A Nash Threats Folk Theorem for Repeated Games with Local Monitoring*

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Abstract

This paper characterises the equilibrium payoff set of a repeated game with local interaction and local monitoring. A sequentially rational Nash threats folk theorem holds without any restrictions on the network structure when players are arbitrarily patient, i.e. any feasible payoff above the Nash equilibrium point can be approximated arbitrarily well in sequential equilibrium. No form of communication or coordination device is required. When players discount the future, the folk theorem cannot hold unless further restrictions are made either on payoffs or the network structure.

1 Introduction

In this paper we will prove a Nash threats folk theorem for infinitely repeated games with local monitoring and interaction. We assume that the monitoring and the interaction structure are both determined by an undirected network. This means that for each player, her stage game payoffs only depend on the actions of a subset of players, and these actions are the only actions that she observes. We also assume that players are patient, and repeated game payoffs are given by the Banach-Mazur limit of the sequence of average stage game payoffs. This latter assumption is crucial for our results. Indeed, we show that under discounting, a folk theorem cannot hold in our setting without further assumptions on the network structure or the payoffs. In particular, we show without further

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assumptions on payoffs, a necessary condition for a folk theorem to hold under discounting is that every connected component of the network is 2-connected.

Local monitoring prevents standard results from applying. For example, in a star network, the core player may not wish to punish a single defecting peripheral player if that involves ending cooperation in all relationships forever. Thus, grim trigger strategies may not be an equilibrium. One way to view this problem is as follows. With local monitoring, it can be difficult for players to distinguish between punishment and defection. Suppose that Player j defects, and Player j's only neighbour is Player i. Since Player i's neighbours do not observe whether Player j has defected, if Player i punishes Player j, then Player i's neighbours will punish Player i. Now if Player i had strict incentives to play on-path, she may prefer not to punish Player j to avoid getting punished herself. A natural approach would be to construct strategies such that cooperating is only slightly preferred to defecting. Indeed, in the equilibria we construct, players are indifferent between cooperating and any finite sequence of defections.

The proof will be constructive: for any payoff above the Nash equilibrium point, we will construct a strategy profile that achieves the target payoff. The strategies will be stable in the sense that after any arbitrary history, everyone goes back to playing on-path after a finite number of periods. Since players are arbitrarily patient, short term incentives are irrelevant, which means that they can always achieve their equilibrium payoff by following the strategy. Thus, the challenge is to show that no deviation can achieve a payoff greater than the equilibrium payoff (note that the one-shot deviation principle does not apply).

Stability is not a trivial requirement under local monitoring because different players may have different beliefs about when a punishment phase is supposed to end. We circumvent this problem by constructing strategies that exploit the common knowledge of time—that is, rather than requiring players to punish deviations for a fixed number of periods, we require punishment to last until certain dates. Roughly speaking, we will divide the repeated game into T period blocks, and each T period block into 2 parts. Deviations in the first part of a block are punished until the end of the block, and deviations in the second part of the block are punished until the end of the next block. As long as the second part is long enough, deviations will be punished sufficiently harshly, and as long as the first part is long enough, the strategies will be stable.

Here is the intuition behind stability. Suppose that we are trying to sustain mutual cooperation in the prisoners' dilemma, so that C is played on-path and D is played if a deviation is observed (by both the deviating player and her neighbours). At the end of the block following an arbitrary history, it may be the case that some players are supposed to play C, whereas others have to play D for one more block. But those who are supposed to play C and observe D from a neighbour will start playing D themselves. In this way, the punishment will spread throughout the network, and as long as the first part of the block is long enough, everyone will have seen D by the end of the first part, and respond by playing D until the end of the block. Thus, everyone will end punishment at the same time.

A different type of coordination problem is that even though players are punished effectively when all of their neighbours play the Nash equilibrium action, they may benefit from being 'punished' by only some of their neighbours. In order to prevent players from exploiting this, our strategies ensure that the number of periods in each length T block in which a player is punished by only a proper subset of her neighbours is small relative to the number of periods in which she is punished by every neighbour. This is achieved by requiring punishment to continue for an extra block in some cases, and to end prematurely in others.

One final difficulty that could arise under local monitoring is that after observing a deviation, players may try to infer the spread of the deviation in the network and the beliefs of others about future play. We circumvent this difficult by constructing strategies that are optimal for every belief that players might have about play outside their neighbourhood.

2 Related Literature

This paper continues the tradition of providing limiting results for repeated games on networks. Nava (2016) provides a survey of of this literature, which has largely found that very weak conditions on the network structure are required for a folk theorem to hold. In this paper, we impose no restrictions on the network structure. An earlier and related strand of literature, which focused on a random matching environment, was pioneered by Kandori (1992) and Ellison (1994). They establish that even when players are unable to recognise their opponents, cooperation can be sustained as a sequential equilibrium supported by contagious punishments. In our setting, interaction takes place on a stable network, and players know their neighbours and observe their actions.

Ben-Porath and Kahneman (1996) showed that under the assumption that public communication is possible, a necessary and sufficient condition for a folk theorem to hold is that every player is observed by at least two other players. Renault and Tomala (1998) and Tomala (2011) establish Nash folk theorems under similar conditions without any explicit communication, and Laclau (2012) and Laclau (2014) establish Nash and sequentially rational folk theorems, respectively, under different assumptions about the communication possibilities. In this paper, we consider a restricted class of games, which allows us to establish a sequentially rational folk theorem without any form of communication for any network structure.

Cho (2011) and Cho (2014) construct stable and sequentially rational equilibria that sustain cooperation in the repeated prisoners' dilemma. Cho (2011) assumes that players have access to a public randomisation device, and Cho (2014) allows players to communicate with their neighbours. In this paper, players do not have access to a public randomisation device or any method of communication.

This paper builds on Nava and Piccione (2014), who show that mutual cooperation is possible for a broad class of two-action games. They also consider discounted payoffs as well as the case of arbitrarily patient players. However, we extend their results to a folk theorem, and we construct different strategies to support the equilibria.

The model is described in Section 3. Our main result, that any feasible payoff above the Nash equilibrium point can be (approximately) sustained in sequential equilibrium, is stated as Proposition 1 in Section 4. The strategies used to prove Proposition 1 are described in Section 5, and the proof of Proposition 1 appears in Section 6. Section 7 concludes.

3 Model

A set N contains n players who interact according to an undirected graph (N, G). We assume that (N, G) is common knowledge. For each $i \in N$, define $N_i = \{j \in N \setminus \{i\} : ij \in G\}$. Note that $j \in N_i$ if and only if $i \in N_j$. We interpret N_i as the neighbourhood of Player i. Define a path as $(j_1, \ldots, j_m)$ such that $j_{k+1} \in N_{j_k}, k = 1, \ldots, m-1$, and $j_k \neq j_l$ for $k \neq l$.

For $M \subset N$, define $G_M = \{ij \in G : i, j \in M\}$. That is, (M, G_M) is the subgraph with vertices in M. (M, G_M) is connected if for each $j_1, j_m \in M$, there is a path $(j_1, \ldots, j_m)$ such that $j_{k+1} \in N_{j_k} \cap M$. (M, G_M) is 2-connected if for each $i \in M$, $(M \setminus \{i\}, G_{M \setminus \{i\}})$ is connected. For $M \subset N$, (M, G_M) is a connected component of (N, G) if (M, G_M) is connected and for all $i \in N \setminus \{M\}$, $(M \cup \{i\}, G_{M \cup \{i\}})$ is not connected.

The action set of Player i is a finite set A_i. Let $A_M = \times_{j \in M} A_j$. The stage game payoff for Player i is given by the function $v_i : A_{N_i \cup \{i\}} \mapsto R$, and we denote an element of $A_{N_i \cup \{i\}}$ by a_i, a_{N_i}.

Assumption 1. *For each A_i, there exists an action $D \in A_i$ such that $v_i(D, D, \ldots, D) > v_i(a_i', D, \ldots, D)$ for all $a_i' \in A_i \setminus \{D\}$.*

This assumption says that there exists a strict Nash equilibrium of the stage game, which we label as $(D, \ldots, D)$. Without loss of generality, let $v_i(D, \ldots, D) = 0$ for all $i \in N$.

The stage game is repeated infinitely many times. Each player observes the past play of her neighbours. The set of possible histories for Player i is:

$$H_i = \{\varnothing\} \cup \left\{ \cup_{t=1}^{\infty} \left[\times_{s=1}^{t} A_{N_i \cup \{i\}} \right] \right\}.$$

A (pure) strategy for Player i is a function $\sigma_i : H_i \mapsto A_i$. The set of all strategies available to Player i is given by Σ_i.

Given a strategy profile $\sigma_N = (\sigma_1, \ldots, \sigma_n)$, let $\{a_N^t\}_{t=1}^{\infty}$ be the sequence of actions generated by σ_N, and let $\{v_i(a_i^t, a_{N_i}^t)\}_{t=1}^{\infty}$ be the sequence of stage game payoffs for Player i. Players discount the future without common discount factor $\delta \leq 1$, and repeated game payoffs are defined as:

$$\mathcal{V}_i(\sigma_N) = \begin{cases} \Lambda\left(\left\{\frac{1}{t}\sum_{s=1}^{t} v_i(a_i^s, a_{N_i}^s)\right\}_{t=1}^{\infty}\right) & \text{when } \delta = 1 \\ (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1} v_i(a_i^t, a_{N_i}^t) & \text{when } \delta < 1, \end{cases}$$

where $\Lambda(\cdot)$ denotes the Banach-Mazur limit of a sequence. A Banach-Mazur limit is a positive linear functional $\Lambda : \ell_\infty \to \mathbb{R}$, where ℓ_∞ is the space of all bounded sequences, such that $\Lambda(\mathbf{e}) = 1$, where $\mathbf{e} = (1, 1, \ldots)$, and $\Lambda(x_1, x_2, \ldots) = \Lambda(x_2, x_3, \ldots)$ for each $(x_1, x_2, \ldots) \in \ell_\infty$ (see Aliprantis and Border (2006), pp. 550-551).

The set of histories for the entire game is:

$$H = \{\varnothing\} \cup \left\{\cup_{t=1}^{\infty}\left[\times_{s=1}^{t} A_N\right]\right\}.$$

For each observed history $\bar{h}_i$, we can define the information set of Player i as $\mathcal{I}(\bar{h}_i) = \{h \in H : h_i = \bar{h}_i\}$. Let $\beta(h|h_i)$ denote the belief of Player i that the history is h, conditional on observing h_i.

4 Folk Theorem

Let $v(a_N) = (v_1(a_1, a_{N_1}), \ldots, v_n(a_n, a_{N_n}))$. The set of feasible payoffs is $F = \text{co}\{v(a_N) : a_N \in A_N\}$, and the set of feasible payoffs such that each player receives strictly more than her stage game Nash equilibrium payoff is $F^{IR} = F \cap \{v : v_i > 0, \forall i\}$. Let F^* be the subset of F^{IR} where the weights used in the convex combinations are rational numbers.

Proposition 1. *If $\delta = 1$, any $v^* \in F^*$ can be supported as a sequential equilibrium payoff.*

In the next section, we will construct a strategy profile that supports v^* as a sequential equilibrium payoff for any v^*. For the remainder of this section, we discuss the necessity of the assumption that $\delta = 1$.

Remark 1. The folk theorem does not hold in this setting when payoffs are discounted. In order for a folk theorem to hold under discounting, additional restrictions must be made either on payoffs or the network structure. The following example shows that when $\delta < 1$, there exists a stage game satisfying Assumption 1, a network structure, and a payoff $v^* \in F^*$ such that v^* is not the payoff in any sequential equilibrium.

Example 1. Let four players connected on a line play the following stage game with common action space $A = \{C, D\}$, and payoff function:

- $v(D, D, D, D) = (0, 0, 0, 0)$
- $v(C, C, C, C) = (1, 1, 1, 1)$
- $v(C, D, C, C) = (-1, 2, 1, 1)$
- $v(D, D, C, C) = (0, 2, 1, 1)$
- $v(D, C, C, C) = (1, 1, 1, 1)$

Otherwise the payoffs are such that Player 3 gets at most zero, Assumption 1 is satisfied (i.e. (D, D, D, D) is a strict Nash equilibrium of the stage game), and for each i, Player i's payoffs depends only on the actions of Player i and Player i's neighbours. A complete specification of the stage game payoffs are given in the Appendix.

The efficient payoff $(1, 1, 1, 1)$ (which requires Player 2, Player 3, and Player 4 to play C) cannot be sustained in any sequential equilibrium under discounting. To see this, note that in any equilibrium in which Player 3 and Player 4 play C forever on the equilibrium path, Player 3 can guarantee the payoff 1 by always playing C. Moreover, 1 is the highest payoff Player 3 can get in the stage game. In order to prevent Player 2 from deviating from C, Player 2 must be punished for playing D. However, the only way Player 2 can be punished for playing D is if Player 3 plays D (since if Player 3 plays C, Player 2 gets at least 1). But in this case, Player 3 will get strictly less than 1, so Player 3 will not find it optimal to trigger this punishment. Thus, Player 2 will have an incentive to deviate from C, and there cannot exist an equilibrium in which Player 2, Player 3, and Player 4 play C forever.

Note that under perfect monitoring, the payoff $(1,1,1,1)$ can be sustained in sequential equilibrium for high enough δ, for example by using grim trigger strategies. In this case, the problem under network monitoring is that Player 4 does not observe Player 2, and fixing the action of Player 4 to be C, Player 3 can guarantee her maximum stage game payoff (which is 1). In other words, the feasible and individually rational payoff set of Player 2's neighbourhood has empty interior when Player 4's action is fixed at C.

This example generalises to the following Proposition, which establishes that a necessary condition on the network structure for the folk theorem to hold under discounting is that every connected component is 2-connected.

Proposition 2. *For any (N, G) such that there exists a connected component that is not 2-connected, there exists a stage game satisfying Assumption 1 and a payoff $v^* \in F^*$ such that v^* is not the payoff in any sequential equilibrium for any $\delta < 1$.*

Proof. Without loss of generality, assume that the entire network (N, G) is connected (if not, we can just consider each connected component separately). Now suppose that the network is not 2-connected. Let i^* be a player such that $(N \setminus \{i^*\}, G_{N\setminus\{i^*\}})$ is not connected. First, we will define $N^L_{i^*}$ and $N^R_{i^*}$ as two nonempty, disjoint subsets of N_{i^*} with the property that for $i \in N^L_{i^*}$ and $j \in N^R_{i^*}$, every path $(i, \ldots, j)$ contains i^*. Let (N^L, G_{N^L}) and (N^R, G_{N^L}) be two distinct connected components of $(N \setminus \{i^*\}, G_{N\setminus\{i^*\}})$, and let $N^L_{i^*} = N^L \cap N_{i^*}$ and $N^R_{i^*} = N^R \cap N_{i^*}$. We argue that $N^L_{i^*}$ and $N^R_{i^*}$ are non empty. By the assumption that (N, G) is connected, for $i \in N^L \subset N$ and $j \in N^R \subset N$, there must be a path $(i, \ldots, j)$, where each element of the path is in N. Since (N^L, G_{N^L}) and (N^R, G_{N^L}) are two disjoint connected components of $(N \setminus \{i^*\}, G_{N\setminus\{i^*\}})$, there is no path from i to j that does not contain i^*. Therefore, there must exist a path $(i, \ldots, i^L, i^*, i^R, \ldots, j)$. Now we show that $i^L \in N^L \cap N_{i^*}$. The argument that $i^R \in N^R \cap N_{i^*}$ is analogous. By definition $i^L \in N_{i^*}$. If $i^L = i \in N^L$, then we are done. Otherwise, there is a path $(i, i_2, \ldots, i_{m-1}, i^L)$, where each element is in $N \setminus \{i^*\}$. Since (N^L, G_{N^L}) is a connected component of $(N \setminus \{i^*\}, G_{N\setminus\{i^*\}})$, (N^L, G_{N^L}) is connected. Since $i \in N^L$ and there is a path (i, i_2), $(N^L \cup \{i_2\}, G_{N^L\cup\{i_2\}})$ is also connected. But the definition of a connected component then implies that $i_2 \notin N \setminus (\{i^*\} \cup N^L)$. Since $i_2 \neq i^*$, this implies that $i_2 \in N^L$. Repeating this argument yields $i^L \in N^L$.

Now we define a stage game satisfying Assumption 1 such that there exists a payoff $v^* \in F^*$ that is not the payoff in any sequential equilibrium for any $\delta < 1$. For each player, let $A_i = \{C, D\}$. Without loss of generality, let $v_i(D, \ldots, D) = 0$ and $v_i(C, D, \ldots, D) = -1$ for all i.

For all $i \notin \{i^*\} \cup N_{i^*}^L \cup N_{i^*}^R$:

- $v_i(D, a_{N_i}) = 0$ for $a_{N_i} = (D, \ldots, D)$
- $v_i(C, a_{N_i}) = -1$ for $a_{N_i} = (D, \ldots, D)$
- $v_i(C, a_{N_i}) = 1$ for all $a_{N_i} \neq (D, \ldots, D)$
- $v_i(D, a_{N_i}) = 1$ for all $a_{N_i} \neq (D, \ldots, D)$.

For $i \in N_{i^*}^L \cup N_{i^*}^R$:

- $v_i(D, a_{N_i}) = 0$ for $a_{N_i} = (D, \ldots, D)$
- $v_i(C, a_{N_i}) = -1$ for $a_{N_i} = (D, \ldots, D)$
- $v_i(C, a_{N_i}) = 1$ whenever $a_{i^*} = C$
- $v_i(C, a_{N_i}) = -1$ whenever $a_{i^*} = D$
- $v_i(D, a_{N_i}) = 2$ whenever $a_{i^*} = C$
- $v_i(D, a_{N_i}) = 1$ otherwise.

For i^*, let

- $v_i(D, a_{N_i}) = 0$ for $a_{N_i} = (D, \ldots, D)$
- $v_i(C, a_{N_i}) = -1$ for $a_{N_i} = (D, \ldots, D)$
- $v_{i^*}(D, a_{N_{i^*}}) = 0$ for all $a_{N_{i^*}}$
- $v_{i^*}(C, a_{N_{i^*}}) = 1$ whenever $a_{N_{i^*}^R} = (C, \ldots, C)$
- $v_{i^*}(C, a_{N_{i^*}}) = -1$ otherwise.

Note that for each player i, $0 < 1 \leq \max_{(a_i, a_{N_i})} v_i(a_i, a_{N_i})$. Suppose for a contradiction that there exists a sequential equilibrium σ_N such that $\mathcal{V}_i(\sigma_N) = 1$ for all i. This requires that for every on-path history h_i, $\sigma_i(h_i) = C$ for all $i \in \{i^*\} \cup N^R_{i^*}$, since $v_{i^*}(a_{i^*}, a_{N_{i^*}}) < 1$ for all $(a_{i^*}, a_{N_{i^*}})$ such that $(a_{i^*}, a_{N^R_{i^*}}) \neq (C, \ldots, C)$. As long as no player $i \in \{i^*\} \cup N^R_{i^*}$ has deviated, Player i^* can guarantee the stage game payoff 1 in every period by always playing C. Now, following a history in which no player has deviated, consider the one shot deviation for some player $i \in N^L_{i^*}$ to D. For $i \in N^L_{i^*}$, the stage game payoff to D when i^* is playing C is 2, and $i \in N^L_{i^*}$ can only receive a stage game payoff less than 1 if player i^* plays D.

Without loss of generality, let $1 \in N^L_{i^*}$. Let h be the one period history where $a_i = \sigma_i(\emptyset)$ for each $i \neq 1$, and $a_1 = D$. Let σ^h_N be the strategy profile induced by the σ_N after history h. Let σ'_1 be the strategy profile where Player 1 plays D in period 1, and follows σ_1 from period 2 onwards, and let $\{\bar{a}^t_N\}^\infty_{t=1}$ be the sequence of action profiles induced by $(\sigma'_1, \sigma_{N\setminus\{1\}})$. Note that in any sequential equilibrium, $\beta(h|h_1) = 1$. Then Player 1's payoff is:

$$\mathcal{V}_1(\sigma'_1, \sigma_{N\setminus\{1\}}) = (1-\delta)2 + \delta\mathcal{V}_1(\sigma^h_N)$$

For this one shot deviation not to be profitable, we need $\mathcal{V}_1(\sigma^h_N) < 1$, which involves Player i^* playing D at least once following the deviation. To see this, suppose that $a^t_{i^*} \neq D$ for all $t \geq 2$. Then $v_i(\bar{a}^t_1, \bar{a}^t_{N_1}) \geq 1$ for all $t \geq 2$ and:

$$\begin{aligned}\mathcal{V}_1(\sigma^h_N) &= (1-\delta)\sum_{t=2}^{\infty}\delta^{t-2}v_i(\bar{a}^t_1, \bar{a}^t_{N_1}) \\ &\geq (1-\delta)\sum_{t=2}^{\infty}\delta^{t-2} \geq 1\end{aligned}$$

But consider the strategy σ'_{i^*} where Player i^* plays C in every period. In this case, all $i \in N^R_{i^*}$ will also play C and:

$$\mathcal{V}_{i^*}(\sigma'^h_{i^*}, \sigma^h_{N\setminus\{i^*\}}) = 1$$

On the other hand, if $a^t_{i^*} = D$ for some $t \geq 2$, then $v_i(\bar{a}^t_{i^*}, \bar{a}^t_{N_{i^*}}) \leq 1$ for all t with a strict inequality for the t such that $a^t_{i^*} = D$. Then:

$$\begin{aligned}\mathcal{V}_{i^*}(\sigma^h_N) &= (1-\delta)\sum_{t=2}^{\infty}\delta^{t-2}v_i(\bar{a}^t_{i^*}, \bar{a}^t_{N_{i^*}}) \\ &< (1-\delta)\sum_{t=2}^{\infty}\delta^{t-2} = 1\end{aligned}$$

Therefore, it must be the case that $\bar{a}^t_{i^*} \neq D$ for all $t \geq 2$, and so the one shot deviation must be profitable for Player 1. This contradicts the assumption that σ_N is a sequential equilibrium. □

5 Strategies

For any payoff $v^* \in F^*$, construct a finite deterministic sequence of stage game actions $\{a^s_N\}^{T_*}_{s=1}$ such that $\frac{1}{T_*}\sum_{s=1}^{T_*} v(a^s_N) = v^*$. There must be a finite sequence that achieves this payoff because of the assumption that the weights used in the convex combinations are rational numbers. Now for $T > nT_*$, let $\{a^s_N\}^T_{s=1}$ be T/T_* repetitions of the length T_* sequence (we will require $T - nT_*$ to be sufficiently large). Note that this implies that T is a multiple of T_*, i.e. $T \mod T_* = 0$.

Define $s(t) \equiv ((t-1) \mod T) + 1$ (i.e. $s(t) = t \mod T$ except when $t \mod T = 0$, in which case $s(t) = T$). In any period t, let $C_{is(t)}$ denote the i-th element of the vector a^s_N that is s-th term of the sequence $\{a^s_N\}^T_{s=1}$. We will refer to $C_{is(t)}$ as the on-path action for player i in period t. By construction $C_{is(t)} = C_{i(s(t)+T_*)}$ (as long as $s(t) + T_* \leq T$, otherwise $C_{i(s(t)+T_*)}$ is not defined), which ensures that on-path actions are repeated at least every T_* periods. For $M \subset N$, let $C_{Ms(t)} = (C_{js(t)})_{j\in M}$.

5.1 The strategy profile $\zeta_i : H_i \mapsto A_i$

The strategy profile $\zeta_i : H_i \mapsto A_i$ can be described with the help of:

- A set of states $\mathcal{S} = \{\mathcal{A}, \mathcal{B}(1), \ldots, \mathcal{B}(2T), \mathcal{A}^1, \ldots, \mathcal{A}^n\}$

- For each state, an output function $f_i : \mathcal{S} \times \mathbb{N} \mapsto A_i$, where:

$$f_i(\mathcal{A}, t) = C_{is(t)}$$
$$f_i(\mathcal{B}(p), t) = D$$
$$f_i(\mathcal{A}^j, t) = C_{is(t)}$$

- For each history h_i, a state function $P_i : H_i \mapsto \mathcal{S}$

In the next subsection, we will define $P_i(h_i)$ recursively. Then for each h_i of length $t-1$:

$$\zeta_i(h_i) = f_i(P_i(h_i), t)$$

That is, if Player i is in state $\mathcal{A}$ or $\mathcal{A}^j$ at time t, then Player i plays $C_{is(t)}$. If Player i is in state $\mathcal{B}(p)$, then Player i plays D. Note that our state function depends on the history, unlike a conventional transition function that depends on the state and the action profile. Thus, our 'states' are not true states. Moreover, our output function depends on time. Of course it is possible to describe the strategy profile using a standard automaton representation, but we would need many more states.

5.2 The function P_i

After the initial history:

$$P_i(\varnothing) = \mathcal{A}$$

Let h_i^t denote a length t history for Player i. Suppose that $P_i(h_i^{t-1}) = \mathcal{A}$.

- If $a_j^t = C_{js(t)}$ for all $j \in N_i \cup \{i\}$, then $P_i(h_i^t) = \mathcal{A}$
- Otherwise $P_i(h_i^t) = \mathcal{B}(p(t))$, where:

$$p(t) \equiv \begin{cases} T - s(t) & \text{for } s(t) \in \{1, \ldots, nT_*\} \\ 2T - s(t) & \text{for } s(t) \in \{nT_* + 1, \ldots, T\} \end{cases}$$

Suppose that $P_i(h_i^{t-1}) = \mathcal{B}(p)$. $P_i(h_i^t)$ depends on two functions $d_{1i} : H_i \mapsto \{0,1\}$ and $d_{2i} : H_i \mapsto \{0,1\}$, which we define in the next subsection to preserve continuity.

- If $d_{1i}(h_i^t) = 1$, then $P_i(h_i^t) = \mathcal{A}^j$
- If $d_{2i}(h_i^t) = 1$, then $P_i(h_i^t) = \mathcal{B}(2T - s(t))$
- Otherwise, $P_i(h_i^t) = \begin{cases} \mathcal{B}(p-1) & \text{if } p > 1 \\ \mathcal{A} & \text{if } p = 1 \end{cases}$

Suppose that $P_i(h_i^{t-1}) = \mathcal{A}^j$.

- If $a_j^t = C_{js(t)}$ and $s(t) < T$, then $P_i(h_i^t) = \mathcal{A}^j$
- If $a_j^t = C_{js(t)}$ and $s(t) = T$, then $P_i(h_i^t) = \mathcal{A}$
- Otherwise, $P_i(h_i^t) = B(2T - s(t))$

5.3 The functions d_{1i} and d_{2i}

Define $d_{1i} : H_i \mapsto \{0,1\}$ and $d_{2i} : H_i \mapsto \{0,1\}$ as follows:

$d_{1i}(h_i^t) = 1$ if and only if:

a) $s(t) = nT_*$

b) $a_i^\tau \neq C_{is(\tau)}$ for some $\tau \in \{t - s(t) + 1, \ldots, t - s(t) + (n-1)T_*\}$

c) There is a unique player $j \in N_i$ such that:

  i) $a_j^\tau = C_{js(\tau)}$ for all $\tau \in \{t - s(t) + 1, \ldots, t - s(t) + nT_*\}$,

  ii) $a_j^\tau \neq C_{js(\tau)}$ for all $\tau \in \{t - s(t) - T + 1, \ldots, t - s(t) - T + nT_*\}$

$d_{2i}(h_i^t) = 1$ if and only if:

a) $s(t) = T$

b) There is a player $j \in N_i$ such that:

i) $a_j^\tau = C_{js(\tau)}$ for all $\tau \in \{t - s(t) + 1, \ldots, t - s(t) + nT_*\}$

ii) $a_j^\tau \neq C_{js(t)}$ for some $\tau \in \{t - s(t) + nT_* + 1, \ldots, t - s(t) + T\}$

5.4 Intuition

After a history h_i^{t-1} in which Player i is in state $\mathcal{A}$, if Player i and her neighbours all play the on-path action $C_{js(t)}$, then Player i remains in state $\mathcal{A}$ after history h_t. If anyone deviates, then after history h^t Player i will be in state $\mathcal{B}(T - s(t))$ if t is in the first nT_* periods of a length T block or $\mathcal{B}(2T - s(t))$ if t is in the last $T - nT_*$ periods of a length T block. An interpretation of $\mathcal{B}(p)$ is that Player i should play D for p periods; however, this interpretation is not strictly correct because after a history where $d_{1i}(h^t) = 1$, which is possible only when $s(t) = nT_*$, Player i could 'transition' from $\mathcal{B}(T - nT_* + 1)$ to $\mathcal{A}^j$ (by 'transition', we mean that $P_i(h_i^{t-1}) = \mathcal{B}(T - nT_* + 1)$ and $P_i(h_i^t) = \mathcal{A}^j$, and after a history where $d_{2i}(h^t) = 1$, which is possible only when $s(t) = T$, Player i could transition from $\mathcal{B}(1)$ to $\mathcal{B}(T)$.

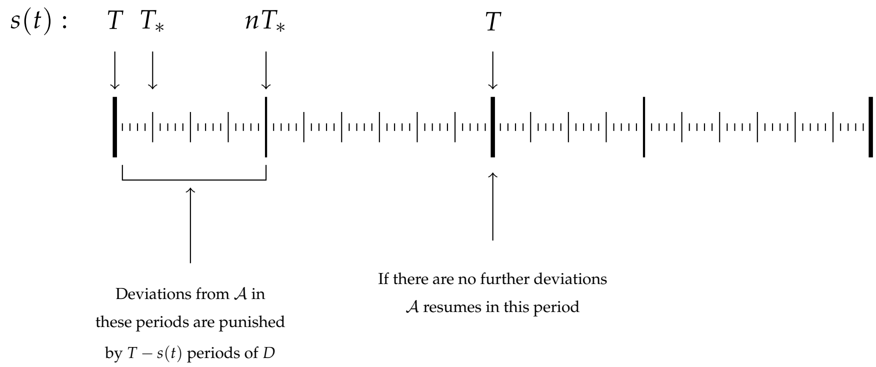

Figure 1: Transitions in $\mathcal{A}$, $s(t) \in \{1, \ldots, nT_*\}$

Loosely speaking, deviations from $\mathcal{A}$ in the first part of each length T block are punished until the end of the block, and deviations in the second part of

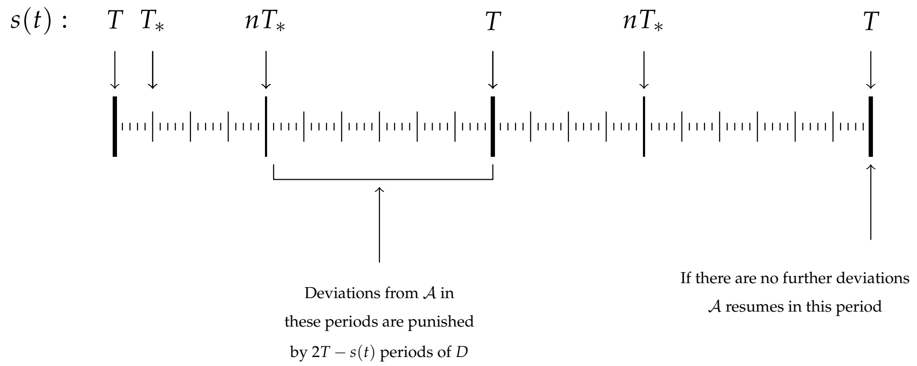

Figure 2: Transitions in $\mathcal{A}$, $s(t) \in \{nT_* + 1, \ldots, T\}$

each length T block are punished until the end of the next block. The idea is that any player who deviates from $\mathcal{A}$ is punished by D for a sufficiently long time (at least $T - nT_*$ periods). The reason that first part of the length T block is nT_* periods is to ensure that ζ_N is stable, in the sense that if every player follows the strategy after an arbitrary history h, after finitely many periods, every player will return to state $\mathcal{A}$.

To see the intuition behind this result, suppose that the network is connected and for all i, $C_{is(t)} \neq D$ for some $s(t)$ (that is, every player has to play an action other than D in some period on the equilibrium path). If Player 1 is in state $\mathcal{B}(T)$ at the start of a length T block, then she will play D at least for the first nT_* periods. To see this, note that for any h_i^t such that $s(t) \notin \{nT_*, T\}$, $d_{1i}(h_i^t) = 0$ and $d_{2i}(h_i^t) = 0$, and so if $P(h_i^{t-1}) = \mathcal{B}(p)$, then $P(h_i^t) = \mathcal{B}(p-1)$. Thus, if Player 1 is in state $\mathcal{B}(T)$ at the start of a length T block, after the first period of the length T block (when $s(t) = 1$), she will be in state $\mathcal{B}(T-1)$, and after after $nT_* - 1$ periods, she will be in state $\mathcal{B}(T - nT_* + 1)$. So she will play D in at least the first nT_* periods.

Since the on-path actions are repeated every T_* periods, after T_* periods Player 1 must have played D in some period t where $D \neq C_{1s(t)}$, and hence her neighbours will also be in state $\mathcal{B}(T - T_*)$. By the same argument as before, they will also play D until at least the nT_*-th period of the length T block. Then after $2T_*$ periods, all of their neighbours will be in state $\mathcal{B}(T - T_*)$, and after $(n-1)T_*$ periods, everyone in the network will be in state $\mathcal{B}(T - (n-1)T_*)$.

After the nT_*-th period of the block, it is possible that some Player i may be in state $\mathcal{A}^j$ for some $j \in N_i$; in the Appendix, we will deal with this case, but under our current assumption that for all i, $C_{is(t)} \neq D$ for some $s(t)$, it is easy to verify that part (i) of condition (c) in the definition of d_{1j} will not be satisfied by any $j \in N_i$ since everyone in the network is playing D after $(n-1)T_*$ periods. Thus, everyone will continue playing D until the end of the length T block.

After the last period of the length T block, when $s(t) = T$, everyone will be in state $\mathcal{A}$ unless d_{2j} is equal to 1 after that history for some i. However, since everyone was playing D after $(n-1)T_*$ periods, part (i) of condition (b) in the definition of d_{2j} cannot be satisfied; hence after T periods, everyone will be in state $\mathcal{A}$. In the next subsection we will discuss stability in more detail.

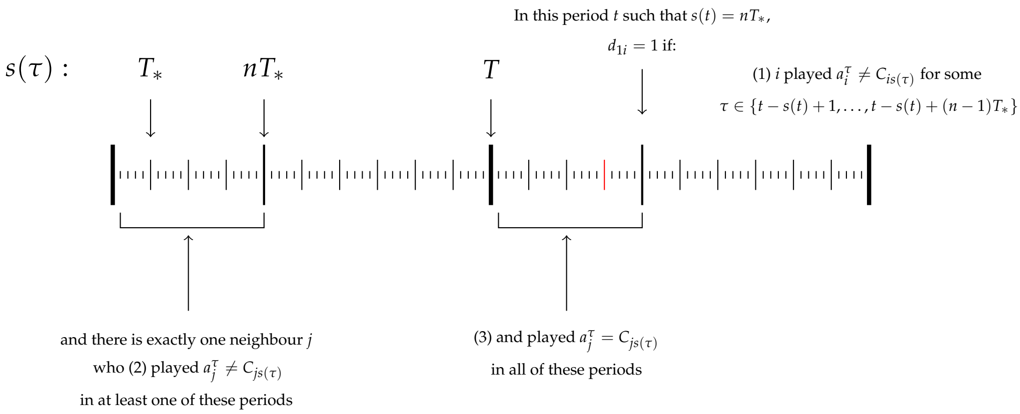

Figure 3: Transitions in $\mathcal{B}$: d_1

Now we discuss the purpose of d_{1i} and d_{2i}. If we consider the strategy without the transitions out of $\mathcal{B}(p)$ defined by these two functions, there are two types of deviations that are possible for certain network structures that could be profitable. If Player j deviates towards the end of the first part of a length T block, it is possible that in the next length T block some neighbours are in state $\mathcal{A}$ and some neighbours are in state $\mathcal{B}(p)$ in the second part of that block, and having some neighbours play C and others play D may yield a high payoff to Player j. Then repeatedly deviating in this way could improve Player j's repeated game payoff (note that the one shot deviation principle does not apply since $\delta = 1$, and any finite sequence of deviations will have no effect on

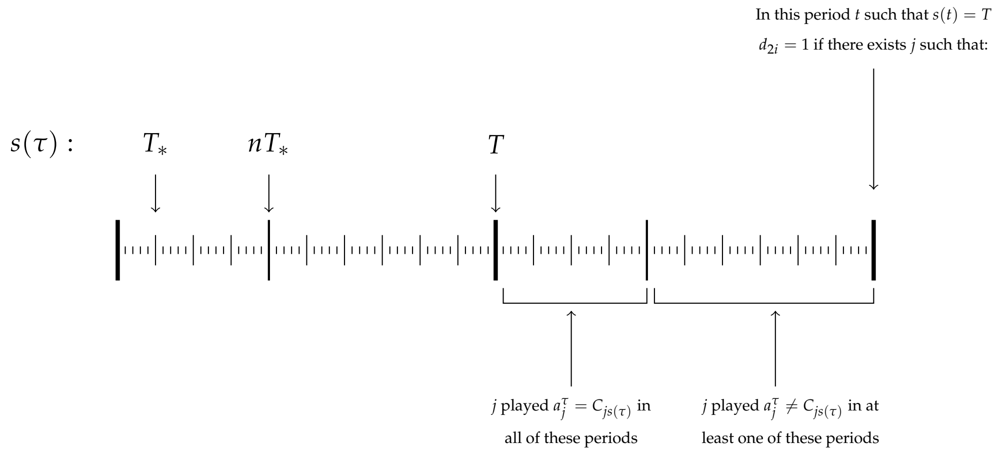

Figure 4: Transitions in $\mathcal{B}$: d_2

repeated game payoffs owing to stability).

For a concrete example of this type of deviation, consider 4 players connected on a line. Let $A_i = \{C, D\}$, and suppose that on the equilibrium path, C is played in every period (so we can let $T_* = 1$, $nT_* = 4$). Assume that every player $i \neq 2$ will play according to the strategy profile ζ_i. Now suppose that there are no deviations up to period 3, so that each player is in state $\mathcal{A}$ after history h^3, and Player 2 deviates for the first time in period 4, so that Players 1, 2, and 3 are in state $\mathcal{B}(T-4)$ after history h^4. Note that Player 4 then does not observe D until period 5, and so Player 4 is in state $\mathcal{B}(2T-5)$ after h^5. Thus, after period T, Players 1, 2, and 3 are in state $\mathcal{A}$, but Player 4 is in state $\mathcal{B}(T)$.

In period $T+1$, Player 4 will play D, so Player 3 will play D from period $T+2$ onwards. Note however, that if Player 2 does not play D, Player 1 will remain in state $\mathcal{A}$. Thus, from periods $T+2$ to $2T$, Player 1 will play C and if Player 3 remains in state $\mathcal{B}(p)$ she will will play D, which may be beneficial for Player 2. The transition out of $\mathcal{B}(p)$ when d_{1i} is equal to 1 is designed to rule out exactly this type of deviation. In this case, if Player 2 plays C in every period from $T+1$ to $T+4$, then $d_{1_3}(h_3^{T+4}) = 1$, and Player 3 will be in state $\mathcal{A}^2$ after h^{T+4}, and revert to playing C (unless Player 2 deviates to D again). Intuitively, if a player has already been punished for a deviation, and refuses to match another player's punishment, then the latter player should end the punishment in case it is actually benefiting the former player.

The second type of deviation occurs if a player finds herself being punished by one group of neighbours but not another. By deviating near the end of the length T block, she can reverse the pattern, which may also be profitable. Using the same set up as the previous example, suppose that after some arbitrary history, at the start of a length T block, Player 1 is in state $\mathcal{B}(T)$, and Player 3 is in state $\mathcal{A}$. Now as long as Player 2 plays C in this length T block, Player 1 will play D and Player 3 will play C, which may be beneficial for Player 2. It may also be beneficial for Player 2 if Player 1 plays C and Player 3 plays D. Suppose that Player 2 plays D in the last period of the length T block. Then Player 3 will be in state $\mathcal{B}(T)$ after the last period of the length T block, and play D in every period in the next block. Note that Player 1 is in state $\mathcal{B}(1)$ in the penultimate period of the length T block. If she is in state $\mathcal{A}$ after the last period of the length T block, she will play C in the next block. The transition out of $\mathcal{B}(p)$ when $d_{2i} = 1$ is designed to ensure that in this situation, Player 1 will instead be in state $\mathcal{B}(T)$ after the last period of the length T block and play D for an additional block. Intuitively, if a player deviates in the second part of a block, playing on-path actions in the first part, all of her neighbours should punish her until the end of the next block even if they were already in state $\mathcal{B}(p)$. This ensures that even when some neighbours are in state $\mathcal{B}(p)$, any deviation from the on-path actions will be punished with D *by every neighbour* for at least $T - nT_*$ periods.

If Player j is in state $\mathcal{A}^k$, for $k \neq i$, then Player j will not punish any deviation by Player i. However, it is shown in the Appendix that if all players other than i have been playing according to $\zeta_{N\setminus\{i\}}$ for a sufficiently long time, then Player j cannot be in state $\mathcal{A}^k$, for $k \neq i$.

5.5 Stability

If everyone plays according to the strategy ζ_N, $C_{Ns(t)}$ will be played for all $t \geq 1$. An important feature of the strategy profile ζ_N is that if every player plays according to ζ_N^h after an arbitrary history h of length z, after at most $2T$ periods, every player will play as if there are no deviations in h. This is established in the following definition and Lemma.

Definition 1. For any arbitrary history h of length z, let $\hat{h}^t$, $t \geq z$, be the length t history generated by ζ_N after h. The strategy profile ζ_N is *stable* if for any h there exists a T such that $\zeta_N(\hat{h}^t) = C_{Ns(t)}$ for all $t > T$.

Lemma 3. *The strategy profile ζ_N is stable. Moreover, for any h of length z, $\zeta_N(\hat{h}^t) = C_{Ns(t)}$ for all $t > z + 2T - s(z)$.*

Proof of Lemma 3. Appendix. □

The proof proceeds as follows. Let N_C be the set of all players such that $C_{is(t)} \neq D$ for some $s(t)$, and let $(N_C^1, \ldots, N_C^L)$ be a partition of N_C into connected components, i.e. each $(N_C^l, G_{N_C^l})$ is a connected component of (N_C, G_{N_C}).

First it is shown that for any arbitrary history of length z, for any $i \in N$, either $P_i(\hat{h}_i^{z+T-s(z)}) = \mathcal{A}$ or $P_i(\hat{h}_i^{z+T-s(z)}) = \mathcal{B}(T)$. If for all $i \in N_C^l$, $P_i(\hat{h}_i^{z+T-s(z)}) = \mathcal{A}$, then in period $z + 2T - s(z)$, for each $i \in N_C^l$, $P_i(\hat{h}_i^{z+T-s(z)}) = \mathcal{A}$.

Suppose that in period $z + T - s(z)$ there is at least one $i \in N_C^l$ such that $P_i(\hat{h}_i^{z+T-s(z)}) = \mathcal{B}(T)$. Using the fact that the on-path actions are repeated every T_* periods, it is then shown that by period $z + T - s(z) + nT_*$, for all $i \in N_C^l$, either $P_i(\hat{h}_i^{z+T-s(z)+nT_*}) = \mathcal{B}(T - nT_*)$ or $P_i(\hat{h}_i^{z+T-s(z)+nT_*}) = \mathcal{A}^j$ for some $j \notin N_C$. In either case, it follows that in period $z + 2T - s(z)$, $P_i(\hat{h}_i^{z+2T-s(z)}) = \mathcal{A}$ for all $i \in N_C^l$.

Since this is true for any N_C^l and $N_C = \cup_{l=1}^{L} N_C^l$, it follows that for any $i \in N_C$, i will play $\hat{a}_i^t = C_{is(t)}$ for all $t > z + 2T - s(z)$. Since any $i \notin N_C$ always plays $C_{is(t)} = D$ regardless of history, it follows that for all $i \in N$, $\hat{a}_i^t = C_{is(t)}$ for all $t > z + 2T - s(z)$.

6 Proof of Proposition 1

Now we will prove Proposition 1 by showing that the strategy profile ζ_N is a sequential equilibrium and supports the desired payoff v^*.

Proof of Proposition 1. First, we show that for any $i \in N$, $\mathcal{V}_i(\zeta_N) = v_i^*$. Let $\{a_N^t\}_{t=1}^{\infty}$ be the sequence of stage game actions generated by ζ_N, and note that:

$$\begin{aligned}
\mathcal{V}_i(\zeta_N) &= \Lambda\left(\left\{\frac{1}{K}\sum_{t=1}^{K} v_i(a_i^t, a_{N_i}^t)\right\}_{K=1}^{\infty}\right) \\
&= \lim_{K\to\infty} \frac{1}{K}\sum_{t=1}^{K} v_i(a_i^t, a_{N_i}^t) \\
&= v_i^*.
\end{aligned}$$

The second equality follows from the fact that $\Lambda(x) = \lim_{n\to\infty} x_n$ for each $x \in c$, the space of all convergent sequences (Lemma 16.45 in Aliprantis and Border (2006) p. 550). To see that the limit of $\frac{1}{K}\sum_{t=1}^{K} v_i(a_i^t, a_{N_i}^t)$ exists and is indeed equal to v_i^*, note that for $K > T$ the following expression is valid:

$$\begin{aligned}
&\left|\frac{1}{K}\sum_{t=1}^{K} v_i(a_i^t, a_{N_i}^t) - v_i^*\right| \\
&= \left|\frac{1}{K}\sum_{t=1}^{K-s(K)} v_i(C_{is(t)}, C_{N_is(t)}) - v_i^* + \frac{1}{K}\sum_{t=K-s(K)+1}^{K} v_i(C_{is(t)}, C_{N_is(t)})\right| \\
&\leq \left|\left(\frac{1}{K} - \frac{1}{K-s(K)}\right)\sum_{t=1}^{K-s(K)} v_i(C_{is(t)}, C_{N_is(t)}) + \frac{1}{K}\sum_{t=K-s(K)+1}^{K} v_i(C_{is(t)}, C_{N_is(t)})\right| \\
&< \frac{KT\max_{a_i,a_{N_i}} v_i(a_i, a_{N_i})}{K(K-T)} + \frac{T\max_{a_i,a_{N_i}} v_i(a_i, a_{N_i})}{K},
\end{aligned}$$

which can be made arbitrarily small for large K. In the third line, we have used the fact that when on-path actions are played over complete length T blocks, the average payoff over those blocks is v^*, i.e. $\frac{1}{K-s(K)}\sum_{t=1}^{K-s(K)} v_i(C_{is(t)}, C_{N_is(t)}) = v_i^*$, which is true by construction.

Given a strategy profile $\sigma_N = (\sigma_1, \ldots, \sigma_n)$ and a history h, let $\sigma_N^h = (\sigma_1^h, \ldots, \sigma_n^h)$ be the profile induced by the history h. We show that for any history h and any strategy $\theta_i \in \Sigma_i$,

$$\mathcal{V}_i(\zeta_N^h) \geq \mathcal{V}_i(\theta_i, \zeta_{-i}^h).$$

Since this is true for any history h, it is optimal for player i to follow the strategy at each of her information sets $\mathcal{I}(h_i)$, whatever her beliefs about h conditional on h_i may be.

Consider any history $h \in H$ of length z. Let $\{\hat{a}_N^t\}_{t=z+1}^{\infty}$ be the sequence of stage game actions generated by ζ_N^h after history h. By Lemma 3 and the properties of Banach-Mazur limits, $\mathcal{V}_i(\zeta_N^h) = v_i^*$. To see this, first note that for $K \geq 2T - s(z)$:

$$\frac{1}{K}\sum_{t=z+1}^{z+K} v_i(\hat{a}_i{}^t, \hat{a}_{N_i}^t) = \frac{1}{K}\sum_{t=z+1}^{z+2T-s(z)} v_i(\hat{a}_i{}^t, \hat{a}_{N_i}^t) + \frac{1}{K}\sum_{t=z+2T-s(z)+1}^{z+K} v_i(\hat{a}_i{}^t, \hat{a}_{N_i}^t).$$

Then by the property that $\Lambda(x_1, x_2, \ldots) = \Lambda(x_2, x_3, \ldots)$ for each $(x_1, x_2, \ldots) \in \ell_\infty$:

$$\begin{aligned}
\mathcal{V}_i(\zeta_N^h) &= \Lambda\left(\left\{\frac{1}{K}\sum_{t=z+1}^{z+K} v_i(\hat{a}_i{}^t, \hat{a}_{N_i}^t)\right\}_{K=2T-s(z)}^{\infty}\right) \\
&= \Lambda\left(\left\{\frac{1}{K}\sum_{t=z+1}^{z+2T-s(z)} v_i(\hat{a}_i{}^t, \hat{a}_{N_i}^t) + \frac{1}{K}\sum_{t=z+2T-s(z)+1}^{z+K} v_i(\hat{a}_i{}^t, \hat{a}_{N_i}^t)\right\}_{K=2T-s(z)}^{\infty}\right) \\
&= \lim_{K\to\infty}\left(\frac{1}{K}\sum_{t=z+1}^{z+2T-s(z)} v_i(\hat{a}_i{}^t, \hat{a}_{N_i}^t) + \frac{1}{K}\sum_{t=z+2T-s(z)+1}^{z+K} v_i(\hat{a}_i{}^t, \hat{a}_{N_i}^t)\right) \\
&= 0 + v_i^*,
\end{aligned}$$

where third equality follows since the sequence converges (which we will show), and the fourth equality follows from Lemma 3 (which implies that for all $t > z + 2T - s(z)$, $\hat{a}_j^t = C_{js(t)}$ for all $j \in N$) and the fact that:

$$\lim_{K\to\infty}\frac{1}{K}\sum_{t=z+2T-s(z)+1}^{z+K} v_i(C_{is(t)}, C_{N_is(t)}) = v_i^*.$$

To see this, note that for $K > 3T - s(z)$ the following expression is valid:

$$\begin{aligned}
&\left| \frac{1}{K} \sum_{t=z+2T-s(z)+1}^{z+K} v_i(C_{is(t)}, C_{N_i s(t)}) - v_i^* \right| \\
&= \left| \frac{1}{K} \sum_{t=z+2T-s(z)+1}^{z+K-s(z+K)} v_i(C_{is(t)}, C_{N_i s(t)}) - v_i^* + \frac{1}{K} \sum_{t=z+K-s(z+K)+1}^{z+K} v_i(C_{is(t)}, C_{N_i s(t)}) \right| \\
&\leq \left| \left(\frac{1}{K} - \frac{1}{K-\kappa} \right) \sum_{t=z+2T-s(z)+1}^{z+K-s(z+K)} v_i(C_{is(t)}, C_{N_i s(t)}) \right| \\
&\quad + \left| \frac{1}{K} \sum_{t=z+K-s(z+K)+1}^{z+K} v_i(C_{is(t)}, C_{N_i s(t)}) \right| \\
&< \frac{3KT \max_{a_i, a_{N_i}} v_i(a_i, a_{N_i})}{K(K-3T)} + \frac{T \max_{a_i, a_{N_i}} v_i(a_i, a_{N_i})}{K},
\end{aligned}$$

where $\kappa \equiv s(z+K) + 2T - s(z)$. For the last inequality, we multiply the maximum number of terms in each summation by the maximum value of the summand and replace κ by its maximum value, noting that since $\kappa < K$, $1/K - 1/(K-\kappa)$ is less than zero and decreasing in κ. This bound can be made arbitrarily small for large K. Since $z + 2T - s(z) + 1$ is the beginning of a length T block, $\frac{1}{K-\kappa} \sum_{t=z+2T-s(z)+1}^{z+K-s(z+K)} v_i(C_{is(t)}, C_{N_i s(t)}) = v_i^*$, which explains the third line.

Thus, we need to show that for any player $i \in N$ and any strategy $\theta_i \in \Sigma_i$,

$$v_i^* \geq \mathcal{V}_i(\theta_i, \zeta_{-i}^h).$$

Let $\{\bar{a}_N^t\}_{t=z+1}^{\infty}$ be the sequence of stage game actions generated by (θ_i, ζ_{-i}^h) after history h, and let $\bar{h}^t$, $t \geq z$, be the length t history generated by (θ_i, ζ_{-i}^h) after h. Note the payoff for player i satisfies:

$$\begin{aligned}
\sum_{t=z+1}^{z+K} v_i(\bar{a}_i^t, \bar{a}_{N_i}^t) &= \sum_{t=z+1}^{z+K} v_i(C_{is(t)}, C_{N_i s(t)}) + \sum_{t=z+1}^{z+K} \left[v_i(\bar{a}_i^t, \bar{a}_{N_i}^t) - v_i(C_{is(t)}, C_{N_i s(t)}) \right] \\
&= \sum_{t=z+1}^{z+K} v_i(C_{is(t)}, C_{N_i s(t)}) + \Pi(K)
\end{aligned}$$

where:

$$\Pi(K) \equiv \sum_{t=z+1}^{z+K} \left[v_i(\bar{a}_i^t, \bar{a}_{N_i}^t) - v_i(C_{is(t)}, C_{N_i s(t)}) \right] \tag{1}$$
$$\leq 6T \max_{a_i a_{N_i}, a'_i a'_{N_i}} |v_i(a_i, a_{N_i}) - v_i(a'_i, a'_{N_i})|$$

for all $K \geq 1$ (when $T - nT_*$ is chosen to be sufficiently large). We establish inequality 1 as a Lemma:

Lemma 4. *For an appropriately chosen value of T, $\Pi(K) \leq 6TM$ for all $K \geq 1$, where $M \equiv \max_{a_i a_{N_i}, a'_i a'_{N_i}} |v_i(a_i, a_{N_i}) - v_i(a'_i, a'_{N_i})|$.*

Proof. Appendix. □

Note also:

$$\lim_{K \to \infty} \frac{1}{K} \sum_{t=z+1}^{z+K} v_i(C_{is(t)}, C_{N_i s(t)}) = v_i^*.$$

To see this, note that for $K > 2T$ the following expression is valid:

$$\begin{aligned}
&\left| \frac{1}{K} \sum_{t=z+1}^{z+K} v_i(C_{is(t)}, C_{N_i s(t)}) - v_i^* \right| \\
&\leq \left| \frac{1}{K} \sum_{t=z+1}^{z+T-s(z)} v_i(C_{is(t)}, C_{N_i s(t)}) + \frac{1}{K} \sum_{t=z+K-s(z+K)+1}^{z+K} v_i(C_{is(t)}, C_{N_i s(t)}) \right| \\
&\quad + \left| \frac{1}{K} \sum_{t=z+T-s(z)+1}^{z+K-s(z+K)} v_i(C_{is(t)}, C_{N_i s(t)}) - v_i^* \right| \\
&= \left| \frac{1}{K} \sum_{t=z+1}^{z+T-s(z)} v_i(C_{is(t)}, C_{N_i s(t)}) + \frac{1}{K} \sum_{t=z+K-s(z+K)+1}^{z+K} v_i(C_{is(t)}, C_{N_i s(t)}) \right| \\
&\quad + \left| \left(\frac{1}{K} - \frac{1}{K-\nu} \right) \sum_{t=z+T-s(z)+1}^{z+K-s(z+K)} v_i(C_{is(t)}, C_{N_i s(t)}) \right| \\
&< \frac{2T \max_{a_i, a_{N_i}} v_i(a_i, a_{N_i})}{K} + \frac{2KT \max_{a_i, a_{N_i}} v_i(a_i, a_{N_i})}{K(K-2T)},
\end{aligned}$$

where $\nu \equiv s(z+K) + T - s(z)$. This last line can be made arbitrarily small for large K. The argument here is similar to when we showed that $\lim_{K\to\infty} \frac{1}{K} \sum_{t=1}^{K} v_i(a_i^t, a_{N_i}^t) = v_i^*$, the difference being that now we have to 'trim' the beginning as well as the end of the summation.

Combining the previous results, we have that:

$$\limsup_{K\to\infty} \frac{1}{K} \sum_{t=z+1}^{z+K} v_i(\bar{a}_i^t, \bar{a}_{N_i}^t) \le v_i^*.$$

Since $\Lambda(x) \le \limsup x_n$ for any $x = (x_1, x_2, \ldots) \in \ell_\infty$ (Lemma 16.45 in Aliprantis and Border (2006) p. 550), $\mathcal{V}_i(\theta_i, \zeta_{-i}^h) = \Lambda\left(\left\{\frac{1}{K+1} \sum_{t=z}^{z+K} v_i(\bar{a}_i^t, \bar{a}_{N_i}^t)\right\}_{K=0}^{\infty}\right) \le v_i^*$. □

7 Conclusion

We have shown that it is possible to sustain as a sequential equilibrium any payoff that can be achieved by a finite deterministic sequence of stage game actions such that each player receives strictly more than her Nash equilibrium payoff. We provide incentives to play the equilibrium strategies by punishing deviations for at least $T - nT_*$ periods. In order to achieve stability, the exact number of periods for which each deviation is punished depends on when it occurs. Similar strategies can be used to prove a folk theorem without the assumption of local interaction as long as the entire network is a connected component. In this case, we would also need a signalling action to ensure that deviations spread to the entire network. It may be possible to extend the result to payoffs that require infinite sequences using a trick similar to Fudenberg and Maskin (1991). For stability, we would need the non-punishing action to be played sufficiently often, so that punishment can spread to the entire network quickly enough, and it seems plausible that sequential rationality would be maintained if the continuation payoff at each date remained close to the target payoff.

A Proof of Lemma 3

Lemma 5. *For an arbitrary history h_i^t of length t:*

1. *If $P_i(h_i^t) = \mathcal{B}(p)$, $p > 0$, then $p = T - s(t)$ or $p = 2T - s(t)$*

2. *If $s(t) \in \{1, \ldots, nT_*\}$, then $P_i(h_i^{t-1}) \neq \mathcal{A}^j$*

Proof. We prove part 1 by induction on the history length. For the empty history, the antecedent is false, and so the statement is true. Now suppose that for all h_i^{t-1}, $P_i(h_i^{t-1}) = \mathcal{B}(p)$ implies $p = T - s(t-1)$ or $p = 2T - s(t-1)$. We show that $P_i(h_i^t) = \mathcal{B}(p)$ implies $p = \mathcal{B}(T - s(t))$ or $p = \mathcal{B}(2T - s(t))$. If $P_i(h_i^{t-1}) = \mathcal{A}$ or $P_i(h_i^{t-1}) = \mathcal{A}^j$, and $P_i(h_i^t) = B(p)$, then $p = p(t) = T - s(t)$ or $p = p(t) = 2T - s(t)$ by definition. Suppose that $P_i(h_i^{t-1}) = \mathcal{B}(T - s(t-1))$ or $P_i(h_i^{t-1}) = \mathcal{B}(2T - s(t-1))$. Note that $P_i(h_i^t)$ is either $\mathcal{A}^j$, $\mathcal{B}(2T - s(t))$, or $\mathcal{B}(p-1)$. For $s(t-1) < T$, $s(t) = s(t-1) + 1$, so $p - 1 = T - s(t)$ or $p - 1 = 2T - s(t)$. For $s(t-1) = T$, $p - 1 = T - 1 = T - s(t)$. Thus, in each case, $P_i(h_i^t) = \mathcal{B}(p)$ implies $p = \mathcal{B}(T - s(t))$ or $p = \mathcal{B}(2T - s(t))$.

For part 2, note that $P_i(h_i^t) = \mathcal{A}^j$ and $P_i(h_i^{t-1}) \neq \mathcal{A}^j$ if and only if $t = nT_*$, and when $s(t) = T$, $P_i(h_i^t) \neq \mathcal{A}^j$. □

Lemma 6. *Take any h, hh' of lengths t and $t + t'$ such that $s(t), s(t+t') \in \{1, \ldots, nT_* - 1\}$. For $\tau \leq t$, let a_N^τ be the period τ action profile in h. Then:*

1. *For any $i \in N$, $P_i(h) = \mathcal{A}$ or $P_i(h) = \mathcal{B}(T - s(t))$*

2. *If $P_i(h) = \mathcal{B}(T - s(t))$, then $P_i(hh') = \mathcal{B}(T - s(t + t'))$*

3. *If for any $\tau \in \{t - s(t) + 1, \ldots, t\}$, $(a_i^\tau, a_{N_i}^\tau) \neq (C_{is(\tau)}, C_{N_i s(\tau)})$, $P_i(h) = \mathcal{B}(T - s(t))$*

Proof. For any history h of length t such that $s(t) = T$, if $P_i(h_i) = \mathcal{B}(p)$, then p is at most T by Lemma 5. For any h of length t such that $s(t) \in \{1, \ldots, nT_* - 1\}$, if $P_i(h_i^{t-1}) = \mathcal{A}$, $P_i(h_i) = \mathcal{A}$ or $P_i(h_i) = \mathcal{B}(T - s(t))$, and if $P_i(h_i^{t-1}) = \mathcal{B}(p)$, $P_i(h_i) = \mathcal{B}(p-1)$. Thus, if $P_i(h_i) = \mathcal{B}(p)$, $p < T$. By Lemma 5, $P_i(h_i) = \mathcal{A}$, $P_i(h_i) = \mathcal{B}(T - s(t))$, or $P_i(h) = \mathcal{B}(2T - s(t))$, but by the previous claim, it cannot be $\mathcal{B}(2T - s(t))$.

Part 2 follows immediately from the definition of P_i after noting that for any h_i^{t-1} such that $s(t) \in \{1, \ldots, nT_* - 1\}$, if $P_i(h_i^{t-1}) = \mathcal{B}(p)$ then $P_i(h_i^t) = \mathcal{B}(p-1)$. For part 3, the definition of P_i implies that $P_i(h^\tau) \neq \mathcal{A}$, and the claim follows from parts 1 and 2. □

Proof of Lemma 3. For an arbitrary history h of length t, let $\hat{h}^\tau$, $\tau \geq t$ be the length τ history generated by (ζ_N^h) after h, and let $\{\hat{a}_N^\tau\}_{\tau=t}^\infty$ be the corresponding sequence of stage game action profiles. Let N_C denote the set of players who have to play an action other than D in some period on the equilibrium path. Note that for each player $i \notin N_C$, $\hat{a}_i^\tau = C_{is(\tau)}$ for all $\tau \geq t+1$. Thus, we need to show that for all $i \in N_C$, $P_i(\hat{h}^\tau) = \mathcal{A}$ for all $\tau \geq t + 2T - s(t)$, which implies that $\hat{a}_i^\tau = C_{is(\tau)}$ for all $\tau > t + 2T - s(t)$ for all $i \in N_C$.

Let $(N^1, \ldots, N^L)$ be a partition of N_C into L connected components. That is, each $(N_C^l, G_{N_C^l})$ is connected component of (N_C, G_{N_C}). Note that a player $i \in N_C^l$ cannot have a neighbour $j \in N_C^k, k \neq l$, because in that case $(N_C^l \cup \{j\}, G_{N_C^l \cup \{j\}})$ would be connected, contradicting the definition of a connected component. We will consider the connected component N_C^l, where l is arbitrary, and show that for all $i \in N_C^l$, $P_i(\hat{h}_i^{t+2T-s(t)}) = \mathcal{A}$.

First, we argue that if for all $i \in N_C^l$, $P_i(\hat{h}_i^k) = \mathcal{A}$ for some period k, then for all $i \in N_C^l$, $P_i(\hat{h}_i^\tau) = \mathcal{A}$ for all $\tau \geq k$. To see this, consider the set $B \equiv N_C^l \cup \{j : j \in N_i \text{ for some } i \in N_C^l\}$, and note that if $P_i(\hat{h}_i^{\tau-1}) = \mathcal{A}$ for all $i \in N_C^l$, then $P_i(\hat{h}_i^\tau) = \mathcal{A}$ for all $i \in N_C^l$. This is because each $j \in B$ is either in N_C^l or not in N_C (since N_C^l is a connected component of N_C). Each $j \in N_i \cap N_C^l$ will play $C_{js(\tau)}$ since $P_j(\hat{h}_j^{\tau-1}) = \mathcal{A}$, each $j \notin N_C$ will play $D = C_{js(\tau)}$ regardless of $P_j(\hat{h}_j^{\tau-1})$, and i will play $C_{is(\tau)}$ since $P_i(\hat{h}_i^{\tau-1}) = \mathcal{A}$. Thus, $\hat{a}_j^\tau = C_{js(\tau)}$ for all $j \in N_i \cup \{i\}$, and so $P_i(\hat{h}_i^\tau) = \mathcal{A}$.

By Lemma 5, for any $i \in N_C^l$, we have either $P_i(\hat{h}_i^{t+T-s(t)}) = \mathcal{A}$ or $P_i(\hat{h}_i^{t+T-s(t)}) = \mathcal{B}(T)$. If $P_i(\hat{h}_i^{t+T-s(t)}) = \mathcal{A}$ for all $i \in N_C$, then for all $i \in N_C^l$, $P_i(\hat{h}_i^\tau) = \mathcal{A}$ for all $\tau \geq t + T - s(t)$, so assume that for some $i \in N_C^l$, $P_i(\hat{h}_i^{t+T-s(t)}) = \mathcal{B}(T)$.

Without loss of generality, let Player 1 belong to N_C^l and let $P_1(\hat{h}_l^{t+T-s(t)}) = \mathcal{B}(T)$. First, we argue that for all $i \in N_C^l$, $P_i(\hat{h}_i^{t+T-s(t)+nT_*-1}) = \mathcal{B}(T - nT_* + 1)$. For any player $i \in N_C^l$, there exists a path $(1, j_2, \ldots, j_{m-1}, i)$ such that each player in the path belongs to N_C^l (since N_C^l is connected), and $m \leq n$. Note that player 1 will play D in every period $\tau \in \{t+T-s(t)+1,\ldots,t+T-s(t)+T_*\}$ according to ζ_1^h. Since player 1 belongs to N_C, it must be the case that $C_{1s(\tau)} \neq D$ for at least one $\tau \in \{t+T-s(t)+1,\ldots,t+T-s(t)+T_*\}$ (recall that the sequence of on-path actions repeat every T_* periods). By Lemma 6, $P_{j_2}(\hat{h}_{j_2}^{t+T-s(t)+T_*}) = B(T-T_*)$. Now j_2 will play D in every period $\tau \in \{t+T-s(t)+T_*+1,\ldots,t+T-s(t)+2T_*\}$, and since $j_2 \in N_C$, the previous argument implies that $P_{j_3}(\hat{h}_{j_3}^{t+T-s(t)+2T_*}) = B(T-2T_*)$. Thus, by period $t+T-s(t)+(m-1)T_*$, we have $P_i(\hat{h}_i^{t+T-s(t)+(m-1)T_*}) = B(T-(m-1)T_*)$. Since, $m \leq n$, Lemma 6 implies that $P_i(\hat{h}_i^{t+T-s(t)+nT_*-1}) = B(T-nT_*+1)$.

Now for each $i \in N_C^I$, $P_i(\hat{h}_i^{t+T-s(t)+nT_*})$ depends on whether $d_{1i}(\hat{h}_i^{t+T-s(t)+nT_*}) = 1$. If there does not exist a unique j satisfying parts (i) and (ii) of condition (c) in the definition of d_{1i}, then $P_i(\hat{h}_i^{t+T-s(t)+nT_*}) = \mathcal{B}(T-nT_*)$. Now we argue that $d_{2i}(h_i^\tau) = 0$ for all $\tau \in \{t+T-s(t)+nt_*+1,\ldots,t+2T-s(t)\}$, and hence $P_i(\hat{h}_i^{t+2T-s(t)}) = \mathcal{A}$. First note that condition (a) in the definition of d_{2i} implies that $d_{2i}(h_i^\tau) = 0$ for all τ such that $s(\tau) \neq T$. Thus, we only need to show that $d_{2i}(h_i^{t+2T-s(t)}) = 0$. To see this, note that for any $j \in N_i$, there is a path $(1, j_2, \ldots, j_{m-1}, j)$, where $m \leq n$, which implies that $P_j(\hat{h}_j^{t+T-s(t)+(n-1)T_*}) = B(T-(n-1)T_*)$, and thus, $\hat{a}_j^\tau = D$ for all $\tau \in \{t+T-s(t)+(n-1)T_*+1,\ldots,t+T-s(t)+nT_*\}$. Thus, for j to satisfy part (i) of condition (b) in the definition of d_{2i}, it must be the case that $j \notin N_C$. But this means that j will play $D = C_{js(\tau)}$ for each $\tau \in \{t+T-s(t)+nT_*+1,\ldots,t+2T-s(t)\}$, and so will not satisfy part (ii) of the condition. Therefore, for each $\tau \in \{t+T-s(t)+nt_*+1,\ldots,t+2T-s(t)\}$, if $P_i(h_i^{\tau-1}) = \mathcal{B}(p)$, then $P_i(h_i^\tau) = \mathcal{B}(p-1)$ for $p > 1$, and $P_i(h_i^\tau) = \mathcal{A}$ for $p = 1$. Since $P_i(\hat{h}_i^{t+T-s(t)+nT_*}) = \mathcal{B}(T-nT_*)$, $P_i(\hat{h}_i^{t+2T-s(t)}) = \mathcal{A}$.

If for some i, $d_{1i}(h_i^{t+T-s(t)+nT_*}) = 1$, then $P_i(\hat{h}_i^{t+T-s(t)+nT_*}) = A^{j^*}$, where j^* is the unique neighbour satisfying condition (c) in the definition of d_{1i}. Note that j^* cannot be in N_C, and therefore j^* will play $\hat{a}_{j^*}^\tau = C_{j^*s(\tau)}$ for each $\tau \in \{t+T-s(t)+nT_*+1,\ldots,t+2T-s(t)\}$, which implies that $P_i(\hat{h}_i^{t+2T-s(t)}) = \mathcal{A}$. □

B Proof of Lemma 4

Let $\zeta_i'(h_i) = C_{is(t)}$ for any h_i of length $t-1$. That is, ζ_i' is the strategy where player i plays the on-path actions after every history. For any arbitrary history h of legnth t, let $\tilde{h}_i^\tau$, $\tau \geq t$, be the length τ history generated by $(\zeta_i', \zeta_{N\setminus\{i\}})$ after h.

Lemma 7. *The strategy profile $(\zeta_i', \zeta_{N\setminus\{i\}})$ is stable. Moreover for all $i \in N_C \setminus \{i\}$, $P_i(\tilde{h}_i^\tau) = \mathcal{A}$ for all $\tau \geq t + 2T - s(t)$.*

Proof. Replace N_C with $N_C \setminus \{i\}$ in the proof of Lemma 3. □

Recall that h is an arbitrary history of length z, $\{\bar{a}_N^t\}_{t=z+1}^\infty$ is the sequence of stage game actions generated by (θ_i, ζ_{-i}^h) after history h, and $\bar{h}^t$, $t \geq z$ is the length t history generated by (θ_i, ζ_{-i}^h) after h.

Lemma 8. *Suppose that for $\bar{h}_j^t$, $j \in N_i$, $t > z + 2T - s(t)$, conditions (a) and (b) in the definition of d_{1j} are satisfied. Then $d_{1j}(\bar{h}_j^t) = 1$ if and only if Player i satisfies parts (i) and (ii) of condition (c) in the definition of d_{1j}.*

Proof. We will show that no $k \in N_j \setminus \{i\}$ can satisfy parts (i) and (ii) of condition (c) in the definition of d_{1j}, when conditions (a) and (b) are satisfied. Assume that $s(t) = nT_*$, and take any $k \in N_j \setminus \{i\}$, and note that for k to satisfy (i), $\bar{a}_k^\tau \neq C_{ks(\tau)}$ for some $\tau \in \{t - s(t) - T + 1, \ldots, t - s(t) - T + nT_*\}$. But this implies that $k \in N_C$. Condition (b) requires that $\bar{a}_j^\tau \neq C_{js(\tau)}$ for some $\tau \in \{t - s(t) + 1, \ldots, t - s(t) + (n-1)T_*\}$, which implies that then $\bar{a}_k^\tau = D$ for all $\tau \in \{t - s(t) + (n-1)T_* + 1, \ldots, t - s(t) + nT_*\}$, and in at least one of these periods $C_{ks(\tau)} \neq D$, and so k cannot satisfy condition (ii). Thus, if Player i satisfies parts (i) and (ii) of condition (c), then Player i is the unique player satisfying this condition, and $d_{1j}(\bar{h}_j^t) = 1$. If Player i does not satisfy parts (i) and (ii) of condition (c), then no player satisfies the condition, and $d_{1j}(\bar{h}_j^t) = 0$. □

Lemma 9 says that when i's opponents play D in every period in a length T_* block, the value of $\Pi(K)$ decreases by at least $T_* v_i^*$.

Lemma 9. *Assume that for all $j \in N_i$, $\bar{a}_j^t = D$ for all $t \in \{k+1, \ldots, k+T_*\}$, where k is such that $k \mod T_* = 0$. Then:*

$$\Pi(k - z + T_*) - \Pi(k - z) \leq -T_* v_i^*.$$

Proof.

$$\begin{aligned}
&\Pi(k-z+T_*) - \Pi(k-z) \\
&= \sum_{t=z+1}^{k+T_*} \left[v_i(\bar{a}_i^t, \bar{a}_{N_i}^t) - v_i(C_{is(t)}, C_{N_is(t)}) \right] - \sum_{t=z+1}^{k} \left[v_i(\bar{a}_i^t, \bar{a}_{N_i}^t) - v_i(C_{is(t)}, C_{N_is(t)}) \right] \\
&= \sum_{t=k+1}^{k+T_*} \left[v_i(\bar{a}_i^t, \bar{a}_{N_i}^t) - v_i(C_{is(t)}, C_{N_is(t)}) \right] \\
&= \sum_{t=k+1}^{k+T_*} \left[v_i(\bar{a}_i^t, D, \ldots, D) - v_i(C_{is(t)}, C_{N_is(t)}) \right] \\
&\leq - \sum_{t=k+1}^{k+T_*} v_i(C_{is(t)}, C_{N_is(t)}) \\
&= -T_* v_i^*.
\end{aligned}$$

□

Lemma 10 says that if each player other than i has been playing according to $\zeta_{N\setminus\{i\}}$ for a sufficiently long time, and i plays an action other than $C_{is(t)}$ in some period t in the first part of a length T block, then all of i's neighbours will punish i with D until the end of the length T block.

Lemma 10. *For any $k_m \geq z + 2T - s(z)$ such that $s(k_m) = T$, if $\bar{a}_i^{t^*} \neq C_{is(t^*)}$, where $t^* \in \{k_m + 1, k_m + nT_*\}$, then for all $j \in N_i$, $\bar{a}_j^t = D$ for all $t \in \{t^* + 1 \ldots, k_m + T\}$.*

Proof. In period $t^* - 1$, for each $j \in N_i$, either $P_j(\bar{h}_j^{t^*-1}) = \mathcal{A}$ or $P_j(\bar{h}_j^{t^*-1}) = \mathcal{B}(T - s(t^* - 1))$ by Lemma 6. Since $\bar{a}_i^{t^*} \neq C_{is(t^*)}$, in both cases $P_j(\bar{h}_j^{t^*}) = \mathcal{B}(T - s(t^*))$. Since $\bar{a}_i^{t^*} \neq C_{is(t^*)}$, Lemma 8 implies that $d_j(\bar{h}_j^{k_m+nT_*}) = 0$, and so $P_j(\bar{h}_j^{k_m+nT_*}) = \mathcal{B}(T - nT_*)$. Thus, for all $t \in \{t^* + 1 \ldots, k_m + T\}$, $P_j(\bar{h}_j^{t-1}) = \mathcal{B}(T - s(t-1))$ or $P_j(\bar{h}_j^{t-1}) = \mathcal{B}(2T - s(t-1))$, depending on whether $d_{2i}(\bar{h}_j^{t-1}) = 1$, but in either case for all $j \in N_i$, $\bar{a}_j^t = D$ for all $t \in \{t^* + 1 \ldots, k_m + T\}$. □

Lemma 11 says that if each player other than i has been playing according to $\zeta_{N\setminus\{i\}}$ for a sufficiently long time, and i plays an action other than $C_{is(t)}$ in some period t in the first part of a length T block, then plays $C_{is(t)}$ in every period t in the first part of the next length T block, then for every period t starting from the beginning of the second part of that length T block, each of of i's neighbours, $j \in N_i$, will play $C_{js(t)}$ until i plays something other than $C_{is(t)}$.

Lemma 11. *For any $k_m \geq z + 2T - s(z)$ such that $s(k_m) = T$, if $\bar{a}_i^t \neq C_{is(t)}$ for some $t \in \{k_m - T + 1, k_m - T + nT_*\}$ and $\bar{a}_i^t = C_{is(t)}$ for all $t \in \{k_m + 1, \ldots, t^* - 1\}$, where $t^* > k_m + nT_*$, then for all $j \in N_i$, $\bar{a}_j^t = C_{js(t)}$ for all $t \in \{k_m + nT_* + 1 \ldots, t^*\}$.*

Proof. Take an arbitrary $j \in N_i$. Let M be the maximal connected component of $N_C \setminus \{i\}$ containing j. As in the proof of Lemma 3, replacing N_C with $N_C \setminus \{i\}$, if for all $k \in M$, $P_k(\bar{h}_i^{k_m}) = \mathcal{A}$, then for all $k \in M$, $P_k(\bar{h}_i^t) = \mathcal{A}$, for all $t \in \{k_m, \ldots, t^*\}$.

If for some $k \in M$, $P_k(\bar{h}_i^{k_m}) = \mathcal{B}(T)$, then after period $k_m + (n-2)T_*$, $P_j(\bar{h}_i^{k_m+(n-2)T_*}) = \mathcal{B}(T - (n-2)T_*)$. This means that $\bar{a}_j^t \neq C_{js(t)}$ for some $t \in \{k_m + (n-2)T_* + 1, \ldots, k_m + (n-1)T_*\}$, and therefore j satisfies condition (b) in the definition of d_{1j}. Note that i satisfies parts (i) and (ii) of condition (c) in the definition of d_{1j} after history $\bar{h}_j^{k_m+nT_*}$. Since, $s(k_m + nT_*) = nT_*$, condition (a) of the definition of d_{1j} is also satisfied, and by Lemma 8, for all $j \in N_i$, $P_j(\bar{h}_j^{k_m+nT_*}) = \mathcal{A}^i$. □

Lemma 12 says that if each player other than i has been playing according to $\zeta_{N\setminus\{i\}}$ for a sufficiently long time, and i plays $C_{is(t)}$ in every period t in the first part of a length T block, and then plays an action other than $C_{is(t)}$ in some period t in the second part of a length T block, then all of i's neighbours will punish i with D until the end of the following length T block.

Lemma 12. *For any $k_m \geq z + 2T - s(z)$ such that $s(k_m) = T$, if $\bar{a}_i^t = C_{is(t)}$ for all $t \in \{k_m + 1, \ldots, t^* - 1\}$ and $\bar{a}_i^{t^*} \neq C_{is(t^*)}$, where $t^* \in \{k_m + nT_* + 1, \ldots, k_m + T\}$, then for all $j \in N_i$, $\bar{a}_j^t = D$ for all $t \in \{t^* + 1, \ldots, k_m + 2T\}$.*

Proof. Take an arbitrary $j \in N_i$. If $P_j(\bar{h}_j^{t^*-1}) = \mathcal{A}$ or $P_j(\bar{h}_j^{t^*-1}) = \mathcal{A}^i$, then $P_j(\bar{h}_j^{t^*}) = \mathcal{B}(2T - s(t))$ (note that by Lemma 8, $P_j(\bar{h}_j^{t^*-1}) \neq \mathcal{A}^k$ for $k \in N_j \setminus \{i\}$)

and $\bar{a}_j^t = D$ for all $t \in \{t^* + 1, \ldots, k_m + 2T\}$, unless $d_{1j}(\bar{h}_j^{k_m+T+nT_*}) = 1$. Note that i cannot satisfy (i) and (ii) of condition (c) in the definition of d_{1j} because $\bar{a}_i^t = C_{is(t)}$ for all $t \in \{k_m + 1, \ldots, k_m + nT_*\}$, so by Lemma 8, $d_{1j}(\bar{h}_j^{k_m+T+nT_*}) = 0$.

If $P_j(\bar{h}_j^{t^*-1}) = \mathcal{B}(p)$, $p > 0$, then we immediately have $\bar{a}_j^t = D$ for all $t \in \{t^* + 1, \ldots, k_m + T\}$. Note, however, that i satisfies (i) and (ii) of condition (b) in the definition of d_{2j} after history $\bar{h}_j^{k_m+T}$ and $s(k_m + T) = T$, so $P_j(\bar{h}_j^{k_m+T}) = \mathcal{B}(T)$. Thus $\bar{a}_j^t = D$ for all $t \in \{t^* + 1, \ldots, k_m + 2T\}$, since $d_{1j}(\bar{h}_j^{k_m+T+nT_*}) = 0$ by the same argument as in the previous paragraph. □

For Lemmas 13 and 14, assume that T is chosen such that $Tv_i^* > (n+1)T_*M$.

Lemma 13. *Take some $\Pi(k_m - z), k_m \geq z + 2T - s(z)$, such that $s(k_m) = T$, and $P_j(\bar{h}_j^{k_m}) = \mathcal{A}$ for all $j \in N_C \setminus \{i\}$. For either $k_{m+1} = k_m + T$ or $k_{m+1} = k_m + 2T$, it is either the case that:*

1. $\Pi(k_{m+1} - z) \leq \Pi(k_m - z)$
2. $P_j(\bar{h}_j^{k_{m+1}}) = \mathcal{A}$ *for all* $j \in N_C \setminus \{i\}$

or

1. $\Pi(k_{m+1} - z) \leq \Pi(k_m - z) - (T - nT_*)v_i^* + T_*M$

Proof. Assume that $k_m \geq z + 2T - s(z)$, $s(k_m) = T$, and $P_j(\bar{h}_j^{k_m}) = \mathcal{A}$ for all $j \in N_C \setminus \{i\}$. We need to show that for either $k_{m+1} = k_m + T$ or $k_{m+1} = k_m + 2T$, it is the case that either $\Pi(k_{m+1} - z) \leq \Pi(k_m - z)$ and $P_j(\bar{h}_j^{k_m}) = \mathcal{A}$ for all $j \in N_C \setminus \{i\}$, or $\Pi(k_{m+1} - z) \leq \Pi(k_m - z) - (T - nT_*)v_i^* + T_*M$.

Case 1: $\bar{a}_i^t = C_{is(t)}$ for all $t \in \{k_m + 1, \ldots, k_m + T\}$. Note that in this case, $(\bar{a}_i^t, \bar{a}_{N_i}^t) = (C_{is(t)}, C_{N_is(t)})$ for all $t \in \{k_m + 1, \ldots, k_m + T\}$. Therefore, there will be no change in $\Pi(K)$. By Lemma 7, $P_j(\bar{h}_j^{k_m+T}) = \mathcal{A}$ for all $j \in N \setminus \{i\}$. Thus, $k_{m+1} = k_m + T$ will do.

Case 2: $\bar{a}_i^t = C_{is(t)}$ for all $t \in \{k_m + 1, \ldots, k_m + nT_*\}$, $\bar{a}_i^t \neq C_{is(t)}$ for some $t \in \{k_m + nT_* + 1, k_m + T\}$. Let t^* be the first $t \in \{k_m + nT_* + 1, k_m + T\}$ such that $\bar{a}_i^t \neq C_{is(t)}$. There will be no change in $\Pi(K)$ over the periods in $\{k_m + 1, \ldots, t^* - 1\}$. By Lemma 12, all of i's neighbours will play D in every period $t \in \{t^* + 1, k_m + 2T\}$. In the length T_* block containing t^*, $\Pi(K)$ can can go up by at most T_*M. By Lemma 9, $\Pi(K)$ must go down by at least Tv_i^* over the periods in $\{k_m+T+1, k_m+2T\}$. So as long as $Tv_i^* > T_*M$, $k_{m+1}=k_m+2T$ will do.

Case 3: $\bar{a}_i^t \neq C_{is(t)}$ for some $t \in \{k_m+1,\dots,k_m+nT_*\}$. $\Pi(K)$ can go up by at most T_*M over the periods in $\{k_m+1,\dots,k_m+nT_*\}$. Lemma 10 and Lemma 9 imply that $\Pi(K)$ must go down by $(T-nT_*)v_i^*$ over the periods in $\{k_m+nT_*+1,\dots,k_m+T\}$. So as long as $(T-nT_*)v_i^* > T_*M$, $k_{m+1}=k_m+T$ will do. □

Lemma 14. *Take some $\Pi(k_m-z), k_m \geq z+2T-s(z)$, such that $s(k_m)=T$, and $P_j(\bar{h}_j^{k_m})=\mathcal{B}(T)$ for some $j \in N_C \setminus \{i\}$. For either $k_{m+1}=k_m+T$ or $k_{m+1}=k_m+2T$, it is either the case that:*

1. $\Pi(k_{m+1}-z) \leq \Pi(k_m-z)$

or

1. $\Pi(k_{m+1}-z) \leq \Pi(k_m-z)+nT_*M$
2. $P_j(\bar{h}_j^{k_{m+1}})=\mathcal{A}$ *for all* $j \in N_C \setminus \{i\}$

Proof. Assume that $k_m \geq z+2T-s(z)$, $s(k_m)=T$, and $P_j(\bar{h}_j^{k_m})=\mathcal{B}(T)$ for some $j \in N_C \setminus \{i\}$. We need to show that for either $k_{m+1}=k_m+T$ or $k_{m+1}=k_m+2T$, it is the case that either $\Pi(k_{m+1}-z) \leq \Pi(k_m-z)$, or $\Pi(k_{m+1}-z) \leq \Pi(k_m-z)+nT_*M$ and $P_j(\bar{h}_j^{k_{m+1}})=\mathcal{A}$ for all $j \in N_C \setminus \{i\}$.

Note that if $\bar{a}_i^t = C_{is(t)}$ for all $t \in \{k_m-T+1,\dots,k_m\}$, then by Lemma 7 $P_j(\bar{h}_j^{k_m})=\mathcal{A}$ for all $j \in N_C \setminus \{i\}$. Thus, we only have to consider what happens when $\bar{a}_i^t \neq C_{is(t)}$ for some $t \in \{k_m-T+1,\dots,k_m\}$.

First, suppose that $\bar{a}_i^t = C_{is(t)}$ for all $t \in \{k_m-T+1,\dots,k_m-T+nT_*\}$, but $\bar{a}_i^t \neq C_{is(t)}$ for some $t \in \{k_m-T+nT_*+1,\dots,k_m\}$. Thus, $P_i(\bar{h}_i^{k_m})=\mathcal{B}(T)$ for each $j \in N_i$, and all of i's neighbours will play D in every period $t \in \{k_m+1,\dots,k_m+T\}$, as long as for all $j \in N_i \cap N_C$, $d_{1j}(\bar{h}_j^{k_m+nT_*})=0$. Take any $j \in N_i \cap N_C$. Note that $\bar{a}_i^t = C_{is(t)}$ for all $t \in \{k_m-T+1,\dots,k_m-T+nT_*\}$, and so i does not satisfy (i) and (ii) of condition (c) in the definition of d_{1j}. Then Lemma 8 implies that $d_{1j}(\bar{h}_j^{k_m+nT_*})=0$.

By Lemma 9, $\Pi(K)$ must fall by at least Tv_i^* over the periods in $\{k_m+1,\dots,k_m+T\}$, and so $k_{m+1}=k_m+T$ will do.

Now suppose that $\bar{a}_i^t \neq C_{is(t)}$ for some $t \in \{k_m - T + 1, \ldots, k_m - T + nT_*\}$. There are three cases.

Case 1: $\bar{a}_i^t = C_{is(t)}$ for all $t \in \{k_m + 1, \ldots, k_m + T\}$. By Lemma 11, each $j \in N_i$ will play $C_{js(t)}$ for each $t \in \{k_m + nT_* + 1, \ldots, k_m + T\}$. Note that if every $j \in N_i \cup \{i\}$ plays $C_{js(t)}$ in period t, the value of $\Pi(K)$ does not change in that period. Thus, $\Pi(K)$ cannot increase over the periods in $\{k_m + nT_* + 1, \ldots, k_m + T\}$. The most $\Pi(K)$ can go up by over the periods in $\{k_m + 1, \ldots, k_m + nT_*\}$ is nT_*M. By Lemma 7, $P_j(\bar{h}_j^{k_m+T}) = \mathcal{A}$ for all $j \in N_C \setminus \{i\}$. Thus, $k_{m+1} = k_m + T$ will do.

Case 2: $\bar{a}_i^t = C_{is(t)}$ for all $t \in \{k_m + 1, \ldots, k_m + nT_*\}$, $\bar{a}_i^t \neq C_{is(t)}$ for some $t \in \{k_m + nT_* + 1, \ldots, k_m + T\}$. Let t^* be the first $t \in \{k_m + nT_* + 1, \ldots, k_m + T\}$ such that $\bar{a}_i^t \neq C_{is(t)}$. By Lemma 11, each $j \in N_i$ will play $C_{js(t)}$ for each $t \in \{k_m + nT_* + 1, \ldots, t^*\}$. By Lemma 12, all of i's neighbours will play D in every period $t \in \{t^* + 1, \ldots, k_m + 2T\}$. This implies that $\Pi(K)$ can increase by at most $(n+1)T_*M$ over the periods in $\{k_m + 1, \ldots, k_m + T\}$. By Lemma 9, $\Pi(K)$ must fall by at least Tv^* over the periods in $\{k_m + T + 1, \ldots, k_m + 2T\}$. Thus, $k_{m+1} = k_m + 2T$ will do

Case 3: $\bar{a}_i^t \neq C_{is(t)}$ for some $t \in \{k_m + 1, \ldots, k_m + nT_*\}$. The maximum $\Pi(K)$ can go up by over the periods $t \in \{k_m + 1, \ldots, k_m + nT_*\}$ is nT_*M, but Lemma 10 implies that every $j \in N_i$ will play D in every period in $\{k_m + nT_* + 1, \ldots, k_m + T\}$. By Lemma 9, $\Pi(K)$ must go down by $(T - nT_*)v_i^*$ over the periods in $\{k_m + nT_* + 1, \ldots, k_m + T\}$. For sufficiently large T, $(T - nT_*)v_i^* > nT_*M$, and so $k_{m+1} = k_m + T$ will do. □

Proof of Lemma 4. First, note that for any $K \in \{1, \ldots, 2T - s(z)\}$, $\Pi(K)$ can be at most $2TM$. Let $k_0 = z + 2T - s(z)$. Lemmas 14 and 13 imply that we can find a sequence $(k_1, k_2, \ldots)$, $k_m < k_{m+1} \leq k_m + 2T$ such that $\Pi(k_m - z) < 4TM$ for all $m \geq 0$ (for sufficiently large T). Since the maximum amount $\Pi(K)$ can change in $2T$ periods is $2TM$, this implies that $6TM$ is an upper bound for $\Pi(K)$ for all $K \geq 1$. □

C Complete payoffs and proof for Example 1

$$v(D,D,D,D) = (0,0,0,0)$$

$$v(C,D,D,D) = (-\varepsilon,0,0,0)$$

$$v(D,C,D,D) = (0,-\varepsilon,0,0)$$

$$v(D,D,C,D) = (0,2,-\varepsilon,0)$$

$$v(D,D,D,C) = (0,0,0,-\varepsilon)$$

$$v(C,C,D,D) = (1,0,0,0)$$

$$v(D,C,C,D) = (1,1,0,0)$$

$$v(D,D,C,C) = (0,2,1,1)$$

$$v(C,D,C,D) = (-1,2,-\varepsilon,0)$$

$$v(D,C,D,C) = (0,-\varepsilon,0,-\varepsilon)$$

$$v(C,D,D,C) = (-\varepsilon,0,0,-\varepsilon)$$

$$v(C,C,C,D) = (1,1,0,0)$$

$$v(C,C,C,C) = (1,1,1,1)$$

$$v(D,C,C,C) = (1,1,1,1)$$

$$v(C,C,D,C) = (1,0,0,-\varepsilon)$$

$$v(C,D,C,C) = (-1,2,1,1)$$

Proposition 15. *If $\delta < 1$, there exists a stage game satisfying Assumption 1, a network structure, and a payoff $v^* \in F^*$ such that v^* is not the payoff in any sequential equilibrium.*

Proof. Suppose for a contradiction that there exists a sequential equilibrium σ_N such that $\mathcal{V}_i(\sigma_N) = 1$ for all i. This requires that for every on-path history h_i, $\sigma_i(h_i) = C$. As long as Player 3 and Player 4 have not deviated, Player 3 can guarantee the payoff 1 by always playing C. Now consider the one shot deviation where Player 2 plays D. For this not to be profitable, Player 3 must play D at least once following the deviation. But if Player 3 plays D, her payoff is strictly less than 1, which cannot be optimal. Thus, the one shot deviation must be profitable, contradicting the assumption that σ_N is a sequential equilibrium. □

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