A Deferred Acceptance Algorithm for Large Marriage Markets*

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Abstract

We consider a version of Gale and Shapley's (1962) marriage market featuring a continuum of women and men based on Greinecker and Kah (2021). We define a deferred acceptance algorithm for this setting and show that it terminates after a finite number of iterations to yield a side-optimal stable matching.

1 Introduction

Matching theory is widely used to address questions such as who gets which jobs, which school places and who marries whom. Gale and Shapley (1962) (GS henceforth) provided the first model of two-sided matching and its key solution concept, stability, which requires that no one is matched with an unacceptable person and that no man and woman who are not matched to each other would both prefer to be.

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GS also introduced the so-called deferred acceptance algorithm (DAA henceforth) which converges in finitely many steps to a stable matching. Thus, the DAA establishes the existence of stable matchings and can also be used to compute the side-optimal (i.e. man-optimal and woman-optimal) stable matchings.

There have been many extensions and variations of GS's original work. One such line of research that has been particularly active recently focuses on large matching markets, motivated by the observation that, in reality, many such markets feature a large number of participants.[1] In this paper we extend GS's DAA to the setting of Greinecker and Kah (2021) (GK henceforth), which provides a formalization of GS's marriage market for the case where there is a continuum of men and women.

Specifically, we consider a version of GK's setting where there is a finite set of contracts, a finite set of types of men, each having preferences over types of women and contracts, and a finite set of types of women, each having preferences over types of men and contracts; in particular, for each contract, women are indifferent between different men of the same type and vice versa.[2] Instead of having an explicit set of men and women, we only specify the type distribution of each gender. We then define a DAA algorithm for this setting and show that, when preferences are linear orders and each distribution has rational coefficients, it converges in finitely many steps to a stable matching, which is side-optimal.

One interpretation of GK's framework is that there is a continuum of men and women, and the type distributions specify the measure of each type in the population. Another, which is possible when each distribution has rational coefficients, is that each man (resp. woman) type corresponds to a man (resp. woman). Thus, GS's DAA can be applied to the setting we consider, something which is already known since GK, who use Crawford and Knoer's (1981) version of GS's DAA to show the existence of

[1]This literature includes, among others, Azevedo and Leshno (2016), Chiappori and Reny (2016), Fisher and Hafalir (2016), Ashlagi, Kanoria, and Leshno (2017), Eeckhout and Kircher (2018), Fuentes and Tohmé (2018), Nöldeke and Samuelson (2018), Che, Kim, and Kojima (2019), Che and Tercieux (2019) and Greinecker and Kah (2021).

[2]Distributional marriage models with finitely many types have also been considered in Baïou and Balinski (2002), Echenique, Lee, and Yenmez (2010) and Echenique, Lee, Shum, and Yenmez (2013).

stable matchings in discrete instances of their framework.

Our version of the DAA has an important advantage over the above as it solves the following dimensionality problem. Suppose that there are 100 women, all of the same type, and 100 men. If all of the men are of the same type, this market can be scaled down to the case where there is just one man and one woman and the stable matching is easy to obtain. If, instead, 99 of the men are of type m_1 and 1 is of type m_2, then we can no longer scale it down. If we represent this marriage market in GS's setting, then, despite there being just one type of woman, each man has to linearly order each of 100 women, each of whom he regards as equivalent. This is in contrast with our algorithm in which we just need the single type of woman to linearly order the two types of men. In short, our DAA does not require that certain ties be broken and, thus, allows for a more parsimonious description of individual preferences. We illustrate this advantage in an example and present a python code to compute stable allocations using our version of the DAA.

A similar algorithm to ours is the column-greedy algorithm of Baïou and Balinski (2002). The main contribution of our algorithm over theirs is that it allows for contracts.

The paper is organized as follows. The framework we consider is introduced in Section 2. Section 3 illustrates our version of the DAA. Its formal definition and our main result showing that it converges in finitely many steps to a side-optimal stable matching is in Section 4. Appendix A contains the proof of our main result and a description of how to use a (rudimentary) python code to implement our DAA; the code itself is available here.[3]

2 Large marriage markets

We consider a simplified version of GK's marriage market framework with finitely many types and contracts. Such a marriage market is defined by the following elements. There are nonempty and finite sets W and M of types describing, respectively,

[3] https://drive.google.com/file/d/1dttfikCLlERLYPztlvRq_qNT4rk0VRvF/view

the types of women and men in the marriage market. To represent unmatched individuals, there is a dummy type $\emptyset \notin W \cup M$; let $W_\emptyset = W \cup \{\emptyset\}$ and $M_\emptyset = M \cup \{\emptyset\}$. The distributions of women and men are described by nonzero, finite measures ν_W and ν_M on W and M, respectively.

There is a nonempty and finite set C of contracts. Women have preferences over men (including the "empty man", i.e. being unmatched) and contracts, and analogously for men. Preferences are then described by a linear order $\succ_w$ on $M_\emptyset \times C$ for each $w \in W$ and a linear order $\succ_m$ on $W_\emptyset \times C$ for each $m \in M$.

A *marriage matching* is a Borel measure $\mu \in M(M_\emptyset \times W_\emptyset)$ such that

1. $\mu(M_\emptyset \times \{w\} \times C) = \nu_W(w)$ for each $w \in W$,
2. $\mu(\{m\} \times W_\emptyset \times C) = \nu_M(m)$ for each $m \in M$, and
3. $\mu(\{(w, m, c) : w = m = \emptyset\}) = 0.$[4]

A marriage matching describes, for each $(m, w, c) \in M_\emptyset \times W_\emptyset \times C$, the measure $\mu(m, w, c)$ of people that are matched, whose type profile is (m, w) and whose contract is c; the unmatched are those who are matched with $\emptyset$ so that, in this sense, everyone is matched. Condition 1 then says that the measure of those in a match where the female type is $w \in W$ is, since everyone is matched, exactly the measure of women whose type is w. Condition 2 is the analogous condition for men. Condition 3 says that almost all matches have a woman, or a man or both.

The definition of a stable marriage matching we use is that in Carmona and Laohakunakorn (2022), which is equivalent to that in GK. For each $m \in M_\emptyset$, define

$$\begin{aligned} T_m(\mu) = \;& \{(w, c) \in W \times C : \text{ there exists } (m', c') \in M_\emptyset \times C \\ & \text{such that } (w, m', c') \in \text{supp}(\mu) \text{ and } (m, c) \succ_w (m', c')\} \\ & \cup (\{\emptyset\} \times C)\,. \end{aligned}$$

The set $T_m(\mu)$ consists of the pairs of women types and contracts $(w, c) \in W_\emptyset \times C$ that a man of type m can target at μ in the sense that either $w = \emptyset$ or there are

[4]Throughout the paper, whenever Y is a finite set, $M(Y)$ denotes the set of finite measures on Y and is identified with $\mathbb{R}_+^{|Y|}$. We write $\mu(y)$ for $\mu(\{y\})$ whenever $y \in Y$ and $\mu \in M(Y)$.

women of type $w \in W$ that prefers (m, c) to their current match in μ. Analogously, define, for each $w \in W_\emptyset$,

$$
\begin{aligned}
T_w(\mu) = \quad & \{(m, c) \in M \times C : \text{ there exists } (w', c') \in W_\emptyset \times C \\
& \text{such that } (w', m, c') \in \text{supp}(\mu) \text{ and } (w, c) \succ_m (w', c')\} \\
& \cup (\{\emptyset\} \times C) .
\end{aligned}
$$

Finally, define

$$
\begin{aligned}
S_M(\mu) = \quad & \{(m, w, c) \in M_\emptyset \times W_\emptyset \times C : \text{ there does not exist } (w', c') \in T_m(\mu) \\
& \text{such that } (w', c') \succ_m (w, c)\}, \\
S_W(\mu) = \quad & \{(m, w, c) \in M_\emptyset \times W_\emptyset \times C : \text{ there does not exist } (m', c') \in T_w(\mu) \\
& \text{such that } (m', c') \succ_w (m, c)\}, \\
S(\mu) = \quad & S_M(\mu) \cap S_W(\mu), \text{ and} \\
IR_W(\mu) = \quad & \{(m, w, c) \in M_\emptyset \times W_\emptyset \times C : \text{ there does not exist } c' \in C \\
& \text{such that } (\emptyset, c') \succ_w (m, c)\}.
\end{aligned}
$$

A marriage matching μ is *stable* if $\text{supp}(\mu) \subseteq S(\mu)$. As shown in Carmona and Laohakunakorn (2022), a matching is stable if and only if $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR_W(\mu)$.

3 An Example

We introduce our version of the DAA using a simple example without contracts. In this example, women prefer men of the same type and men prefer women of a different type. Let $M = \{\alpha, \beta\}$, $W = \{A, B\}$ and preferences be $B \succ_\alpha A \succ_\alpha \emptyset$, $A \succ_\beta B \succ_\beta \emptyset$, $\alpha \succ_A \beta \succ_A \emptyset$ and $\beta \succ_B \alpha \succ_B \emptyset$. Type distributions are $\nu_M(\alpha) = 1$, $\nu_M(\beta) = 2$, $\nu_W(A) = 2$ and $\nu_W(B) = 2$.

The idea of our DAA is analogous to that of GS: Each woman proposes to her favorite man. Each man who receives more than one proposal rejects all but his favorite from among those who have proposed to him and keeps his favorite on a string to allow for the possibility that someone better may come along later.

We start by specifying an initial distribution μ_0 so that every (nonempty) man is matched with the empty woman, $\mu_0(m, \emptyset) = \nu_M(m)$ for each $m \in M$, and no (nonempty) woman is matched, $\mu_0(m, w) = 0$ for each $m \in M_\emptyset$ and $w \in W$. The interpretation is that μ_0 is the distribution of men and women on the string and initially there is no (nonempty) woman on the string. We also specify $\mu_0(\emptyset, \emptyset) = 0$ to satisfy condition 3 in the definition of a matching, and we set $\nu_W(\emptyset) = \nu_M(M) = 3$ which ensures that there is always enough measure of the empty woman to be added to the string with any (nonempty) man.

In the first stage of the DAA, we let the set of rejected women be $R_1 = \{A, B\}$ to reflect that $\sum_{m \in M_\emptyset} \mu_0(m, w) < \nu_W(w)$ for each $w \in W$. Since there have been no proposals yet, we let the set of men who have rejected women of type A be empty, $M_{1,A} = \emptyset$, and likewise for type B, $M_{1,B} = \emptyset$. Women propose to their favorite men: We let $P_w(k)$ be the kth favorite man according to w's preferences, let $r_1(A) = 1$ and $r_1(B) = 1$, and specify that women of type w propose to men of type $P_w(r_1(w))$. Thus, women of type A propose to men of type α and women of type B propose to men of type β. Men can choose between women who have proposed to them, with the convention that the empty woman proposes to each (nonempty) man. Hence, men of type α can choose between women of type $w \in W^*_{1,\alpha} = \{A, \emptyset\}$, and men of type β can choose between women of type $w \in W^*_{1,\beta} = \{B, \emptyset\}$. We also let $W^*_{1,\emptyset} = \emptyset$ to indicate that no woman (nonempty or otherwise) proposed to the empty man. We then allocate all men of type α to women of type A since $A \succ_\alpha \emptyset$, $\mu_1(\alpha, A) = 1$, and all men of type β to women of type B, $\mu_1(\beta, B) = 2$. All the remaining entries of μ_1 are equal to zero.

In stage 2, we obtain that $R_2 = \{A\}$ since $1 = \sum_{m \in M_\emptyset} \mu_0(m, A) < \nu_W(A) = 2$ and $\sum_{m \in M_\emptyset} \mu_0(m, B) = \nu_W(B)$. Some women of type A have been rejected by men of type α, thus $M_{2,A} = \{\alpha\}$; in contrast, no women of type B has been rejected: $M_{2,B} = \emptyset$. We let $r_2(A) = 2 = 1 + |M_{2,A}|$ and $r_2(B) = 1 = 1 + |M_{2,B}|$, hence women of type A now propose to their second favorite type of man, which is β. Proposals are made by the women who were rejected, thus by a measure $\nu^2_W(A) = \nu_W(A) - \sum_{m \in M_\emptyset} \mu_0(m, A) = 1$ of women of type A and a measure $\nu^2_W(B) = 0$ of women of type B. Men of type

$m \in M_\emptyset$ can choose from $W^*_{2,\alpha} = \{A, \emptyset\}$, $W^*_{2,\beta} = \{A, B, \emptyset\}$ and $W^*_{2,\emptyset} = \emptyset$; for instance, men of type β can always choose to be unmatched (i.e. choose $\emptyset$), can choose from those women of type B who are on the string with them and also from those women of type A that just proposed to them. We then allocate men of type β as follows: Since A is their favorite type of women,

$$\mu_2(\beta, A) = \min\{\mu_1(\beta, A) + \nu^2_W(A), \nu_M(\beta)\} = 1$$

and

$$\mu_2(\beta, B) = \min\{\mu_1(\beta, B), \nu_M(\beta) - \mu_2(\beta, A)\} = 1;$$

note that $\mu_1(\beta, A) = 0$ is the measure of women of type A who were on the string with men of type β in stage 1 and $\nu^2_W(A) = 1$ is the measure of rejected women of type A who proposed to type β in stage 2; thus men of type β can choose from a measure 1 of women of type A; they can also choose from a measure 1 of women of type B which consist only of those who were on the string with men of type β in stage 1. Nothing changed regarding men of type α, thus $\mu_2(\alpha, A) = 1$.

Continuing to apply the above procedure, we obtain the following.

Stage 3: It follows that $R_3 = \{B\}$, $r_3(B) = 2$, $M_{3,A} = \{\alpha\}$, $M_{3,B} = \{\beta\}$, $r_3(A) = 2$, $r_3(B) = 2$, $\nu^3_W(B) = 1$, $W^*_{3,\alpha} = \{A, B, \emptyset\}$, $W^*_{3,\beta} = \{A, B, \emptyset\}$, $\mu_3(\alpha, B) = 1$, $\mu_3(\beta, A) = 1$ and $\mu_3(\beta, B) = 1$.

Stage 4: We have that $R_4 = \{A\}$, $M_{4,A} = \{\alpha\}$, $M_{4,B} = \{\beta\}$, $r_4(A) = 2$, $r_4(B) = 2$, $\nu^4_W(A) = 1$, $W^*_{4,\alpha} = \{B, \emptyset\}$, $W^*_{4,\beta} = \{A, B, \emptyset\}$, $\mu_4(\alpha, B) = 1$ and $\mu_4(\beta, A) = 2$.

Stage 5: It follows that $R_5 = \{B\}$, $M_{5,A} = \{\alpha\}$, $M_{5,B} = \{\beta\}$, $r_5(A) = 2$, $r_5(B) = 2$, $\nu^5_W(B) = 1$, $W^*_{5,\alpha} = \{B, \emptyset\}$, $W^*_{5,\beta} = \{A, \emptyset\}$, $\mu_5(\alpha, B) = 1$ and $\mu_5(\beta, A) = 2$.

Stage 6: We obtain that $R_6 = \{B\}$, $M_{6,A} = \{\alpha\}$, $M_{6,B} = \{\alpha, \beta\}$, $r_6(A) = 2$, $r_6(B) = 3$, $\nu^6_W(B) = 1$, $W^*_{6,\alpha} = \{B, \emptyset\}$, $W^*_{6,\beta} = \{A, \emptyset\}$, $W^*_{6,\emptyset} = \{B, \emptyset\}$, $\mu_6(\alpha, B) = 1$, $\mu_6(\beta, A) = 2$ and $\mu_6(\emptyset, B) = 1$.

Since $R_7 = \emptyset$, the algorithm stops after 6 iterations and μ_6 is a stable matching.

4 A DAA for large marriage markets

4.1 The algorithm

The key concept in the construction of the algorithm is what GS call the string, which consists in their setting of the best woman that has proposed to a man. Here we have a distribution of men and women on the string. The initial distribution μ_0 is such that there is no woman on the string, so that $\mu_0(m, \emptyset, c_m) = \nu_M(m)$ for each $m \in M$, where $c_m \in C$ is such that $(\emptyset, c_m) \succ_m (\emptyset, c)$ for each $c \in C \setminus \{c_m\}$, $\mu_0(m, w, c) = 0$ for each $m \in M_\emptyset$, $w \in W$ and $c \in C$ and $\mu_0(\emptyset, \emptyset, c) = 0$ for each $c \in C$. Set $\nu_W(\emptyset) = \nu_M(M)$.

Stage 1: The set of rejected women is

$$R_1 = \{w \in W : \sum_{(m,c) \in M_\emptyset \times C} \mu_0(m, w, c) < \nu_W(w)\} = W$$

to reflect that no woman is on the string and define the new measure of women by setting, for each $w \in W_\emptyset$,

$$\nu_W^1(w) = \nu_W(w) - \sum_{(m,c) \in M_\emptyset \times C} \mu_0(m, w, c).$$

We then have that $\nu_W^1(w) = \nu_W(w)$ for each $w \in W$.

For each $w \in W$, the set of man-contract pairs who have rejected w is

$$M_{1,w} = \emptyset$$

to reflect that no offer and, thus, no rejections have been made yet. Let

$$r_1(w) = |M_{1,w}| + 1 = 1$$

be 1+the number of man-contract pairs that have rejected a woman of type w. The reason why we add 1 to the latter number in the definition of $r_1(w)$ is as follows. For each $w \in W$, order $M_\emptyset \times C$ according to $\succ_w$; thus,

$$M_\emptyset \times C = \{(m_1, c_1), \ldots, (m_{|M_\emptyset \times C|}, c_{|M_\emptyset \times C|})\}$$

with $(m_1, c_1) \succ_w \cdots \succ_w (m_{|M_\emptyset \times C|}, c_{|M_\emptyset \times C|})$ and, for each $1 \leq k \leq |M_\emptyset \times C|$, let $P_w(k) = (m_k, c_k)$ be the kth favorite man-contract pair for any woman of type w.

Each $(m,c) \in M_\emptyset \times C$ receives proposals from $w \in R_1$ such that $P_w(1) = P_w(r_1(w)) = (m,c)$.

For each $(m,c) \in M \times C$, let

$$W_{1,m,c} = \{w \in R_1 : P_w(r_1(w)) = (m,c)\} \cup \{\emptyset\}$$

be the set of women types that men of type m have to decide upon (with contract c), and which equals those w that have proposed to match with m under contract c as well as the empty woman type $\emptyset$. In general, men of type m need also consider those women on the string with m under contract c, hence, the set of women types in

$$W^*_{1,m,c} = W_{1,m,c} \cup \text{supp}(\mu_0(m,\cdot,c));$$

since initially there are no women on the string, we have that $\text{supp}(\mu_0(m,\cdot,c))$ equals $\{\emptyset\}$ if $c = c_m$ or $\emptyset$ otherwise; in either case, $W^*_{1,m,c} = W_{1,m,c}$. Let

$$W^*_{1,m} = \cup_{c \in C}(W^*_{1,m,c} \times \{c\})$$

be the set of woman-contract pairs such that the women have proposed to m and order it according to $\succ_m$: $W^*_{1,m} = \{(w^{1,m}_1, c^{1,m}_1), \ldots, (w^{1,m}_{j^{1,m}}, c^{1,m}_{j^{1,m}})\}$ with $j^{1,m} = |W^*_{1,m}|$ and $(w^{1,m}_k, c^{1,m}_k) \succ_m (w^{1,m}_{k+1}, c^{1,m}_{k+1})$ for each $k \in \{1, \ldots, j^{1,m} - 1\}$. The distribution of those on the string is:

$$\begin{aligned}
\mu_1(m, w^{1,m}_1, c^{1,m}_1) = \min \quad & \{\mu_0(m, w^{1,m}_1, c^{1,m}_1) + \nu^1_W(w^{1,m}_1) 1_{W_{1,m,c^{1,m}_1}}(w^{1,m}_1), \\
& \nu_M(m)\} \text{ and} \\
\mu_1(m, w^{1,m}_k, c^{1,m}_k) = \min \quad & \{\mu_0(m, w^{1,m}_k, c^{1,m}_k) + \nu^1_W(w^{1,m}_k) 1_{W_{1,m,c^{1,m}_k}}(w^{1,m}_k), \\
& \nu_M(m) - \sum_{j<k} \mu_1(m, w^{1,m}_j, c^{1,m}_j)\}.
\end{aligned}$$

In short, men of type m add women outside the string in accordance with their preferences up to capacity.

If $m = \emptyset$, let, for each $c \in C$,

$$\begin{aligned}
& W_{1,\emptyset,c} = \{w \in R_1 : P_w(r_1(w)) = (\emptyset, c)\} \text{ and} \\
& W^*_{1,\emptyset,c} = W_{1,\emptyset,c} \cup \text{supp}(\mu_0(\emptyset, \cdot, c)).
\end{aligned}$$

For each $w \in W^*_{1,\emptyset,c}$, let

$$\mu_1(\emptyset, w, c) = \mu_0(\emptyset, w, c) + \nu^1_W(w) 1_{W_{1,\emptyset,c}}(w).$$

The case of the empty man type $\emptyset$ differs from the previous case in that there is no limit to the measure of women that can be added to the string with $\emptyset$.

Stage $n > 1$: The set of rejected women is

$$R_n = \{w \in W : \sum_{(m,c) \in M_\emptyset \times C} \mu_{n-1}(m, w, c) < \nu_W(w)\},$$

i.e. the set of women types with some of its members being outside the string. For each $w \in W$, let

$$M_{n,w} = M_{n-1,w} \cup \{(m,c) \in M \times C : \mu_{n-1}(m,w,c) - \mu_{n-2}(m,w,c) < 0\}$$
$$\cup \{(m,c) \in M \times C : w \in W_{n-1,m,c} \text{ and } \mu_{n-1}(m,w,c) - \mu_{n-2}(m,w,c) < \nu^{n-1}_W(w)\}$$

be the set of (m, c) who have rejected w before the start of stage n. This rejection could have happened before stage $n-1$ or it could happen in stage $n-1$ when some women of type w are removed from the string or when some of them outside the string had proposed to some men of type m but were not added to the string with such men. Set

$$r_n(w) = |M_{n,w}| + 1$$

and, for each $w \in W_\emptyset$,

$$\nu^n_W(w) = \nu_W(w) - \sum_{(m,c) \in M_\emptyset \times C} \mu_{n-1}(m, w, c)$$

be the measure of women of type w who are outside the string.

For each $m \in M$ and $c \in C$, let

$$W_{n,m,c} = \{w \in R_n : P_w(r_n(w)) = (m,c)\} \cup \{\emptyset\} \text{ and}$$
$$W^*_{n,m,c} = W_{n,m,c} \cup \text{supp}(\mu_{n-1}(m, \cdot, c)).$$

Order

$$W^*_{n,m} = \cup_{c \in C}(W^*_{n,m,c} \times \{c\})$$

according to $\succ_m$: $W^*_{n,m} = \{(w_1^{n,m}, c_1^{n,m}), \ldots, (w_{j^{n,m}}^{n,m}, c_{j^{n,m}}^{n,m})\}$ with $j^{n,m} = |W^*_{n,m}|$ and $(w_k^{n,m}, c_k^{n,m}) \succ_m (w_{k+1}^{n,m}, c_{k+1}^{n,m})$ for each $k \in \{1, \ldots, j^{1,m} - 1\}$. The distribution of those on the string is:

$$\begin{aligned}
\mu_n(m, w_1^{n,m}, c_1^{n,m}) = \min \quad & \{\mu_{n-1}(m, w_1^{n,m}, c_1^{n,m}) + \nu_W^n(w_1^{n,m}) 1_{W_{n,m,c_1^{n,m}}}(w_1^{n,m}), \\
& \nu_M(m)\} \text{ and} \\
\mu_n(m, w_k^{n,m}, c_k^{n,m}) = \min \quad & \{\mu_{n-1}(m, w_k^{n,m}, c_k^{n,m}) + \nu_W^n(w_k^{n,m}) 1_{W_{n,m,c_k^{n,m}}}(w_k^{n,m}), \\
& \nu_M(m) - \sum_{j<k} \mu_n(m, w_j^{n,m}, c_j^{n,m})\}.
\end{aligned}$$

If $m = \emptyset$, let, for each $c \in C$,

$$\begin{aligned}
W_{n,\emptyset,c} &= \{w \in R_n : P_w(r_n(w)) = (\emptyset, c)\} \text{ and} \\
W^*_{n,\emptyset,c} &= W_{n,\emptyset,c} \cup \text{supp}(\mu_{n-1}(\emptyset, \cdot, c)).
\end{aligned}$$

Then, for each $w \in W^*_{n,\emptyset,c}$, let

$$\mu_n(\emptyset, w, c) = \mu_{n-1}(\emptyset, w, c) + \nu_W^n(w) 1_{W_{n,\emptyset,c}}(w).$$

4.2 Convergence of the DAA

Let N be the first n such that $R_n = \emptyset$. Then we say that the algorithm *terminates after* $N - 1$ *iterations*. Our main result shows that when the type distributions have rational coefficients, such an N always exists and μ_{N-1} is a stable matching. In addition, μ_{N-1} is woman-optimal in the sense that, for each $w \in W$ and each stable matching μ',

$$\sum_{(m',c') \in M_\emptyset \times C : (m',c') \succeq_w (m,c)} \mu'(m', w, c') \leq \sum_{(m',c') \in M_\emptyset \times C : (m',c') \succeq_w (m,c)} \mu_{N-1}(m', w, c')$$

for each $(m, c) \in M_\emptyset \times C$, where $(m, c) \succeq_w (m', c')$ stands for either $(m, c) = (m', c')$ or $(m, c) \succ_w (m', c')$ (and analogously for $(w, c) \succeq_m (w', c')$).[5]

[5]For a discussion of this notion of woman-optimality, which is defined in analogy to first order stochastic dominance, see, for example, Echenique, Lee, and Yenmez (2010).

Theorem 1 *Let E be a marriage market. If preferences are linear orders, $\nu_M(m) \in \mathbb{Q}$ for each $m \in M$ and $\nu_W(w) \in \mathbb{Q}$ for each $w \in W$, then there is $N \in \mathbb{N}$ such that the DAA terminates after $N-1$ iterations and μ_{N-1} is a woman-optimal stable matching.*

Theorem 1 can be used in marriage markets in which preferences are only acyclic. In this case, since M and W are finite, each preference relation can be extended to a linear order to which our DAA applies. Since extending preferences can only make blocking easier, the stable matching for the extended preferences is a stable matching of the original marriage market.

The assumption that the distributions of men and women have rational coefficients is needed to guarantee that the DAA terminates in finitely many steps. Nevertheless, marriage markets whose distributions of men and women do not have rational coefficients can be approximated by a sequence of marriage markets to which Theorem 1 can be applied and, thus, a stable matching for the original marriage market can be obtained as the limit of a sequence of stable matchings for the marriage markets in the sequence.

Our DAA can be easily modified to produce a man-optimal stable matching. All it takes is to reverse the role of men and women in the algorithm, i.e. to make men be the ones who propose.

The proof of Theorem 1 is in Section A.1; here we provide an outline of it.

Note first that, by construction, $\mu_n(\emptyset, \emptyset, c) = 0$ for each $n \in \mathbb{N}_0$ and $c \in C$ since $\emptyset \notin W^*_{n,\emptyset,c}$; thus, we obtain condition 3 in the definition of a matching.

Condition 2 for a matching is also obtained by construction since the algorithm allocates all men. Indeed, since the measure of the empty woman is effectively unbounded, it can absorb all men of a given type that are not on the string with women that they prefer to the empty woman; thus, we actually have that

$$\sum_{(w,c) \in W_\emptyset \times C : (w,c) \succeq_m (\emptyset, c_m)} \mu_n(m, w, c) = \sum_{(w,c) \in W_\emptyset \times C} \mu_n(m, w, c) = \nu_M(m) \qquad (1)$$

for each $n \in \mathbb{N}_0$ and $m \in M$ (recall that $c_m \in C$ is the best contract for unmatched men of type m).

The marginal condition for women (i.e. condition 1 for a matching) holds when the algorithm stops. Nevertheless, along the sequence, we have that

$$\sum_{(m,c)\in M_\emptyset\times C} \mu_n(m,w,c) \le \nu_W(w)$$

for each $n \in \mathbb{N}_0$ and $w \in W_\emptyset$. This, in turn, implies that $\nu_W^n(w) \ge 0$ and $\mu_n(m,w,c) \ge 0$ for each $m \in M_\emptyset$, $w \in W_\emptyset$, $c \in C$ and $n \in \mathbb{N}$; hence μ_n is a measure for each $n \in \mathbb{N}$. When the algorithm stops, i.e. when $R_N = \emptyset$, we have from the definition of R_N that $\sum_{(m,c)\in M_\emptyset\times C} \mu_{N-1}(m,w,c) \ge \nu_W(w)$ for all $w \in W$. Combined with the above, this implies that $\sum_{(m,c)\in M_\emptyset\times C} \mu_{N-1}(m,w,c) = \nu_W(w)$ for all $w \in W$.

The algorithm is guaranteed to stop when the type distributions have rational coefficients. The importance of this feature is that there is a common denominator K for $\nu_M(m)$ and $\nu_W(w)$ for each $m \in M$ and $w \in W$; from this, it can be shown that K is in fact a common denominator for $\{\nu_W^n(w)\}_{n=1}^\infty$, and hence $\nu_W^n(w) > 0$ implies $\nu_W^n(w) \ge \frac{1}{K}$.

To see how the above implies that the algorithm stops, note first that $\max_n r_n(w)$ exists since $r_n(w) \in \{1,\ldots,|M_\emptyset \times C|\}$. Thus, $r_n(w)$ is eventually constant, i.e. there is k such that $r_n(w) = r$ for each $n \ge k$. Then for $n \ge k$, we must have $P_w(r) \notin M_{n,w}$. Since $r_{n+1}(w) > r_n(w)$ if $\mu_n(P_w(r),w) < \mu_{n-1}(P_w(r),w) + \nu_W^n(w)$,[6,7] a contradiction, it follows that for $n \ge k$

$$\mu_n(P_w(r),w) \ge \mu_{n-1}(P_w(r),w) + \nu_W^n(w)$$

and, in fact (from the argument in the previous paragraph),

$$\mu_n(P_w(r),w) \ge \mu_{n-1}(P_w(r),w) + \frac{1}{K}$$

provided that $\nu_W^n(w) > 0$ i.e. $w \in R_n$. Since $\mu_n(P_w(r),w)$ is bounded, it follows that there is N_w such that $w \notin R_n$ for all $n \ge N_w$. Then, $R_n = \emptyset$ for all $n \ge \max_w N_w$.

[6] The notation $\mu_n(P_w(r),w)$ stands for $\mu_n(m,w,c)$ with $(m,c) = P_w(r)$.

[7] By definition, $r_{n+1}(w) > r_n(w)$ if for some $(m,c) \notin M_{n,w}$, $\mu_n(m,w,c) < \mu_{n-1}(m,w,c)$ or $\mu_n(m,w,c) < \mu_{n-1}(m,w,c) + \nu_W^n(w)$ and $w \in W_{n,m,c}$. But since $P_w(r) \notin M_{n,w}$ and $w \in W_{n,P_w(r)}$ whenever $\nu_W^n(w) > 0$, we have that $r_{n+1}(w) > r_n(w)$ if $\mu_n(P_w(r),w) < \mu_{n-1}(P_w(r),w) + \nu_W^n(w)$.

Let N be the first $n \in \mathbb{N}$ such that $R_n = \emptyset$. Then the algorithm stops after $N-1$ iterations.

We now outline the argument for why $\mu := \mu_{N-1}$ is stable; in Section A.1, we show that in addition μ is woman-optimal. It follows from (1) that μ is individually rational for the men. Individual rationality for the women follows because one can show that $\mu_n(P_w(r), w) = 0$ for each $n \in \mathbb{N}$ and $r > r_w$, where $r_w \in \{1, \ldots, |M_\emptyset \times C|\}$ is such that $P_w(r_w) = (\emptyset, c_w)$, where $c_w \in C$ is the best contract for unmatched women of type w, i.e. $(\emptyset, c_w) \succ_w (\emptyset, c)$ for each $c \in C \setminus \{c_w\}$.

It can be shown that for each $n \in \mathbb{N}$, $m \in M$, $w \in W$ and $c \in C$ such that $(m, w, c) \in \text{supp}(\mu_n)$,

$$\begin{gathered}\sum_{(w',c'):(w',c') \succeq_m (w,c)} \mu_n(m, w', c') = \nu_M(m) \text{ or} \\ \sum_{(m',c'):(m',c') \succeq_w (m,c)} \mu_n(m', w, c') = \sum_{(m',c') \in M_\emptyset \times C} \mu_n(m', w, c').\end{gathered} \tag{2}$$

In words, men of type m are on the string only with women that are not worse than w or women of type w are on the string only with men that are not worse than m.

This then shows that there are no blocking pairs as follows. Let $m \in M$, $w \in W$ and $c \in C$ be such that $(m, w, c) \in \text{supp}(\mu)$ and there exists $(w', c', \bar{m}, \bar{c})$ such that $\mu(\bar{m}, w', \bar{c}) > 0$, $(w', c') \succ_m (w, c)$ and $(m, c') \succ_{w'} (\bar{m}, \bar{c})$. Since $\mu(m, w, c) > 0$, it follows that $\sum_{(\tilde{w},\tilde{c}):(\tilde{w},\tilde{c}) \succeq_m (w',c')} \mu(m, \tilde{w}, \tilde{c}) < \nu_M(m)$. Using this inequality, it can be shown that $\mu(m, w', c') > 0$. Then (2) and $R_N = \emptyset$ imply that

$$\sum_{(\tilde{m},\tilde{c}):(\tilde{m},\tilde{c}) \succeq_{w'} (m,c')} \mu(\tilde{m}, w', \tilde{c}) = \sum_{(\tilde{m},\tilde{c}) \in M_\emptyset \times C} \mu(\tilde{m}, w', \tilde{c}) = \nu_W(w').$$

Hence, $\mu(\bar{m}, w', \bar{c}) = 0$, a contradiction to $\mu(\bar{m}, w', \bar{c}) > 0$. This then shows that there are no blocking pairs and that μ is stable.

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A Appendix

A.1 Proof of Theorem 1

We establish Theorem 1 in the following claims.

Claim 1 *For each $n \in \mathbb{N}_0$ and $c \in C$, $\mu_n(\emptyset, \emptyset, c) = 0$.*

Proof. Let $c \in C$. The conclusion is clear when $n = 0$ and, for $n > 0$, it follows because $\emptyset \notin W^*_{n,\emptyset,c}$ for each $n \in \mathbb{N}$. ■

Claim 2 *For each $n \in \mathbb{N}_0$ and $w \in W_\emptyset$, $\sum_{(m,c)\in M_\emptyset \times C} \mu_n(m, w, c) \leq \nu_W(w)$.*

Proof. Let $w \in W_\emptyset$ and note that the conclusion is clear when $n = 0$. Let $n > 0$, $w \in W$ and $(m^*, c^*) = P_w(r_n(w))$; then $1_{W_{n,m,c}}(w) = 0$ for each $(m, c) \neq (m^*, c^*)$. It then follows that

$$\begin{aligned} &\sum_{(m,c)\in M_\emptyset \times C} \mu_n(m, w, c) \leq \sum_{(m,c)\in M_\emptyset \times C} \mu_{n-1}(m, w, c) + \nu^n_W(w) \\ &= \sum_{(m,c)\in M_\emptyset \times C} \mu_{n-1}(m, w, c) + \nu_W(w) - \sum_{(m,c)\in M_\emptyset \times C} \mu_{n-1}(m, w, c) = \nu_W(w). \end{aligned}$$

When $w = \emptyset$, using Claim 1,

$$\sum_{(m,c)\in M_\emptyset \times C} \mu_n(m, \emptyset, c) = \sum_{(m,c)\in M \times C} \mu_n(m, \emptyset, c) \leq \sum_{m\in M} \nu_M(m) = \nu_M(M) \leq \nu_W(\emptyset).$$

■

Claim 2 implies that $\nu^n_W(w) \geq 0$ for each $w \in W_\emptyset$ and $n \in \mathbb{N}$. This, in turn, implies that $\mu_n(m, w, c) \geq 0$ for each $m \in M_\emptyset$, $w \in W_\emptyset$, $c \in C$ and $n \in \mathbb{N}$.

Write $(m, c) \succeq_w (m', c')$ if either $(m, c) = (m', c')$ or $(m, c) \succ_w (m', c')$ and analogously for $(w, c) \succeq_m (w', c')$.

Claim 3 *For each $n \in \mathbb{N}_0$ and $m \in M$,*

$$\sum_{(w,c)\in W_\emptyset \times C : (w,c) \succeq_m (\emptyset, c_m)} \mu_n(m, w, c) = \sum_{(w,c)\in W_\emptyset \times C} \mu_n(m, w, c) = \nu_M(m).$$

Proof. Let $m \in M$ be given and note that the conclusion is clear when $n = 0$. Let $n > 0$ and assume that the conclusion holds for each $j = 0, \ldots, n-1$. We have that $(\emptyset, c_m) \in W^*_{n,m}$ and $\nu^n_W(\emptyset) \geq \nu_M(M) - \sum_{(m',c) \in M_\emptyset \times C} \mu_{n-1}(m', \emptyset, c) = \nu_M(M) - \sum_{(m',c) \in M \times C} \mu_{n-1}(m', \emptyset, c) = \nu_M(M) - \sum_{m' \in M} \mu_{n-1}(m', \emptyset, c_{m'})$ by Claim 1 and the inductive hypothesis. Thus,

$$\begin{aligned} \mu_{n-1}(m, \emptyset, c_m) + \nu^n_W(\emptyset) &\geq \nu_M(M) - \sum_{m' \in M \setminus \{m\}} \mu_{n-1}(m', \emptyset, c_{m'}) \\ &\geq \nu_M(M) - \sum_{m' \in M \setminus \{m\}} \nu_M(m') = \nu_M(m). \end{aligned}$$

Hence, $\mu_n(m, \emptyset, c_m) = \nu_M(m) - \sum_{(w,c) \in W^*_{n,m} : (w,c) \succ_m (\emptyset, c_m)} \mu_n(m, w, c)$ and, therefore, $\sum_{(w,c) \in W_\emptyset \times C : (w,c) \succeq_m (\emptyset, c_m)} \mu_n(m, w) = \nu_M(m)$. ■

Claim 4 *If $n \in \mathbb{N}$, $m \in M_\emptyset$, $w \in W$, $c \in C$ and $(m, w, c) \in \text{supp}(\mu_n)$, then $(m, c) \in \{P_w(r) : r \in \{1, \ldots, r_n(w)\}\}$.*

Proof. The conclusion clearly holds for μ_1. Suppose that the conclusion holds for $\mu_1, \ldots, \mu_{n-1}$ and that $(m, w, c) \in \text{supp}(\mu_n)$. Then

$$\mu_{n-1}(m, w, c) > 0 \text{ or } \nu^n_W(w) 1_{W_{n,m,c}}(w) > 0.$$

In the former case, $(m, c) = P_w(r)$ for some $r \leq r_{n-1}(w)$ by the inductive hypothesis; hence, $r \leq r_n(w)$ since $M_{n-1,w} \subseteq M_{n,w}$ and, thus, $r_{n-1}(w) \leq r_n(w)$. In the latter case, it follows by $w \in W_{n,m,c}$ that $(m, c) = P_w(r_n(w))$. ■

For each $n \in \mathbb{N}$, $w \in W$ and $r \in \{1, \ldots, |M_\emptyset \times C|\}$, let $\mu_n(P_w(r), w) = \mu_n(m, w, c)$ where $(m, c) = P_w(r)$.

Claim 5 *For each $n \in \mathbb{N}$ and $w \in W$, $M_{n,w} = \{P_w(1), \ldots, P_w(r_n(w) - 1)\}$ (with the convention that $\{P_w(1), P_w(0)\} = \emptyset$) and $r_{n+1}(w) - r_n(w) \in \{0, 1\}$.*

Proof. The claim holds for $M_{1,w}$ and $M_{2,w}$. For the latter, note that $M_{2,w} = \{(m, c) \in M \times C : w \in W_{1,m,c} \text{ and } \mu_1(m, w, c) < \nu_W(w)\}$. Since $w \in W_{1,m,c}$ if and only if $P_w(1) = (m, c)$, $M_{2,w} = \{P_w(1)\}$ if $\mu_1(P_w(1), w) < \nu_W(w)$ and $M_{2,w} = \emptyset$ if $\mu_1(P_w(1), w) = \nu_W(w)$. In the former case, $r_2(w) = 2$ so $M_{2,w} = \{P_w(1)\}$ and

$r_2(w) - r_1(w) = 1$. In the latter case, $r_2(w) = 1$ so $M_{2,w} = \{P_w(1), P_w(0)\} = \emptyset$ and $r_2(w) - r_1(w) = 0$.

Suppose that it holds for $M_{1,w}, \ldots, M_{n-1,w}$. Then

$$M_{n-1,w} = \{P_w(1), \ldots, P_w(r_{n-1}(w) - 1)\}.$$

If $r > r_{n-1}(w) \geq r_{n-2}(w)$ (since $M_{n-2,w} \subseteq M_{n-1,w}$), then $\mu_{n-2}(P_w(r), w) = 0$ by Claim 4. Hence,

$$\{(m, c) \in M \times C : \mu_{n-1}(m, w, c) < \mu_{n-2}(m, w, c)\} \subseteq$$
$$\{P_w(1), \ldots, P_w(r_{n-1}(w) - 1), P_w(r_{n-1}(w))\}.$$

Furthermore, $\{(m, c) \in M \times C : w \in W_{n-1,m,c}\} \subseteq \{P_w(r_{n-1}(w))\}$.

Thus, either $r_n(w) = r_{n-1}(w)$ and

$$M_{n,w} = \{P_w(1), \ldots, P_w(r_{n-1}(w) - 1)\} = \{P_w(1), \ldots, P_w(r_n(w) - 1)\}$$

or $r_n(w) = r_{n-1}(w) + 1$ and

$$M_{n,w} = \{P_w(1), \ldots, P_w(r_{n-1}(w))\} = \{P_w(1), \ldots, P_w(r_n(w) - 1)\}.$$

Since the conclusion of the claim holds in both cases, this completes the proof. ■

For each $w \in W$, let $c_w \in C$ be such that $(\emptyset, c_w) \succ_w (\emptyset, c)$ for each $c \in C \setminus \{c_w\}$.

Claim 6 *For each $w \in W$, let $r_w \in \{1, \ldots, |M \times C| + 1\}$ be such that $P_w(r_w) = (\emptyset, c_w)$. Then $\mu_n(P_w(r), w) = 0$ for each $n \in \mathbb{N}$ and $r \in \{r_w + 1, \ldots, |M_\emptyset \times C|\}$.*

Proof. Let $w \in W$, $n \in \mathbb{N}$ and $r \in \{r_w + 1, \ldots, |M_\emptyset \times C|\}$. Note first that $r_n(w) \leq r_w$ for each $n \in \mathbb{N}$ since otherwise, by Claim 5, $(\emptyset, c_w) \in \{P_w(1), \ldots, P_w(r_w)\} \subseteq M_{n,w} \subseteq M \times C$, a contradiction as $\emptyset \notin M$. It then follows that $w \notin W_{n,m,c}$ for each $n \in \mathbb{N}$ and $(m, c) \notin \{P_w(1), \ldots, P_w(r_w)\}$.

It follows from the above that $\mu_n(P_w(r), w) = 0$ for all $n \in \mathbb{N}$. To see this, note that $\mu_0(P_w(r), w) = 0$. Assume that $\mu_{n-1}(P_w(r), w) = 0$. Then $w \notin \text{supp}(\mu_{n-1}(P_w(r), \cdot)) \cup W_{n,P_w(r)} = W^*_{n,P_w(r)}$. Thus, $\mu_n(P_w(r), w) = 0$. ■

For each $w \in W$ and $r \in \{1, \ldots, |M_\emptyset \times C|\}$, write $P_w(r) = (m_w(r), c_w(r))$. Also, for each $m \in M$, $w \in W_\emptyset$ and $c \in C$, write $S_m(w, c) = \{(w', c') \in W_\emptyset \times C : (w', c') \succ_m (w, c)\}$ and $S_w(m, c) = \{(m', c') \in M_\emptyset \times C : (m', c') \succ_w (m, c)\}$.

Claim 7 *For each* $n \in \mathbb{N}$ *and* $w \in W$,

$$\mu_n(P_w(r), w) = \nu_M(m_w(r)) - \sum_{(w',c') \in S_{m_w(r)}(w, c_w(r))} \mu_n(m_w(r), w', c')$$

if $r < r_n(w)$

$$\mu_n(P_w(r_n(w)), w) = \min\{\mu_{n-1}(P_w(r_n(w)), w) + \nu_W^n(w), \\ \nu_M(m_w(r_n(w))) - \sum_{(w',c') \in S_{m_w(r_n(w))}(w, c_w(r_n(w)))} \mu_n(m_w(r_n(w)), w', c')\}$$

and, if $r_n(w) > r_{n-1}(w)$*, then*

$$\mu_{n-1}(P_w(r_{n-1}(w)), w) = \\ \nu_M(m_w(r_{n-1}(w))) - \sum_{(w',c') \in S_{m_w(r_{n-1}(w))}(w, c_w(r_{n-1}(w)))} \mu_{n-1}(m_w(r_{n-1}(w)), w', c').$$

Proof. The conclusion holds for $n = 1$ since $r_1(w) = 1$, $w \in W_{1, P_w(1)}$ and

$$\mu_1(P_w(1), w) = \min\{\mu_0(P_w(1), w) + \nu_W^1(w), \\ \nu_M(m_w(1)) - \sum_{(w',c') \in S_{m_w(1)}(w, c_w(1))} \mu_n(m_w(1), w', c')\}.$$

Suppose it holds for $\mu_1, \ldots, \mu_{n-1}$ and consider first the case where $r_n(w) = r_{n-1}(w)$. Let $r < r_n(w)$. Then $r < r_{n-1}(w)$ and, by the inductive hypothesis,

$$\mu_{n-1}(P_w(r), w) = \nu_M(m_w(r)) - \sum_{(w',c') \in S_{m_w(r)}(w, c_w(r))} \mu_{n-1}(m_w(r), w', c').$$

Moreover, $P_w(r_n(w)) \neq P_w(r)$, hence $w \notin W_{n, P_w(r)}$. The latter implies that

$$\mu_{n-1}(P_w(r), w) + \nu_W^n(w) 1_{W_{n, P_w(r)}}(w) = \mu_{n-1}(P_w(r), w)$$

and we may assume that $\mu_n(P_w(r), w) = \mu_{n-1}(P_w(r), w)$ since otherwise there is nothing to prove.

If $\mu_{n-1}(P_w(r), w) = 0$, then

$$\sum_{(w',c') \in S_{m_w(r)}(w, c_w(r))} \mu_{n-1}(m_w(r), w', c') = \nu_M(m_w(r)).$$

This then implies that $\text{supp}(\mu_{n-1}(m_w(r),\cdot)) \subseteq S_{m_w(r)}(w, c_w(r))$ and, hence,

$$\text{supp}(\mu_n(m_w(r),\cdot)) \subseteq S_{m_w(r)}(w, c_w(r))$$

as follows: This is clear if there is $(w', c') \in \text{supp}(\mu_{n-1}(m_w(r),\cdot))$ such that

$$\mu_n(m_w(r), w', c') = \nu_M(m_w(r)) - \sum_{(\tilde{w},\tilde{c}) \in S_{m_w(r)}(w',c')} \mu_n(m_w(r), \tilde{w}, \tilde{c});$$

if this condition does not hold, then

$$\mu_n(m_w(r), w', c') = \mu_{n-1}(m_w(r), w', c') + \nu_W^n(w') 1_{W_{n,m_w(r),c'}}(w')$$

for each $(w', c') \in \text{supp}(\mu_{n-1}(m_w(r),\cdot))$ and it follows by Claim 3 that

$$\begin{aligned} \nu_M(m_w(r)) &\geq \sum_{(w',c') \in \text{supp}(\mu_{n-1}(m_w(r),\cdot))} \mu_n(m_w(r), w', c') \\ &\geq \sum_{(w',c') \in \text{supp}(\mu_{n-1}(m_w(r),\cdot))} \mu_{n-1}(m_w(r), w', c') = \nu_M(m_w(r)); \end{aligned}$$

thus, indeed, $\text{supp}(\mu_n(m_w(r),\cdot)) \subseteq S_{m_w(r)}(w, c_w(r))$. This, together with Claim 3, then implies that $\sum_{(w',c') \in S_{m_w(r)}(w,c_w(r))} \mu_n(m_w(r), w', c') = \nu_M(m_w(r))$ and

$$\mu_n(P_w(r), w) = 0 = \nu_M(m_w(r)) - \sum_{(w',c') \in S_{m_w(r)}(w,c_w(r))} \mu_n(m_w(r), w', c').$$

If $\mu_{n-1}(P_w(r), w) > 0$, then

$$\mu_n(P_w(r), w) = \mu_{n-1}(P_w(r), w) = \nu_M(m_w(r)) - \sum_{(w',c') \in S_{m_w(r)}(w,c_w(r))} \mu_{n-1}(m_w(r), w', c').$$

Because $\mu_n(P_w(r), w) > 0$, for each $(w', c') \in S_{m_w(r)}(w, c_w(r))$, it cannot be that

$$\mu_n(m_w(r), w', c') = \nu_M(m_w(r)) - \sum_{(\tilde{w},\tilde{c}) \in S_{m_w(r)}(w',c')} \mu_n(m_w(r), \tilde{w}, \tilde{c}).$$

Thus, for each $(w', c') \in S_{m_w(r)}(w, c_w(r))$, $\mu_n(m_w(r), w', c') = \mu_{n-1}(m_w(r), w', c') + \nu_W^n(w') 1_{W_{n,m_w(r),c'}}(w')$. Hence,

$$\sum_{(w',c') \in S_{m_w(r)}(w,c_w(r))} \mu_n(m_w(r), w', c') \geq \sum_{(w',c') \in S_{m_w(r)}(w,c_w(r))} \mu_{n-1}(m_w(r), w', c');$$

if this inequality holds strictly, then

$$\mu_n(P_w(r), w) \leq \nu_M(m_w(r)) - \sum_{(w',c') \in S_{m_w(r)}(w, c_w(r))} \mu_n(m_w(r), w', c')$$
$$< \nu_M(m_w(r)) - \sum_{(w',c') \in S_{m_w(r)}(w, c_w(r))} \mu_{n-1}(m_w(r), w', c') = \mu_{n-1}(P_w(r), w)$$
$$= \mu_n(P_w(r), w),$$

a contradiction. Thus,

$$\sum_{(w',c') \in S_{m_w(r)}(w, c_w(r))} \mu_n(m_w(r), w', c') = \sum_{(w',c') \in S_{m_w(r)}(w, c_w(r))} \mu_{n-1}(m_w(r), w', c')$$

and

$$\begin{aligned} \mu_n(P_w(r), w) &= \nu_M(m_w(r)) - \sum_{(w',c') \in S_{m_w(r)}(w, c_w(r))} \mu_{n-1}(m_w(r), w', c') \\ &= \nu_M(m_w(r)) - \sum_{(w',c') \in S_{m_w(r)}(w, c_w(r))} \mu_n(m_w(r), w', c') \end{aligned}$$

as claimed.

For $r = r_n(w)$, $w \in W_{n, P_w(r_n(w))}$ if and only if $\nu_W^n(w) > 0$, hence

$$\mu_n(P_w(r_n(w)), w) = \min\{\mu_{n-1}(P_w(r_n(w)), w) + \nu_W^n(w),$$
$$\nu_M(m_w(r_n(w))) - \sum_{(w',c') \in S_{m_w(r_n(w))}(w, c_w(r_n(w)))} \mu_n(m_w(r_n(w)), w', c')\}$$

as claimed.

Consider the remaining (by Claim 5) case where $r_n(w) = r_{n-1}(w) + 1$. When $r < r_{n-1}(w)$ or $r = r_n(w)$, the above argument yields the desired conclusion. Thus, let $r = r_{n-1}(w)$. If

$$\mu_{n-1}(P_w(r_{n-1}(w)), w) =$$
$$\nu_M(m_w(r_{n-1}(w))) - \sum_{(w',c') \in S_{m_w(r_{n-1}(w))}(w, c_w(r_{n-1}(w)))} \mu_{n-1}(m_w(r_{n-1}(w)), w', c'), \tag{3}$$

then the above argument applies as well. Thus, in this case, it suffices to show that (3) holds.

By Claim 5,

$$M_{n-1,w} = \{P_w(1), \ldots, P_w(r_{n-1}(w) - 1)\} \text{ and } M_{n,w} = \{P_w(1), \ldots, P_w(r_{n-1}(w))\}$$

since $r_n(w) - 1 = r_{n-1}(w)$. For convenience, let $(m, c) = P_w(r_{n-1}(w))$. Since $(m, c) \in M_{n,w} \setminus M_{n-1,w}$, we have that (a) $\mu_{n-1}(m, w, c) < \mu_{n-2}(m, w, c)$ or (b) $w \in W_{n-1,m,c}$ and $\mu_{n-1}(m, w, c) < \mu_{n-2}(m, w, c) + \nu_W^{n-1}(w)$. If (3) does not hold, then

$$\mu_{n-1}(m, w, c) = \mu_{n-2}(m, w, c) + 1_{W_{n-1,m,c}}(w)\nu_W^{n-1}(w)$$

and neither (a) nor (b) can hold. Thus, (3) must hold and the claim follows. ■

Claim 8 *For each $n \in \mathbb{N}$, $m \in M$, $w \in W$ and $c \in C$, if $(m, w, c) \in \operatorname{supp}(\mu_n)$ then*

$$\sum_{(w',c'):(w',c') \succeq_m (w,c)} \mu_n(m, w', c') = \nu_M(m)$$

or

$$\sum_{(m',c'):(m',c') \succeq_w (m,c)} \mu_n(m', w, c') = \sum_{(m',c') \in M_\emptyset \times C} \mu_n(m', w, c').$$

Proof. Let $n \in \mathbb{N}$, $m \in M$, $w \in W$ and $c \in C$ be such that $(m, w, c) \in \operatorname{supp}(\mu_n)$. Then $(m, c) = P_w(r)$ for some $1 \leq r \leq r_n(w)$ by Claim 4. The conclusion of the claim follows immediately from Claim 7 except when $(m, c) = P_w(r_n(w))$ and $\mu_n(m, w, c) = \mu_{n-1}(m, w, c) + \nu_W^n(w)$.

Hence, suppose that $(m, c) = P_w(r_n(w))$ and $\mu_n(m, w, c) = \mu_{n-1}(m, w, c) + \nu_W^n(w)$. By Claim 5, there are two possible cases: (a) $r_n(w) = r_{n-1}(w) + 1$ and (b) $r_n(w) = r_{n-1}(w)$.

In case (a), $\mu_{n-1}(P_w(r_n(w)), w) = 0$ since

$$\operatorname{supp}(\mu_{n-1}(\cdot, w)) \subseteq \{P_w(1), \ldots, P_w(r_n(w) - 1)\},$$

thus $\mu_n(P_w(r_n(w)), w) = \nu_W^n(w)$ and

$$\nu_W^n(w) = \nu_W(w) - \sum_{r=1}^{r_{n-1}(w)} \mu_{n-1}(P_w(r), w) = \nu_W(w) - \sum_{r=1}^{r_n(w)-1} \mu_{n-1}(P_w(r), w).$$

Hence,

$$\sum_{r=1}^{r_n(w)} \mu_n(P_w(r), w) = \nu_W(w) - \sum_{r=1}^{r_n(w)-1} \mu_{n-1}(P_w(r), w) \\ + \sum_{r=1}^{r_n(w)-1} \left(\nu_M(m_w(r)) - \sum_{(w',c') \in S_{m_w(r)}(w, c_w(r))} \mu_n(m_w(r), w', c') \right). \tag{4}$$

In case (b),

$$\nu_W^n(w) = \nu_W(w) - \sum_{r=1}^{r_{n-1}(w)} \mu_{n-1}(P_w(r), w) = \nu_W(w) - \sum_{r=1}^{r_n(w)} \mu_{n-1}(P_w(r), w)$$

and, hence,

$$\sum_{r=1}^{r_n(w)} \mu_n(P_w(r), w) = \mu_{n-1}(P_w(r_n(w)), w) + \nu_W(w) - \sum_{r=1}^{r_n(w)} \mu_{n-1}(P_w(r), w) \\ + \sum_{r=1}^{r_n(w)-1} \left(\nu_M(m_w(r)) - \sum_{(w',c') \in S_{m_w(r)}(w, c_w(r))} \mu_n(m_w(r), w', c') \right) \\ = \nu_W(w) - \sum_{r=1}^{r_n(w)-1} \mu_{n-1}(P_w(r), w) \\ + \sum_{r=1}^{r_n(w)-1} \left(\nu_M(m_w(r)) - \sum_{(w',c') \in S_{m_w(r)}(w, c_w(r))} \mu_n(m_w(r), w', c') \right).$$

Thus, (4) holds.

It then follows from $\mu_n(P_w(r_n(w)), w) = \mu_{n-1}(P_w(r_n(w)), w) + \nu_W^n(w)$, (4) and

Claim 7 that

$$\sum_{(m',c'):(m',c')\succeq_w (m,c)} \mu_n(m',w,c') = \sum_{r=1}^{r_n(w)} \mu_n(P_w(r),w)$$

$$= \nu_W(w) - \sum_{r=1}^{r_n(w)-1} \mu_{n-1}(P_w(r),w)$$

$$+ \sum_{r=1}^{r_n(w)-1} \left(\nu_M(m_w(r)) - \sum_{(w',c')\in S_{m_w(r)}(w,c_w(r))} \mu_n(m_w(r),w',c') \right)$$

$$= \nu_W(w) - \sum_{r=1}^{r_n(w)-1} \mu_{n-1}(P_w(r),w) + \sum_{r=1}^{r_n(w)-1} \mu_n(P_w(r),w)$$

$$= \nu_W(w) - \sum_{r=1}^{r_n(w)-1} \mu_{n-1}(P_w(r),w) + \sum_{r=1}^{r_n(w)-1} \mu_n(P_w(r),w)$$

$$- \mu_{n-1}(P_w(r_n(w)),w) + \mu_n(P_w(r_n(w)),w) - \nu_W^n(w)$$

$$= \nu_W(w) - \sum_{r=1}^{r_n(w)} \mu_{n-1}(P_w(r),w) + \sum_{r=1}^{r_n(w)} \mu_n(P_w(r),w) - \nu_W^n(w)$$

$$= \sum_{r=1}^{r_n(w)} \mu_n(P_w(r),w) = \sum_{(m',c')\in M_\emptyset\times C} \mu_n(m',w,c').$$

■

Claim 9 *There exists $\underline{\nu} > 0$ such that if $\nu_W^n(w) > 0$, then $\nu_W^n(w) \geq \underline{\nu}$.*

Proof. We have that $\nu_M(m)$ and $\nu_W(w)$ are rational for all $m \in M$ and $w \in W_\emptyset$, and let $K > 0$ be such that $\nu_M(m) = \frac{q_M(m)}{K}$ and $\nu_W(w) = \frac{q_W(w)}{K}$ for each $m \in M$, $w \in W_\emptyset$, and $q_M(w)$, $q_W(w) \in \mathbb{Z}$. We will show that for all $m \in M_\emptyset$, $w \in W_\emptyset$, $c \in C$ and $n \in \mathbb{N}$, $\mu_n(m,w,c) = \frac{q_\mu(n,m,w,c)}{K}$ and $\nu_W^{n+1}(w) = \frac{q_\nu(n+1,w)}{K}$ for some $q_\mu(n,m,w,c)$, $q_\nu(n+1,w) \in \mathbb{Z}$. Thus, $\underline{\nu} = \frac{1}{K}$ proves the claim.

For any $m \in M_\emptyset$, let $W_{1,m}^* = \{(w_1,c_1),\ldots,(w_{|W_{1,m}^*|}, c_{|W_{1,m}^*|})\}$. Then

$$\mu_1(m,w_1,c) = \min\left\{\frac{q_W(w_1)}{K}, \frac{q_M(m)}{K}\right\} = \frac{q_\mu(1,m,w_1,c_1)}{K},$$

where $q_\mu(1,m,w_1,c_1) \in \mathbb{Z}$. For each $1 < k \leq |W_{1,m}^*|$, assume that for each $j < k$, $\mu_1(m,w_j,c_j) = \frac{q_\mu(1,m,w_j,c_j)}{K}$ for some $q_\mu(1,m,w_j,c_j) \in \mathbb{Z}$. Then

$$\mu_1(m,w_k,c_k) = \min\left\{\frac{q_W(w_k)}{K}, \frac{q_M(m)}{K} - \sum_{j<k} \frac{q_\mu(1,m,w_j,c_j)}{K}\right\} = \frac{q_\mu(1,m,w_k,c_k)}{K},$$

where $q_\mu(1,m,w_k,c_k) \in \mathbb{Z}$, since $q_W(w_1) \in \mathbb{Z}$, $q_M(m) \in \mathbb{Z}$, and $q_\mu(1,m,w_j,c_j) \in \mathbb{Z}$ for each $j < k$. Thus, for each $(w,c) \in W^*_{1,m}$, $\mu_1(m,w,c) = \frac{q_\mu(1,m,w,c)}{K}$ for some $q_\mu(1,m,w,c) \in \mathbb{Z}$. For each $(w,c) \notin W^*_{1,m}$, $\mu_1(m,w,c) = \frac{q_\mu(1,m,w,c)}{K}$ for $q_\mu(1,m,w,c) = 0 \in \mathbb{Z}$. Finally, for each $w \in W_\emptyset$, $\nu^2_W(w) = \frac{q_W(w)}{K} - \sum_{(m,c)\in M_\emptyset \times C} \frac{q_\mu(1,m,w,c)}{K} = \frac{q_\nu(2,w)}{K}$ for some $q_\nu(2,w) \in \mathbb{Z}$.

Now suppose that for all $m \in M_\emptyset$, $w \in W_\emptyset$ and $c \in C$, $\mu_{n-1}(m,w,c) = \frac{q_\mu(n-1,m,w,c)}{K}$ and $\nu^n_W(w) = \frac{q_\nu(n,w)}{K}$ for some $q_\mu(n-1,m,w,c)$, $q_\nu(n,w) \in \mathbb{Z}$. For any $m \in M_\emptyset$, let $W^*_{n,m} = \{(w_1,c_1),\ldots,(w_{|W^*_{n,m}|}, c_{|W^*_{n,m}|})\}$. Then

$$\begin{aligned}\mu_n(m,w_1,c_1) &= \min\left\{\frac{q_\mu(n-1,m,w_1,c_1)}{K} + \frac{q_\nu(n,w_1)}{K} 1_{W_{n,m,c_1}}(w_1), \frac{q_M(m)}{K}\right\} \\ &= \frac{q_\mu(n,m,w_1,c_1)}{K},\end{aligned}$$

where $q_\mu(n,m,w_1,c_1) \in \mathbb{Z}$. For each $1 < k \le |W^*_{n,m}|$, assume that for each $j < k$, $\mu_n(m,w_j,c_j) = \frac{q_\mu(n,m,w_j,c_j)}{K}$ for some $q_\mu(n,m,w_j,c_j) \in \mathbb{Z}$. Then

$$\begin{aligned}\mu_n(m,w_k,c_k) &= \min\left\{\frac{q_\mu(n-1,m,w_k,c_k)}{K} + \frac{q_\nu(n,w_k)}{K} 1_{W_{n,m,c_k}}(w_k),\right. \\ &\qquad\left.\frac{q_M(m)}{K} - \sum_{j<k} \frac{q_\mu(n,m,w_j,c_j)}{K}\right\} \\ &= \frac{q_\mu(n,m,w_k,c_k)}{K},\end{aligned}$$

where $q_\mu(n,m,w_k,c_k) \in \mathbb{Z}$, since $q_\mu(n-1,m,w_k,c_k) \in \mathbb{Z}$, $q_\nu(n,w_k) \in \mathbb{Z}$, $q_M(m) \in \mathbb{Z}$, and $q_\mu(n,m,w_j,c_j) \in \mathbb{Z}$ for each $j < k$. Thus, for each $(w,c) \in W^*_{n,m}$, $\mu_n(m,w,c) = \frac{q_\mu(n,m,w,c)}{K}$ for some $q_\mu(n,m,w,c) \in \mathbb{Z}$. For each $(w,c) \notin W^*_{n,m}$, $\mu_n(m,w,c) = \frac{q_\mu(n,m,w,c)}{K}$ for $q_\mu(n,m,w,c) = 0 \in \mathbb{Z}$. Finally, for each $w \in W_\emptyset$, $\nu^{n+1}_W(w) = \frac{q_W(w)}{K} - \sum_{(m,c)\in M_\emptyset\times C} \frac{q_\mu(n,m,w,c)}{K} = \frac{q_\nu(n+1,w)}{K}$ for some $q_\nu(n+1,w) \in \mathbb{Z}$. ■

Claim 10 *For each $w \in W$ and $k \in \mathbb{N}$, if $|\{n > k : w \in R_n\}| > \frac{\nu_W(w)}{\underline{\nu}}$ then $r_{k'}(w) = r_k(w) + 1$ for some $k' > k$.*

Proof. Let $w \in W$ and $k \in \mathbb{N}$. Suppose for a contradiction that $r_n(w) = r_k(w)$ for all $n \ge k$ and, for convenience, let $r = r_k(w)$. For each $n > k$, it follows that $P_w(r) \notin M_{n+1,w}$ since $M_{n+1,w} = \{P_w(1),\ldots,P_w(r-1)\}$ by Claim 5 and, therefore, $\mu_n(P_w(r),w) \ge \mu_{n-1}(P_w(r),w) + \nu^n_W(w)$ since $w \in W_{n,P_w(r)}$ whenever $\nu^n_W(w) > 0$.

Hence, $\mu_n(P_w(r),w) \geq \mu_{n-1}(P_w(r),w)$ if $w \notin R_n$ and $\mu_n(P_w(r),w) - \mu_{n-1}(P_w(r),w) \geq \underline{\nu}$ if $w \in R_n$ by Claim 9 and because $\nu_W^n(w) > 0$ whenever $w \in R_n$.

It then follows by induction that

$$\mu_{k'}(P_w(r),w) \geq \mu_k(P_w(r),w) + |\{k' \geq n > k : w \in R_n\}|\underline{\nu}.$$

Then for k' such that $|\{k' \geq n > k : w \in R_n\}| > \nu_W(w)/\underline{\nu}$, $\mu_{k'}(P_w(r),w) > \mu_k(P_w(r),w) + \nu_W(w)$, a contradiction to Claim 2. Thus, there exists $k^* > k$ such that $r_{k^*}(w) > r_k(w)$ and it follows by Claim 5 that $r_{k'}(w) = r_k(w) + 1$ for some $k' > k$. ■

Claim 11 *For each $w \in W$ and $k \in \mathbb{N}$, either there exists $k' > k$ such that $r_{k'}(w) = r_k(w) + 1$, or there exists $N_w \in \mathbb{N}$ such that $w \notin R_n$ for all $n > N_w$.*

Proof. By Claim 10, if there does not exist $k' > k$ such that $r_{k'}(w) = r_k(w) + 1$, then $|\{n > k : w \in R_n\}| \leq \nu_W(w)/\underline{\nu}$. Thus, there must exist some N_w such that $w \notin R_n$ for all $n > N_w$. ■

Claim 12 *For each $w \in W$, there exists $N_w \in \mathbb{N}$ such that $w \notin R_n$ for all $n > N_w$.*

Proof. Let $r = \max_{n \in \mathbb{N}} r_n(w)$, which exists since $r_n(w) \in \{1, \ldots, |M_\emptyset \times C|\}$, and $k \in \mathbb{N}$ such that $r_k(w) = r$. It then follows from Claim 11 that $w \notin R_n$ for all $n > N_w$.

■

Claim 13 *There exists $n \in \mathbb{N}$ such that $R_n = \emptyset$.*

Proof. This follows from Claim 12 (take n to be the max N_w over all w). ■

Let $N = \min\{n \in \mathbb{N} : R_n = \emptyset\}$ and define $\mu = \mu_{N-1}$.

Claim 14 *For each $w \in W$, $\sum_{(m,c) \in M_\emptyset \times C} \mu(m,w,c) = \nu_W(w)$.*

Proof. It follows from Claim 2 and from $R_n = \emptyset$ that

$$\sum_{(m,c) \in M_\emptyset \times C} \mu(m,w,c) = \sum_{(m,c) \in M_\emptyset \times C} \mu_{n-1}(m,w,c) = \nu_W(w).$$

■

Claim 15 $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR_W(\mu)$.

Proof. Let $(m, w, c) \in \text{supp}(\mu)$ and suppose that $(m, w, c) \notin IR_W(\mu)$. Thus, there exists $c' \in C$ such that $(\emptyset, c') \succ_w (m, c)$ and, hence, $w \in W$ and $r > r_w$, where $P_w(r) = (m, c)$. Hence, $\mu(m, w, c) = 0$ by Claim 6, a contradiction to $\mu(m, w, c) > 0$.

Suppose next that $(m, w, c) \notin S_M(\mu)$. Then $m \in M$. If $(\emptyset, c') \succ_m (w, c)$, then $\mu(m, w, c) = 0$ by Claim 3, a contradiction to $\mu(m, w, c) > 0$.

Thus, there exists $(w', c', \bar{m}, \bar{c})$ such that $\mu(\bar{m}, w', \bar{c}) > 0$, $(w', c') \succ_m (w, c)$ and $(m, c') \succ_{w'} (\bar{m}, \bar{c})$. Since $\mu(m, w, c) > 0$, it follows that $\sum_{(\tilde{w},\tilde{c}):(\tilde{w},\tilde{c})\succeq_m(w',c')} \mu(m, \tilde{w}, \tilde{c}) < \nu_M(m)$. Letting $r, \bar{r}$ be such that $P_{w'}(r) = (m, c')$ and $P_{w'}(\bar{r}) = (\bar{m}, \bar{c})$, $(m, c') \succ_{w'} (\bar{m}, \bar{c})$ implies that $r < \bar{r}$ and $\mu(\bar{m}, w', \bar{c}) > 0$ implies that $\bar{r} \leq r_{N-1}(w')$ since $\text{supp}(\mu(w', \cdot)) \subseteq \{P_{w'}(1), \ldots, P_{w'}(r_{N-1}(w'))\}$. Hence, by Claim 7,

$$\mu(m, w', c') = \nu_M(m) - \sum_{(\tilde{w},\tilde{c}):(\tilde{w},\tilde{c})\succ_m(w',c')} \mu(m, \tilde{w}, \tilde{c}) > 0.$$

Since $\sum_{(\tilde{w},\tilde{c}):(\tilde{w},\tilde{c})\succeq_m(w',c')} \mu(m, \tilde{w}, \tilde{c}) < \nu_M(m)$ and $\mu(m, w', c') > 0$, Claims 8 and 14 then imply that $\sum_{(\tilde{m},\tilde{c}):(\tilde{m},\tilde{c})\succeq_{w'}(m,c')} \mu(\tilde{m}, w', \tilde{c}) = \sum_{(\tilde{m},\tilde{c})\in M_\emptyset\times C} \mu(\tilde{m}, w', \tilde{c}) = \nu_W(w')$. Hence, $\mu(\bar{m}, w', \bar{c}) = 0$, a contradiction to $\mu(\bar{m}, w', \bar{c}) > 0$. ■

Thus, μ is stable. Claims 16 to 18 establish that μ is woman-optimal.

Claim 16 *For each $w \in W$ and $r' \in \{1, \ldots, |M_\emptyset \times C|\}$, $\mu_n(P_w(r'), w) \leq \nu_W(w) - \sum_{r=1}^{r'-1} \mu_{n-1}(P_w(r), w)$.*

Proof. If $(w, c_w(r')) \notin W^*_{n,m_w(r')}$, then $\mu_n(P_w(r'), w) = 0$. Otherwise, by definition, the distribution must satisfy $\mu_n(P_w(r'), w) \leq \mu_{n-1}(P_w(r'), w) + 1_{W_{n,P_w(r')}} \nu^n_W(w)$. By Claim 2, $\mu_{n-1}(P_w(r'), w) \leq \nu_W(w) - \sum_{r=1}^{r'-1} \mu_{n-1}(P_w(r), w)$. Whenever $w \in W_{n,P_w(r')}$, $r' = r_n(w)$, and by Claim 4, $\text{supp}(\mu_{n-1}(\cdot, w)) \subseteq \{P_w(1), \ldots, P_w(r_{n-1}(w))\} \subseteq \{P_w(1), \ldots, P_w(r_n(w))\}$. Thus, $\nu^n_W(w) = \nu_W(w) - \sum_{r=1}^{r'} \mu_{n-1}(P_w(r), w)$. ■

Claim 17 *For each $n \in \mathbb{N}$, for each $w \in W$, and for each stable matching μ':*

1. $\sum_{r=1}^{k} \mu_n(P_w(r), w) \geq \sum_{r=1}^{k} \mu'(P_w(r), w)$ *for each* $k \in \{1, \ldots, r_n(w) - 1\}$.

2. $\sum_{r=1}^{r_n(w)} \mu_n(P_w(r), w) \geq \sum_{r=1}^{r_n(w)} \mu'(P_w(r), w)$ *if* $r_n(w) < r_{n+1}(w)$.

Proof. For $n = 1$: It is sufficient to show that $\mu'(P_w(1), w) \leq \mu_1(P_w(1), w)$ for each $w \in W$. Fix $w \in W$, and suppose for a contradiction that $\mu'(P_w(1), w) > \mu_1(P_w(1), w)$. This implies that $\mu_1(P_w(1), w) < \nu_W(w) = \mu_0(P_w(1), w) + \nu_W^1(w)$ and hence by Claim 7:

$$\mu_1(P_w(1), w) = \nu_M(m_w(1)) - \sum_{(\hat{w},\hat{c}) \in S_{m_w(1)}(w, c_w(1))} \mu_1(m_w(1), \hat{w}, \hat{c}).$$

Suppose that $\mu'(m_w(1), \hat{w}, \hat{c}) \geq \mu_1(m_w(1), \hat{w}, \hat{c})$ for every $(\hat{w}, \hat{c}) \in S_{m_w(1)}(w, c_w(1))$. Then:

$$\begin{aligned}
\nu_M(m_w(1)) - \mu'(P_w(1), w) &\geq \sum_{(\hat{w},\hat{c}) \in S_{m_w(1)}(w, c_w(1))} \mu'(m_w(1), \hat{w}, \hat{c}) \\
&\geq \sum_{(\hat{w},\hat{c}) \in S_{m_w(1)}(w, c_w(1))} \mu_1(m_w(1), \hat{w}, \hat{c}) \\
&= \nu_M(m_w(1)) - \mu_1(P_w(1), w),
\end{aligned}$$

a contradiction because $\mu_1(P_w(1), w) < \mu'(P_w(1), w)$. Thus, it must be that for some $(\hat{w}, \hat{c}) \in S_{m_w(1)}(w, c_w(1))$, $0 \leq \mu'(m_w(1), \hat{w}, \hat{c}) < \mu_1(m_w(1), \hat{w}, \hat{c}) \leq \nu_W(\hat{w})$. Since $0 < \mu_1(m_w(1), \hat{w}, \hat{c})$, $P_{\hat{w}}(1) = (m_w(1), \hat{c})$ by Claim 4. Since $\mu'(m_w(1), \hat{w}, \hat{c}) < \nu_W(\hat{w})$ and $P_{\hat{w}}(1) = (m_w(1), \hat{c})$, $(\hat{w}, \hat{c}) \in T_{m_w(1)}(\mu')$. But $(\hat{w}, \hat{c}) \in S_{m_w(1)}(w, c_w(1))$ and $(P_w(1), w) \in \text{supp}(\mu')$, a contradiction to the stability of μ'.

Now assume that the Claim holds for $\mu_1, \ldots, \mu_{n-1}$. We need to show that if $k \leq r_n(w)-1$ or if $k = r_n(w) < r_{n+1}(w)$, then $\sum_{r=1}^{k} \mu'(P_w(r), w) \leq \sum_{r=1}^{k} \mu_n(P_w(r), w)$ for each $w \in W$. First consider $k = 1$. Fix $w \in W$, and suppose for a contradiction that $\mu'(P_w(1), w) > \mu_n(P_w(1), w)$. Since $\mu'(P_w(1), w) \leq \mu_{n-1}(P_w(1), w)$, this implies that $\mu_{n-1}(P_w(1), w) > \mu_n(P_w(1), w)$, and hence by Claim 7:

$$\mu_n(P_w(1), w) = \nu_M(m_w(1)) - \sum_{(\hat{w},\hat{c}) \in S_{m_w(1)}(w, c_w(1))} \mu_n(m_w(1), \hat{w}, \hat{c}).$$

Suppose that $\mu'(m_w(1), \hat{w}, \hat{c}) \geq \mu_n(m_w(1), \hat{w}, \hat{c})$ for every $(\hat{w}, \hat{c}) \in S_{m_w(1)}(w, c_w(1))$.

Then:

$$\begin{aligned}\nu_M(m_w(1)) - \mu'(P_w(1),w) &\geq \sum_{(\hat{w},\hat{c})\in S_{m_w(1)}(w,c_w(1))} \mu'(m_w(1),\hat{w},\hat{c}) \\ &\geq \sum_{(\hat{w},\hat{c})\in S_{m_w(1)}(w,c_w(1))} \mu_n(m_w(1),\hat{w},\hat{c}) \\ &= \nu_M(m_w(1)) - \mu_n(P_w(1),w),\end{aligned}$$

a contradiction because $\mu_n(P_w(1),w) < \mu'(P_w(1),w)$. Thus, it must be that for some $(\hat{w},\hat{c}) \in S_{m_w(1)}(w,c_w(1))$, $0 \leq \mu'(m_w(1),\hat{w},\hat{c}) < \mu_n(m_w(1),\hat{w},\hat{c}) \leq \nu_W(\hat{w})$. Since $0 < \mu_n(m_w(1),\hat{w},\hat{c})$, $P_{\hat{w}}(r') = (m_w(1),\hat{c})$ for some $r' \leq r_n(\hat{w})$ by Claim 4. Note that either $r' \leq r_{n-1}(\hat{w})$ or $r_{n-1}(w) < r_n(w)$. By the inductive hypothesis, $\sum_{r=1}^{r'-1} \mu'(P_{\hat{w}}(r),\hat{w}) \leq \sum_{r=1}^{r'-1} \mu_{n-1}(P_{\hat{w}}(r),\hat{w})$. By Claim 16, $\mu_n(P_{\hat{w}}(r'),\hat{w}) \leq \nu_W(\hat{w}) - \sum_{r=1}^{r'-1} \mu_{n-1}(P_{\hat{w}}(r),\hat{w})$. Therefore, $\sum_{r=1}^{r'-1} \mu'(P_{\hat{w}}(r),\hat{w}) + \mu'(m_w(1),\hat{w},\hat{c}) < \sum_{r=1}^{r'-1} \mu_{n-1}(P_{\hat{w}}(r),\hat{w}) + \mu_n(m_w(1),\hat{w},\hat{c}) \leq \nu_W(\hat{w})$, so $\nu_W(\hat{w}) > \sum_{r=1}^{r'-1} \mu'(P_{\hat{w}}(r),\hat{w}) + \mu'(m_w(1),\hat{w},\hat{c})$. Thus, $\mu'(P_{\hat{w}}(r),\hat{w}) > 0$ for some $r > r'$, and so $(\hat{w},\hat{c}) \in T_{m_w(1)}(\mu')$. But $(\hat{w},\hat{c}) \in S_{m_w(1)}(w,c_w(1))$ and $(P_w(1),w) \in \text{supp}(\mu')$, a contradiction to the stability of μ'.

Now assume that $\sum_{r=1}^{k-1} \mu'(P_w(r),w) \leq \sum_{r=1}^{k-1} \mu_n(P_w(r),w)$ for each $w \in W$. Fix $w \in W$, and suppose for a contradiction that $\sum_{r=1}^{k} \mu'(P_w(r),w) > \sum_{r=1}^{k} \mu_n(P_w(r),w)$. This implies that $\mu_n(P_w(k),w) < \mu'(P_w(k),w)$. If $k < r_n(w)$ or if $k = r_n(w) < r_{n+1}(w)$, then by Claim 7:

$$\mu_n(P_w(k),w) = \nu_M(m_w(k)) - \sum_{(\hat{w},\hat{c})\in S_{m_w(k)}(w,c_w(k))} \mu_n(m_w(k),\hat{w},\hat{c}).$$

Suppose that $\mu'(m_w(k),\hat{w},\hat{c}) \geq \mu_n(m_w(k),\hat{w},\hat{c})$ for every $(\hat{w},\hat{c}) \in S_{m_w(k)}(w,c_w(k))$. Then:

$$\begin{aligned}\nu_M(m_w(k)) - \mu'(P_w(k),w) &\geq \sum_{(\hat{w},\hat{c})\in S_{m_w(k)}(w,c_w(k))} \mu'(m_w(k),\hat{w},\hat{c}) \\ &\geq \sum_{(\hat{w},\hat{c})\in S_{m_w(k)}(w,c_w(k))} \mu_n(m_w(k),\hat{w},\hat{c}) \\ &= \nu_M(m_w(k)) - \mu_n(P_w(k),w),\end{aligned}$$

a contradiction because $\mu_n(P_w(k), w) < \mu'(P_w(k), w)$. Thus, it must be that for some $(\hat{w}, \hat{c}) \in S_{m_w(k)}(w, c_w(k))$, $0 \leq \mu'(m_w(k), \hat{w}, \hat{c}) < \mu_n(m_w(k), \hat{w}, \hat{c}) \leq \nu_W(\hat{w})$. Since $0 < \mu_n(m_w(k), \hat{w}, \hat{c})$, $P_{\hat{w}}(r') = (m_w(k), \hat{c})$ for some $r' \leq r_n(\hat{w})$ by Claim 4. Note that either $r' \leq r_{n-1}(\hat{w})$ or $r_{n-1}(w) < r_n(w)$. By the inductive hypothesis, $\sum_{r=1}^{r'-1} \mu'(P_{\hat{w}}(r), \hat{w}) \leq \sum_{r=1}^{r'-1} \mu_{n-1}(P_{\hat{w}}(r), \hat{w})$. By Claim 16, $\mu_n(P_{\hat{w}}(r'), \hat{w}) \leq \nu_W(\hat{w}) - \sum_{r=1}^{r'-1} \mu_{n-1}(P_{\hat{w}}(r), \hat{w})$. Then $\sum_{r=1}^{r'-1} \mu'(P_{\hat{w}}(r), \hat{w}) + \mu'(m_w(k), \hat{w}, \hat{c}) < \sum_{r=1}^{r'-1} \mu_{n-1}(P_{\hat{w}}(r), \hat{w}) + \mu_n(m_w(k), \hat{w}, \hat{c}) \leq \nu_W(\hat{w})$, so $\nu_W(\hat{w}) > \sum_{r=1}^{r'-1} \mu'(P_{\hat{w}}(r), \hat{w}) + \mu'(m_w(k), \hat{w}, \hat{c})$. Thus, $\mu'(P_{\hat{w}}(r), \hat{w}) > 0$ for some $r > r'$, and so $(\hat{w}, \hat{c}) \in T_{m_w(k)}(\mu')$. But $(\hat{w}, \hat{c}) \in S_{m_w(k)}(w, c_w(k))$ and $(P_w(k), w) \in \text{supp}(\mu')$, a contradiction to the stability of μ'. ■

Claim 18 *Let N be the first n such that $R_n = \emptyset$. For each $w \in W$ and each stable matching μ',*

$$\sum_{(m',c') \in M_\emptyset \times C : (m',c') \succeq_w (m,c)} \mu'(m', w, c') \leq \sum_{(m',c') \in M_\emptyset \times C : (m',c') \succeq_w (m,c)} \mu_{N-1}(m', w, c')$$

for each $(m, c) \in M_\emptyset \times C$.

Proof. For each $k \in \{1, \ldots, r_{N-1}(w) - 1\}$,

$$\sum_{r=1}^{k} \mu_{N-1}(P_w(r), w) \geq \sum_{r=1}^{k} \mu'(P_w(r), w)$$

for every stable matching μ' by Claim 17. Also by Claims 4 and 14,

$$\sum_{r=1}^{k} \mu_{N-1}(P_w(r), w) = \sum_{r=1}^{r_{N-1}(w)} \mu_{N-1}(P_w(r), w) = \nu_W(w) \geq \sum_{r=1}^{k} \mu'(P_w(r), w)$$

for any $k \in \{r_{N-1}(w), \ldots, |M_\emptyset \times C|\}$. ■

A.2 Python code

In this section we describe how to use a python code that implements the DAA defined in Section 4; the code itself is available at `https://drive.google.com/file/d/1dttfikCLlERLYPztlvRq_qNT4rk0VRvF/view`. It considers the specification of the example in Section 3 and here we describe how to modify its preamble, i.e. lines 3-18, to obtain any other example.

We first specify the number of men and women types and the number of contracts:

```
#m is the number of men types
m=2
#w is the number of women types
w=2
#c is the number of contracts
c=1
```

Then we specify preferences in the following form: Using $m = |M|$, $w = |W|$ and $c = |C|$ as above and writing $C = \{c_1, \ldots, c_c\}$, $M_\emptyset = \{m_1, \ldots, m_m, m_{m+1}\}$ and $W_\emptyset = \{w_1, \ldots, w_w, w_{w+1}\}$, where $m_{m+1} = w_{w+1} = \emptyset$, preferences as described by

$$p = [[P_{m_1}(w_1, c_1), \ldots, P_{m_1}(w_1, c_c), \ldots, P_{m_1}(w_{w+1}, c_1), \ldots, P_{m_1}(w_{w+1}, c_c)], \ldots,$$
$$[P_{m_m}(w_1, c_1), \ldots, P_{m_m}(w_1, c_c), \ldots, P_{m_m}(w_{w+1}, c_1), \ldots, P_{m_m}(w_{w+1}, c_c)]] \text{ and}$$
$$q = [[P_{w_1}(m_1, c_1), \ldots, P_{w_1}(m_1, c_c), \ldots, P_{w_1}(m_{m+1}, c_1), \ldots, P_{w_1}(m_{m+1}, c_c)], \ldots,$$
$$[P_{w_w}(m_1, c_1), \ldots, P_{w_w}(m_1, c_c), \ldots, P_{w_w}(m_{m+1}, c_1), \ldots, P_{w_w}(m_{m+1}, c_c)]].$$

In the case of the example:

```
#p describes men's preferences, last entry is the empty woman
p=[[2,1,3],
   [1,2,3]]
#q describes women's preferences
q=[[1,2,3],
   [2,1,3]]
```

The final element in the preamble consist of the type distributions. These are described as

$$dm = [\nu_M(m_1), \ldots, \nu_M(m_m)] \text{ and}$$
$$dw = [\nu_W(w_1), \ldots, \nu_W(w_w)]$$

in general, and as

```
#distribution of men
```

```
dm=[1,2]
#distribution of women
dw=[2,2]
```

in the case of example.

Once the above specifications are made, the code computes a stable matching using our DAA. Its output for the example is:

```
mu is: [[0, 1, 0], [2, 0, 0], [0, 1, 0]] number of iterations 6
```

In general, the output is a stable matching μ in the form

$$[[\mu(m_1, w_1, c_1), \ldots, \mu(m_1, w_1, c_c), \ldots, \mu(m_1, w_{w+1}, c_1), \ldots, \mu(m_1, w_{w+1}, c_c)], \ldots,$$
$$[\mu(m_{m+1}, w_1, c_1), \ldots, \mu(m_{m+1}, w_1, c_c), \ldots, \mu(m_{m+1}, w_{w+1}, c_1), \ldots, \mu(m_{m+1}, w_{w+1}, c_c)]]$$

and $N - 1$, where N is the first $n \in \mathbb{N}$ such that $R_n = \emptyset$.