Existence of Stable Matchings in Large Economies with Externalities*

Guilherme Carmona† Krittanai Laohakunakorn‡

University of Surrey University of Surrey

June 14, 2023

Abstract

We extend the two-sided many-to-one matching setting of Che, Kim, and Kojima (2019) by allowing workers' preferences to depend on the matching itself. In finite markets, complementarities and externalities are both known to cause problems for the existence of stable matchings. Che, Kim, and Kojima (2019) find that in a large market with a continuum of workers, a stable matching exists even when the firms' preferences exhibit complementarities. In the same spirit, we show that as long as workers' preferences depend on the matching in a continuous way, a stable matching exists in the presence of both complementarities and externalities.

*We wish to thank Federico Echenique, Michael Greinecker and two anonymous referees for helpful comments. Any remaining errors are, of course, ours.

†Address: University of Surrey, School of Economics, Guildford, GU2 7XH, UK; email: g.carmona@surrey.ac.uk.

‡Address: University of Surrey, School of Economics, Guildford, GU2 7XH, UK; email: k.laohakunakorn@surrey.ac.uk.

1 Introduction

In a recent paper, Che, Kim, and Kojima (2019) (CKK henceforth) solve a longstanding problem in matching theory concerning the existence of a stable matching in a two-sided many-to-one matching market. A typical example of such market is a labor market where firms are matched with workers; this is a many-to-one matching market since each firm can hire multiple workers. It is well known that in such markets, when there are finitely many agents, the existence of a stable matching is not guaranteed when the firms' preferences exhibit complementarities. Since it is natural that a firm may wish to hire workers with complementary skills, for example, the nonexistence of a stable matching in such situations is a serious problem for the applicability of matching theory.

Another challenge for matching theory — one that has recently been tackled in the context of school choice by Cox, Fonseca, and Pakzad-Hurson (2022) and Leshno (2022) — is that agents may have preferences that depend on the matching itself. Beyond the school choice problem, externalities are also a natural feature of more general many-to-one matching markets. For example, a worker may prefer to work for a firm that hires other similarly qualified workers. Like complementarities, externalities can cause problems for the existence of a stable matching in finite economies. Indeed, Pycia and Yenmez (2022) show that in finite markets with externalities, substitutability is a necessary condition for the existence of stable matchings in a maximal domain sense.

CKK solve the existence problem for complementarities by considering matching markets that are large in the sense that there is a continuum of workers. They provide an existence result that is general enough to allow firms' preferences to exhibit complementarities; however, externalities are not allowed on either side of the market. Our aim is to extend CKK's existence result to allow workers' preferences to depend on the matching itself, i.e. to provide a general existence result that allows for both complementarities and externalities. In particular, we show that in the CKK economy, a stable matching exists even when there are externalities on the non-atomic side of the market, as long as preferences depend on the matching in a continuous way.

As well as being key for the existence of stable matchings, non-atomicity is also important conceptually since stability is easy to define when preferences exhibit externalities only on the non-atomic side of the market. Indeed, each worker is negligible and thus her decisions have no impact on the matching. In this case, each worker can believe that she is the only one being hired by some firm and, because she is of negligible size, there is no change to the matching. The definition of stability we use is thus analogous to the one in Fisher and Hafalir (2016) whose stability notion assumes that there is no change to the matching in response to the formation of a blocking coalition; in a setting with non-atomic workers, this notion of stability can be justified, for example, by assuming that each worker who is considering a job offer does not have sufficient knowledge about (any positive measure of) other workers who also received job offers. Moreover, our notion of stability coincides with the one in CKK in the case where workers' preferences are strict and do not depend on the matching. See Section 2.4 for a more detailed discussion of these issues.[1]

In CKK's model, a finite number of firms are matched with a continuum of workers. A matching specifies for each firm a workforce to which the firm is matched such that, for each worker type, the total measure of the workers of that type employed by the firms (including by a dummy firm used to represent unemployment) equals the total measure of that type of workers in the population. We extend CKK's model by allowing workers' preferences to depend on the matching itself. In addition, unlike CKK, we allow workers to be indifferent between firms; in fact, indifferences are unavoidable when workers' preferences depend on the matching in a non-trivial and continuous way.

[1]Several papers address the definition and the existence of stable matchings in economies with finitely many individuals and externalities; these include Sasaki and Toda (1996), Dutta and Massó (1997), Ma (2001), Echenique and Yenmez (2007), Hafalir (2008), Mumcu and Saglam (2010), Bando (2012), Pycia (2012), Fisher and Hafalir (2016) and Pycia and Yenmez (2022). Their results are formally unrelated to our and, in particular, use specific conditions on how preferences depend on externalities to obtain the existence of stable matchings which we dispense with; in contrast, some of our assumptions, such as the convexity of firms' preferences, are not needed in their results.

We show that if the firms' preferences are described by choice correspondences that are upper-hemicontinuous with nonempty, compact and convex values as in CKK, and if the workers' preferences are complete, transitive and continuous, then a stable matching exists even when workers' preferences are allowed to depend on the matching. We then show that this results implies the existence result in CKK; it also implies the existence of stable matchings in the setting of Azevedo and Leshno (2016) without requiring strict preferences for the firms.

In the specific setting of school choice, Cox, Fonseca, and Pakzad-Hurson (2022) and Leshno (2022) also consider large matching markets with a continuum of students and establish the existence of a stable matching when students' preferences may depend on the matching. Although their setting imposes more restrictions on schools' preferences than we do, they do not impose a continuity requirement on individual students' preferences, but instead assume a *diversity of preferences* assumption which ensures that the aggregate demand for schools is continuous. Assuming that the students' type space is separable, which they do not require, we show that a stable matching exists in a general setting under the diversity of preferences assumption. In particular, this result does not require individual students' preferences to depend continuously on the matching, allows for general preferences for the colleges which can exhibit complementarities, and for students' preferences that depend on the entire matching and not just on some of its summary statistics.

The approach of Cox, Fonseca, and Pakzad-Hurson (2022) and Leshno (2022) relies on the specific (cutoff) characterization of stable matchings which holds in their setting but not in general. Thus, our approach is closest to that of CKK. The main contribution of our approach is that it allows us to accommodate workers being indifferent between firms; once this is accomplished, adding externalities under a variety of different assumptions is then relatively easy. As we now explain, accommodating indifference is a challenge because it causes the measure of workers available to a firm to change discontinuously with the matching.

Given a matching, the measure of a worker type available to a firm is the measure of that worker type matched with the firm or some other firm that is strictly worse from the perspective of the worker. In CKK, the correspondence that maps each profile of available workers (one for each firm) to a new profile of available workers resulting from the firms' optimal choices is upper-hemicontinuous and plays a crucial role in the fixed point argument that establishes existence. With externalities, however, the measure of available workers may change discontinuously as the matching changes. This is because, as the matching changes, workers that previously preferred to match with a given firm may become indifferent, causing a jump in the measure of workers available to that firm. Indeed, the problem of discontinuity arises because workers are allowed to be indifferent between firms. Thus, the correspondence used by CKK, suitably generalized to allow for externalities, will fail to be upper-hemicontinuous even when the preferences are continuous.

We address the above discontinuity issue using ideas from the discontinuous games literature that followed Reny (1999) along the lines we used in Carmona (2011). We establish our result in two steps. First, we show that when the set of worker types is finite, a stable matching exists. To deal with the issue of ties, we define an aggregate choice correspondence for the workers as well as for the firms. To ensure that the correspondences are upper hemicontinuous, we use a continuous approximation to the measure of available workers. Using the Kakutani fixed point theorem, we establish the existence of a stable matching when the set of worker types is finite. The final step in our argument is to establish the existence of a stable matching for a general distribution. We approximate such distribution with a sequence of finitely supported distributions, and we show that the limit of the sequence of stable matchings for the finitely supported distributions is a stable matching for the limit distribution.

Our result implies that as long as preferences depend on the matching in a continuous way, externalities on the non-atomic side of the market cause no problems for existence. However, we show in Section 4.2 that a stable matching may fail to exist when firms' preferences are allowed to depend on the matching. Partly motivated by this issue, in Carmona and Laohakunakorn (2023), we establish the existence of a stable matching in a setting with a continuum of firms as well as a continuum of workers. There, we find that as long as preferences are continuous (convexity is not needed), a stable matching exists even when there are externalities on both sides of the market.[2]

Our existence result also weakens the assumptions that CKK place on workers. Specifically, each worker is described by her type, in a way that workers of the same type have the same preferences, and their population is described by a probability measure on this set of types. CKK assume that workers' type space is a compact metric space whereas we assume only that it is a separable metric space and that the type distribution is tight.[3] Our result thus allows for the case where the workers' type space is $\mathbb{R}$ and the type distribution is the normal distribution. More importantly, this extra generality allows us to obtain CKK's existence result from ours.[4]

In summary, the contributions of this paper consist of establishing the existence of stable matchings in:

1. CKK's setting extended by allowing workers' preferences to depend on the matching,

2. Azevedo and Leshno's (2016) setting without strict preferences for the firms,

3. the setting of Cox, Fonseca, and Pakzad-Hurson (2022) and Leshno (2022) if workers' preferences depend continuously on the matching (and not just on some of its summary statistics) with neither strict preferences for the firms nor diversity of preferences being assumed, and

4. when workers' preferences do not depend continuously on the matching and

[2]In Carmona and Laohakunakorn (2023), we also allow occupational choice, i.e. each individual chooses the side of the market to which she belongs. As we show in that paper, the standard two-sided matching model is a special case of the model with occupational choice.

[3]Note that probability measures on compact metric spaces are trivially tight.

[4]A simple example of CKK's setting illustrates this issue: There is one firm, firm 1, in addition to the dummy firm denoted by $\emptyset$, workers' type space is $[0, 1]$ and the type distribution is uniform. Workers with type $\theta \leq 1/2$ strictly prefer firm $\emptyset$ to firm 1 whereas workers with type $\theta > 1/2$ strictly prefer firm 1 to firm $\emptyset$. In this example, workers' preferences are discontinuous in θ at $1/2$; we deal with this by considering $[0, 1/2) \cup (1/2, 1]$ as the workers' type space, which is then neither compact nor complete, but it is separable and the restriction of the type distribution to it is tight.

workers' type space is separable, under the diversity of preferences assumption and general preferences for the firms which need not be strict.

The paper is organized as follows. The framework we consider is introduced in Section 2, which also includes motivating examples, special cases and a discussion of the no blocking condition in our stability notion. Section 3 contains our existence result with continuous individual preferences and an outline of the main argument of its proof. In Section 4.1 we obtain the existence of stable matchings for the case where firms have explicit preferences as a corollary to our existence result. In Section 4.2, we provide a counterexample showing that our result cannot be generalized to allow firms' preferences to exhibit externalities. In Section 4.3, we consider extensions of our results, including to the case of diverse preferences. Omitted proofs are in the Appendix.

2 Model

2.1 Environment and matching

We consider CKK's model but with the addition of externalities in workers' preferences. There is a finite set $F = \{f_1, \ldots, f_n\}$ of firms and a mass of workers; let $\emptyset$ denote the null firm so that a worker matched with $\emptyset$ is unemployed and $\tilde{F} = F \cup \{\emptyset\}$.

Workers are described by their type so that workers with the same type have the same preferences. Let Θ be the set of workers' types, a separable metric space with metric d, and Borel σ-algebra Σ. Let $\bar{\mathcal{X}}$ be the set of all nonnegative measures X such that $X(\Theta) \leq 1$. The type distribution is a tight measure $G \in \bar{\mathcal{X}}$ such that $G(\Theta) = 1$.[5] A subpopulation is $X \in \bar{\mathcal{X}}$ such that $X(E) \leq G(E)$ for all $E \in \Sigma$; let $\mathcal{X}$ be the set of all subpopulations.[6] Feasibility requires each firm to be matched with a subpopulation so that it doesn't hire more workers of a type than those available in the population. More generally, a measure $\tilde{X} \in \mathcal{X}$ is a subpopulation of $X \in \mathcal{X}$,

[5] Recall that G is tight if, for each $\varepsilon > 0$, there exists a compact subset K of Θ such that $G(\Theta \setminus K) < \varepsilon$.

[6] We endow $\mathcal{X}$ with the weak convergence (narrow) topology (see Varadarajan (1958) for details).

denoted $\tilde{X} \sqsubset X$, if $\tilde{X}(E) \leq X(E)$ for all $E \in \Sigma$. The set of all subpopulations of X is denoted by $\mathcal{X}_X$.

Firms' preferences are described indirectly by a choice correspondence as in CKK following Alkan and Gale (2003) (see Section 4.1 for the case where firms have explicit preferences). Each firm $f \in F$ has a choice correspondence $C_f : \mathcal{X} \rightrightarrows \mathcal{X}$ satisfying: (i) $C_f(X) \subseteq \mathcal{X}_X$, (ii) for each $X, X' \in \mathcal{X}$ with $X \sqsubset X'$, if $C_f(X') \cap \mathcal{X}_X \neq \emptyset$, then $C_f(X) = C_f(X') \cap \mathcal{X}_X$, and (iii) C_f is closed (i.e. it has a closed graph) with nonempty and convex values. The choice set $C_f(X)$ describes the set of optimal workforces for firm f when constrained by the subpopulation X, i.e. when firm f cannot hire more that $X(E)$ workers with type in $E \in \Sigma$; thus, (i) requires that this constraint is satisfied in each optimal workforce. Condition (ii) is the revealed preference property in CKK which is automatically satisfied when firms have explicit preferences; in the case of explicit firm preferences, condition (iii) holds when preferences are continuous and convex.[7] Regarding the empty firm, let $C_\emptyset(X) = \{X\}$ for each $X \in \mathcal{X}$. If $C_f(X)$ is singleton for all $X \in \mathcal{X}$, then we sometimes abuse notation and write C_f as a function, e.g. $C_\emptyset(X) = X$ for all $X \in \mathcal{X}$.

Up to this point, the only difference between our setting and that of CKK is that we allow Θ to be separable whereas Θ is compact in CKK. A more important departure from CKK is that we allow workers' preferences to depend on the matching itself.

A matching is $M = (M_f)_{f \in \tilde{F}}$ such that $M_f \in \mathcal{X}$ for each $f \in \tilde{F}$ and $\sum_{f \in \tilde{F}} M_f = G$. Each worker θ has a complete, transitive and continuous preference relation $\succeq_\theta$ on $\tilde{F} \times \mathcal{X}^{n+1}$. We further require that workers' preferences vary continuously with θ so

[7] For an example that satisfies conditions (i)–(iii), let $\Theta = \{\theta_1, \theta_2\}$, $C_f(X) = \{(\min\{X(\theta_1), X(\theta_2)\}, \min\{X(\theta_1), X(\theta_2)\})\}$. Here, C_f is also the solution correspondence to the maximization of an explicit utility function $u_f(X) = \min\{X(\theta_1), X(\theta_2)\} - \frac{X(\theta_1)+X(\theta_2)}{4}$, i.e. $C_f(X) = \{X' \in \mathcal{X} : X' \text{ solves } \max_{\hat{X} \in \mathcal{X}} u_f(\hat{X}) \text{ subject to } \hat{X} \sqsubset X\}$. Since u_f is continuous and quasiconcave, C_f is closed with nonempty and convex values (in fact, C_f is a continuous function). As pointed out in CKK (p. 71), the firm "has a "complementary" (or more precisely, non-substitutable) preference in the sense that availability of one worker causes it to demand the other."

that

$$\Big\{(\theta, f, M, f', M') \in \Theta \times (\tilde{F} \times \mathcal{X}^{n+1})^2 : (f, M) \succ_\theta (f', M')\Big\} \text{ is open.}$$

The following examples illustrate this assumption and the departure of our framework from that of CKK. Let $\Theta = \{\theta\}$ and $\succeq_\theta$ be such that workers are indifferent between all firms, e.g. $\succeq_\theta$ is represented by $u_\theta \equiv 0$. In this example workers' preferences do not depend on the matching but nevertheless are not covered by those allowed in CKK since there workers have strict preferences. Clearly, workers' preferences are continuous in this example. For an example where workers' preferences depend on the matching, let $\Theta = \{\theta_1, \theta_2\}$ and $\succeq_{\theta_i}$ be represented by u_{θ_i} such that $u_{\theta_i}(f, M) = M_f(\theta_i)$ and $u_{\theta_i}(\emptyset, M) = -1$ for each $i = 1, 2$, $f \in F$ and $M \in \mathcal{X}^{n+1}$. Workers' preference are continuous and such that each worker prefers to be employed than to be unemployed and prefers the (non-null) firm which employs the most workers of its own type.

A general example where the continuity assumption is natural is when Θ is the space of bounded and continuous utility functions. Let S be a compact metric space, $s : \mathcal{X}^{n+1} \to S$ be a continuous function and $\Theta = C(\tilde{F} \times S)$, where $C(\tilde{F} \times S)$ denotes the space of bounded and continuous real-valued functions on $\tilde{F} \times S$ endowed with the sup norm, which is then a complete and separable metric space. The utility for a worker of type $\theta \in \Theta$ of choosing firm f when the matching is $M \in \mathcal{X}^{n+1}$ is then $\theta(f, s(M))$.[8] Then the continuity assumption on workers' preferences holds since $\{(\theta, f, M, f', M') \in \Theta \times (\tilde{F} \times \mathcal{X}^{n+1})^2 : (f, M) \succ_\theta (f', M')\} = \{(\theta, f, M, f', M') \in \Theta \times (\tilde{F} \times \mathcal{X}^{n+1})^2 : \theta(f, s(M)) > \theta(f', s(M'))\}$.

2.2 Stability

The choice correspondence C_f induces a preference relation $\succeq_f$ for each $f \in F$, known as the Blair order after Blair (1984), as follows. For each $X, Y \in \mathcal{X}$, let $X \vee Y$ (join)

[8]Workers' preferences are allowed to depend on the entire matching because $\mathcal{X}^{n+1}$ is homeomorphic to a subset of the compact metric space $S = [-1, 1]^{\tilde{F} \times \mathbb{N}}$, in which case we can take s to be such a homeomorphism. See Appendix A.5 for details.

be the supremum of X and Y; it satisfies

$$(X \vee Y)(E) = \sup_{D \in \Sigma} (X(E \cap D) + Y(E \cap D^c))$$

for each $E \in \Sigma$. We have that $X \vee Y \in \mathcal{X}$ and we then write $X \succeq_f Y$ if $X \in C_f(X \vee Y)$.

Let $D^{\preceq f}(M)$ be defined by

$$D^{\preceq f}(M)(E) = M_f(E) + \sum_{f' \in \tilde{F} \setminus \{f\}} M_{f'}(E \cap P(f, f', M))$$

for each $E \in \Sigma$, where

$$P(f, f', M) = \{\theta \in \Theta : (f, M) \succ_\theta (f', M)\}.$$

We have that $D^{\preceq f}(M)$ is the measure of workers assigned to firm f or worse under matching M; it measures the number of workers who are available to match with f.

A matching M is stable if

1. (Individual Rationality) For each $f \in F$, $M_f(P(\emptyset, f, M)) = 0$; and

2. (No Blocking Coalition) No $f \in F$ and $M'_f \in \mathcal{X}$ exist such that $M'_f \sqsubset D^{\preceq f}(M)$ and $M'_f \succ_f M_f$.

This notion of stability is exactly equal to the one in CKK in their setting where workers' preferences are strict and do not depend on the matching. Since, in contrast with CKK, our setting allows for indifference and externalities in the workers' preferences, certain details of the above definition of stability require some discussion; see Section 2.4 below. As we show in the following section, it coincides with the stability notion used in Cox, Fonseca, and Pakzad-Hurson (2022) and Leshno (2022) when applied to their environment which, as ours, also features indifference and externalities in workers' preferences.

2.3 Examples and special cases

A fully specified example of our environment is obtained by setting $F = \{f_1, f_2\}$, $\Theta = \{\theta_1, \theta_2\}$, $G(\theta_1) = G(\theta_2) = 1/2$,

$$C_f(X) = (\min\{X(\theta_1), X(\theta_2)\}, \min\{X(\theta_1), X(\theta_2)\})$$

for each $f \in F$ and $X \in \mathcal{X}$, and $\succeq_{\theta_i}$ be represented by u_{θ_i} such that, for each $i = 1, 2$, $f \in \tilde{F}$ and $M \in \mathcal{X}^{n+1}$,

$$u_{\theta_i}(f, M) = \begin{cases} M_f(\theta_i) & \text{if } f \in F, \\ -1 & \text{if } f = \emptyset. \end{cases}$$

The set of stable matchings for this example is easy to characterize; it equals the set of $M \in \mathcal{X}^{n+1}$ such that $M_\emptyset(\theta_i) = 0$, $M_{f_1}(\theta_i) = \alpha$ and $M_{f_2}(\theta_i) = 1/2 - \alpha$ for each $i = 1, 2$ and $\alpha \in \{0, 1/4, 1/2\}$. Indeed, if M is a stable matching, then $M_f \in C_f(M_f)$ (see Footnote 28 in CKK) and, hence, $M_f(\theta_1) = M_f(\theta_2)$ for each $f \in F$. Then $M_\emptyset(\theta_1) = M_\emptyset(\theta_2)$ since M is a matching and, in fact, $M_\emptyset(\theta_1) = M_\emptyset(\theta_2) = 0$ since M is stable. Thus, $M_{f_1}(\theta_1) = M_{f_1}(\theta_2) = \alpha$ and $M_{f_2}(\theta_1) = M_{f_2}(\theta_2) = 1/2 - \alpha$ for some $\alpha \in [0, 1/2]$. If $\alpha \notin \{0, 1/4, 1/2\}$ and $\alpha > 1/4$, then $D^{\preceq f_1}(M) = (1/2, 1/2)$ and $M'_{f_1} = (1/2, 1/2)$ is such that $M'_{f_1} \sqsubset D^{\preceq f_1}(M)$ and $M'_{f_1} \succ_{f_1} M_{f_1}$. An analogous argument using firm f_2 shows that M fails to be stable when $\alpha \notin \{0, 1/4, 1/2\}$ and $\alpha < 1/4$, thus concluding the proof of the necessity part of the claim. Regarding its sufficiency part, we have that $D^{\preceq f}(M) = (1/4, 1/4)$ for each $f \in F$ when $\alpha = 1/4$, $D^{\preceq f_1}(M) = (1/2, 1/2)$ and $D^{\preceq f_2}(M) = (0, 0)$ when $\alpha = 1$, and $D^{\preceq f_1}(M) = (0, 0)$ and $D^{\preceq f_2}(M) = (1/2, 1/2)$ when $\alpha = 0$. In either case, there is no $f \in F$ and $M'_f \in \mathcal{X}$ such that $M'_f \sqsubset D^{\preceq f}(M)$ and $M'_f \succ_f M_f$.

We next show how the setting of CKK and the school choice framework of Cox, Fonseca, and Pakzad-Hurson (2022) and Leshno (2022) can be captured in our environment.

In CKK, the set Θ of workers' types is a compact metric space and each worker has a strict preference P over $\tilde{F}$. Let $\mathcal{P}$ denote the (finite) set of all possible strict workers' preferences over $\tilde{F}$ and, for each $P \in \mathcal{P}$, let Θ_P denote the set of all worker types whose preference is given by P. CKK assume that $\Theta = \cup_{P \in \mathcal{P}} \Theta_P$ and, for each $P \in \mathcal{P}$, that the set Θ_P is measurable and $G(\partial \Theta_P) = 0$. Firms and their preferences are exactly as in our environment, and so is the definition of stability.

Set $W = \cup_{P \in \mathcal{P}} \text{int}(\Theta_P)$. Because $G(\partial \Theta_P) = 0$ for each $P \in \mathcal{P}$, it follows that $G(\Theta \setminus W) = 0$; thus, only a null set of types of workers are excluded. Thus, up to null sets, there is no difference between the economy with Θ as the workers' type space and the one with W, and we do not distinguish between the two. The advantage of considering the latter is that all our assumptions are satisfied. Indeed, W is a separable metric space and an open, hence, Borel subset of Θ, which is itself a compact (hence, complete and separable) metric space. Thus, the restriction $G|_W$ of G to W is tight by Parthasarathy (1967, Theorem 3.2, p. 29). Furthermore, workers' preferences are continuous since $\left\{(\theta, f, M, f', M') \in W \times (\tilde{F} \times \mathcal{X}^{n+1})^2 : (f, M) \succ_\theta (f', M')\right\} = \left\{(\theta, f, M, f', M') \in W \times (\tilde{F} \times \mathcal{X}^{n+1})^2 : \theta \in \cup_{P \in \mathcal{P}: f \succ_P f'} \mathrm{int}(\Theta_P)\right\}$ is open.

Our framework also includes a setting analogous to those of Cox, Fonseca, and Pakzad-Hurson (2022) and Leshno (2022) under the assumption of continuous workers' preferences (which they do not make). In such setting, firms are interpreted as colleges and workers as students. College $f \in \tilde{F}$ has the capacity to admit a mass of $q_f > 0$ students with $q_\emptyset > 1$. The students' type space is $\Theta = [0,1]^{|F|} \times C(\tilde{F} \times S)$, where S is a compact metric space and $C(\tilde{F} \times S)$ denotes the space of bounded and continuous real-valued functions on $\tilde{F} \times S$. A typical student's type is then $\theta = (r, u)$, r is the student's rank and is used to specify colleges' preferences, and u is the student's utility function. The utility for a student of type $\theta = (r, u)$ of choosing firm f when the matching is $M \in \mathcal{X}^{n+1}$ is $u(f, s(M))$ where $s : \mathcal{X}^{n+1} \to S$ is a continuous function. Thus, students' preferences are allowed to depend on the matching in any (continuous) way.[9] For each $f \in F$, define $C_f : \mathcal{X} \rightrightarrows \mathcal{X}$ by setting, for each $X \in \mathcal{X}$,

$$
\begin{aligned}
C_f(X) = \{\delta \in \mathcal{X} : \quad & \delta \text{ solves } \max_{\delta' \in \mathcal{X}} \int_\Theta \pi_{1,f}(\theta) \mathrm{d}\delta'(\theta) \\
& \text{subject to } \delta' \sqsubset X \text{ and } \delta'(\Theta) \leq q_f\},
\end{aligned}
$$

where $\pi_{1,f} : \Theta \to [0,1]$ is the projection of Θ onto the fth coordinate of $[0,1]^{|F|}$.

We refer to the above setting as a college admission economy. The resulting notion of a stable matching of a college admission economy differs from that in Cox, Fonseca, and Pakzad-Hurson (2022) and Leshno (2022) but only slightly. Indeed, stability for a matching $M = (M_f)_{f \in \tilde{F}}$ of a college admission economy is characterized by the

[9] Students' preferences can depend on the entire matching by letting S be a countable product of $[-1, 1]$; see footnote 8 for details.

following conditions which are analogous to the ones in those papers: For each $f \in F$,

(i) $M_f(\Theta) \leq q_f$,

(ii) $\left(D^{\preceq f}(M) - M_f\right)(\{\theta \in \Theta : \pi_{1,f}(\theta) > 0\}) = 0$ if $M_f(\Theta) < q_f$,

(iii) $\left(D^{\preceq f}(M) - M_f\right)(\{\theta \in \Theta : \pi_{1,f}(\theta) > \inf_{\theta' \in \text{supp}(M_f)} \pi_{1,f}(\theta')\}) = 0$ and

(iv) $M_f(\{\theta \in \Theta : \pi_2(\theta)(\emptyset, s(M)) > \pi_2(\theta)(f, s(M))\}) = 0$,

where $\pi_2 : \Theta \to C(\tilde{F} \times S)$ is the projection of Θ onto $C(\tilde{F} \times S)$.[10]

The assumptions we made in Section 2.1 are satisfied in any college admission economy. Indeed, $C(\tilde{F} \times S)$ is a complete and separable metric space and, thus, G is tight. Moreover, C_f satisfies assumptions (i)–(iii).[11] Finally,

$$\begin{aligned}&\Big\{(\theta, f, M, f', M') \in \Theta \times (\tilde{F} \times \mathcal{X}^{n+1})^2 : (f, M) \succ_\theta (f', M')\Big\} \\ &= \Big\{(\theta, f, M, f', M') \in \Theta \times (\tilde{F} \times \mathcal{X}^{n+1})^2 : \pi_2(\theta)(f, s(M)) > \pi_2(\theta)(f', s(M'))\Big\}\end{aligned}$$

and it follows that this set is open.

2.4 No blocking condition

The no blocking condition in the definition of a stable matching requires that no firm gets strictly better off by hiring a workforce that does not exceed the measure of available workers to it. In this section we argue that different ways of defining the measure of available workers to a firm have different implications for the existence of stable matchings. We first use a simple example to discuss the measure of available workers to a firm we use and to show how this notion in CKK coincides with ours. This is an example where workers' preferences feature indifferences but not externalities. We then discuss the implications of our notion of the measure of available workers to a firm in the presence of externalities.

[10] See Appendix A.3 for a proof of this claim.

[11] The proof of this claim is analogous to the proof of Corollary 1 in Appendix A.2.

2.4.1 Weak vs strong domination

Consider then the following example with two firms, f_1 and f_2, one type of workers and a measure one of workers of this single type in the population. Workers are indifferent between each of the two firms and prefer to work than to be unemployed. The two firms have the same preferences that are strictly increasing in the measure of workers employed. In particular, when faced with some measure $X \in [0,1]$ of available workers, each firm will choose to employ the entire measure X.

There is no stable matching in this example if the measure of available workers to a firm is defined using weak domination, i.e. no one in the blocking coalition can be made worse off and only one agent in the blocking coalition needs to be made strictly better off. Indeed, with this notion, the measure of available workers to each firm is one since the workers are indifferent between each firm. Consequently, in any stable matching, each firm must employ a measure one of workers since its preferences are strictly increasing in the measure of workers it employs. But this is impossible since there is only a measure one of workers.

If we define the measure of available workers using strong domination, i.e. by requiring that every agent in the blocking coalition be strictly better off, then stable matchings exist. Indeed, these are easily characterized as follows: for each $\alpha \in [0,1]$, firm f_1 hires a measure α of workers and firm f_2 hires a measure $1-\alpha$ of workers.

Thus, existence of stable matchings (under general assumptions) is not guaranteed when stability is defined using weak domination but, as Theorem 1 shows, it is guaranteed when stability is defined using strong domination. In general, we can do better by defining the measure of available workers to a firm and stability using a notion of domination that is between weak and strong domination. Namely, the measure of available workers to a firm f in the stability notion that we consider is the measure of those who are matched with the firm f itself plus the measure of those who are matched with firms that they regard as worse than f.

The above notion of the measure of available workers to a firm coincides with the one used by CKK. In their setting, because workers' preferences are strict, it also coincides with the notion of the measure of available workers to a firm defined via weak domination. Thus, CKK obtain the existence of stable matchings when stability is defined via weak domination. As the above example shows, this conclusion does not generalize beyond the strict preference case, something already present in the discrete markets of Kelso and Crawford (1982). The result that does extend is the one for the stability notion defined via the above notion of the measure of available workers to a firm.

The notion of the measure of available workers to a firm we use and resulting stability notion is unusual in that workers of the same type, matched with firms over which they are indifferent, are treated differently depending on which of them they are currently matched with. Nevertheless, this notion of stability is appealing because it yields a smaller set of stable matchings as compared to the one obtained when stability is defined via strong domination and, as we show in this paper, its existence is guaranteed under standard assumptions.

The following example illustrates. There are two types of workers and one firm so that $\Theta = \{\theta_1, \theta_2\}$ and $X \in \mathcal{X}$ can be written as $(X(\theta_1), X(\theta_2))$. The firm's preferences are given by $C_{f_1}(X) = \{X' \in \mathcal{X}_X : \min\{X'(\theta_1), X'(\theta_2)\} = \min\{X(\theta_1), X(\theta_2)\}\}$. Workers prefer working for the firm to being unmatched. Let $G(\theta_1) = G(\theta_2) = 1/2$. When stability is defined via strong domination, M such that $M_{f_1} = (1/2, 0)$ is stable; the only profitable deviations involve both types of workers,[12] but workers of type 1 are already working for the firm so will not be strictly better off. Also since the firm is indifferent between all M_{f_1} such that $M_{f_1}(\theta_2) = 0$, $M_{f_1} \in C_{f_1}(M_{f_1})$.[13] On the other hand, by our notion of stability, which allows the firm to form blocking coalitions that combine workers who would be strictly bettter off with its existing workers, the only stable matching is M such that $M_{f_1} = (1/2, 1/2)$. This example shows that the notion of stability we use can yield a strictly smaller set of stable matchings as compared to the one obtained when stability is defined via strong domination.

[12]Indeed, $M'_{f_1} \succ_{f_1} M_{f_1}$ requires $\min\{M'_{f_1}(\theta_1), M'_{f_1}(\theta_2)\} > 0$.

[13]The individual rationality requirement that $M_f \in C_f(M_f)$ is implied by our no blocking condition but not by the one defined via strong domination and, thus, should be explicitly added to the latter.

2.4.2 Externalities

The main issue regarding the definition of stability in economies with externalities concerns how a blocking coalition will affect the matching and, thus, the preferences of its members via the externality. In our setting this concerns the workers since they are the ones whose preferences depend on externalities.

The presence of externalities has been incorporated in stability notions in several ways. One approach, along the lines of Echenique and Yenmez (2007), is to assume that the workers in a blocking coalition will evaluate the matching according to the one that will result if the formation of the coalition is the only change to the matching. A difficulty with this approach is, however, that stable matchings may fail to exist under our assumptions as the following example shows.

Consider the following example with two firms and two types of workers, so that $F = \{f_1, f_2\}$ and $\Theta = \{\theta_1, \theta_2\}$ with $G(\theta_1) = G(\theta_2) = 1/2$. Firm 1 has preferences given by $C_{f_1}(X) = (\min\{X(\theta_1), X(\theta_2)\}, \min\{X(\theta_1), X(\theta_2)\})$ and firm 2 has preferences given by $C_{f_2}(X) = (X(\theta_1), X(\theta_2))$. Workers' preferences are given by:

$$u_{\theta_1}(f, M) = \begin{cases} -4M_{f_2}(\theta_2) & \text{if } f \in F, \\ 1 - 4M_{f_2}(\theta_2) & \text{if } f = \emptyset, \end{cases}$$

$$u_{\theta_2}(f, M) = \begin{cases} 1 & \text{if } f = f_1, \\ 0 & \text{if } f = f_2, \text{ and} \\ -1 & \text{if } f = \emptyset. \end{cases}$$

Now consider an alternative no blocking condition that requires for M to be stable that there does not exist $f \in \{f_1, f_2\}$ and $(\delta_k)_{k \in \{f_1, f_2, \emptyset\}} \in \mathcal{X}^3$ such that, for each $k \in \{f_1, f_2, \emptyset\}$ and $\theta \in \{\theta_1, \theta_2\}$, (i) $0 \leq \delta_k(\theta) \leq M_k(\theta)$ and $\delta_k(\theta) > 0$ only if $u_\theta(k, M) < u_\theta(f, M')$, and (ii) $M'_f \succ_f M_f$, where

$$M'_f = \sum_{k \in \{f_1, f_2, \emptyset\}} \delta_k,$$

$$M'_\emptyset = M_\emptyset + M_f - \delta_f - \delta_\emptyset, \text{ and}$$

$$M'_k = M_k - \delta_k \text{ for each } k \in \{f_1, f_2\} \setminus \{f\}.$$

In words, a blocking coalition consists of a firm, for example f_1, who keeps δ_{f_1} of its existing workers and hires δ_{f_2} and $\delta_\emptyset$ from firm f_2 and the unemployed. If the initial matching was M, then, in the matching M' that results from the formation of this coalition, firm f_1's workforce is $M'_{f_1} = \delta_{f_1} + \delta_{f_2} + \delta_\emptyset$, firm f_2's workforce is $M'_{f_2} = M_{f_2} - \delta_{f_2}$ and the measure of the unemployed workers become $M'_\emptyset = M_\emptyset + M_{f_1} - \delta_{f_1} - \delta_\emptyset$. This alternative no blocking condition then requires that it is not the case that firm f_1 prefers M'_{f_1} to M_{f_1} and every worker hired under M'_{f_1} prefers working for firm f_1 given matching M' to working for their old employer given the original matching M.

In any stable matching M, individual rationality requires that $M_\emptyset(\theta_1) = 1/2$, which implies that $M_{f_1} = (0, 0)$ and hence $M_{f_2} = (0, 1/2)$. However, under this alternative no blocking condition, $M'_{f_1} = (1/2, 1/2)$ blocks. Firm f_1 prefers M'_{f_1} to M_{f_1}; all workers of type θ_2 prefer working for firm f_1 over working for firm f_2; and all workers of θ_1 prefer working for firm f_1 when no one is working for firm f_2 over being unemployed when all workers of type θ_2 are working for firm f_2.

The above example establishes in our setting the conclusion in Echenique and Yenmez (2007) that stable matchings may fail to exist in their setting. Alternative notions of stability have been proposed in settings which, as in Echenique and Yenmez (2007), feature finitely many individuals with the goal of restoring existence of stable matchings, e.g. strong and weak stability in Bando (2012), the stability notion in Mumcu and Saglam (2010) and the notion of prudent stability in Fisher and Hafalir (2016). These notions still postulate that a specific matching will result in response to a blocking coalition but impose some degree of far-sightedness in individuals' forecast of the ultimate matching that will result. These stability notions involve varying degrees of far-sightedness and raise some conceptual issues about the appropriate level of sophistication that should be attributed to agents.

In this paper – as in Cox, Fonseca, and Pakzad-Hurson (2022) and Leshno (2022) – our approach is to assume that the workers in a blocking coalition evaluate their preferences holding the rest of the matching fixed. This can be justified, for example, if each worker is only aware that some firm is trying to hire her, without any knowledge about who else the firm is trying to hire. Hence, each worker can believe that she is the only one moving and, because she is of negligible size, there is no change to the matching. Thus, the conceptual issues regarding how a blocking coalition will ultimately affect the matching can be avoided if the set of individuals whose preferences depend on the matching is non-atomic.

In many situations, it seems reasonable to suppose that workers in a blocking coalition do not know who else is part of the coalition other than the firm trying to hire them. One story consistent with our assumption is that blocking coalitions are formed when a firm sends job offers to the workers that it wishes to hire. If job offers are confidential, then each worker may not be able to find out who else the firm intends to hire. In such situations, it is reasonable for each worker to evaluate her offer assuming that she would be the only one moving, i.e. with the overall matching remaining fixed as she is a negligible agent.[14] The resulting notion of stability is thus analogous to the one in Fisher and Hafalir (2016) whose no blocking condition assumes that there is no change to the matching in response to the formation of such coalition.

An alternative approach, proposed by Sasaki and Toda (1996) and Dutta and Massó (1997), is to define the no blocking condition to require that members of the coalition be better off for each possible matching in which they are matched; a somewhat milder requirement is proposed in Hafalir (2008). Of course, our existence result also implies the existence of stable matchings in this approach. Indeed, by making the requirement for blocking more demanding by allowing more matchings to arise in response to a blocking coalition and by requiring its members to be better off in all of them, existence becomes easier to guarantee under the resulting stability notion.

[14] As well as causing issues for existence, allowing workers to anticipate changes to the matching raises some conceptual issues. In the above example, should the relevant comparison for θ_1 really be between $(\emptyset, M)$ and (f_1, M')? Workers are small and cannot affect the matching, so their choice of firm should be made with the matching held fixed. But which matching? Resolving such issues is beyond the scope of this paper but our point is that in a market with a continuum of workers, these issues can be avoided if we assume that workers lack information about other workers in the economy.

3 Existence of stable matchings

Our assumptions guarantee the existence of stable matchings.

Theorem 1 *A stable matching exists.*

We remark that Theorem 1 has CKK's existence result as a special case by what we have shown in Section 2.3. Similarly, Theorem 1 also implies that any college admission economy has a stable matching.

We note that the only assumption needed for the application of Theorem 1 to college admission economies is that students' preferences depend continuously on the matching. In the particular case where students' preferences do not depend on the matching as in Azevedo and Leshno (2016), this assumption is trivially satisfied and thus we obtain the existence of a stable matching without any assumption besides those defining a college admission economy; in particular, the assumption of strict preferences in Azevedo and Leshno (2016) can be dispensed with as far as existence of stable matchings is concerned.

Cox, Fonseca, and Pakzad-Hurson (2022) and Leshno (2022) do not assume that students' preferences depend continuously on the matching; however, when students' preferences do depend continuously on the matching, then their other assumptions can also be dispensed with for the existence of stable matchings. We consider the case where students' preferences are not required to depend continuously on the matching in Section 4.3.

The importance of the continuity of workers' preferences for the existence can be illustrated with the following example, which modifies the one in Section 2.4.1. There is only one firm, one type of workers and a measure one of workers of this single type in the population. The firm f_1 has preferences that are strictly increasing in the measure of workers employed: thus, when faced with some measure $X \in [0, 1]$ of available workers, each firm will choose to employ the entire measure X. Workers' preferences are represented by the following utility function $u_\theta : \tilde{F} \times \mathcal{X}^2 \to \mathbb{R}$, defined by setting, for each $(f, M) \in \tilde{F} \times \mathcal{X}^2$,

$$u_\theta(f, M) = \begin{cases} 2 & \text{if } f = f_1 \text{ and } M_f < 1/2, \\ 0 & \text{if } f = f_1 \text{ and } M_f \geq 1/2, \\ 1 & \text{if } f = \emptyset. \end{cases}$$

Then there is no stable matching. Indeed, if M is a matching such that $M_{f_1} < 1/2$, then $D^{\preceq_{f_1}}(M) = 1$ and $M'_{f_1} = 1$ blocks; if $M_{f_1} \geq 1/2$, then $M_{f_1}(P(\emptyset, f_1, M)) = M_{f_1} \geq 1/2 > 0$. All our assumptions are satisfied except the continuity of workers' preferences. Moreover, if workers' utility function is changed in such a way that $u_\theta(f_1, M) = 2(1 - M_{f_1})$, then u_θ is continuous and a stable matching exists. Indeed, in this case, the matching M such that $M_{f_1} = 1/2$ is a stable matching (the unique one in fact).

Non-atomicity of workers is also an important condition. As discussed in Section 2.4.2, this property is important for the definition of stability itself. In addition, we present in Section 4.2 a counterexample for existence of stable matchings for the case where the atomic individuals, i.e. the firms, have preferences that depend on the matching. As CKK have pointed out (see Section 2 of their paper), the non-atomicity of the workers cannot be dispensed with for the existence of stable matchings when firms' preferences are non-substitutable; since our setting includes that of CKK, the same holds here.

We next present a brief outline of the argument we use to establish Theorem 1 before its proof, which is in Section A.1.

3.1 Outline

The proof of Theorem 1 follows that of CKK's existence result. We first establish its conclusion in the special case where Θ is finite and we use the following example to illustrate our approach: There is only one firm, one type of workers and a measure one of workers of this single type in the population. The firm f_1 has preferences that are strictly increasing in the measure of workers employed: $C_{f_1}(X) = X$ for each $X \in [0,1]$. Workers' preferences are represented by the following utility function $u_\Theta : \tilde{F} \times \mathcal{X}^2 \to \mathbb{R}$, defined by setting, for each $(f, M) \in \tilde{F} \times \mathcal{X}^2$,

$$u_\Theta(f, M) = \begin{cases} 1 + 2(1 - M_{f_1}) & \text{if } f = f_1, \\ 2 & \text{if } f = \emptyset. \end{cases}$$

The main idea is that firms choose workforces and, thus, a matching μ' given available workers κ and workers choose an allocation τ given the matching μ; this allocation τ specifies the measure of workers of each type that will work in each firm and determines the measure of available workers κ'. We then obtain a mapping $(\mu, \kappa) \mapsto (\mu', \kappa')$ and the goal is to obtain a stable matching via a fixed point of such mapping.

The firms' and workers' problems are as follows. For each $f \in \tilde{F}$ and given (μ, κ) where $\mu = (\mu_f)_{f \in \tilde{F}}$ and $\kappa = (\kappa_f)_{f \in \tilde{F}}$, firm f's problem is to choose μ'_f optimally from the available workers κ_f. As in CKK, the solution is

$$D_f(\mu, \kappa) = C_f(\kappa_f).$$

In the example,

$$D_f(\mu, \kappa) = \kappa_f.$$

The workers' problem is to choose the measure of workers of each type that will work in each firm. It depends on the matching because workers' preferences depend on the matching and also because it describes the demand of labor by firms. First, represent workers' preferences by a (bounded and continuous) utility function $u_\Theta : \tilde{F} \times \Theta \times \mathcal{X}^{n+1} \to \mathbb{R}$, normalized so that $u_\Theta \geq 1$. Given a matching μ, the solution to the workers' problem is

$$\begin{aligned} D_\Theta(\mu) = \{\tau \in \mathcal{X}^{n+1} : \tau \in \arg \max_{\tau' \in \mathcal{X}^{n+1}} & \sum_{f \in \tilde{F}, \theta \in \Theta} u_\Theta(f, \theta, \mu) \tau'_f(\theta) \\ & \text{subject to } \sum_{f \in \tilde{F}} \tau'_f(\theta) \leq G(\theta) \text{ and} \\ & \tau'_f(\theta) \leq \mu_f(\theta) \text{ for each } f \in \tilde{F} \text{ and } \theta \in \Theta\}. \end{aligned}$$

In the example, where u_Θ does not depend on θ because there is only a single type of worker,

$$D_\Theta(\mu) = \{\tau \in \mathcal{X}^{n+1} : \tau \in \arg\max_{\tau' \in \mathcal{X}^{n+1}} 2\tau'_\emptyset + (1 + 2(1 - \mu_{f_1}))\tau'_{f_1}$$
$$\text{subject to } \tau'_\emptyset + \tau'_{f_1} \leq 1, \tau'_\emptyset \leq \mu_\emptyset \text{ and } \tau'_{f_1} \leq \mu_{f_1}\}.$$

Each solution τ to the workers' problem determines the measure of available workers: the workers of type θ available to firm f are those allocated to f or to firms that workers of type θ regard as worse than f. Thus, letting

$$B(f, \theta, \mu) = \{f' \in \tilde{F} : (f', \mu) \succeq_\theta (f, \mu)\} \text{ and}$$
$$B_-(f, \theta, \mu) = B(f, \theta, \mu) \setminus \{f\},$$

the measure of available workers $g(\mu, \tau)(f, \theta)$ of type θ to firm f is

$$G(\theta) - \sum_{f' \in B_-(f,\theta,\mu)} \tau_{f'}(\theta) = \tau_f(\theta) + G(\theta) - \sum_{f' \in B(f,\theta,\mu)} \tau_{f'}(\theta).$$

In the example, for the single type of worker $\theta \in \Theta$,

$$g(\mu, \tau)(f, \theta) = \begin{cases} 1 & \text{if } f = f_1 \text{ and } \mu_{f_1} < 1/2, \\ 1 - \tau_\emptyset & \text{if } f = f_1 \text{ and } \mu_{f_1} \geq 1/2, \\ 1 - \tau_{f_1} & \text{if } f = \emptyset \text{ and } \mu_{f_1} \leq 1/2, \\ 1 & \text{if } f = \emptyset \text{ and } \mu_{f_1} > 1/2. \end{cases}$$

When workers' preferences are strict and do not depend on externalities, the above corresponds essentially to CKK's approach. Indeed, in this case, the allocation of workers to firms is uniquely determined (i.e. D_Θ is a function) and, thus, so is the measure of available workers. This measure of available workers depends on the matching only indirectly through the dependence of τ on μ; indeed, in this case, the set $B_-(f, \theta, \mu)$ does not depend on μ and equals the set of firms that workers of type θ strictly prefer to f. Hence, we then obtain a continuous function $\mu \mapsto g(\mu, D_\Theta(\mu))$ that gives us the measure of available workers. Then we obtain a well-behaved correspondence

$$(\mu, \kappa) \mapsto \prod_{f \in \tilde{F}} D_f(\mu, \kappa) \times \{g(\mu, D_\Theta(\mu))\}$$

which has a fixed point and is such that its fixed points are stable.

When workers' preferences depend on externalities, the above uniqueness no longer holds and this has forced us to consider explicitly the workers' problem D_Θ. More importantly, the measure of available workers is no longer (directly) independent of μ. The main difficulty is that this dependence is discontinuous since $f' \neq f$ may belong to $B_-(f, \theta, \mu)$ but not to $B_-(f, \theta, \mu')$ for some μ' in a neighborhood of μ. In the example, note that g is discontinuous at (μ, τ) such that $\mu_{f_1} = 1/2$ and $\tau_\emptyset + \tau_{f_1} > 0$, in particular when $\tau \in D_\Theta(\mu)$.

We deal with the above difficulty by considering a continuous approximation to g. The key idea is that $g(\mu, \tau)(f, \theta)$ can be written as

$$g(\mu, \tau)(f, \theta) = \tau_f(\theta) + G(\theta) - \sum_{f' \in \tilde{F}} \alpha_{(f,\theta)}(f', \mu)\tau_{f'}(\theta)$$

with

$$\alpha_{(f,\theta)}(f', \mu) = \begin{cases} 1 & \text{if } u_\Theta(f', \theta, \mu) \geq u_\Theta(f, \theta, \mu), \\ 0 & \text{otherwise.} \end{cases}$$

The discontinuity of g can then be tracked back to the weights $\alpha_{(f,\theta)}$ and, hence, we approximate these weights with continuous ones, namely

$$\alpha_{j,(f,\theta)}(f', \mu) = j \max\left\{0, \min\left\{\frac{1}{j} + u_\Theta(f', \theta, \mu) - u_\Theta(f, \theta, \mu), \frac{1}{j}\right\}\right\}.$$

We then let

$$g_j(\mu, \tau)(f, \theta) = \tau_f(\theta) + G(\theta) - \sum_{f' \in \tilde{F}} \alpha_{j,(f,\theta)}(f', \mu)\tau_{f'}(\theta).$$

In the example, for the single type of worker $\theta \in \Theta$,

$$g_j(\mu, \tau)(f_1, \theta) = \begin{cases} 1 & \text{if } \mu_{f_1} \leq \frac{1}{2} - \frac{1}{2j}, \\ 1 - (1 + j(2\mu_{f_1} - 1))\tau_\emptyset & \text{if } \frac{1}{2} - \frac{1}{2j} < \mu_{f_1} < \frac{1}{2}, \\ 1 - \tau_\emptyset & \text{if } \mu_{f_1} \geq \frac{1}{2}, \end{cases}$$

and

$$g_j(\mu, \tau)(\emptyset, \theta) = \begin{cases} 1 - \tau_{f_1} & \text{if } \mu_{f_1} \leq \frac{1}{2}, \\ 1 - (1 + j(1 - 2\mu_{f_1}))\tau_{f_1} & \text{if } \frac{1}{2} < \mu_{f_1} < \frac{1}{2} + \frac{1}{2j}, \\ 1 & \text{if } \mu_{f_1} \geq \frac{1}{2} + \frac{1}{2j}. \end{cases}$$

The continuity of g_j implies that the mapping

$$(\mu, \kappa) \mapsto \prod_{f \in \tilde{F}} D_f(\mu, \kappa) \times \{g_j(\mu, \tau) : \tau \in D_\Theta(\mu)\}$$

is well-behaved and, thus, has a fixed point (μ_j, κ_j) for each $j \in \mathbb{N}$. The function g_j proves to be a good approximation to g, in the sense that $g_j(\mu, \tau)(f, \theta) \le g(\mu, \tau)(f, \theta)$ and $\liminf_j g_j(\mu_j, \tau_j)(f, \theta) \ge g(\mu, \tau)(f, \theta)$ whenever $(\mu_j, \tau_j) \to (\mu, \tau)$; these properties allow us to prove that if (μ, κ) is a limit point of the sequence $\{(\mu_j, \kappa_j)\}_{j=1}^{\infty}$, then μ is a stable matching and, thus, to establish the existence of a stable matching in discrete economies.

We then use a limit argument to extend the existence result from discrete to general economies. The limit argument builds upon analogous results in CKK (namely, their Lemma 7) but extended to the case of a separable Θ and tight G. The main issue concerns again the measure of available workers. To see this, let $\{G_k\}_{k=1}^{\infty}$ such that $G_k \to G$ and $\text{supp}(G_k)$ is finite for each k define a sequence of discrete economies converging to the one defined by G. Furthermore, let (μ_k, κ_k) be obtained via the above fixed point argument and (μ, κ) be a limit point of $\{(\mu_k, \kappa_k)\}_{k=1}^{\infty}$. Despite the lack of continuity of $\mu' \mapsto D^{\preceq f}(\mu')$, we establish that $D^{\preceq f}(\mu_k) \sqsubset \kappa_{k,f}$ for each k implies that $D^{\preceq f}(\mu) \sqsubset \kappa_f$. Using this result, we show that there are no blocking coalitions in the economy defined by G and, in fact, show that μ is a stable matching.

4 Discussion

In this section, we show that Theorem 1 applies to the case where firms have explicit preferences (although not without difficulties). Its conclusion, however, does not extend to the case where firms' preferences depend on the matching as we show by an example. The section then concludes with two extensions of our approach, including to the case of diverse worker preferences.

4.1 Existence of stable matchings with explicit preferences

In this section we assume that firms have explicit preferences. Intuitively, this yields a particular case of the model in Section 2 and we will show that this intuition is indeed correct but not without difficulties.

The main difficulty in establishing this result consists in showing that the choice correspondence C_f obtained from preference maximization is closed. This is relatively simple when Θ is finite and can be shown by proving, in particular, that the constraint correspondence in the definition of C_f below is lower hemicontinuous. Extending this result to the case where Θ is separable or even compact is challenging. Nevertheless, by following an approach that requires only a weaker form of lower hemicontinuity of the constraint correspondence, we show that C_f is indeed closed and, thus, that Theorem 1 applies to establish that stable matchings exist for the setting of this section with explicit firms' preferences.

The model with explicit firms' preferences is as in Section 2 with the following differences. Each firm $f \in F$ has a complete, transitive and continuous preference relation $\succeq_f$ on $\mathcal{X}$ which is convex.[15]

In this setting, a matching M is stable if

1. (Individual Rationality) For each $f \in F$, $M_f(P(\emptyset, f, M)) = 0$; and
2. (No Blocking Coalition) No $f \in F$ and $M'_f \in \mathcal{X}$ exist such that $M'_f \sqsubset D^{\preceq f}(M)$ and $M'_f \succ_f M_f$.

Condition 1 in the above definition is the same as in Section 2. Although condition 2 above looks exactly like condition 2 of Section 2, they are different because $\succ_f$ in the former is a primitive concept whereas in the latter is derived from C_f in the specific way described in Section 2.

However, as we next show, condition 2 above is just condition 2 of Section 2 specialized to the setting of this section. To see this, note that $\mathcal{X}$ is compact since G is tight and let $u_F : F \times \mathcal{X} \to \mathbb{R}$ be a bounded and continuous utility function that

[15]Convexity of preferences means that, for each $\lambda \in (0, 1)$ and $M_f, M'_f \in \mathcal{X}$, if $M_f \succeq_f M'_f$, then $\lambda M_f + (1-\lambda)M'_f \succeq_f M'_f$.

represents firms' preferences (see, e.g. Debreu (1964)). Then, for each $f \in F$ and $X \in \mathcal{X}$, define

$$C_f(X) = \{\delta \in \mathcal{X} : \delta \text{ solves } \max_{\delta' \in \mathcal{X}} u_F(f, \delta') \text{ subject to } \delta' \sqsubset X\}.$$

Let $\geq_f$ be obtained from C_f as in Section 2. The following simple lemma relates $>_f$ with $\succ_f$.

Lemma 1 *Let $f \in F$ and $X, X', \bar{X} \in \mathcal{X}$. Then:*

1. *If $X' >_f X$, then $X' \succ_f X$.*

2. *If $X' \sqsubset \bar{X}$, $X \sqsubset \bar{X}$ and $X' \succ_f X$, then there exists $X^* \sqsubset \bar{X}$ such that $X^* >_f X$.*

Proof. Part 1: Since $X \sqsubset (X \vee X')$ and $X' \sqsubset (X \vee X')$, it follows from $X' \in C_f(X \vee X')$ and $X \notin C_f(X \vee X')$ that $u_F(f, X') > u_F(f, X)$.

Part 2: Take $X^* \in C_f(\bar{X})$ to obtain $u_F(f, X^*) \geq u_F(f, X') > u_F(f, X)$. This then implies $X^* >_f X$. ■

Part 2 of the above lemma implies that if there is a blocking pair (f, X') according to $\succ_f$, then there is a blocking pair (f, X^*) according to $>_f$ (set $\bar{X} = D^{\preceq f}(M)$). Thus, if M is stable according to Section 2, then M is stable according to the definition in this section; the converse also holds by part 1 of the above lemma.

It is straightforward to show that C_f defined above satisfies assumptions (i) and (ii) in Section 2 and that it has nonempty and convex values. Thus, once we show that C_f is closed, using the above consequence of Lemma 1, it follows from Theorem 1 that stable matchings exist for the setting of this section.

Corollary 1 *A stable matching exists when firms have explicit preferences.*

4.2 Nonexistence of stable matchings with externalities in firms' preferences

We present an example in this section showing that Theorem 1 does not extend to the case where firms' preferences depend on the matching.

As discussed in Section 2.4.2, there are several definitions of stability in the case where atomic individuals have preferences that depend on externalities. The main issue concerns how a blocking coalition will affect the matching and, thus, the preferences of its members via the externality. The stability notion we use in this section postulates that a specific matching will result in response to a blocking coalition. Namely, that the matching changes only due the actions of the firm in the blocking coalition of hiring workers from other firms or of firing some of its workers. The stability notion in our nonexistence example is then analogous to strong stability in Bando (2012) and, in the special case of one-to-one matching, to the one in Mumcu and Saglam (2010) and to the notion of prudent stability in Fisher and Hafalir (2016).[16]

The setting for our nonexistence example is as in Section 2 except that each firm $f \in F$ has a continuous utility function $u_f : \mathcal{X}^{n+1} \to \mathbb{R}$; we focus on the case where firms have explicit preferences since our counterexample to the existence of a stable matching has this property too. The function u_f is assumed to have enough concavity to imply an analogous condition to C_f being convex-valued as in Section 2, as described below.

The presence of externalities in firms' preferences require a change to the definition of stability. The reason is that each firm can unilaterally change the coordinates of a matching other than its own by hiring workers who are matched with firms they find worse or by firing some of its workers. Unlike in the case of Section 2, such changes matter for its preferences. This yields the following definition of stability: A matching M is stable if

1. (Individual Rationality) For each $f \in F$, $M_f(P(\emptyset, f, M)) = 0$; and

[16]Bando (2012) also defines *weak stability*, which is motivated by the idea that certain no blocking coalitions may not be credible if workers anticipate that the firm may make further deviations, e.g. a firm may wish to hire some workers from its competitor just to fire them. Weak stability does not allow blocking coalitions where the firm has an incentive to make certain kinds of further deviations, as workers may anticipate such deviations. We do not wish to attribute such strategic considerations to our workers, and so our stability notion in this section is analogous to Bando's (2012) strong stability. However, the counterexample we provide in this section continues to hold under weak stability — see Appendix A.6 for details.

2. (No Blocking Coalition) There does not exist $f \in F$, $\delta'_\emptyset \in \mathcal{X}$ and $(\delta_k)_{k \in \tilde{F} \setminus \{f\}} \in \mathcal{X}^n$ such that $\delta_k \sqsubset D_k^{\preceq f}(M)$ for each $k \in \tilde{F} \setminus \{f\}$, $\delta'_\emptyset \sqsubset M_f$, and $u_f(M') > u_f(M)$, where

$$M'_f = M_f + \sum_{k \in \tilde{F} \setminus \{f\}} \delta_k - \delta'_\emptyset,$$

$$M'_\emptyset = M_\emptyset - \delta_\emptyset + \delta'_\emptyset, \text{ and}$$

$$M'_k = M_k - \delta_k \text{ for each } k \in F \setminus \{f\}$$

and, for each $k \in \tilde{F} \setminus \{f\}$, $D_k^{\preceq f}(M)$ is defined by setting, for each $E \in \Sigma$,

$$D_k^{\preceq f}(M)(E) = M_k(E \cap P(f, k, M)).$$

Condition 1 is exactly as in Section 2. Condition 2 requires M to solve each firm's problem, which is how to change the matching to its own advantage by hiring workers that prefer it to the firms to whom they are matched and by firing its current workers.

In Section 2 we required the set of solution to each firm's problem to be convex. Thus, the convexity assumption we make in this section is that, for each $M \in \mathcal{X}^{n+1}$, the set

$$\begin{aligned} C_f(M) := \Big\{ M^* \in \mathcal{X}^{n+1} : M^* \text{ solves } & \max_{M' \in \mathcal{X}^{n+1}} u_f(M') \\ \text{subject to } M'_f &= M_f + \sum_{k \in \tilde{F} \setminus \{f\}} \delta_k - \delta'_\emptyset, \\ M'_\emptyset &= M_\emptyset - \delta_\emptyset + \delta'_\emptyset, \\ M'_k &= M_k - \delta_k \text{ for each } k \in F \setminus \{f\}, \\ & 0 \sqsubset \delta_k \sqsubset D_k^{\preceq f}(M) \text{ for each } k \in \tilde{F} \setminus \{f\} \text{ and} \\ & 0 \sqsubset \delta'_\emptyset \sqsubset M_f \Big\} \end{aligned}$$

is convex.

The following is our counterexample for the existence of stable matchings when firms' preference depend on the matching. There are two firms and only one type of workers. Each worker prefers firm 1 to firm 2 and prefers to work rather than to be unemployed. Because there is only one type of workers, for each $f \in \tilde{F}$, M_f is fully described by $M_f(\Theta)$, which we also write as M_f. For each $f \in F$, firm f's utility function is, for each $M \in \mathcal{X}^{n+1}$,

$$u_f(M) = (M_\emptyset - \alpha)M_f,$$

where $0 < \alpha < 1$.

We will show that no stable matching exists. Suppose that there is a matching M that is stable. We consider three cases: (a) $M_\emptyset = \alpha$, (b) $M_\emptyset < \alpha$ and (c) $M_\emptyset > \alpha$.

Case (a): $M_\emptyset = \alpha$. Since $\alpha < 1$, there exists $f \in F$ such that $M_f > 0$ and let $-f \in F$ be such that $f \neq -f$. Let $M'_f = M_f - \varepsilon$, $M'_\emptyset = M_\emptyset + \varepsilon$ and $M'_{-f} = M_{-f}$ for some $0 < \varepsilon < M_f$. Then $u_f(M') = \varepsilon(M_f - \varepsilon) > 0 = u_f(M)$, a contradiction to the stability of M.

Case (b): $M_\emptyset < \alpha$. Then there exists $f \in F$ such that $M_f > 0$ and, hence, $u_f(M) < 0$. Let $M'_f = 0$, $M'_\emptyset = M_\emptyset + M_f$ and $M'_{-f} = M_{-f}$. Then $u_f(M') = 0 > u_f(M)$, a contradiction to the stability of M.

Case (c): $M_\emptyset > \alpha$. Note first that $M_f > 0$ for each $f \in F$. Indeed, if $M_f = 0$ for some $f \in F$, then $u_f(M) = 0$. Let $M'_f = \varepsilon$, $M'_\emptyset = M_\emptyset - \varepsilon$ and $M'_{-f} = M_{-f}$ for some $0 < \varepsilon < M_\emptyset - \alpha$. Then $u_f(M') = (M_\emptyset - \alpha - \varepsilon)\varepsilon > 0 = u_f(M)$, a contradiction to the stability of M.

Thus, consider $M'_1 = M_1 + \varepsilon$, $M'_2 = M_2 - \varepsilon$ and $M'_\emptyset = M_\emptyset$ for some $0 < \varepsilon < M_2$. Then $u_1(M') = u_1(M) + (M_\emptyset - \alpha)\varepsilon > u_1(M)$, a contradiction to the stability of M.

It follows by what has been shown above that no stable matching exists in this example. We conclude its analysis by showing that the convexity assumption on preferences is satisfied. Firm 1's problem is

$$\begin{aligned} \max_{(\delta_\emptyset, \delta_2, \delta'_\emptyset)} & (M_\emptyset - \delta_\emptyset + \delta'_\emptyset - \alpha)(M_1 + \delta_\emptyset + \delta_2 - \delta'_\emptyset) \\ \text{subject to } & 0 \leq \delta_\emptyset \leq M_\emptyset, \\ & 0 \leq \delta_2 \leq M_2 \text{ and} \\ & 0 \leq \delta'_\emptyset \leq M_1. \end{aligned}$$

It is easy to see that, for each $M \in \mathcal{X}^{n+1}$, the set $C_f(M)$ of $M' = (M'_\emptyset, M'_1, M'_2)$

obtained from the solutions to this problem is

$$C_1(M) = \begin{cases} \{(\min\{\frac{\sum_{f\in\tilde{F}} M_f+\alpha}{2}, M_\emptyset + M_1\}, \sum_{f\in\tilde{F}} M_f - M'_\emptyset, 0)\} & \text{if } M_\emptyset + M_1 > \alpha, \\ \{(\alpha, \delta_2, M_2 - \delta_2) : \delta_2 \in [0, M_2]\} & \text{if } M_\emptyset + M_1 = \alpha, \\ \{(M_\emptyset + M_1, 0, M_2)\} & \text{if } M_\emptyset + M_1 < \alpha. \end{cases}$$

Thus, $C_1(M)$ is convex.[17]

Firm 2's problem is

$$\max_{(\delta_\emptyset,\delta_1,\delta'_\emptyset)} (M_\emptyset - \delta_\emptyset + \delta'_\emptyset - \alpha)(M_2 + \delta_\emptyset + \delta_1 - \delta'_\emptyset)$$

$$\text{subject to } 0 \le \delta_\emptyset \le M_\emptyset,$$

$$\delta_1 = 0 \text{ and}$$

$$0 \le \delta'_\emptyset \le M_2$$

and we conclude that $C_2(M)$ is convex for each M similarly to the case of firm 1.

4.3 Extensions

Our approach can be easily extended to obtain the existence of stable matchings in a setting analogous to that of Fisher and Hafalir (2016). Consider as in Section 4.2 that firms' preferences are represented by a continuous utility function $u_f : \mathcal{X}^{n+1} \to \mathbb{R}$ such that $M_f \mapsto u_f(M_f, M_{-f})$ is quasi-concave for each $M_{-f} \in \mathcal{X}^n$. In addition, define stability in an analogous way as Fisher and Hafalir (2016) by replacing the no blocking condition in Section 2 with

> No $f \in F$ and $M'_f \in \mathcal{X}$ exist such that $M'_f \sqsubset D^{\preceq f}(M)$ and $u_f(M'_f, M_{-f}) > u_f(M)$.

Then, Theorem 1 readily extends to establish the existence of such stable matchings. Indeed, all it takes is to define, for each $f \in F$, $X \in \mathcal{X}$ and $M_{-f} \in \mathcal{X}^n$,

$$C_f(X, M_{-f}) = \{\delta \in \mathcal{X} : \delta \text{ solves } \max_{\delta' \in \mathcal{X}} u_f(\delta', M_{-f}) \text{ subject to } \delta' \sqsubset X\}$$

[17] The second and third cases in the characterization of $C_f(M)$ are clear and for the first note that the solution must have $M'_\emptyset > \alpha$, $\delta_2 = M_2$ and $M'_\emptyset$ must solve $\max_{0\le M'_\emptyset \le M_\emptyset + M_1} (M'_\emptyset - \alpha)(M_\emptyset + M_1 + M_2 - M'_\emptyset)$.

and let, in the proof of Theorem 1 and for each $(\mu, \kappa) \in \mathcal{X}^{2(n+1)}$,

$$D_f(\mu, \kappa) = C_f(\kappa_f, \mu_{-f}).$$

The proof of Theorem 1 also readily extends to obtain the existence of stable matchings under the assumption of diversity of preferences in Leshno (2022) and Cox, Fonseca, and Pakzad-Hurson (2022) in place of the continuity assumption we made in Section 2 that $\left\{(\theta, f, M, f', M') \in \Theta \times (\tilde{F} \times \mathcal{X}^{n+1})^2 : (f, M) \succ_\theta (f', M')\right\}$ is open. We will establish this in a setting analogous to the one in Section 2 and show how it applies to a setting analogous to one considered in those papers.

Consider the setting of Section 2 with the above continuity assumption replaced with the following weak continuity assumption: $P(f, f', M)$ is open for each $f, f' \in \tilde{F}$ and $M \in \mathcal{X}^{n+1}$. In addition, we make the diversity of preferences assumption which, in our setting, requires that, for each $\mu \in \mathcal{X}^{n+1}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$G(\{\theta \in \Theta : \succeq_{\theta|\mu} \neq \succeq_{\theta|\mu'}\}) < \varepsilon$$

for each $\mu' \in \mathcal{X}^{n+1}$ such that $\rho(\mu, \mu') < \delta$, where $\succeq_{\theta|\mu}$ is the preference relation of θ over $\tilde{F}$ given μ, i.e. $f \succeq_{\theta|\mu} f'$ if and only if $(f, \mu) \succeq_\theta (f', \mu)$ and ρ is a metric on $\mathcal{X}^{n+1}$.[18] The following existence result shows that the continuity assumption made in Section 2 can be weakened provided preferences are diverse (see Appendix A.4 for its proof).

Theorem 2 *A stable matching exists in weakly continuous economies with diverse preferences.*

Theorem 2 applies to a college admission setting analogous to the one considered by Cox, Fonseca, and Pakzad-Hurson (2022) and Leshno (2022) in a similar way to what has been described in Section 2.3. The only changes concerns (i) the students' type space Θ, which is now a separable subset of $[0,1]^{|F|} \times B(\tilde{F} \times S)$, where $B(\tilde{F} \times S)$ denotes the space of bounded real-valued functions on $\tilde{F} \times S$ and (ii) the function

[18] The space of measures on Θ with the weak convergence of measures is metrizable by Varadarajan (1958, Theorem 3.1).

$s : \mathcal{X}^{n+1} \to S$ which is no longer assumed to be continuous. Rather, it is assumed that the diversity of preferences assumption holds when $\succeq_{\theta|\mu}$ is defined by $f \succeq_{\theta|\mu} f'$ if and only if $u(f, s(\mu)) \geq u(f', s(\mu))$ for each $\theta = (r, u) \in \Theta$, $f, f' \in \tilde{F}$ and $\mu \in \mathcal{X}^{n+1}$. We note that neither Leshno (2022) nor Cox, Fonseca, and Pakzad-Hurson (2022) make any such separability assumption.[19] Weak continuity is satisfied since $P(f, f', M) = \{\theta \in \Theta : \pi_2(\theta)(f, s(M)) > \pi_2(\theta)(f', s(M))\}$ for each $f, f' \in \tilde{F}$ and $M \in \mathcal{X}^{n+1}$. Since $[0,1]^{|F|} \times B(\tilde{F} \times S)$ is complete, the closure of Θ is complete and separable and it follows by Parthasarathy (1967, Theorem 3.2, p. 29) that G is tight. Thus, a stable matching exists in any college admission economy with diverse preferences.

A Appendix

A.1 Proof of Theorem 1

The first step in our existence proof is to show the existence of a stable matching when Θ is finite.

A.1.1 Finite case

Lemma 2 *If Θ is finite and* $\operatorname{supp}(G) = \Theta$*, then a stable matching exists.*

Proof. Note that in this case

$$\mathcal{X} = \{\delta \in \mathbb{R}^{\Theta} : 0 \leq \delta(\theta) \leq G(\theta)\}$$

is a nonempty, convex, and compact subset of a Euclidean space.

For each $f \in \tilde{F}$, let $D_f : \mathcal{X}^{n+1} \times \mathcal{X}^{n+1} \rightrightarrows \mathcal{X}$ be defined by setting, for each $(\mu, \kappa) \in \mathcal{X}^{n+1} \times \mathcal{X}^{n+1}$, where $\mu = (\mu_f)_{f \in \tilde{F}}$ and $\kappa = (\kappa_f)_{f \in \tilde{F}}$,

$$D_f(\mu, \kappa) = C_f(\kappa_f).$$

The following claim follows by assumption (iii) on C_f.

[19] Our separability assumption holds, for instance, if there exists a subset Γ of a Euclidean space and a continuous function $U : \Gamma \to B(\tilde{F} \times S)$ such that $\Theta = [0,1]^{|F|} \times U(\Gamma)$. While Leshno (2022) makes a similar assumption, he does not require U to be continuous.

Claim 1 *For each $f \in \tilde{F}$, D_f is upper hemicontinuous with nonempty, compact and convex values.*

Let $u_\Theta : \tilde{F} \times \Theta \times \mathcal{X}^{n+1} \to \mathbb{R}$ be a bounded and continuous utility function that represents workers' preferences (see, e.g., Debreu (1964)). We normalize so that $u_\Theta \geq 1$. Let $D_\Theta : \mathcal{X}^{n+1} \rightrightarrows \mathcal{X}^{n+1}$ be defined by setting, for each $\mu \in \mathcal{X}^{n+1}$,

$$\begin{aligned} D_\Theta(\mu) =& \{\tau \in \mathcal{X}^{n+1} : \tau \in \arg\max_{\tau' \in \mathcal{X}^{n+1}} \sum_{f \in \tilde{F}, \theta \in \Theta} u_\Theta(f, \theta, \mu)\tau'_f(\theta) \\ & \text{subject to } \sum_{f \in \tilde{F}} \tau'_f(\theta) \leq G(\theta) \text{ and} \\ & \tau'_f(\theta) \leq \mu_f(\theta) \text{ for each } f \in \tilde{F} \text{ and } \theta \in \Theta\}. \end{aligned}$$

Claim 2 *D_Θ is upper hemicontinuous with nonempty, compact and convex values.*

Proof. It follows by the linearity of the objective function together with the convexity of the constraint set that D_Θ has convex values. It follows from Berge's maximum theorem that D_Θ is upper hemicontinuous with nonempty and compact values. To see this, first note that the objective function is continuous and that the constraint set, denoted by $\Phi_\Theta(\mu)$, is contained in the compact set $\mathcal{X}^{n+1}$. It is clear that Φ_Θ is upper hemicontinuous with compact and nonempty values; for the latter, note that $0 \in \Phi_\Theta(\mu)$ for each $\mu \in \mathcal{X}^{n+1}$. Finally, to see that Φ_Θ is lower hemicontinuous, let $\mu \in \mathcal{X}^{n+1}$, $O \subseteq \mathcal{X}^{n+1}$ be an open set such that $\Phi_\Theta(\mu) \cap O \neq \emptyset$, and $\tau \in \Phi_\Theta(\mu) \cap O$. Let $\hat{\tau} = \lambda\tau \in O$ for some $\lambda \in (0, 1)$. Then $\sum_{f \in \tilde{F}} \hat{\tau}_f(\theta) < G(\theta)$ for each $\theta \in \Theta$ and $\hat{\tau}_f(\theta) < \mu_f(\theta)$ for each $(f, \theta) \in \tilde{F} \times \Theta$ such that $\tau_f(\theta) > 0$, hence, $\hat{\tau} \in \Phi_\Theta(\mu')$ for each μ' in a neighborhood of μ. ■

For each $\mu \in \mathcal{X}^{n+1}$ and $(f, \theta) \in \tilde{F} \times \Theta$, let

$$\begin{aligned} W(f, \theta, \mu) &= \{f' \in \tilde{F} : (f, \mu) \succ_\theta (f', \mu)\}, \\ I(f, \theta, \mu) &= \{f' \in \tilde{F} : (f', \mu) \sim_\theta (f, \mu)\}, \\ S(f, \theta, \mu) &= \{f' \in \tilde{F} : (f', \mu) \succ_\theta (f, \mu)\}, \\ B(f, \theta, \mu) &= \{f' \in \tilde{F} : (f', \mu) \succeq_\theta (f, \mu)\} \text{ and} \\ B_-(f, \theta, \mu) &= B(f, \theta, \mu) \setminus \{f\}. \end{aligned}$$

Claim 3 *If $\mu \in \mathcal{X}^{n+1}$, $\tau \in D_\Theta(\mu)$ and $(f,\theta) \in \tilde{F} \times \Theta$ is such that*

$$\sum_{f' \in B_-(f,\theta,\mu)} \tau_{f'}(\theta) < G(\theta)$$

(in particular, if $\tau_f(\theta) > 0$), then $\tau_{f'}(\theta) = \mu_{f'}(\theta)$ for each $f' \in S(f,\theta,\mu)$.

Proof. If not, then $\tau_{f'}(\theta) < \mu_{f'}(\theta)$ for some $f' \in \tilde{F}$ such that $(f',\mu) \succ_\theta (f,\mu)$. Thus, increase $\tau_{f'}(\theta)$ while decreasing, if $\sum_{\hat{f} \in \tilde{F}} \tau_{\hat{f}}(\theta) = G(\theta)$, $\tau_{\tilde{f}}(\theta)$ by the same amount $\varepsilon \in (0, \tau_{\tilde{f}}(\theta))$, for some $\tilde{f} \in W(f,\theta,\mu) \cup \{f\}$; note that this is possible because if $\sum_{\hat{f} \in \tilde{F}} \tau_{\hat{f}}(\theta) = G(\theta)$, then $\sum_{\hat{f} \in W(f,\theta,\mu) \cup \{f\}} \tau_{\hat{f}}(\theta) = G(\theta) - \sum_{f' \in B_-(f,\theta,\mu)} \tau_{f'}(\theta) > 0$. This increases the objective function in $D_\Theta(\mu)$ while satisfying the constraints.

Finally, note that if $\tau_f(\theta) > 0$, then $\sum_{f' \in B_-(f,\theta,\mu)} \tau_{f'}(\theta) \leq G(\theta) - \tau_f(\theta) < G(\theta)$. ■

Claim 4 *Let $\mu \in \mathcal{X}^{n+1}$, $\tau \in D_\Theta(\mu)$ and $\theta \in \Theta$. If $\sum_{f \in \tilde{F}} \mu_f(\theta) > G(\theta)$, then $\sum_{f \in \tilde{F}} \tau_f(\theta) = G(\theta)$.*

Proof. If not, then $\sum_{f \in \tilde{F}} \tau_f(\theta) < G(\theta)$ and there exists $f' \in \tilde{F}$ such that $\tau_{f'}(\theta) < \mu_{f'}(\theta)$ since, otherwise, $\sum_{f \in \tilde{F}} \tau_f(\theta) = \sum_{f \in \tilde{F}} \mu_f(\theta) > G(\theta) > \sum_{f \in \tilde{F}} \tau_f(\theta)$. Thus, increase $\tau_{f'}(\theta)$ to increase the objective function while satisfying the constraints. ■

Claim 5 *Let $\mu \in \mathcal{X}^{n+1}$, $\tau \in D_\Theta(\mu)$, $\theta \in \Theta$ and $\bar{f} \in \tilde{F}$ be such that $\tau_{\bar{f}}(\theta) > 0$ and $\tau_f(\theta) = 0$ for each $f \in W(\bar{f},\theta,\mu)$.*

If

$$\sum_{f \in I(\bar{f},\theta,\mu)} \mu_f(\theta) > G(\theta) - \sum_{f \in S(\bar{f},\theta,\mu)} \mu_f(\theta), \tag{1}$$

then

$$\sum_{f \in I(\bar{f},\theta,\mu)} \tau_f(\theta) = G(\theta) - \sum_{f \in S(\bar{f},\theta,\mu)} \tau_f(\theta).$$

Proof. This claim is a corollary of Claim 4. Indeed, since $\sum_{f \in \tilde{F}} \mu_f(\theta) \geq \sum_{f \in B(\bar{f},\theta,\mu)} \mu_f(\theta) > G(\theta)$, Claim 4 implies that $\sum_{f \in \tilde{F}} \tau_f(\theta) = G(\theta)$. Since $\tau_f(\theta) = 0$ for each $f \in W(\bar{f},\theta,\mu)$, we obtain that $\sum_{f \in B(\bar{f},\theta,\mu)} \tau_f(\theta) = \sum_{f \in \tilde{F}} \tau_f(\theta) = G(\theta)$. ■

Claim 6 *Let $\mu \in \mathcal{X}^{n+1}$, $\tau \in D_\Theta(\mu)$, $\theta \in \Theta$ and $\bar{f} \in \tilde{F}$ be such that $\tau_{\bar{f}}(\theta) > 0$ and $\tau_f(\theta) = 0$ for each $f \in W(\bar{f},\theta,\mu)$.*

If

$$\sum_{f\in I(\bar{f},\theta,\mu)} \mu_f(\theta) \le G(\theta) - \sum_{f\in S(\bar{f},\theta,\mu)} \mu_f(\theta), \tag{2}$$

then $\tau_f(\theta) = \mu_f(\theta)$ *for each* $f \in B(\bar{f},\theta,\mu)$.

Proof. Note that the conclusion holds for each $f \in S(\bar{f},\theta,\mu)$ by Claim 3. We next show that it also holds for each $f \in I(\bar{f},\theta,\mu)$. Indeed, if not, then $\tau_f(\theta) < \mu_f(\theta)$ for some $f \in I(\bar{f},\theta,\mu)$. Then, by (2) and $\tau_{f'}(\theta) = 0$ for each $f' \in W(\bar{f},\theta,\mu)$, it follows that

$$\sum_{f'\in\tilde{F}} \tau_{f'}(\theta) = \sum_{f'\in B(\bar{f},\theta,\mu)} \tau_{f'}(\theta) < \sum_{f'\in B(\bar{f},\theta,\mu)} \mu_{f'}(\theta) \le G(\theta).$$

Thus increase $\tau_f(\theta)$ to increase the objective function while satisfying the constraints.

■

Let $g : \mathcal{X}^{n+1} \times D_\Theta(\mathcal{X}^{n+1}) \to \mathcal{X}^{n+1}$ be defined by setting, for each $(\mu,\tau) \in \mathcal{X}^{n+1} \times D_\Theta(\mathcal{X}^{n+1})$ and $(f,\theta) \in \tilde{F}\times\Theta$,

$$g(\mu,\tau)(f,\theta) = G(\theta) - \sum_{f'\in B_-(f,\theta,\mu)} \tau_{f'}(\theta) = \tau_f(\theta) + G(\theta) - \sum_{f'\in B(f,\theta,\mu)} \tau_{f'}(\theta).$$

For each $j \in \mathbb{N}$ and $(f,\theta) \in \tilde{F}\times\Theta$, let $\alpha_{j,(f,\theta)} : \tilde{F}\times\mathcal{X}^{n+1} \to [0,1]$ be defined by setting, for each $(f',\mu) \in \tilde{F}\times\mathcal{X}^{n+1}$,

$$\alpha_{j,(f,\theta)}(f',\mu) = j \max\left\{0, \min\left\{\frac{1}{j} + u_\Theta(f',\theta,\mu) - u_\Theta(f,\theta,\mu), \frac{1}{j}\right\}\right\}.$$

For each $j \in \mathbb{N}$, let $g_j : \mathcal{X}^{n+1} \times D_\Theta(\mathcal{X}^{n+1}) \to \mathcal{X}^{n+1}$ be defined by setting, for each $(\mu,\tau) \in \mathcal{X}^{n+1} \times D_\Theta(\mathcal{X}^{n+1})$ and $(f,\theta) \in \tilde{F}\times\Theta$,

$$g_j(\mu,\tau)(f,\theta) = \tau_f(\theta) + G(\theta) - \sum_{f'\in\tilde{F}} \alpha_{j,(f,\theta)}(f',\mu)\tau_{f'}(\theta).$$

Claim 7 *The following holds:*

1. g_j *is continuous for each* $j \in \mathbb{N}$.

2. *For each* $\mu \in \mathcal{X}^{n+1}$ *and* $j \in \mathbb{N}$, $\tau \mapsto g_j(\mu,\tau)$ *is linear.*

3. For each $(\mu,\tau) \in \mathcal{X}^{n+1} \times D_\Theta(\mathcal{X}^{n+1})$*,* $\{(\mu_j,\tau_j)\}_{j=1}^\infty \subseteq \mathcal{X}^{n+1} \times D_\Theta(\mathcal{X}^{n+1})$ *such that* $(\mu_j,\tau_j) \to (\mu,\tau)$ *and* $(f,\theta) \in \tilde{F} \times \Theta$*,*

$$g_j(\mu,\tau)(f,\theta) \leq g(\mu,\tau)(f,\theta) \text{ for each } j \in \mathbb{N}, \text{ and} \tag{3}$$

$$\liminf_j g_j(\mu_j,\tau_j)(f,\theta) \geq g(\mu,\tau)(f,\theta). \tag{4}$$

Proof. Let $j \in \mathbb{N}$. We have that g_j is continuous since $\alpha_{j,(f,\theta)}$ is continuous for each $(f,\theta) \in \tilde{F} \times \Theta$. It is clear that $\tau \mapsto g_j(\mu,\tau)$ is linear.

For each $f, f' \in \tilde{F}$, $\theta \in \Theta$, $\mu \in \mathcal{X}^{n+1}$ and $\tau \in \mathcal{X}^{n+1}$,

$$\alpha_{j,(f,\theta)}(f',\mu) \in \begin{cases} \{1\} & \text{if } u_\Theta(f',\theta,\mu) \geq u_\Theta(f,\theta,\mu), \\ (0,1) & \text{if } u_\Theta(f',\theta,\mu) - \frac{1}{j} < u_\Theta(f',\theta,\mu) < u_\Theta(f,\theta,\mu), \\ \{0\} & \text{if } u_\Theta(f',\theta,\mu) \leq u_\Theta(f,\theta,\mu) - \frac{1}{j}. \end{cases}$$

Hence, it follows that $g_j(\mu,\tau)(f,\theta) \leq g(\mu,\tau)(f,\theta)$ since

$$g(\mu,\tau)(f,\theta) = \tau_f(\theta) + G(\theta) - \sum_{f' \in \tilde{F}} \alpha_{(f,\theta)}(f',\mu)\tau_{f'}(\theta)$$

with

$$\alpha_{(f,\theta)}(f',\mu) = \begin{cases} 1 & \text{if } u_\Theta(f',\theta,\mu) \geq u_\Theta(f,\theta,\mu), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, for each $(\mu,\tau) \in \mathcal{X}^{n+1} \times D_\Theta(\mathcal{X}^{n+1})$, $(f,\theta) \in \tilde{F} \times \Theta$ and sequence $\{(\mu_j,\tau_j)\}_{j=1}^\infty \subseteq \mathcal{X}^{n+1} \times D_\Theta(\mathcal{X}^{n+1})$ such that $\lim_j(\mu_j,\tau_j) = (\mu,\tau)$,

$$\liminf_j g_j(\mu_j,\tau_j)(f,\theta) \geq g(\tau,\mu)(f,\theta).$$

Indeed, there is $J \in \mathbb{N}$ such that the following holds for each $j \geq J$: (i) if $u_\Theta(f',\theta,\mu) > u_\Theta(f,\theta,\mu)$, then $u_\Theta(f',\theta,\mu_j) > u_\Theta(f,\theta,\mu_j)$, and (ii) if $u_\Theta(f',\theta,\mu) < u_\Theta(f,\theta,\mu)$, then

$u_\Theta(f',\theta,\mu_j) < u_\Theta(f,\theta,\mu_j) - \frac{1}{j}$. Hence, for each $j \geq J$,

$$\begin{aligned}
&g_j(\mu_j,\tau_j)(f,\theta) - g(\tau,\mu)(f,\theta) = \\
&\sum_{f'\in\tilde{F}} \tau_{f'}(\theta)\big(\alpha_{(f,\theta)}(f',\mu) - \alpha_{j,(f,\theta)}(f',\mu)\big) + \\
&\sum_{f'\in\tilde{F}} \alpha_{j,(f,\theta)}(f',\mu)\big(\tau_{f'}(\theta) - \tau_{j,f'}(\theta)\big) = \\
&\sum_{f':u_\Theta(f',\theta,\mu)=u_\Theta(f,\theta,\mu)} \tau_{f'}(\theta)\big(\alpha_{(f,\theta)}(f',\mu) - \alpha_{j,(f,\theta)}(f',\mu)\big) + \\
&\sum_{f'\in\tilde{F}} \alpha_{j,(f,\theta)}(f',\mu)\big(\tau_{f'}(\theta) - \tau_{j,f'}(\theta)\big) \geq \\
&\sum_{f'\in\tilde{F}} \alpha_{j,(f,\theta)}(f',\mu)\big(\tau_{f'}(\theta) - \tau_{j,f'}(\theta)\big)
\end{aligned}$$

since $\alpha_{(f,\theta)}(f',\mu) - \alpha_{j,(f,\theta)}(f',\mu) = 1 - \alpha_{j,(f,\theta)}(f',\mu) \geq 0$ for each $f' \in \tilde{F}$ such that $u_\Theta(f',\theta,\mu) = u_\Theta(f,\theta,\mu)$. Since

$$\lim_j \sum_{f'\in\tilde{F}} \alpha_{j,(f,\theta)}(f',\mu)\big(\tau_{f'}(\theta) - \tau_{j,f'}(\theta)\big) = 0,$$

it follows that $\liminf_j g_j(\mu_j,\tau_j)(f,\theta) \geq g(\tau,\mu)(f,\theta)$. ■

For each $j \in \mathbb{N}$, let $H_j : \mathcal{X}^{n+1} \rightrightarrows \mathcal{X}^{n+1}$ be defined by setting, for each $\mu \in \mathcal{X}^{n+1}$,

$$H_j(\mu) = \{g_j(\mu,\tau) : \tau \in D_\Theta(\mu)\}.$$

Claim 8 *For each $j \in \mathbb{N}$, H_j is upper hemicontinuous with nonempty, compact and convex values.*

Proof. Since g_j is continuous and D_Θ is upper hemicontinuous with nonempty and compact values, H_j is upper-hemicontinuous with nonempty and compact values. We have that H_j is convex-valued as follows. Let $\mu \in \mathcal{X}^{n+1}$, $\kappa,\kappa' \in H_j(\mu)$ and $\lambda \in (0,1)$. Furthermore, let $\tau,\tau' \in D_\Theta(\mu)$ be such that $\kappa = g_j(\mu,\tau)$ and $\kappa' = g_j(\mu,\tau')$. Then $\lambda\tau + (1-\lambda)\tau' \in D_\Theta(\mu)$ and $g_j(\mu,\lambda\tau + (1-\lambda)\tau') = \lambda\kappa + (1-\lambda)\kappa'$ by Claim 7. Hence, $\lambda\kappa + (1-\lambda)\kappa' \in H_j(\mu)$. ■

For each $j \in \mathbb{N}$, let $\Psi_j : \mathcal{X}^{n+1} \times \mathcal{X}^{n+1} \rightrightarrows \mathcal{X}^{n+1} \times \mathcal{X}^{n+1}$ be defined by setting, for each $(\mu,\kappa) \in \mathcal{X}^{n+1} \times \mathcal{X}^{n+1}$,

$$\Psi_j(\mu,\kappa) = \prod_{f\in\tilde{F}} D_f(\mu,\kappa) \times H_j(\mu).$$

It follows from Claims 1 and 8 that Ψ_j is upper hemicontinuous with nonempty, compact and convex values. Hence, by the Kakutani fixed point theorem, let (μ_j, κ_j) be a fixed point of Ψ_j and $\tau_j \in D_\Theta(\mu_j)$ be such that $\kappa_j = g_j(\mu_j, \tau_j)$.

Since $\mathcal{X}^{n+1} \times \mathcal{X}^{n+1} \times D_\Theta(\mathcal{X}^{n+1})$ is compact, taking a subsequence if necessary, we may assume that $\{(\mu_j, \kappa_j, \tau_j)\}_{j=1}^{\infty}$ converges; let $(\mu, \kappa, \tau) = \lim_{j\to\infty}(\mu_j, \kappa_j, \tau_j)$. We have that $\tau \in D_\Theta(\mu)$ since D_Θ is compact-valued and upper-hemicontinuous, and $\mu \in \prod_{f\in\tilde{F}} D_f(\mu, \kappa)$ since $\prod_{f\in\tilde{F}} D_f$ is compact-valued and upper-hemicontinuous.

Claim 9 *For each $\theta \in \Theta$ and $j \in \mathbb{N}$, there exists $\bar{f}_j \in \tilde{F}$ such that $\tau_{j,\bar{f}_j}(\theta) > 0$ and $\tau_{j,f}(\theta) = 0$ for each $f \in W(\bar{f}_j, \theta, \mu_j)$.*

Proof. Let $\theta \in \Theta$ and $j \in \mathbb{N}$. First note that $\sum_{f\in\tilde{F}} \tau_{j,f}(\theta) > 0$. Suppose not; then $\tau_{j,f}(\theta) = 0$ for each $f \in \tilde{F}$. Hence, $\mu_{j,f}(\theta) = 0$ for each $f \in \tilde{F}$ since $\tau_j \in D_\Theta(\mu_j)$ and $\kappa_{j,\emptyset}(\theta) = g_j(\mu_j, \tau_j)(\emptyset, \theta) = G(\theta)$. But this contradicts $\mu_{j,\emptyset}(\theta) = \kappa_{j,\emptyset}(\theta) = G(\theta) > 0$, which follows since $\mu_{j,\emptyset} \in C_\emptyset(\kappa_{j,\emptyset})$.

Partition $\tilde{F}$ using $\cup_{l=1}^{L} I(f_l, \theta, \mu_j)$ for some $L \in \{1, \ldots, n+1\}$ and $f_1, \ldots, f_L \in \tilde{F}$, where $(f, \mu_j) \succ_\theta (f', \mu_j)$ whenever $f \in I(f_l, \theta, \mu_j)$, $f' \in I(f_{l'}, \theta, \mu_j)$ and $l < l'$. Define $L^* = \min\{l \in \{1, \ldots, L\} : \sum_{f\in I(f_l,\theta,\mu_j)} \tau_{j,f}(\theta) = 0\}$ with the convention that $L^* = L + 1$ if $\{l \in \{1, \ldots, L\} : \sum_{f\in I(f_l,\theta,\mu_j)} \tau_{j,f}(\theta) = 0\} = \emptyset$. Since $\sum_{f\in\tilde{F}} \tau_{j,f}(\theta) > 0$, it follows that $L^* > 1$ and, thus, $\sum_{f\in I(f_{L^*-1},\theta,\mu_j)} \tau_{j,f}(\theta) > 0$ by the definition of L^*. Thus, let $f \in I(f_{L^*-1}, \theta, \mu_j)$ be such that $\tau_{j,f}(\theta) > 0$ and set $\bar{f}_j = f$. ■

Claim 10 *μ is a matching.*

Proof. Let $\theta \in \Theta$ and $j \in \mathbb{N}$. Let, by Claim 9, $\bar{f}_j \in \tilde{F}$ be such that $\tau_{j,\bar{f}_j}(\theta) > 0$ and $\tau_{j,f}(\theta) = 0$ for each $f \in W(\bar{f}_j, \theta, \mu_j)$.

We first show that (1) with μ_j in place of μ cannot hold. Suppose for a contradiction that

$$\sum_{f\in I(\bar{f}_j,\theta,\mu_j)} \mu_{j,f}(\theta) > G(\theta) - \sum_{f\in S(\bar{f}_j,\theta,\mu_j)} \mu_{j,f}(\theta). \tag{5}$$

Since $\mu_j \in \prod_{f\in\tilde{F}} D_f(\mu_j, \kappa_j)$ and $\kappa_j = g_j(\mu_j, \tau_j)$, assumption (i) on $C_{\bar{f}_j}$ and Claim 7 imply that

$$\mu_{j,\bar{f}_j}(\theta) \le g_j(\mu_j, \tau_j)(\bar{f}_j, \theta) \le g(\mu_j, \tau_j)(\bar{f}_j, \theta) = G(\theta) - \sum_{f\in B_-(\bar{f}_j,\theta,\mu_j)} \tau_{j,f}(\theta).$$

Hence,

$$G(\theta) \geq \mu_{j,\bar{f}_j}(\theta) + \sum_{f \in B_-(\bar{f}_j,\theta,\mu_j)} \tau_{j,f}(\theta). \tag{6}$$

Let $\beta = |I(\bar{f}_j, \theta, \mu_j)|$. If $\beta > 1$, since (6) holds for each $f' \in I(\bar{f}_j, \theta, \mu_j)$, it follows that

$$\beta\big(G(\theta) - \sum_{f \in S(\bar{f}_j,\theta,\mu_j)} \tau_{j,f}(\theta)\big) \geq \sum_{f \in I(\bar{f}_j,\theta,\mu_j)} \big(\mu_{j,f}(\theta) + (\beta - 1)\tau_{j,f}(\theta)\big).$$

Since $\tau_{j,\bar{f}_j}(\theta) > 0$, Claim 3 implies that $\tau_{j,f}(\theta) = \mu_{j,f}(\theta)$ for each $f \in S(\bar{f}_j, \theta, \mu_j)$ and this, together with (5), implies that

$$\beta\big(G(\theta) - \sum_{f \in S(\bar{f}_j,\theta,\mu_j)} \mu_{j,f}(\theta)\big) > G(\theta) - \sum_{f \in S(\bar{f}_j,\theta,\mu_j)} \mu_{j,f}(\theta) + (\beta - 1) \sum_{f \in I(\bar{f}_j,\theta,\mu_j)} \tau_{j,f}(\theta).$$

Thus,

$$G(\theta) - \sum_{f \in S(\bar{f}_j,\theta,\mu_j)} \tau_{j,f}(\theta) = G(\theta) - \sum_{f \in S(\bar{f}_j,\theta,\mu_j)} \mu_{j,f}(\theta) > \sum_{f \in I(\bar{f}_j,\theta,\mu_j)} \tau_{j,f}(\theta),$$

a contradiction to Claim 5.

If $\beta = 1$, then by (5)

$$\mu_{j,\bar{f}_j}(\theta) > G(\theta) - \sum_{f \in S(\bar{f}_j,\theta,\mu_j)} \mu_{j,f}(\theta) = G(\theta) - \sum_{f \in S(\bar{f}_j,\theta,\mu_j)} \tau_{j,f}(\theta)$$

since $\tau_{j,f}(\theta) = \mu_{j,f}(\theta)$ for each $f \in S(\bar{f}_j, \theta, \mu_j)$ by Claim 3. Since

$$\sum_{f \in B_-(\bar{f}_j,\theta,\mu_j)} \tau_{j,f}(\theta) = \sum_{f \in S(\bar{f}_j,\theta,\mu_j)} \tau_{j,f}(\theta)$$

due to $\beta = 1$, it follows from (6) that

$$G(\theta) > G(\theta) - \sum_{f \in S(\bar{f}_j,\theta,\mu)} \tau_{j,f}(\theta) + \sum_{f \in B_-(\bar{f}_j,\theta,\mu_j)} \tau_{j,f}(\theta) = G(\theta),$$

a contradiction.

It follows from the above argument that

$$\sum_{f \in I(\bar{f}_j,\theta,\mu_j)} \mu_{j,f}(\theta) \leq G(\theta) - \sum_{f \in S(\bar{f}_j,\theta,\mu_j)} \mu_{j,f}(\theta) \tag{7}$$

for each $j \in \mathbb{N}$. Fix $j \in \mathbb{N}$. It follows by Claim 6 that

$$\tau_{j,f}(\theta) = \mu_{j,f}(\theta) \text{ for each } f \in B(\bar{f}_j, \theta, \mu_j). \tag{8}$$

We have that $\mu_{j,\emptyset}(\theta) = g_j(\mu_j, \tau_j)(\emptyset, \theta)$ for each $j \in \mathbb{N}$ since $\mu_{j,\emptyset} \in D_\emptyset(\mu_j, \kappa_j)$ and $\kappa_j = g_j(\mu_j, \tau_j)$. Hence, by Claim 7,

$$\mu_\emptyset(\theta) = \lim_j g_j(\mu_j, \tau_j)(\emptyset, \theta) \geq g(\mu, \tau)(\emptyset, \theta) = G(\theta) - \textstyle\sum_{f \in B_-(\emptyset,\theta,\mu)} \tau_f(\theta).$$

We have that $\tau_f(\theta) \leq \mu_f(\theta)$ for each $f \in \tilde{F}$ since $\tau \in D_\Theta(\mu)$, hence

$$\sum_{f \in \tilde{F}} \mu_f(\theta) \geq \sum_{f \in B(\emptyset,\theta,\mu)} \mu_f(\theta) \geq \mu_\emptyset(\theta) + \sum_{f \in B_-(\emptyset,\theta,\mu)} \tau_f(\theta) \geq G(\theta). \tag{9}$$

If $\sum_{f \in \tilde{F}} \mu_f(\theta) > G(\theta)$, then $\sum_{f \in \tilde{F}} \tau_f(\theta) = G(\theta)$ by Claim 4 since $\tau \in D_\Theta(\mu)$. For each j and $f \in W(\bar{f}_j, \theta, \mu_j)$,

$$\begin{aligned} &\mu_{j,f}(\theta) \leq g_j(\mu_j, \tau_j)(f, \theta) \leq g(\mu_j, \tau_j)(f, \theta) = G(\theta) - \sum_{f' \in B_-(f,\theta,\mu_j)} \tau_{j,f'}(\theta) \\ &= G(\theta) - \sum_{f' \in B(\bar{f}_j,\theta,\mu_j)} \tau_{j,f'}(\theta) = G(\theta) - \sum_{f' \in \tilde{F}} \tau_{j,f'}(\theta) \end{aligned}$$

by assumption (i) on C_f, Claim 7 and since $\tau_{j,f'}(\theta) = 0$ for each $f' \in W(\bar{f}_j, \theta, \mu_j)$. Thus, by the above, (8) and $\tau_{j,f'}(\theta) = 0$ for each $f' \in W(\bar{f}_j, \theta, \mu_j)$,

$$\begin{aligned} \textstyle\sum_{f \in \tilde{F}} \mu_{j,f}(\theta) \quad &\leq \sum_{f \in B(\bar{f}_j,\theta,\mu_j)} \mu_{j,f}(\theta) + |W(\bar{f}_j, \theta, \mu_j)| \big(G(\theta) - \sum_{f \in \tilde{F}} \tau_{j,f}(\theta)\big) \\ &= \sum_{f \in \tilde{F}} \tau_{j,f}(\theta) + |W(\bar{f}_j, \theta, \mu_j)| \big(G(\theta) - \sum_{f \in \tilde{F}} \tau_{j,f}(\theta)\big). \end{aligned}$$

Since $\sum_{f \in \tilde{F}} \tau_{j,f}(\theta) \to \sum_{f \in \tilde{F}} \tau_f(\theta) = G(\theta)$, it follows that

$$\sum_{f \in \tilde{F}} \mu_f(\theta) = \lim_j \sum_{f \in \tilde{F}} \mu_{j,f}(\theta) \leq \sum_{f \in \tilde{F}} \tau_f(\theta) = G(\theta),$$

a contradiction. Thus, $\sum_{f \in \tilde{F}} \mu_f(\theta) \leq G(\theta)$ and this, together with (9), implies that $\sum_{f \in \tilde{F}} \mu_f(\theta) = G(\theta)$. ■

Claim 11 *For each* $f \in F$, $\mu_f(P(\emptyset, f, \mu)) = 0$.

Proof. Suppose not; then there exists $f \in F$ and $\theta \in \Theta$ such that $\mu_f(\theta) > 0$ and $(\emptyset, \mu) \succ_\theta (f, \mu)$. For sufficiently large j, $\mu_{j,f}(\theta) > 0$ and $(\emptyset, \mu_j) \succ_\theta (f, \mu_j)$. Since $\mu_{j,f} \in D_f(\mu_j, \kappa_j)$, it follows by assumption (i) on C_f and Claim 7 that

$$
\begin{aligned}
0 < \mu_{j,f}(\theta) &\leq \kappa_{j,f}(\theta) = g_j(\mu_j, \tau_j)(f, \theta) \\
&\leq g(\mu_j, \tau_j)(f, \theta) = G(\theta) - \sum_{f' \in B_-(f,\theta,\mu_j)} \tau_{j,f'}(\theta).
\end{aligned}
$$

This then implies that $\tau_{j,\emptyset}(\theta) = \mu_{j,\emptyset}(\theta)$ by Claim 3 since $\tau_j \in D_\Theta(\mu_j)$.

Since $\tau_{j,\emptyset}(\theta) = \mu_{j,\emptyset}(\theta)$ and $\mu_{j,\emptyset}(\theta) = \kappa_{j,\emptyset}(\theta)$ (since $\mu_{j,\emptyset} \in D_\emptyset(\mu_j, \kappa_j)$) for each j sufficiently large, using Claim 7 it follows that

$$
\begin{aligned}
\tau_\emptyset(\theta) &= \mu_\emptyset(\theta) = \lim_j \mu_{j,\emptyset}(\theta) = \lim_j \kappa_{j,\emptyset}(\theta) = \lim_j g_j(\mu_j, \tau_j)(\emptyset, \theta) \\
&\geq g(\mu, \tau)(\emptyset, \theta) = G(\theta) - \sum_{f' \in B_-(\emptyset,\theta,\mu)} \tau_{f'}(\theta).
\end{aligned}
$$

Then $\sum_{f' \in B(\emptyset,\theta,\mu)} \tau_{f'}(\theta) \geq G(\theta)$ and, since $\tau \in D_\Theta(\mu)$, $\sum_{f' \in B(\emptyset,\theta,\mu)} \mu_{f'}(\theta) \geq G(\theta)$. Since μ is a matching by Claim 10, it follows that $\sum_{f' \in B(\emptyset,\theta,\mu)} \mu_{f'}(\theta) = G(\theta)$ and that $\mu_f(\theta) = 0$. But this contradicts $\mu_f(\theta) > 0$. ■

Claim 12 *For each $f \in F$, $D^{\preceq f}(\mu) \sqsubset \kappa_f$.*

Proof. For each $\theta \in \Theta$,

$$
D^{\preceq f}(\mu)(\theta) = \mu_f(\theta) + \sum_{f' \in W(f,\theta,\mu)} \mu_{f'}(\theta)
$$

and

$$
\begin{aligned}
\kappa_f(\theta) &= \lim_j \kappa_{j,f}(\theta) = \lim_j g_j(\mu_j, \tau_j)(f, \theta) \geq g(\mu, \tau)(f, \theta) \\
&= G(\theta) - \sum_{f' \in B_-(f,\theta,\mu)} \tau_{f'}(\theta) \geq G(\theta) - \sum_{f' \in B_-(f,\theta,\mu)} \mu_{f'}(\theta) \\
&= \mu_f(\theta) + \sum_{f' \in W(f,\theta,\mu)} \mu_{f'}(\theta)
\end{aligned}
$$

where the first inequality follows by Claim 7, the second inequality since $\tau \in D_\Theta(\mu)$ and the last equality since μ is a matching by Claim 10. ■

Claim 13 *There does not exist* $f \in F$ *and* $\delta \in \mathcal{X}$ *such that* $\delta \sqsubset D^{\preceq_f}(\mu)$ *and* $\delta \succ_f \mu_f$.

Proof. Suppose otherwise; then there is $f \in F$ and $\delta \in \mathcal{X}$ such that $\delta \sqsubset D^{\preceq_f}(\mu)$ and $\delta \succ_f \mu_f$. By Claim 12, $D^{\preceq_f}(\mu) \sqsubset \kappa_f$. Hence, $\delta \sqsubset \kappa_f$.

Since $\mu_f \in C_f(\kappa_f)$, we have that $\mu_f \sqsubset \kappa_f$ by assumption (i) on C_f. Thus, assumption (ii) on C_f implies that $\mu_f \in C_f(\mu_f \vee \delta)$. But this is a contradiction to $\delta \succ_f \mu_f$. ■

It follows from Claims 10, 11 and 13 that μ is a stable matching. ■

A.1.2 Limit argument

Let $\{G_k\}_{k=1}^{\infty}$ be such that $G_k \to G$ and, for each k, $\text{supp}(G_k) = \Theta_k$ is a finite subset of Θ and $G_k(\Theta) = 1$ (see, e.g., Parthasarathy (1967, Theorem 6.3, p. 44)). Let $\mathcal{X}_k = \{X \in \bar{\mathcal{X}} : X \sqsubset G_k\}$. For each $k \in \mathbb{N}$ and $f \in \tilde{F}$, define $D_{k,f} : \mathcal{X}_k^{n+1} \times \mathcal{X}_k^{n+1} \rightrightarrows \mathcal{X}_k$ and $D_{k,\Theta} : \mathcal{X}_k^{n+1} \rightrightarrows \mathcal{X}_k^{n+1}$ as in the finite case, with Θ_k in place of Θ, G_k in place of G, and $\mathcal{X}_k$ in place of $\mathcal{X}$. Note that, for each $(\mu, \kappa) \in \mathcal{X}_k^{n+1} \times \mathcal{X}_k^{n+1}$, if $\delta \in D_{k,f}(\mu, \kappa) = C_f(\kappa_f)$, then $\delta \sqsubset \kappa_f \sqsubset G_k$ and, hence, $D_{k,f}(\mu, \kappa) \subseteq \mathcal{X}_k$.

For each $k \in \mathbb{N}$, let $g_{k,j} : \mathcal{X}_k^{n+1} \times D_{k,\Theta}(\mathcal{X}_k^{n+1}) \to \mathcal{X}_k^{n+1}$ be defined for each $j \in \mathbb{N}$ by setting, for each $(\mu, \tau) \in \mathcal{X}_k^{n+1} \times D_{k,\Theta}(\mathcal{X}_k^{n+1})$, $f \in \tilde{F}$, and $\theta \in \Theta_k$,

$$g_{k,j}(\mu, \tau)(f, \theta) = \tau_f(\theta) + G_k(\theta) - \sum_{f' \in \tilde{F}} \alpha_{j,(f,\theta)}(f', \mu)\tau_{f'}(\theta).$$

By Lemma 2, for each k, there exists a stable matching μ_k when the set of types is Θ_k and the distribution is G_k. Furthermore, for each k, we also have $\mu_k \in \prod_{f \in \tilde{F}} D_{k,f}(\mu_k, \kappa_k)$ and $\tau_k \in D_{k,\Theta}(\mu_k)$ as the proof of Lemma 2 shows. In particular, $\mu_k \in \mathcal{X}_k^{n+1}$, $\tau_k \in \mathcal{X}_k^{n+1}$ and $\kappa_k \in \mathcal{X}_k^{n+1}$. Since G is tight, it follows that $\{G_k\}_{k=1}^{\infty}$ is tight and, hence, $\{(\mu_k, \kappa_k, \tau_k)\}_{k=1}^{\infty}$ is tight. Thus, taking a subsequence if necessary, we may assume that $\{(\mu_k, \kappa_k, \tau_k)\}_{k=1}^{\infty}$ converges to some $(\mu, \kappa, \tau) = \lim_{k \to \infty}(\mu_k, \kappa_k, \tau_k)$.[20] It then follows from Lemma 3 below, which is analogous to Lemma 7 in CKK, that $(\mu, \kappa, \tau) \in \mathcal{X}^{n+1} \times \mathcal{X}^{n+1} \times \mathcal{X}^{n+1}$.

[20] This result follows essentially by Aliprantis and Border (2006, Lemma 15.21, p. 518) by reducing the problem to the case of probability measures, which is possible since $\mu_{k,f}(\Theta) \leq 1$ for each $k \in \mathbb{N}$ and $f \in \tilde{F}$ and analogously for κ_k and τ_k.

Lemma 3 *Let X be a metric space, $\mu, \nu \in M(X)$ and $\{\mu_k\}_{k=1}^{\infty}, \{\nu_k\}_{k=1}^{\infty} \subseteq M(X)$ be such that $\mu_k \to \mu$ and $\nu_k \to \nu$. If $\nu_k \sqsubset \mu_k$ for each $k \in \mathbb{N}$, then $\nu \sqsubset \mu$.*

Proof. Suppose first that $\nu_k = 0$ for each $k \in \mathbb{N}$. Then $0 \leq \limsup_k \mu_k(C) \leq \mu(C)$ for each closed subset C of X. Measures on metric spaces are regular (see, e.g., Parthasarathy (1967, Theorem 1.2, p. 27)), hence, for each Borel $B \subseteq X$,

$$\mu(B) = \sup\{\mu(C) : C \text{ is a closed subset of } B\} \geq 0.$$

For the general case, define $\psi_k = \mu_k - \nu_k$. Then $\psi_k \to \mu - \nu$ and $0 \sqsubset \psi_k$ for each $k \in \mathbb{N}$. Thus, by what was shown in the above paragraph, $0 \leq \mu(B) - \nu(B)$ for each Borel $B \subseteq X$ and, hence, $\nu \sqsubset \mu$. ■

Claim 14 *μ is a matching.*

Proof. Since μ_k is a stable matching for G_k, for each $f \in \tilde{F}$, $\mu_{k,f} \sqsubset G_k$. By Lemma 3, $\mu_f \sqsubset G$.

Similarly, we have that $\sum_{f \in \tilde{F}} \mu_f = G$, since $\sum_{f \in \tilde{F}} \mu_f = \lim_{k \to \infty} \sum_{f \in \tilde{F}} \mu_{k,f} = \lim_{k \to \infty} G_k = G$. ■

Claim 15 *For each $f \in F$, $\mu \in D_f(\mu, \kappa)$.*

Proof. Since $\mu_{k,f} \sqsubset \kappa_{k,f}$ for each k, $\mu_f \sqsubset \kappa_f$ by Lemma 3. Furthermore, for each k, $\mu_{k,f} \in D_{k,f}(\mu_k, \kappa_k) = C_f(\kappa_{k,f})$. Since C_f is closed, it follows that $\mu_f \in C_f(\kappa_f) = D_f(\mu, \kappa)$. ■

Claim 16 *For each $f \in F$, $\mu_f(P(\emptyset, f, \mu)) = 0$.*

Proof. Suppose not; then there exists $f \in F$ such that $\mu_f(P(\emptyset, f, \mu)) > 0$. Note that $P(\emptyset, f, \mu)$ is open by the continuity assumption on workers' preferences. Since $\mu_k \to \mu$, we have that $\liminf_{k \to \infty} \mu_{k,f}(P(\emptyset, f, \mu)) \geq \mu_f(P(\emptyset, f, \mu)) > 0$. Thus, for sufficiently large k, there exists $\theta \in P(\emptyset, f, \mu)$ such that $\mu_{k,f}(\theta) > 0$ and $(\emptyset, \mu_k) \succ_\theta (f, \mu_k)$ (the latter again following from continuity of preferences), contradicting the stability of μ_k. ■

Claim 17 *For each $f \in F$, $D^{\preceq f}(\mu) \sqsubset \kappa_f$.*

Proof. For each k, we have $D^{\preceq f}(\mu_k) \sqsubset \kappa_{k,f}$ by Claim 12.

For each $E \in \Sigma$, we have:

$$D^{\preceq f}(\mu)(E) = \mu_f(E) + \sum_{f' \neq f} \mu_{f'}(E \cap P(f, f', \mu)).$$

Note that for each open set $O \subseteq \Theta$, $\mu_f(O) \leq \liminf_{k\to\infty} \mu_{k,f}(O)$. In addition, for each $f' \neq f$, we claim that

$$\mu_{f'}(O \cap P(f, f', \mu)) \leq \liminf_{k\to\infty} \mu_{k,f'}(O \cap P(f, f', \mu_k)).$$

This claim can be established as follows. For each $\theta \in O \cap P(f, f', \mu)$, there are open neighborhoods $U_\theta \subseteq O \cap P(f, f', \mu)$ and V_θ of θ and μ, respectively, such that $(f, \mu') \succ_{\theta'} (f', \mu')$ for each $\theta' \in U_\theta$ and $\mu' \in V_\theta$. Since Θ is separable, there exists a countable subcover $\{U_{\theta_j}\}_{j=1}^\infty$ of $O \cap P(f, f', \mu)$, i.e. $O \cap P(f, f', \mu) = \cup_{j=1}^\infty U_{\theta_j}$. Let $\varepsilon > 0$; then there is $J \in \mathbb{N}$ such that $\mu_{f'}(O \cap P(f, f', \mu)) \leq \mu_{f'}(\cup_{j=1}^J U_{\theta_j}) + \varepsilon$ and $K \in \mathbb{N}$ such that $\mu_k \in \cap_{j=1}^J V_{\theta_j}$ for each $k \geq K$. The latter implies that, for each $k \geq K$, $\cup_{j=1}^J U_{\theta_j} \subseteq O \cap P(f, f', \mu_k)$. Since $\cup_{j=1}^J U_{\theta_j}$ is open,

$$\begin{aligned}&\mu_{f'}(O \cap P(f, f', \mu)) \leq \mu_{f'}(\cup_{j=1}^J U_{\theta_j}) + \varepsilon \leq \liminf_k \mu_{k,f'}(\cup_{j=1}^J U_{\theta_j}) + \varepsilon \\ &\leq \liminf_k \mu_{k,f'}(O \cap P(f, f', \mu_k)) + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the claim follows.

It then follows that for each open set O,

$$\begin{aligned}D^{\preceq f}(\mu)(O) &\leq \liminf_{k\to\infty} \mu_{k,f}(O) + \liminf_{k\to\infty} \sum_{f' \neq f} \mu_{k,f'}(O \cap P(f, f', \mu_k)) \\ &\leq \liminf_{k\to\infty} D^{\preceq f}(\mu_k)(O) \leq \liminf_{k\to\infty} \kappa_{k,f}(O).\end{aligned}$$

We claim that for each closed subset C of Θ,

$$D^{\preceq f}(\mu)(C) \leq \kappa_f(C).$$

Let C be a closed subset of Θ, $\eta > 0$ and, for each $\varepsilon > 0$, let $B_\varepsilon = \{\theta \in \Theta : d(\theta, C) < \varepsilon\}$ and $S_\varepsilon = \{\theta \in \Theta : d(\theta, C) = \varepsilon\}$. Since $B_\varepsilon \downarrow C$, there is $\bar{\varepsilon} > 0$ such that

$\kappa_f(B_\varepsilon) < \kappa_f(C) + \eta$ for each $0 < \varepsilon < \bar{\varepsilon}$. Since the family $\{S_\varepsilon : 0 < \varepsilon < \bar{\varepsilon}\}$ is pairwise disjoint, it follows that $\kappa_f(S_\varepsilon) = 0$ for all but countably many $\varepsilon \in (0, \bar{\varepsilon})$. Thus, let $\varepsilon^* \in (0, \bar{\varepsilon})$ be such that $\kappa_f(S_{\varepsilon^*}) = 0$. Since the boundary of B_{ε^*} is contained in S_{ε^*} and $\kappa_k \to \kappa$, it follows that $\lim_{k\to\infty} \kappa_{k,f}(B_{\varepsilon^*}) = \kappa_f(B_{\varepsilon^*})$. Hence, since B_{ε^*} is open,

$$D^{\preceq f}(\mu)(C) \leq D^{\preceq f}(\mu)(B_{\varepsilon^*}) \leq \liminf_{k\to\infty} \kappa_{k,f}(B_{\varepsilon^*}) = \kappa_f(B_{\varepsilon^*}) < \kappa_f(C) + \eta.$$

Since $\eta > 0$ is arbitrary, this establishes the above claim.

Let $E \in \Sigma$ and $\varepsilon > 0$. By regularity, let C be a closed subset of E such that $D^{\preceq f}(\mu)(E) \leq D^{\preceq f}(\mu)(C) + \varepsilon$. Hence, $\kappa_f(C) \leq \kappa_f(E)$ and

$$\kappa_f(E) - D^{\preceq f}(\mu)(E) \geq \kappa_f(C) - D^{\preceq f}(\mu)(C) - \varepsilon \geq -\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\kappa_f(E) \geq D^{\preceq f}(\mu)(E)$. Thus, $D^{\preceq f}(\mu) \sqsubset \kappa_f$. ■

Claim 18 *There does not exist $f \in F$ and $M_f \sqsubset D^{\preceq f}(\mu)$ such that $M_f \succ_f \mu_f$.*

Proof. Suppose there exists $f \in F$ and $M_f \sqsubset D^{\preceq f}(\mu)$ such that $M_f \succ_f \mu_f$. By Claim 17, $D^{\preceq f}(\mu) \sqsubset \kappa_f$. Hence, $M_f \sqsubset \kappa_f$.

Since $\mu_f \in D_f(\mu, \kappa) = C_f(\kappa_f)$ by Claim 15, we have that $\mu_f \sqsubset \kappa_f$ by assumption (i) on C_f. Thus, assumption (ii) on C_f implies that $\mu_f \in C_f(\mu_f \vee M_f)$. But this is a contradiction to $M_f \succ_f \mu_f$. ■

It follows from Claims 14, 16, and 18 that μ is a stable matching.

A.2 Proof of Corollary 1

Let $f \in F$. It follows from the definition of C_f that assumptions (i) and (ii) in Section 2 hold.

Let $\Phi : \mathcal{X} \rightrightarrows \mathcal{X}$ be defined by setting, for each $X \in \mathcal{X}$, $\Phi(X) = \{\delta \in \mathcal{X} : \delta \sqsubset X\}$. Since Φ is convex-valued and $\succeq_f$ is convex, it follows that C_f is convex-valued. We have that u_F is continuous and Φ is compact-valued, hence C_f is nonempty-valued.

To conclude the proof, we next show that C_f is closed. This requires the following lemma, which builds on Lemma 6 in CKK. Note first that the space of measures on Θ with the weak convergence of measures is metrizable by Varadarajan (1958, Theorem 3.1), hence, let ρ be a metric on this space.

Lemma 4 *Let $\delta \in \mathcal{X}$ and $\kappa \in \mathcal{X}$ be such that $\delta \sqsubset \kappa$, $\{G_k\}_{k=1}^{\infty} \subseteq \bar{\mathcal{X}}$ be such that $G_k \to G$ and* $\text{supp}(G_k)$ *is finite for each $k \in \mathbb{N}$ and $\{\kappa_k\}_{k=1}^{\infty} \subseteq \mathcal{X}$ be such that $\kappa_k \sqsubset G_k$ and $\kappa_k \to \kappa$. Then, there is a subsequence $\{\kappa_{k_j}\}_{j=1}^{\infty}$ of $\{\kappa_k\}_{k=1}^{\infty}$ and a sequence $\{\delta_{k_j}\}_{j=1}^{\infty}$ such that $\delta_{k_j} \to \delta$ and $\delta_{k_j} \sqsubset \kappa_{k_j}$ for each $j \in \mathbb{N}$.*

Proof. We have that G is tight. Thus, for each $j \in \mathbb{N}$, there exists a compact subset K_j of Θ such that $G(\Theta \setminus K_j) < 1/j$; hence, $\delta(\Theta \setminus K_j) \leq \kappa(\Theta \setminus K_j) < 1/j$. Since $\partial K_j = K_j \setminus \text{int}(K_j)$ is closed, by replacing K_j with $\{\theta \in K_j : d(\theta, \partial K_j) \geq \varepsilon\}$ for some appropriately chosen $\varepsilon \geq 0$, we may assume that $G(\partial K_j) = 0$;[21] in particular, $\kappa(\partial K_j) = 0$.

For each $j \in \mathbb{N}$, define G_j, κ_j and δ_j by setting, for each $E \in \Sigma$,

$$G_j(E) = G(E \cap K_j), \kappa_j(E) = \kappa(E \cap K_j) \text{ and } \delta_j(E) = \delta(E \cap K_j).$$

Then, for each $j \in \mathbb{N}$, $\delta_j \sqsubset \kappa_j \sqsubset G_j$, $\text{supp}(G_j) = K_j$ is compact, $\kappa_j \sqsubset \kappa$ and $(\delta_j, \kappa_j) \to (\delta, \kappa)$.

For each $j, k \in \mathbb{N}$, define $G_{j,k}$ and $\kappa_{j,k}$ by setting, for each $E \in \Sigma$,

$$G_{j,k}(E) = G_k(E \cap K_j) \text{ and } \kappa_{j,k}(E) = \kappa_k(E \cap K_j).$$

We have that $\kappa_{j,k} \sqsubset G_{j,k}$, $\kappa_{j,k} \sqsubset \kappa_k$ and $\lim_k(G_{j,k}, \kappa_{j,k}) = (G_j, \kappa_j)$ for each $j, k \in \mathbb{N}$. Indeed, for the latter, $\lim_k \kappa_{j,k}(\Theta) = \lim_k \kappa_k(K_j) = \kappa(K_j) = \kappa_j(\Theta)$ since $\kappa(\partial K_j) = 0$ and $\kappa_k \to \kappa$. Furthermore, for each closed subset C of Θ, $C \cap K_j$ is closed and, hence, $\limsup_k \kappa_{j,k}(C) = \limsup_k \kappa_k(C \cap K_j) \leq \kappa(C \cap K_j) = \kappa_j(C)$. The argument to show that $G_{j,k} \to G_j$ is analogous.

Thus, Lemma 6 in CKK yields, for each $j, k \in \mathbb{N}$, $\delta_{j,k} \in \mathcal{X}_k$ such that $\delta_{j,k} \sqsubset \kappa_{j,k}$ and $\lim_k \delta_{j,k} = \delta_j$. For each $j \in \mathbb{N}$, assuming that $k_1, \ldots, k_{j-1}$ have been chosen, let $k_j \in \mathbb{N}$ be such that $k_j > k_{j-1}$ and $\rho(\delta_{j,k_j}, \delta_j) < 1/j$. Hence, $\delta_{j,k_j} \to \delta$ and, for each $j \in \mathbb{N}$, $\delta_{j,k_j} \in \mathcal{X}_{k_j}$ and $\delta_{j,k_j} \sqsubset \kappa_{j,k_j} \sqsubset \kappa_{k_j}$. ∎

Let $X_k \to X$ and $\delta_k \to \delta$ such that $\delta_k \in C_f(X_k)$ for each $k \in \mathbb{N}$ and suppose that $\delta \notin C_f(X)$. Then there exists $\tau \in \mathcal{X}$ such that $\tau \sqsubset X$ and $u_F(f, \tau) > u_F(f, \delta)$.

[21] Indeed, letting $K_\varepsilon = \{\theta \in K_j : d(\theta, \partial K_j) \geq \varepsilon\}$, we have that $K_\varepsilon \uparrow K_j$ as $\varepsilon \downarrow 0$ and that $\partial K_\varepsilon \subseteq \{\theta \in K_j : d(\theta, \partial K_j) = \varepsilon\}$. Thus, $G(\partial K_\varepsilon) = 0$ for all but countably many $\varepsilon \geq 0$.

Fix $k \in \mathbb{N}$. Since Θ is separable, there exists a sequence $\{(X_{k,j}, \delta_{k,j}, G_j)\}_{j=1}^{\infty}$ such that $\lim_j (X_{k,j}, \delta_{k,j}, G_j) = (X_k, \delta_k, G)$ and, for each $j \in \mathbb{N}$, $X_{k,j}$, $\delta_{k,j}$ and G_j have finite support, and $\delta_{k,j} \sqsubset X_{k,j} \sqsubset G_j$. Since Φ is upper hemicontinuous and u_F is continuous, the value function $X' \mapsto \max_{\delta' \sqsubset X'} u_F(f, \delta')$ is upper semi-continuous. Hence, there is $j_k \in \mathbb{N}$ such that $\rho(X_{k,j_k}, X_k) < 1/k$, $\rho(\delta_{k,j_k}, \delta_k) < 1/k$, $\rho(G_{j_k}, G) < 1/k$ and

$$\max_{\delta' \sqsubset X'} u_F(f, \delta') < u_F(f, \delta_k) + \frac{1}{k}.$$

For each $k \in \mathbb{N}$, define $X'_k = X_{k,j_k}$, $\delta'_k = \delta_{k,j_k}$ and $G_k = G_{j_k}$. By Lemma 4, let $\{X'_{k_l}\}_{l=1}^{\infty}$ be a subsequence of $\{X'_k\}_{k=1}^{\infty}$ and $\{\tau_{k_l}\}_{l=1}^{\infty}$ be such that $\tau_{k_l} \to \tau$ and $\tau_{k_l} \sqsubset X'_{k_l}$ for each $l \in \mathbb{N}$; let $\{\delta'_{k_l}\}_{l=1}^{\infty}$ and $\{G_{k_l}\}_{l=1}^{\infty}$ be corresponding subsequences of $\{\delta'_k\}_{k=1}^{\infty}$ and $\{G_k\}_{k=1}^{\infty}$. Then, given $\varepsilon > 0$ such that $u_F(f, \tau) > u_F(f, \delta) + 2\varepsilon$, there is l sufficiently large such that

$$u_F(f, \tau_{k_l}) > u_F(f, \tau) - \varepsilon > u_F(f, \delta) + \varepsilon > u_F(f, \delta'_{k_l}) + \frac{1}{k_l} > \max_{\delta' \sqsubset X'_{k_l}} u_F(f, \delta').$$

Since $\tau_{k_l} \sqsubset X'_{k_l}$, this is a contradiction. This contradiction shows that $\delta \in C_f(X)$ and that C_f is closed.

A.3 Proof of the characterization of stability in college admission economies

Condition (iv) is clearly equivalent to the individual rationality of M. Hence, it suffices to show that conditions (i)–(iii) are equivalent to the no blocking coalition condition.

(Necessity) Let M be a matching satisfying the no blocking coalition condition and $f \in F$. Then, $M_f \in C_f(M_f)$ (see Che, Kim, and Kojima (2019, Footnote 28)) and, hence, $M_f(\Theta) \leq q_f$. Thus, (i) holds.

To establish (ii), let $E = \{\theta \in \Theta : \pi_{1,f}(\theta) > 0\}$ and assume that $M_f(\Theta) < q_f$ and $\left(D^{\preceq f}(M) - M_f\right)(E) > 0$. Let δ' be such that $\delta'(B) = \left(D^{\preceq f}(M) - M_f\right)(E \cap B)$ for each Borel $B \subseteq \Theta$, $\varepsilon \in (0, 1)$ such that $M_f(\Theta) + \varepsilon \left(D^{\preceq f}(M) - M_f\right)(E) < q_f$ and $M'_f = M_f + \varepsilon\delta'$. Then $M'_f(\Theta) = M_f(\Theta) + \varepsilon \left(D^{\preceq f}(M) - M_f\right)(E) < q_f$ and

$M_f' \sqsubset D^{\preceq f}(M)$, the latter since, for each Borel $B \subseteq \Theta$,

$$\begin{aligned} M_f'(B) &= M_f(B \cap E^c) + M_f(B \cap E) + \varepsilon \left(D^{\preceq f}(M) - M_f \right)(B \cap E) \\ &\le D^{\preceq f}(M)(B \cap E^c) + M_f(B \cap E) + D^{\preceq f}(M)(B \cap E) - M_f(B \cap E) \\ &= D^{\preceq f}(M)(B). \end{aligned}$$

Moreover,

$$\begin{aligned} &\int_\Theta \pi_{1,f}(\theta) \mathrm{d} M_f'(\theta) = \int_\Theta \pi_{1,f}(\theta) \mathrm{d} M_f(\theta) + \varepsilon \int_E \pi_{1,f}(\theta) \mathrm{d} \left(D^{\preceq f}(M) - M_f \right)(\theta) \\ &> \int_\Theta \pi_{1,f}(\theta) \mathrm{d} M_f(\theta) \end{aligned}$$

since $\left(D^{\preceq f}(M) - M_f \right)(E) > 0$ and $\pi_{1,f}(\theta) > 0$ for each $\theta \in E$. But this is a contradiction since M satisfies no blocking coalition condition; thus, (ii) holds.

We turn to (iii). Let $E = \{\theta \in \Theta : \pi_{1,f}(\theta) > \inf_{\theta' \in \mathrm{supp}(M_f)} \pi_{1,f}(\theta')\}$ and assume that $\left(D^{\preceq f}(M) - M_f \right)(E) > 0$.

For each $\theta \in E$, let $\varepsilon_\theta > 0$ and $\theta'_\theta \in \mathrm{supp}(M_f)$ be such that

$$\pi_{1,f}(\theta'_\theta) < \inf_{\theta \in \mathrm{supp}(M_f)} \pi_{1,f}(\theta) + \varepsilon_\theta < \pi_{1,f}(\theta).$$

Let V_θ be an open neighborhood of θ'_θ such that $\pi_{1,f}(\theta) > \sup_{\hat\theta \in V_\theta} \pi_{1,f}(\hat\theta)$ and $E_\theta = \{\tilde\theta \in \Theta : \pi_{1,f}(\tilde\theta) > \sup_{\hat\theta \in V_\theta} \pi_{1,f}(\hat\theta)\}$; then $\theta \in E_\theta \subseteq E$, E_θ is open, $E_\theta \cap V_\theta = \emptyset$ and $M_f(V_\theta) > 0$, the latter since $\theta'_\theta \in \mathrm{supp}(M_f)$. Since Θ is separable, there exists $\{\theta_k\}_{k=1}^\infty$ such that $E = \cup_k E_{\theta_k}$. Hence, there exists $k \in \mathbb{N}$ such that $\left(D^{\preceq f}(M) - M_f \right)(E_{\theta_k}) > 0$.

Let δ, δ' be such that $\delta(B) = M_f(V_{\theta_k} \cap B)$ and $\delta'(B) = \left(D^{\preceq f}(M) - M_f \right)(E_{\theta_k} \cap B)$ for each Borel $B \subseteq \Theta$ and $M_f' = M_f + \varepsilon\delta' - \eta\delta$ where $\varepsilon > 0$, $\eta > 0$ are such that $\varepsilon \left(D^{\preceq f}(M) - M_f \right)(E_{\theta_k}) = \eta M_f(V_{\theta_k})$ and $\varepsilon + \eta = 1$.[22] Then $M_f'(V_{\theta_k}) = 0$, $0 \sqsubset M_f'$, $M_f'(\Theta) \le M_f(\Theta) \le q_f$, $M_f' \sqsubset M_f + \varepsilon\delta' \sqsubset D^{\preceq f}(M)$ (the latter as in part (ii)) and

$$\begin{aligned} &\int_\Theta \pi_{1,f}(\theta) \mathrm{d} M_f'(\theta) - \int_\Theta \pi_{1,f}(\theta) \mathrm{d} M_f(\theta) \\ &= \varepsilon \int_{E_{\theta_k}} \pi_{1,f}(\theta) \mathrm{d} \left(D^{\preceq f}(M) - M_f \right)(\theta) - \eta \int_{V_{\theta_k}} \pi_{1,f}(\theta) \mathrm{d} M_f(\theta) \\ &> \left[\varepsilon \left(D^{\preceq f}(M) - M_f \right)(E_{\theta_k}) - \eta M_f(V_{\theta_k}) \right] \sup_{\theta \in V_{\theta_k}} \pi_{1,f}(\theta) = 0. \end{aligned}$$

[22]Such ε and η exist: $\eta = \frac{\left(D^{\preceq f}(M) - M_f \right)(E_{\theta_k})}{(D^{\preceq f}(M) - M_f)(E_{\theta_k}) + M_f(V_{\theta_k})}$ and $\varepsilon = \frac{M_f(V_{\theta_k})}{(D^{\preceq f}(M) - M_f)(E_{\theta_k}) + M_f(V_{\theta_k})}$.

But this is a contradiction since M satisfies no blocking coalition condition and, thus, (iii) holds.

(Sufficiency) Let M be a matching satisfying conditions (i)–(iii) and suppose that the no blocking coalition condition fails. Then let $M'_f \in \mathcal{X}$ be such that $M'_f \sqsubset D^{\preceq f}(M)$ and $M'_f \succ_f M_f$. Hence, $M'_f(\Theta) \leq q_f$ and $\int \pi_{1,f} \mathrm{d}M'_f > \int \pi_{1,f} \mathrm{d}M_f$.

Consider first the case where $M_f(\Theta) < q_f$. Let $E = \{\theta \in \Theta : \pi_{1,f}(\theta) > 0\}$; (ii) and $M'_f \sqsubset D^{\preceq f}(M)$ imply that $M'_f(E) \leq M_f(E)$. Since $\Theta = E \cup \{\theta \in \Theta : \pi_{1,f}(\theta) = 0\}$, it follows that

$$\int \pi_{1,f} \mathrm{d}M'_f = \int_E \pi_{1,f} \mathrm{d}M'_f \leq \int_E \pi_{1,f} \mathrm{d}M_f = \int \pi_{1,f} \mathrm{d}M_f,$$

a contradiction.

Consider next the remaining case where $M_f(\Theta) = q_f$. Let $\bar{\alpha} = \inf_{\theta \in \mathrm{supp}(M_f)} \pi_{1,f}(\theta)$ and $E = \{\theta \in \Theta : \pi_{1,f}(\theta) > \bar{\alpha}\}$. Fix $k \in \mathbb{N}$ and define, for each $j \in \{1, \ldots, 2^k\}$,

$$E_j = \{\theta \in \Theta : \bar{\alpha} + (j-1)(1-\bar{\alpha})2^{-k} < \pi_{1,f}(\theta) \leq \bar{\alpha} + j(1-\bar{\alpha})2^{-k}\}.$$

Then $E = \cup_{j=1}^{2^k} E_j$ and $\{E_j\}_j$ is pairwise disjoint. Condition (iii) and $M'_f \sqsubset D^{\preceq f}(M)$ imply that

$$M'_f(E_j) \leq M_f(E_j) \text{ for each } j \in \{1, \ldots, 2^k\}. \tag{10}$$

Since $M_f(E^c) = 0$, it follows that

$$\begin{aligned}\sum_{j=1}^{2^k} M_f(E_j) &= M_f(E^c) + \sum_{j=1}^{2^k} M_f(E_j) = M_f(\Theta) = q_f \geq M'_f(\Theta) \\ &= M'_f(E^c) + \sum_{j=1}^{2^k} M'_f(E_j);\end{aligned}$$

hence,

$$M'_f(E^c) \leq \sum_{j=1}^{2^k} (M_f(E_j) - M'_f(E_j)). \tag{11}$$

Define $g_k : \Theta \to [0,1]$ by setting, for each $\theta \in \Theta$,

$$g_k(\theta) = \begin{cases} \bar{\alpha} + (j-1)(1-\bar{\alpha})2^{-k} & \text{if } \theta \in E_j, j = 1, \ldots, 2^k, \\ (j-1)\bar{\alpha}2^{-k} & \text{if } (j-1)\bar{\alpha}2^{-k} < \pi_{1,f}(\theta) \leq j\bar{\alpha}2^{-k}, j = 1, \ldots, 2^k, \\ 0 & \text{if } \pi_{1,f}(\theta) = 0. \end{cases}$$

Then

$$\begin{aligned}
\int g_k \mathrm{d}M'_f &\leq \bar{\alpha} M'_f(E^c) + \sum_{j=1}^{2^k} (\bar{\alpha} + (j-1)(1-\bar{\alpha})2^{-k}) M'_f(E_j) \\
&\leq \bar{\alpha} \sum_{j=1}^{2^k} M_f(E_j) + \sum_{j=1}^{2^k} (j-1)(1-\bar{\alpha})2^{-k} M'_f(E_j) \\
&\leq \sum_{j=1}^{2^k} (\bar{\alpha} + (j-1)(1-\bar{\alpha})2^{-k}) M_f(E_j) \\
&= \int g_k \mathrm{d}M_f,
\end{aligned}$$

where the first inequality follows because $g_k(\theta) < \bar{\alpha}$ for each $\theta \in E^c$, the second by (11) and the third by (10).

We thus have that $\int g_k \mathrm{d}M'_f \leq \int g_k \mathrm{d}M_f$ for each $k \in \mathbb{N}$. Since $\{g_k\}_{k=1}^{\infty}$ converges (uniformly) to $\pi_{1,f}$, it follows that $\int \pi_{1,f} \mathrm{d}M'_f \leq \int \pi_{1,f} \mathrm{d}M_f$, a contradiction. This completes the proof.

A.4 Proof of Theorem 2

The proof of Theorem 1 extends without change provided that (i) there is, when Θ is finite, a function $u_\Theta : \tilde{F} \times \Theta \times \mathcal{X}^{n+1} \to \mathbb{R}$ representing workers' preferences such that the function $U : \mathcal{X}^{2(n+1)} \to \mathbb{R}$ defined by setting, for each $(\mu, \tau) \in \mathcal{X}^{2(n+1)}$,

$$U(\mu, \tau) = \sum_{f \in \tilde{F}, \theta \in \Theta} u_\Theta(f, \theta, \mu) \tau_f(\theta)$$

is continuous and (ii) $\mu_f(O \cap P(f', f, \mu)) \leq \liminf_k \mu_{k,f}(O \cap P(f', f, \mu_k))$ for each open subset O of Θ, $f, f' \in \tilde{F}$, $\mu \in \mathcal{X}^{n+1}$ and $\{\mu_k\}_{k=1}^{\infty} \subseteq \mathcal{X}^{n+1}$ such that $\mu = \lim_k \mu_k$. Hence, it suffices to show that (i) and (ii) hold under the assumptions of Theorem 2.

To establish (i), we normalize the workers' utility function so that $1 \leq u_\Theta \leq 2$ and show that U is continuous. Let $(\mu, \tau) \in \mathcal{X}^{2(n+1)}$, $\{(\mu_k, \tau_k)\}_{k=1}^{\infty} \subseteq \mathcal{X}^{2(n+1)}$ such that $(\mu_k, \tau_k) \to (\mu, \tau)$ and $\varepsilon > 0$. Let $\delta > 0$ be given by the diversity of preferences assumption corresponding to $\varepsilon/|\tilde{F}|$ and $K \in \mathbb{N}$ such that $\rho(\mu_k, \mu) < \delta$ and $|U(\mu, \tau_k) - U(\mu, \tau)| < \varepsilon/2$ for each $k \geq K$. Fix $k \geq K$, let $E_k = \{\theta \in \Theta : u_\Theta(f, \theta, \mu_k) \neq u_\Theta(f, \theta, \mu)$ for some $f \in \tilde{F}\}$ and note that $\tau_{k,f}(E_k) \leq G(E_k) < \varepsilon/|\tilde{F}|$

for each $f \in \tilde{F}$ since $\tau_{k,f} \in \mathcal{X}$ (and, hence, $\tau_{k,f} \sqsubset G$) for each $f \in \tilde{F}$ and due to the diversity of preferences assumption. Thus,

$$\begin{aligned}|U(\mu_k, \tau_k) - U(\mu, \tau_k)| &\leq \sum_{f \in \tilde{F}, \theta \in \Theta} \tau_{k,f}(\theta)|u_\Theta(f, \theta, \mu_k) - u_\Theta(f, \theta, \mu)| \\ &\leq \sum_{f \in \tilde{F}} \tau_{k,f}(E_k) \leq |\tilde{F}|G(E_k) < \frac{\varepsilon}{2}\end{aligned}$$

since $|u_\Theta(f, \theta, \mu_k) - u_\Theta(f, \theta, \mu)| = 0$ if $\theta \in E_k^c$ and $|u_\Theta(f, \theta, \mu_k) - u_\Theta(f, \theta, \mu)| \leq 1$ if $\theta \in E_k$ due to $1 \leq u_\Theta \leq 2$. Hence, for each $k \geq K$,

$$|U(\mu_k, \tau_k) - U(\mu, \tau)| \leq |U(\mu_k, \tau_k) - U(\mu, \tau_k)| + |U(\mu, \tau_k) - U(\mu, \tau)| < \varepsilon.$$

We now establish (ii). Since $P(f', f, \mu)$ is open, it follows that $\mu_f(O \cap P(f', f, \mu)) \leq \liminf_k \mu_{k,f}(O \cap P(f', f, \mu))$. Let $\varepsilon > 0$ and let $\delta > 0$ be given by the diversity of preferences assumption corresponding to ε. For each $k \in \mathbb{N}$, let $E_k = \{\theta \in \Theta : \succeq_{\theta|\mu} \neq \succeq_{\theta|\mu_k}\}$ and let $K \in \mathbb{N}$ be such that $\mu_k \in B_\delta(\mu)$ for each $k \in \mathbb{N}$. Then, for each $k \geq K$,

$$\begin{aligned}\mu_{k,f}(O \cap P(f', f, \mu_k)) &\geq \mu_{k,f}(O \cap P(f', f, \mu_k) \cap E_k^c) = \mu_{k,f}(O \cap P(f', f, \mu) \cap E_k^c) \\ &\geq \mu_{k,f}(O \cap P(f', f, \mu)) - \varepsilon.\end{aligned}$$

It then follows that

$$\mu_f(O \cap P(f', f, \mu)) \leq \liminf_k \mu_{k,f}(O \cap P(f', f, \mu)) \leq \liminf_k \mu_{k,f}(O \cap P(f', f, \mu_k)) + \varepsilon$$

and, since $\varepsilon > 0$ is arbitrary, that $\mu_f(O \cap P(f', f, \mu)) \leq \liminf_k \mu_{k,f}(O \cap P(f', f, \mu_k))$.

A.5 Details for an example in Section 2.1

We provide some details for the example in Section 2.1 in which the workers' type space Θ is the space of bounded and continuous utility functions. Recall that, in such example, S is a compact metric space, $s : \mathcal{X}^{n+1} \to S$ is a continuous function and $\Theta = C(\tilde{F} \times S)$, where $C(\tilde{F} \times S)$ denotes the space of bounded and continuous real-valued functions on $\tilde{F} \times S$ endowed with the sup norm. The utility of a worker of type θ of being matched with firm $f \in \tilde{F}$ when the matching is $M \in \mathcal{X}^{n+1}$ is then $\theta(f, s(M))$.

In Footnote 8 we claimed that it is possible to specify S and s such that workers' preferences are allowed to depend on the entire matching. The intuition for this claim is that $\mathcal{X}^{n+1}$ is homeomorphic to a subset of a compact metric space S, in which case we can take s to be such a homeomorphism.

The details are as follows. Let $S = [-1, 1]^{\tilde{F} \times \mathbb{N}}$ be the countable product of the $[-1, 1]$ interval endowed with the product topology, which is then a compact metric space. Indeed, S is compact by Tychonoff Theorem e.g. Kelley (1955, Theorem 13, p. 143) and by writing $\tilde{F} \times \mathbb{N} = \{\alpha_k\}_{k=1}^{\infty}$, $d : S \times S \to \mathbb{R}_+$ defined by $d(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} |x_{\alpha_k} - y_{\alpha_k}|$ for each $x, y \in S$ is a metric on S that metrizes the product topology.

Let $U(\Theta)$ be the space of real-valued bounded uniformly continuous functions on Θ and $U_1(\Theta)$ be the subset of $U(\Theta)$ consisting of those functions $p \in U(\Theta)$ such that its sup norm $||p||_\infty$ is no greater than 1. The argument from now on is essentially the one in the proof of the if part of Theorem 3.1 in Varadarajan (1958). Indeed, the space $U(\Theta)$ is separable and, hence, so is $U_1(\Theta)$. Let $\{p_i\}_{i \in \mathbb{N}}$ be a countable dense subset of $U_1(\Theta)$ and define $s : \mathcal{X}^{n+1} \to S$ by setting, for each $M \in \mathcal{X}^{n+1}$, $s(M) = (\int_\Theta p_i \mathrm{d}M_f)_{f \in \tilde{F}, i \in \mathbb{N}}$. We have that $s(M) \in S$ for each $M \in \mathcal{X}^{n+1}$ since, for each $M \in \mathcal{X}^{n+1}$ and $f \in \tilde{F}$, $\left|\int_\Theta p_i \mathrm{d}M_f\right| \leq \int_\Theta |p_i| \mathrm{d}M_f \leq ||p_i||_\infty \leq 1$ since $M_f(\Theta) \leq 1$. It follows by the if part of proof of Theorem 3.1 in Varadarajan (1958) together with the following two lemmas that s is injective and continuous, and that its inverse is continuous.

Lemma 5 *Let ϕ be a signed measure on Θ. Then $\int_\Theta p \mathrm{d}\phi = 0$ for each $p \in U(\Theta)$ if and only if $\int_\Theta p \mathrm{d}\phi = 0$ for each $p \in U_1(\Theta)$.*

Proof. The necessity part follows because $U_1(\Theta) \subseteq U(\Theta)$. To see the sufficiency part, let $p \in U(\Theta)$ and $B > 0$ be such that $||p||_\infty \leq B$. Thus, $p/B \in U_1(\Theta)$. Hence, $\int_\Theta p \mathrm{d}\phi = B \int_\Theta \frac{p}{B} \mathrm{d}\phi = 0$. ■

Lemma 6 *Let* $\{\mu_\alpha\}$ *be a net of measures on* Θ *and* μ *be a measure on* Θ*. Then* $\lim_\alpha \int_\Theta p \mathrm{d}\mu_\alpha = \int_\Theta p \mathrm{d}\mu$ *for each* $p \in U(\Theta)$ *if and only if* $\lim_\alpha \int_\Theta p \mathrm{d}\mu_\alpha = \int_\Theta p \mathrm{d}\mu$ *for each* $p \in U_1(\Theta)$*.*

Proof. The necessity part follows because $U_1(\Theta) \subseteq U(\Theta)$. To see the sufficiency part, let $p \in U(\Theta)$ and $B > 0$ be such that $||p||_\infty \leq B$. Thus, $p/B \in U_1(\Theta)$. Hence, $\lim_\alpha \int_\Theta p \mathrm{d}\mu_\alpha = B \lim_\alpha \int_\Theta \frac{p}{B} \mathrm{d}\mu_\alpha = B \int_\Theta \frac{p}{B} \mathrm{d}\mu = \int_\Theta p \mathrm{d}\mu$. ■

A.6 Counterexample under weak stability

Consider the following stability notion (analogous to weak stability in Bando (2012)), where for simplicity workers' preferences do not depend on the matching: a matching M is stable if

1. (Individual Rationality) For each $f \in F$, $M_f(P(\emptyset, f)) = 0$; and

2. (No Strongly Blocking Coalition) There does not exist $f \in F$, $\delta'_\emptyset \in \mathcal{X}$ and $(\delta_k)_{k \in \tilde{F} \setminus \{f\}} \in \mathcal{X}^n$ such that $\delta_k \sqsubset D_k^{\preceq f}(M)$ for each $k \in \tilde{F} \setminus \{f\}$, $\delta'_\emptyset \sqsubset M_f$, and $u_f(M') > u_f(M)$, where

$$M'_f = M_f + \sum_{k \in \tilde{F} \setminus \{f\}} \delta_k - \delta'_\emptyset,$$

$$M'_\emptyset = M_\emptyset - \delta_\emptyset + \delta'_\emptyset, \text{ and}$$

$$M'_k = M_k - \delta_k \text{ for each } k \in F \setminus \{f\}$$

   and, for each $k \in \tilde{F} \setminus \{f\}$, $D_k^{\preceq f}(M)$ is defined by setting, for each $E \in \Sigma$,

$$D_k^{\preceq f}(M)(E) = M_k(E \cap P(f, k)),$$

   and in addition $u_f(M') \geq u_f(\hat{M})$ for every matching $\hat{M}$ such that $\hat{M}_f \sqsubset \delta'_\emptyset + M'_f$ and $\hat{M}_k = M'_k$ for all $k \in F \setminus \{f\}$.

We will show that the counterexample from Section 4.2 continues to hold. Suppose, in the context of that example, that there is a matching M that is stable. We consider three cases: (a) $M_\emptyset = \alpha$, (b) $M_\emptyset < \alpha$ and (c) $M_\emptyset > \alpha$.

Case (a): $M_\emptyset = \alpha$. Since $\alpha < 1$, there exists $f \in F$ such that $M_f > 0$ and let $-f \in F$ be such that $f \neq -f$. Let $M'_f = M_f - \varepsilon^*$, $M'_\emptyset = M_\emptyset + \varepsilon^*$ and $M'_{-f} = M_{-f}$, where $0 < \varepsilon^* < M_f$ solves $\max_\varepsilon \varepsilon(M_f - \varepsilon)$. Then $u_f(M') = \varepsilon^*(M_f - \varepsilon^*) > 0 = u_f(M)$ and in addition, for any matching $\hat{M}$ such that $\hat{M}_f \leq \varepsilon^* + M_f - \varepsilon^* = M_f$ and $\hat{M}_{-f} = M'_{-f} = M_{-f}$, $u_f(\hat{M}) = (M_f - \hat{M}_f)(M_f - (M_f - \hat{M}_f)) \leq \varepsilon^*(M_f - \varepsilon^*) = u_f(M')$, a contradiction to the stability of M.

Case (b): $M_\emptyset < \alpha$. Then there exists $f \in F$ such that $M_f > 0$ and, hence, $u_f(M) < 0$. Let $M'_f = M_f - \varepsilon^*$, $M'_\emptyset = M_\emptyset + \varepsilon^*$ and $M'_{-f} = M_{-f}$, where ε^* solves $\max_{0 \leq \varepsilon \leq M_f}(M_\emptyset - \alpha + \varepsilon)(M_f - \varepsilon)$. Since $(M_\emptyset - \alpha + M_f)(M_f - M_f) = 0$, $u_f(M') = (M_\emptyset - \alpha + \varepsilon^*)(M_f - \varepsilon^*) \geq 0 > u_f(M)$ and in addition for any matching $\hat{M}$ such that $\hat{M}_f \leq \varepsilon^* + M_f - \varepsilon^* = M_f$ and $\hat{M}_{-f} = M'_{-f} = M_{-f}$, $u_f(\hat{M}) = (M_\emptyset - \alpha + M_f - \hat{M}_f)(M_f - (M_f - \hat{M}_f)) \leq u_f(M')$, a contradiction to the stability of M.

Case (c): $M_\emptyset > \alpha$. Note first that for each f, it must be the case that $M_\emptyset - \alpha = M_f$. Suppose that $M_\emptyset - \alpha \neq M_f$ and consider $M'_f = M_f + \varepsilon^*$, $M'_\emptyset = M_\emptyset - \varepsilon^*$ and $M'_{-f} = M_{-f}$, where $\varepsilon^* = \frac{M_\emptyset - \alpha - M_f}{2}$. Note that $-M_f < \varepsilon^* < M_\emptyset$ and for each $\tilde{M}$ such that $\tilde{M}_f = M_f + \varepsilon$, $\tilde{M}_\emptyset = M_\emptyset - \varepsilon$ and $\tilde{M}_{-f} = M_{-f}$, $u(\tilde{M}) = (M_\emptyset - \alpha - \varepsilon)(M_f + \varepsilon)$, which is strictly maximized by ε^*. Hence, $u_f(M') > u_f(M)$ and in addition, for each matching $\hat{M}$ such that $\hat{M}_{-f} = M_{-f}$, $u_f(\hat{M}) \leq u_f(M')$, a contradiction to the stability of M.

Thus, we must have $M_1 = M_2 = \frac{1-\alpha}{3}$. Consider $M'_1 = \frac{1-\alpha}{2}$, $M'_2 = 0$ and $M'_\emptyset = \frac{1+\alpha}{2}$, achieved by firm 1 hiring all of firm 2's workers and then firing a measure $\frac{1-\alpha}{6}$ of its own workers. Then $u_1(M') > u_1(M)$ and in addition, $u_1(M') \geq u_1(\hat{M})$ for all $\hat{M}$, a contradiction to the stability of M.

References

Aliprantis, C., and K. Border (2006): *Infinite Dimensional Analysis*. Springer, Berlin, 3rd edn.

Alkan, A., and D. Gale (2003): "Stable Schedule Matching Under Revealed Preferences," *Journal of Economic Theory*, 112, 289–306.

Azevedo, E., and J. Leshno (2016): "A Supply and Demand Framework for Two-Sided Matching Markets," *Journal of Political Economy*, 124, 1235–1268.

Bando, K. (2012): "Many-To-One Matching Markets with Externalities among Firms," *Journal of Mathematical Economics*, 48, 14–20.

Blair, C. (1984): "Every Finite Distributive Lattice Is a Set of Stable Matchings," *Journal of Combinatorial Theory*, 37, 353–356.

Carmona, G. (2011): "Understanding Some Recent Existence Results for Discontinuous Games," *Economic Theory*, 48, 31–45.

Carmona, G., and K. Laohakunakorn (2023): "Stable Matching in Large Markets with Occupational Choice," University of Surrey.

Che, Y.-K., J. Kim, and F. Kojima (2019): "Stable Matching in Large Economies," *Econometrica*, 87, 65–110.

Cox, N., R. Fonseca, and B. Pakzad-Hurson (2022): "Do Peer Preferences Matter in School Choice Market Design? Theory and Evidence," Princeton University and Brown University.

Debreu, G. (1964): "Continuity Properties of Paretian Utility," *International Economic Review*, 5, 285–293.

Dutta, B., and J. Massó (1997): "Stability of Matchings When Individuals Have Preferences over Colleagues," *Journal of Economic Theory*, 75, 464–475.

Echenique, F., and M. B. Yenmez (2007): "A Solution to Matching with Preferences over Colleagues," *Games and Economic Behavior*, 59, 46–71.

Fisher, J., and I. Hafalir (2016): "Matching with Aggregate Externalities," *Mathematical Social Sciences*, 81, 1–7.

Hafalir, I. (2008): "Stability of Marriage with Externalities," *International Journal of Game Theory*, 37, 353–369.

Kelley, J. (1955): *General Topology.* Springer, New York.

Kelso, A., and V. Crawford (1982): "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, 50, 1483–1504.

Leshno, J. (2022): "Stable Matching with Peer-Dependent Preferences in Large Markets: Existence and Cutoff Characterization," University of Chicago.

Ma, J. (2001): "Job Matching and Coalition Formation with Utility or Disutility of Co-workers," *Games and Economic Behavior*, 34, 83–103.

Mumcu, A., and I. Saglam (2010): "Stable One-to-One Matchings with Externalities," *Mathematical Social Sciences*, 60, 154–159.

Parthasarathy, K. (1967): *Probability Measures on Metric Spaces.* Academic Press, New York.

Pycia, M. (2012): "Stability and Preference Alignment in Matching and Coalition Formation," *Econometrica*, 80, 323–362.

Pycia, M., and M. B. Yenmez (2022): "Matching with Externalities," *Review of Economic Studies*, forthcoming.

Reny, P. (1999): "On the Existence of Pure and Mixed Strategy Equilibria in Discontinuous Games," *Econometrica*, 67, 1029–1056.

Sasaki, H., and M. Toda (1996): "Two-Sided Matching Problems with Externalities," *Journal of Economic Theory*, 70, 93–108.

Varadarajan, V. S. (1958): "Weak Convergence of Measures on Separable Metric Spaces," *Sankhyā: The Indian Journal of Statistics*, 19, 15–22.