution ψ_n satisfies $\psi_n = \alpha_n\sigma + (1-\alpha_n)\mu_n$.*

Proof. For each $n \in \mathbb{N}$, define an anonymous direct mechanism $\Psi_n : T^n \to M(Y_n)$ as follows. Let K_n denote the set of all bijections on $\{1, \ldots, n\}$ and $|K_n| = n!$ be its cardinality. Then, for each $(t_1, \ldots, t_n) \in T^n$ and $(x_1, \ldots, x_n) \in X_0^n$, set

$$\Psi_n(t_1, \ldots, t_n)(x_1, \ldots, x_n) = \frac{1}{|K_n|} \sum_{\kappa \in K_n} \prod_{i=1}^{m_n} \sigma(t_{\kappa(i)})(x_{\kappa(i)}) \\ \times 1_{z_n(x_{\kappa(1)}, \ldots, x_{\kappa(m_n)})}(x_{\kappa(m_n+1)}, \ldots, x_{\kappa(n)}).$$

For each $i \in \{1, \ldots, m_n\}$, $\sigma(t_{\kappa(i)})(x_{\kappa(i)}) > 0$ implies $x_{\kappa(i)} \in \operatorname{supp}(\sigma(t_{\kappa(i)})) \subseteq C$; hence $(x_{\kappa(1)}, \ldots, x_{\kappa(m_n)}, z_n(x_{\kappa(1)}, \ldots, x_{\kappa(m_n)})) \in Y_n$. If $(x_1, \ldots, x_n) \in \operatorname{supp}(\Psi_n(t_1, \ldots, t_n))$, then for some $\kappa \in K_n$, $(x_{\kappa(1)}, \ldots, x_{\kappa(n)}) \in Y_n$. Hence, $(x_{\kappa^{-1} \circ \kappa(1)}, \ldots, x_{\kappa^{-1} \circ \kappa(n)})) \in Y_n$ since $\kappa^{-1} \in K_n$. Thus, $\Psi_n(t_1, \ldots, t_n) \in M(Y_n)$.

If $k \in K_n$ then $K_n = \{\kappa \circ k^{-1} : \kappa \in K_n\}$ since the inverse of a bijection is a bijection and so is the composition of two bijections. Hence, Ψ_n is anonymous since, for each $(t_1, \ldots, t_n) \in A^n$ and $(x_1, \ldots, x_n) \in X_0^n$,

$$\begin{aligned}
&\Psi_n(t_{k(1)}, \ldots, t_{k(n)})(x_{k(1)}, \ldots, x_{k(n)}) \\
&= \frac{1}{|K_n|} \sum_{\kappa \in K_n} \prod_{i=1}^{m_n} \sigma(t_{\kappa \circ k(i)})(x_{\kappa \circ k(i)}) 1_{z_n(x_{\kappa \circ k(1)}, \ldots, x_{\kappa \circ k(m_n)})}(x_{\kappa \circ k(m_n+1)}, \ldots, x_{\kappa \circ k(n)}) \\
&= \frac{1}{|K_n|} \sum_{\hat{\kappa} \in K_n} \prod_{i=1}^{m_n} \sigma(t_{\hat{\kappa} \circ k^{-1} \circ k(i)})(x_{\hat{\kappa} \circ k^{-1} \circ k(i)}) \\
&\times 1_{z_n(x_{\hat{\kappa} \circ k^{-1} \circ k(1)}, \ldots, x_{\hat{\kappa} \circ k^{-1} \circ k(m_n)})}(x_{\hat{\kappa} \circ k^{-1} \circ k(m_n+1)}, \ldots, x_{\hat{\kappa} \circ k^{-1} \circ k(n)}) \\
&= \frac{1}{|K_n|} \sum_{\hat{\kappa} \in K_n} \prod_{i=1}^{m_n} \sigma(t_{\hat{\kappa}(i)})(x_{\hat{\kappa}(i)}) 1_{z_n(x_{\hat{\kappa}(1)}, \ldots, x_{\hat{\kappa}(m_n)})}(x_{\hat{\kappa}(m_n+1)}, \ldots, x_{\hat{\kappa}(n)}) \\
&= \Psi_n(t_1, \ldots, t_n)(x_1, \ldots, x_n).
\end{aligned}$$

For each $(t, \pi) \in T \times M_{n-1}(T)$, let $t_1 = t$ and $(t_2, \ldots, t_n) \in T^n$ be such that

$\pi = \sum_{j=2}^{n} 1_{t_j}/(n-1)$. Then, for each $x_1 \in X_0$,

$$\psi_n(t,\pi)(x_1) = \sum_{(x_2,\ldots,x_n)\in X_0^{n-1}} \Psi_n(t_1,\ldots,t_n)(x_1,\ldots,x_n)$$
$$= \frac{1}{|K_n|}\sum_{\kappa\in K_n}\sum_{(x_2,\ldots,x_n)\in X_0^{n-1}}\prod_{i=1}^{m_n}\sigma(t_{\kappa(i)})(x_{\kappa(i)})1_{z_n(x_{\kappa(1)},\ldots,x_{\kappa(m_n)})}(x_{\kappa(m_n+1)},\ldots,x_{\kappa(n)})$$
$$= \frac{1}{|K_n|}\Bigg(\sum_{\kappa\in K_n:\kappa^{-1}(1)\in\{1,\ldots,m_n\}}\sigma(t)(x_1)$$
$$+\sum_{j=m_n+1}^{n}\sum_{\kappa\in K_n:\kappa^{-1}(1)=j}\sum_{(x_{\kappa(1)},\ldots,x_{\kappa(m_n)})\in X_0^{m_n}}\prod_{i=1}^{m_n}\sigma(t_{\kappa(i)})(x_{\kappa(i)})1_{z_{n,j-m_n}(x_{\kappa(1)},\ldots,x_{\kappa(m_n)})}(x_1)\Bigg)$$
$$= \frac{|\{\kappa\in K_n:\kappa^{-1}(1)\in\{1,\ldots,m_n\}\}|}{|K_n|}\sigma(t)(x_1)+\frac{|\{\kappa\in K_n:\kappa^{-1}(1)>m_n\}|}{|K_n|}\mu_n(\pi)(x_1)$$
$$= \frac{m_n}{n}\sigma(t)(x_1)+\left(1-\frac{m_n}{n}\right)\mu_n(\pi)(x_1),$$

with

$$\mu_n(\pi)(x_1) = \frac{1}{|\{\kappa\in K_n:\kappa^{-1}(1)>m_n\}|}\sum_{j=m_n+1}^{n}\sum_{\kappa\in K_n:\kappa^{-1}(1)=j}\sum_{(x_{\kappa(1)},\ldots,x_{\kappa(m_n)})\in X_0^{m_n}}$$
$$\prod_{i=1}^{m_n}\sigma(t_{\kappa(i)})(x_{\kappa(i)})1_{z_{n,j-m_n}(x_{\kappa(1)},\ldots,x_{\kappa(m_n)})}(x_1).$$

Note that for $(t_2,\ldots,t_n),(t_2',\ldots,t_n')\in T^n$ such that $\pi=\sum_{j=2}^{n}1_{t_j}/(n-1)=\sum_{j=2}^{n}1_{t_j'}/(n-1)$, there exists a bijection $k:\{1,\ldots,n\}\to\{1,\ldots,n\}$ such that $k(1)=1$ and $(t_2',\ldots,t_n')=(t_{k(2)},\ldots,t_{k(n)})$. Also note that, for each j, $\{k\circ\kappa:\kappa\in K_n$ and $\kappa(j)=1\}=\{\hat\kappa\in K_n:\hat\kappa(j)=1\}$. Hence, for each $j>m_n$,

$$\sum_{\kappa\in K_n:\kappa^{-1}(1)=j}\sum_{(x_{\kappa(1)},\ldots,x_{\kappa(m_n)})\in X_0^{m_n}}\prod_{i=1}^{m_n}\sigma(t'_{\kappa(i)})(x_{\kappa(i)})1_{z_{n,j-m_n}(x_{\kappa(1)},\ldots,x_{\kappa(m_n)})}(x_1)$$
$$=\sum_{\kappa\in K_n:\kappa^{-1}(1)=j}\sum_{(x_{k\circ\kappa(1)},\ldots,x_{k\circ\kappa(m_n)})\in X_0^{m_n}}\prod_{i=1}^{m_n}\sigma(t_{k\circ\kappa(i)})(x_{k\circ\kappa(i)})1_{z_{n,j-m_n}(x_{k\circ\kappa(1)},\ldots,x_{k\circ\kappa(m_n)})}(x_1)$$
$$=\sum_{\hat\kappa\in K_n:\hat\kappa^{-1}(1)=j}\sum_{(x_{\hat\kappa(1)},\ldots,x_{\hat\kappa(m_n)})\in X_0^{m_n}}\prod_{i=1}^{m_n}\sigma(t_{\hat\kappa(i)})(x_{\hat\kappa(i)})1_{z_{n,j-m_n}(x_{\hat\kappa(1)},\ldots,x_{\hat\kappa(m_n)})}(x_1),$$

i.e. $\mu_n : M_{n-1}(T)\to X$ is well-defined. Thus, it follows that $\psi_n=\alpha_n\sigma+(1-\alpha_n)\mu_n$. □

Let $\varepsilon'\in(0,\varepsilon)$ and, for each $n\in\mathbb{N}$, define $\Phi_n'=(1-\varepsilon')\Phi_n+\varepsilon'\Psi_n$. Let, for each $n\in\mathbb{N}$, γ_n be the marginal of Φ_n described in Lemma 1 and define $\gamma_n'=(1-\varepsilon')\gamma_n+\varepsilon'\psi_n$.

Then Φ'_n is anonymous and the marginal of Φ'_n is γ'_n. Furthermore,

$$\gamma'_n = \lambda_n(\theta_n \sigma + (1-\theta_n)\gamma_n) + (1-\lambda_n)\mu_n$$

with

$$\lambda_n = 1 - \varepsilon' + \varepsilon'\alpha_n \text{ and}$$
$$\theta_n = \frac{\varepsilon'\alpha_n}{\lambda_n}.$$

Letting

$$\lambda = 1 - \varepsilon' + \varepsilon'\alpha > 0 \text{ and}$$
$$\theta = \frac{\varepsilon'\alpha}{1-\varepsilon'+\varepsilon'\alpha} > 0,$$

it follows that $\lambda_n \to \lambda$, $\theta_n \to \theta$ and $\theta\sigma + (1-\theta)\gamma \in \mathcal{S}^*$ since $\theta > 0$, $\sigma \in \mathcal{S}^*$ and $\gamma \in \mathcal{S}$. We also have that μ_n is strategy-proof since it is constant in t. It then follows from (the proof of) Lemma 3 that Φ'_n is strictly strategy-proof for all n sufficiently large. Finally, we clearly have that $||\Phi_n - \Phi'_n|| < \varepsilon$.

A.8 Envy-freeness in the large

In this section we show that, under a continuity condition, strategy-proof in the large implies envy-free in the large. This result, which will be used in the proof of Theorem 4, has some independent interest as it provides a converse to Theorem 1 in Azevedo and Budish (2019).

A sequence $\langle (Y_n, \Phi_n)\rangle_{n\in\mathbb{N}}$ of direct mechanism is *envy-free in the large* if, for each $\varepsilon > 0$ and $m \in M^0(T)$, there exists $N \in \mathbb{N}$ and $\delta > 0$ such that

$$u(t, \gamma_n(t,\pi)) \geq u(t, \gamma_n(t', \pi + (1_t - 1_{t'})/(n-1))) - \varepsilon.$$

for each $n \geq N$, $t, t' \in T$ and $\pi \in M_{n-1}(T)$ with $\pi(t') > 0$ and $||\pi - m|| \leq \delta$. The difference between envy-free and envy-free in the large is that, in the latter, (1) its requirement is only for sufficiently large n, (2) the utility from own allocation must exceeds the utility from the allocation of other agent minus ε and (3) the distribution of reports must be close to a distribution with full support.

For each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $||x|| = \max_{1 \leq i \leq n} |x_i|$. Define, for each $\varepsilon > 0$,

$\tilde{\omega}_n(\varepsilon) = \sup\{||\gamma_n(t, \pi) - \gamma_n(t, \pi')|| : t \in T, \pi, \pi' \in M_{n-1}(T)$ and $||\pi - \pi'|| \leq \varepsilon\}$ and

$\omega_n(\varepsilon) = \sup\{|u(t, x) - u(t, x')| : t \in T, x, x' \in X$ and $||x - x'|| \leq \tilde{\omega}_n(\varepsilon)\}$.

We consider the case where $\lim_{\varepsilon \to 0} \sup_n \tilde{\omega}_n(\varepsilon) = 0$; the interpretation of this condition is that the effect of an action of any single agent on the outcome of any agent is negligible. Since u is continuous, it follows that $\lim_{\varepsilon \to 0} \sup_n \tilde{\omega}_n(\varepsilon) = 0$ implies that $\lim_{\varepsilon \to 0} \sup_n \omega_n(\varepsilon) = 0$.

Lemma 4. *If $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ is strategy-proof in the large and $\lim_{\varepsilon \to 0} \sup_n \tilde{\omega}_n(\varepsilon) = 0$, then $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ is envy-free in the large.*

Proof. Let $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ be strategy-proof in the large and consider $\varepsilon > 0$ and $m \in M^0(T)$. Let $\alpha = 1/4$ and $\delta > 0$ be such that $\delta < \min_{t \in T} m(t)$, in which case $||\pi - m|| \leq \delta$ implies that $\pi(t) > 0$ for each $t \in T$, and

$$2 \sup_n \omega_n(2\delta) < \alpha\varepsilon.$$

Fix $t, t' \in T$. We have that

$$\begin{aligned}
&u(t, \phi^n(t', m)) - u(t, \phi^n(t, m)) = \\
&\sum_{\pi \in M_{n-1}(T) \setminus \overline{B}_\delta(m)} m^{n-1} \circ e_{n-1}^{-1}(\pi) \Big(u(t, \gamma_n(t', \pi)) - u(t, \gamma_n(t, \pi)) \Big) + \\
&\sum_{\pi \in M_{n-1}(T) \cap \overline{B}_\delta(m)} m^{n-1} \circ e_{n-1}^{-1}(\pi) \Big(u(t, \gamma_n(t', \pi + (1_t - 1_{t'})/(n-1))) - u(t, \gamma_n(t, \pi)) \Big) + \\
&\sum_{\pi \in M_{n-1}(T) \cap \overline{B}_\delta(m)} m^{n-1} \circ e_{n-1}^{-1}(\pi) \Big(u(t, \gamma_n(t', \pi)) - u(t, \gamma_n(t', \pi + (1_t - 1_{t'})/(n-1))) \Big).
\end{aligned}$$

Lemma 4 in Kalai (2004) implies that

$$m^{n-1} \circ e_{n-1}^{-1}(M_{n-1}(T) \setminus \overline{B}_\delta(m)) \leq 2|T| e^{-2\delta^2 n},$$

where $\overline{B}_\delta(m) = \{\pi \in M_{n-1}(T) : ||\pi - m|| \leq \delta\}$ is the closed ball of radius δ around m. Thus,

$$\begin{aligned}
&\sum_{\pi \in M_{n-1}(T) \cap \overline{B}_\delta(m)} m^{n-1} \circ e_{n-1}^{-1}(\pi) \Big(u(t, \gamma_n(t', \pi + (1_t - 1_{t'})/(n-1))) - u(t, \gamma_n(t, \pi)) \Big) < \\
&u(t, \phi^n(t', m)) - u(t, \phi^n(t, m)) + \omega_n(1/(n-1)) + 4 \max_{\tilde{t} \in T, x_0 \in X_0} |u(\tilde{t}, x_0)| |T| e^{-2\delta^2 n}.
\end{aligned}$$

Since $\langle (Y_n, \Phi_n) \rangle_{n\in\mathbb{N}}$ is strategy-proof in the large, $\omega_n(1/(n-1)) \to 0$ and $e^{-2\delta^2 n} \to 0$, there is $N_1 \in \mathbb{N}$ such that

$$\sum_{\pi \in M_{n-1}(T)\cap \overline{B}_\delta(m)} m^{n-1} \circ e_{n-1}^{-1}(\pi) \Big(u(t, \gamma_n(t', \pi + (1_t - 1_{t'})/(n-1))) - u(t, \gamma_n(t, \pi)) \Big) < \alpha\varepsilon.$$

For each $n \geq N_1$, let $\pi_n \in \overline{B}_\delta(m)$ and

$$e_n = u(t, \gamma_n(t', \pi_n + (1_t - 1_{t'})/(n-1))) - u(t, \gamma_n(t, \pi_n)).$$

Then

$$\begin{aligned} & e_n m^{n-1} \circ e_{n-1}^{-1}(\overline{B}_\delta(m)) = \\ & \sum_{\pi \in M_{n-1}(T)\cap \overline{B}_\delta(m)} m^{n-1} \circ e_{n-1}^{-1}(\pi) \Big(e_n - u(t, \gamma_n(t', \pi + (1_t - 1_{t'})/(n-1))) + u(t, \gamma_n(t, \pi)) \Big) + \\ & \sum_{\pi \in M_{n-1}(T)\cap \overline{B}_\delta(m)} m^{n-1} \circ e_{n-1}^{-1}(\pi) \Big(u(t, \gamma_n(t', \pi + (1_t - 1_{t'})/(n-1))) - u(t, \gamma_n(t, \pi)) \Big) < \\ & 2\omega_n(2\delta) + \alpha\varepsilon < 2\alpha\varepsilon. \end{aligned}$$

Hence,

$$e_n < \frac{2\alpha\varepsilon}{m^{n-1} \circ e_{n-1}^{-1}(\overline{B}_\delta(m))} \leq \frac{2\alpha\varepsilon}{1 - 2|T|e^{-2\delta^2 n}}.$$

Thus, there is $N_2 \in \mathbb{N}$ such that $N_2 > N_1$ and $e_n < 3\alpha\varepsilon$ for each $n \geq N_2$.

Finally, for each $n \geq N_2$ and $\pi \in M_{n-1}(T) \cap \overline{B}_\delta(m)$,

$$u(t, \gamma_n(t', \pi + (1_t - 1_{t'})/(n-1))) - u(t, \gamma_n(t, \pi)) \leq e_n + 2\omega_n(2\delta) < 4\alpha\varepsilon = \varepsilon.$$

Thus, $\langle (Y_n, \Phi_n) \rangle_{n\in\mathbb{N}}$ is envy-free in the large. $\square$

A.9 Proof of Theorem 4

The proof of Theorem 4 requires the following lemma.

Lemma 5. *If $\gamma_n \to \gamma$ and $\gamma \in \mathcal{L}$ then* $\lim_{\varepsilon\to 0} \sup_n \tilde{\omega}_n(\varepsilon) = 0$.

Proof. Note first that it is enough to show that, for each $\eta > 0$, there exists $\bar{\varepsilon} > 0$ and $N \in \mathbb{N}$ such that $\sup_{n\geq N} \tilde{\omega}_n(\varepsilon) < \eta$ for each $0 < \varepsilon \leq \bar{\varepsilon}$. Indeed, let $\varepsilon^* = \min\{\bar{\varepsilon}, \frac{1}{N}\} > 0$. Thus, for each $\varepsilon \leq \varepsilon^*$ and $n < N$,

$$||\pi - \pi'|| \geq \frac{1}{n-1} > \frac{1}{N-1} > \varepsilon^* \geq \varepsilon$$

for each $\pi, \pi' \in M_{n-1}(T)$ with $\pi \neq \pi'$, implying that $\tilde{\omega}_n(\varepsilon) = 0$. Thus, for each $0 < \varepsilon \leq \varepsilon^*$, $\sup_n \tilde{\omega}_n(\varepsilon) = \sup_{n \geq N} \tilde{\omega}_n(\varepsilon) < \eta$.

Let $\eta > 0$ be given. Since γ is continuous, let $\varepsilon' > 0$ be such that $\varepsilon' < \eta$ and $||\gamma(t,\pi) - \gamma(t,\pi')|| < \eta/2$ whenever $t \in T$ and $\pi, \pi' \in M(T)$ are such that $||\pi - \pi'|| < \varepsilon'$. Let $\bar{\varepsilon} = \varepsilon'/2$, $N \in \mathbb{N}$ be such that $d(\text{graph}(\gamma_n), \text{graph}(\gamma)) < \varepsilon'/4$ for each $n \geq N$, $0 < \varepsilon \leq \bar{\varepsilon}$ and $n \geq N$.

Let $t \in T$ and $\pi, \pi' \in M_{n-1}(T)$ be such that $||\pi - \pi'|| \leq \varepsilon \leq \bar{\varepsilon} = \varepsilon'/2$. Then there are $\tilde{\pi}$ and $\tilde{\pi}'$, both in $M(T)$, such that $||\pi - \tilde{\pi}|| < \varepsilon'/4$, $||\pi' - \tilde{\pi}'|| < \varepsilon'/4$, $||\gamma_n(t,\pi) - \gamma(t,\tilde{\pi})|| < \varepsilon'/4$ and $||\gamma_n(t,\pi') - \gamma(t,\tilde{\pi}')|| < \varepsilon'/4$. Thus, $||\tilde{\pi} - \tilde{\pi}'|| < \varepsilon'/2 + 2\varepsilon'/4 = \varepsilon'$ and, hence, $||\gamma(t,\tilde{\pi}) - \gamma(t,\tilde{\pi}')|| < \eta/2$. This then implies that

$$||\gamma_n(t,\pi) - \gamma_n(t,\pi')|| < \frac{2\varepsilon'}{4} + \frac{\eta}{2}.$$

Thus, $\sup_{n \geq N} \tilde{\omega}_n(\varepsilon) \leq \frac{\varepsilon' + \eta}{2} < \eta$. □

We now turn to the proof of Theorem 4. Let γ, $\langle \gamma_n \rangle_{n \in \mathbb{N}}$ and $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ be as in the statement of the theorem and let $t, t' \in T$ and $\pi \in M^0(T)$ be given. By Lemmas 4 and 5, it follows that $\langle (Y_n, \Phi_n) \rangle_{n \in \mathbb{N}}$ is envy-free in the large. Fix $\varepsilon > 0$ and let $N \in \mathbb{N}$ and $\delta > 0$ be given as in the definition of envy-free in the large; we may assume that $\delta < \varepsilon$, $||\gamma(t',\tilde{\pi}) - \gamma(t',\pi)|| < \varepsilon$ whenever $||\pi - \tilde{\pi}|| < 3\delta$ and $\delta < \min_{t \in T} \pi(t)$, the latter implying that $\min_{t \in T} \pi'(t) > 0$ for each $\pi' \in M(T)$ with $||\pi' - \pi|| \leq \delta$.

Let $N' \in \mathbb{N}$ be such that $N' \geq N$, $1/(n-1) < \delta$ and $d(\text{graph}(\gamma_n), \text{graph}(\gamma)) < \delta$ for each $n \geq N'$. Thus, for each $n \geq N'$, there exists $\pi' \in M_{n-1}(T)$ and $\tilde{\pi} \in M(T)$ such that $||\pi - \pi'|| < \delta$, $||\pi' + \frac{1_t - 1_{t'}}{n-1} - \tilde{\pi}|| < \delta$, $||\gamma_n(t,\pi') - \gamma(t,\pi)|| < \delta$ and $||\gamma_n(t', \pi' + \frac{1_t - 1_{t'}}{n-1}) - \gamma(t',\tilde{\pi})|| < \delta$. Then $||\pi - \tilde{\pi}|| < 2\delta + \frac{1}{n-1} < 3\delta$ and, therefore, $||\gamma(t',\tilde{\pi}) - \gamma(t',\pi)|| < \varepsilon$. Thus,

$$\left\| \gamma_n \left(t', \pi' + \frac{1_t - 1_{t'}}{n-1} \right) - \gamma(t',\pi) \right\| < \varepsilon + \delta < 2\varepsilon$$

Letting $n \geq N'$ and, for each $\eta > 0$,

$$\omega(\eta) = \sup\{|u(\hat{t}, m) - u(\hat{t}, m')| : \hat{t} \in T, m, m' \in X \text{ such that } ||m - m'|| \leq \eta\},$$

it follows that,

$$u(t, \gamma(t, \pi)) \geq u(t, \gamma_n(t, \pi')) - \omega(\delta) \geq$$
$$u\left(t, \gamma_n\left(t', \pi' + \frac{1_t - 1_{t'}}{n-1}\right)\right) - \varepsilon - \omega(\delta) \geq u(t, \gamma(t', \pi)) - \varepsilon - 2\omega(2\varepsilon).$$

Letting $\varepsilon \to 0$, it follows that $u(t, \gamma(t, \pi)) \geq u(t, \gamma(t', \pi))$.

Finally, for each $t, t' \in T$ and $\pi \in M(T)$, there exists $\pi_k \to \pi$ such that $\pi_k \in M^0(T)$ for each $k \in \mathbb{N}$. The above argument implies that $u(t, \gamma(t, \pi_k)) \geq u(t, \gamma(t', \pi_k))$ for each $k \in \mathbb{N}$ and, hence, we obtain from the continuity of γ that $u(t, \gamma(t, \pi)) \geq u(t, \gamma(t', \pi))$. Thus, γ is strategy-proof.

A.10 A continuous auction perturbation

The auction in Section 5.3 has a discontinuous limiting reduced-form because the cutoff can jump at distributions on the boundary of $M(T)$. This appendix describes a perturbation that removes this discontinuity. The idea is to add a small number of fake reports of each type. These fake reports enter the auction used to determine the cutoff, but units assigned to fake reports are discarded and are not allocated to real bidders.

Fix $\delta > 0$. For each n, let $N_n = n + m\lfloor \delta n \rfloor$ and $g_n = \lfloor qn \rfloor$. For each type $r \in T = \{1, \ldots, m\}$, add $\lfloor \delta n \rfloor$ fake reports equal to r. Let

$$\chi = \frac{1}{m} \sum_{r=1}^{m} 1_r$$

be the uniform distribution on T.

Consider the following anonymous finite mechanism: given the reports of the n real bidders, form the augmented profile consisting of the real reports and the fake reports. The auction rule described in Section 5.3 is applied to this augmented profile with g_n units. If a fake report is selected, the corresponding unit is discarded. The outcome of the mechanism is the allocation and payments of the real bidders only.[15]

[15]Thus fake reports may affect the cutoff and may absorb units in the perturbed auction, but they do not receive goods in the real allocation.

If a real bidder reports t and the empirical distribution of the other real bidders' reports is $\pi \in M_{n-1}(T)$, then the empirical distribution of all reports, including fake reports, is

$$\bar{\pi}_n(t,\pi) = \frac{n}{N_n}\left[\frac{1}{n}1_t + \left(1 - \frac{1}{n}\right)\pi\right] + \frac{m\lfloor \delta n \rfloor}{N_n}\chi.$$

Let $\lambda_n = \frac{g_n}{N_n}$. For $\rho \in M(T)$ and $r \in \{1,\dots,m+1\}$, write

$$F_r(\rho) = \sum_{s\in T: s\geq r} \rho(s),$$

with the convention that $F_{m+1}(\rho) = 0$. Define

$$p_n^\delta(t,\pi) = \min\left\{r \in \{2,\dots,m+1\} : \lambda_n \geq F_r(\bar{\pi}_n(t,\pi))\right\}$$

and

$$\alpha_n^\delta(t,\pi) = \frac{\lambda_n - F_{p_n^\delta(t,\pi)}(\bar{\pi}_n(t,\pi))}{\bar{\pi}_n(t,\pi)(p_n^\delta(t,\pi)-1)}.$$

By the definition of $p_n^\delta(t,\pi)$, the denominator is positive and $\alpha_n^\delta(t,\pi) \in [0,1)$.

The reduced-form of the perturbed auction is

$$\gamma_n^\delta(t,\pi) = \begin{cases} (1-\alpha_n^\delta(t,\pi))1_{(1,p_n^\delta(t,\pi))} + \alpha_n^\delta(t,\pi)1_{(1,p_n^\delta(t,\pi)-1)} & \text{if } t \geq p_n^\delta(t,\pi), \\ (1-\alpha_n^\delta(t,\pi))1_{(0,0)} + \alpha_n^\delta(t,\pi)1_{(1,p_n^\delta(t,\pi)-1)} & \text{if } t = p_n^\delta(t,\pi)-1, \\ 1_{(0,0)} & \text{otherwise.} \end{cases}$$

We now define the limiting reduced-form. Let

$$\lambda = \frac{q}{1+m\delta}$$

and, for each $\pi \in M(T)$, define

$$\bar{\pi} = \frac{1}{1+m\delta}\pi + \frac{m\delta}{1+m\delta}\chi.$$

Since $\delta > 0$, every coordinate of $\bar{\pi}$ is strictly positive. Define

$$p^\delta(\pi) = \min\left\{r \in \{2,\dots,m+1\} : \lambda \geq F_r(\bar{\pi})\right\}$$

and

$$\alpha^\delta(\pi) = \frac{\lambda - F_{p^\delta(\pi)}(\bar{\pi})}{\bar{\pi}(p^\delta(\pi)-1)}.$$

Finally, define $\gamma^\delta : T \times M(T) \to X$ by

$$\gamma^\delta(t,\pi) = \begin{cases} (1-\alpha^\delta(\pi))1_{(1,p^\delta(\pi))} + \alpha^\delta(\pi)1_{(1,p^\delta(\pi)-1)} & \text{if } t \geq p^\delta(\pi), \\ (1-\alpha^\delta(\pi))1_{(0,0)} + \alpha^\delta(\pi)1_{(1,p^\delta(\pi)-1)} & \text{if } t = p^\delta(\pi)-1, \\ 1_{(0,0)} & \text{otherwise.} \end{cases}$$

The reduced-form mechanism γ^δ is strategy-proof by the same cutoff argument used in Section 5.3. We now show that γ^δ is continuous and that $\gamma_n^\delta \to \gamma^\delta$.

Claim 3. *The reduced-form mechanism γ^δ is continuous.*

Proof. Fix $t \in T$ and let $\pi^k \to \pi$ in $M(T)$. Write

$$\bar{\pi}^k = \frac{1}{1+m\delta}\pi^k + \frac{m\delta}{1+m\delta}\chi.$$

Then $\bar{\pi}^k \to \bar{\pi}$, and every coordinate of $\bar{\pi}$ is strictly positive. Let $p = p^\delta(\pi)$ and $\alpha = \alpha^\delta(\pi)$. By definition,

$$F_p(\bar{\pi}) \leq \lambda < F_{p-1}(\bar{\pi}).$$

First suppose that

$$F_p(\bar{\pi}) < \lambda < F_{p-1}(\bar{\pi}).$$

Then, for all sufficiently large k,

$$F_p(\bar{\pi}^k) < \lambda < F_{p-1}(\bar{\pi}^k),$$

so $p^\delta(\pi^k) = p$. Therefore

$$\alpha^\delta(\pi^k) = \frac{\lambda - F_p(\bar{\pi}^k)}{\bar{\pi}^k(p-1)} \to \frac{\lambda - F_p(\bar{\pi})}{\bar{\pi}(p-1)} = \alpha.$$

It follows that $\gamma^\delta(t,\pi^k) \to \gamma^\delta(t,\pi)$.

It remains to consider the boundary case

$$F_p(\bar{\pi}) = \lambda < F_{p-1}(\bar{\pi}).$$

In this case $\alpha = 0$. Since every coordinate of $\bar{\pi}$ is positive,

$$F_{p+1}(\bar{\pi}) = F_p(\bar{\pi}) - \bar{\pi}(p) < \lambda,$$

where the case $p = m+1$ cannot occur because $\lambda > 0$. Hence, for all sufficiently large k, the cutoff $p^\delta(\pi^k)$ is either p or $p+1$.

Along any subsequence such that $p^\delta(\pi^k) = p$, we have

$$\alpha^\delta(\pi^k) = \frac{\lambda - F_p(\bar{\pi}^k)}{\bar{\pi}^k(p-1)} \to 0.$$

Along any subsequence such that $p^\delta(\pi^k) = p+1$, we have

$$\alpha^\delta(\pi^k) = \frac{\lambda - F_{p+1}(\bar{\pi}^k)}{\bar{\pi}^k(p)} \to \frac{\lambda - F_{p+1}(\bar{\pi})}{\bar{\pi}(p)} = 1.$$

In both cases, the resulting lottery converges to the same limit: bidders with reports at least p receive the object and pay p, while bidders with reports below p receive no object and pay zero. This is exactly $\gamma^\delta(t,\pi)$. Therefore, $\gamma^\delta(t,\pi^k) \to \gamma^\delta(t,\pi)$. □

Claim 4. *The reduced-forms γ_n^δ converge to γ^δ.*

Proof. We prove convergence of graphs. Since T is finite, it is enough to verify the following two properties.

First, let $n_k \to \infty$, let $\pi_{n_k} \in M_{n_k-1}(T)$, and suppose that $\pi_{n_k} \to \pi$. Then

$$\bar{\pi}_{n_k}(t,\pi_{n_k}) \to \bar{\pi} \qquad \text{and} \qquad \lambda_{n_k} \to \lambda.$$

The argument in the proof of Claim 3, with λ_{n_k} in place of λ, implies that

$$\gamma_{n_k}^\delta(t,\pi_{n_k}) \to \gamma^\delta(t,\pi).$$

Thus every limit point of the graphs of γ_n^δ belongs to the graph of γ^δ.

Second, fix $\pi \in M(T)$. Since the grids $M_{n-1}(T)$ are dense in $M(T)$, there exists a sequence $\pi_n \in M_{n-1}(T)$ such that $\pi_n \to \pi$. Applying the previous paragraph gives

$$\gamma_n^\delta(t,\pi_n) \to \gamma^\delta(t,\pi).$$

Thus every point of the graph of γ^δ is the limit of points in the graphs of γ_n^δ.

These two properties imply $\mathrm{graph}(\gamma_n^\delta) \to \mathrm{graph}(\gamma^\delta)$ in the Hausdorff metric. Hence $\gamma_n^\delta \to \gamma^\delta$. □

A.11 Finite-Population Strategy-Proofness of the Auction Example

Let γ_n be the reduced-form of the auction in Section 5.3. We show that γ_n is strategy-proof.

Let $\hat{\pi} = \frac{1}{n}1_t + \left(1 - \frac{1}{n}\right)\pi$, $\tilde{\pi} = \frac{1}{n}1_{t'} + \left(1 - \frac{1}{n}\right)\pi$, $p = p_n(\hat{\pi})$, $\alpha = \alpha_n(\hat{\pi})$, $p' = p_n(\tilde{\pi})$ and $\alpha' = \alpha_n(\tilde{\pi})$. Note that $\tilde{\pi} = \hat{\pi} + \frac{1}{n}(1_{t'} - 1_t)$.

Suppose first that $t' > t$. If $t \geq p$, then $\sum_{\hat{t}\geq p} \tilde{\pi}(\hat{t}) = \sum_{\hat{t}\geq p} \hat{\pi}(\hat{t})$ and $\tilde{\pi}(p-1) = \hat{\pi}(p-1)$; hence $p = p'$ and $\alpha = \alpha'$. Thus, $u(t, \gamma_n(t, \pi)) = u(t, \gamma_n(t', \pi))$.

If $t < p$, then $u(t, \gamma_n(t, \pi)) = 0$, $p' \geq p$ and $u(t, \gamma_n(t', \pi)) \leq 0$.

Suppose next that $t' < t$. If $t < p$, then $u(t, \gamma_n(t, \pi)) = 0 = u(t, \gamma_n(t', \pi))$. If $t \geq p$ and $t' \geq p$, then $\sum_{\hat{t}\geq p} \tilde{\pi}(\hat{t}) = \sum_{\hat{t}\geq p} \hat{\pi}(\hat{t})$ and $\tilde{\pi}(p-1) = \hat{\pi}(p-1)$; hence $p = p'$ and $\alpha = \alpha'$. Thus, $u(t, \gamma_n(t, \pi)) = u(t, \gamma_n(t', \pi))$.

Thus, consider $t' < p \leq t$. For each $\theta \in T$ such that $t' < \theta < p$, we have that $q_n < \sum_{\hat{t}\geq\theta} \hat{\pi}(\hat{t})$ by the definition of p and $\sum_{\hat{t}\geq\theta} \tilde{\pi}(\hat{t}) = \sum_{\hat{t}\geq\theta} \hat{\pi}(\hat{t}) - \frac{1}{n}$ since $\tilde{\pi} = \hat{\pi} + \frac{1}{n}(1_{t'} - 1_t)$. Furthermore, $\sum_{\hat{t}\geq t'} \tilde{\pi}(\hat{t}) = \sum_{\hat{t}\geq t'} \hat{\pi}(\hat{t}) > q_n$. Thus, $t' < p' \leq p$.

Thus $u(t, \gamma_n(t', \pi)) = 0 \leq u(t, \gamma_n(t, \pi))$ unless $t' = p' - 1$ and $q_n > \sum_{\hat{t}\geq p'} \tilde{\pi}(\hat{t})$. Thus, assume that $t' = p' - 1$ and $q_n > \sum_{\hat{t}\geq p'} \tilde{\pi}(\hat{t})$.

If $p' < p$, then

$$q_n - \sum_{\hat{t}\geq p'} \tilde{\pi}(\hat{t}) = q_n - \sum_{\hat{t}\geq p'} \hat{\pi}(\hat{t}) + \frac{1}{n} < \frac{1}{n}.$$

This, together with $nq_n \in \mathbb{N}$ and $n\sum_{\hat{t}\geq p'} \tilde{\pi}(\hat{t}) \in \mathbb{N}$, implies that $q_n - \sum_{\hat{t}\geq p'} \tilde{\pi}(\hat{t}) = 0$, contradicting $q_n > \sum_{\hat{t}\geq p'} \tilde{\pi}(\hat{t})$. Thus, it follows that $p = p'$. Hence, $u(t, \gamma_n(t, \pi)) - u(t, \gamma_n(t', \pi)) = (1 - \alpha')(t - t') \geq 0$.