Privately Designed Correlated Equilibrium*

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Abstract

We consider a setting where each player of a simultaneous-move game privately designs an information structure before playing the game. One of these information structures is chosen at random to determine the distribution of the private messages that players receive. These messages allow players to correlate their actions; however, their private design implies a push from correlated to Nash equilibria. Indeed, the sequential equilibrium payoffs of the extensive-form game with privately designed information structures are correlated equilibrium payoffs of the underlying simultaneous-move game, but not all correlated equilibrium payoffs are sequential equilibrium payoffs. In generic 2-player games, the latter are specific convex combinations of two Nash equilibrium payoffs.

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1 Introduction

It is well-known since Aumann (1974) that all players in a normal-form game can obtain a payoff higher than in any of its Nash equilibria by correlating their play, i.e. in a correlated equilibrium.[1] Achieving correlated equilibrium payoffs requires lotteries over a set of messages that are privately observed by the players and which can be thought of as being chosen by an outside mediator. Since the assumption of an impartial mediator may not always be appropriate, there is an interest in the payoffs that can be achieved through unmediated interaction between the players. For example, Aumann and Hart's (2003) results imply that for two player games, pre-play cheap talk can achieve the entire convex hull of Nash payoffs (but no more). Other papers (e.g. Bárány (1992), Ben-Porath (1998), Gerardi (2004), Āzacis, Laclau, and Vida (2025)) attempt to obtain the entire set of correlated equilibrium payoffs, which requires more than two players or richer communication technology than cheap talk (e.g. balls and urns, public verification).[2]

In this paper, we consider this question from a different perspective. We focus on 2-player games and allow players access to fully mediated communication as long as they can agree on the mediation. However, although the *technology* of mediated communication is available, we assume that players can *manipulate* this technology in a general way. In our model of *self-mediated* interaction, we will allow the players to design the information structure by choosing, in principle, any lottery over privately-observed message profiles; however, (i) the true information structure will be exactly as each player chooses only when both players choose the same information structure and (ii) each player's choice of information structure is unobserved by his opponent, allowing for the possibility that each player may secretly manipulate the true information structure.

Our model is guided by the observation that there are many actions that players can take to influence the information structure: for example, one player may antici-

[1]Note, however, that Neyman (1997) defines a class of games having a smooth concave potential such that any correlated equilibrium is a convex combination of pure strategy Nash equilibria.

[2]This literature will be discussed in Section 3.

pate that another will tamper with an agreed upon randomization device and respond by including additional safeguards. The other may anticipate this and secretly hide backdoors in the device. It is difficult to model explicitly each possible manipulation and its effect on the resulting information structure. On the other hand, we do not wish to rule out any kind of manipulation by assumption.

Thus, our aim is to provide a reduced form model that captures the idea that players are able to try to manipulate the information structure in any way they desire. We achieve this by letting each player choose the information structure directly. Our model is also a reduced form model of conflict as it specifies what information structure actually determines message profiles when different players choose different information structures. Our specification is that each player's chosen information structure is the one that actually determines message profiles with a strictly positive probability, i.e. each player i's information structure is chosen with probability $\beta_i > 0$ (with $\sum_i \beta_i = 1$) to determine the message profile that players receive; our specification is also that players do not observe which information structure has been chosen. This specification is a tractable way of obtaining that (i) if all players choose the same information structure, then message profiles are drawn from such common information structure, (ii) each player is successful in attempting to manipulate the information structure however he wishes with a strictly positive probability, which can be thought of as the relative power that each player has in determining the information structure that actually determines message profiles, and (iii) players' deviations are not directly observable.

We focus on 2-player simultaneous-move games and analyze the extensive-form game where players first choose an information structure and then play the simultaneous-move game. We show that the set of Nash and sequential equilibrium payoffs of the extensive-form coincide and such set is a specific subset of the convex hull of the Nash equilibrium payoffs of the simultaneous-move game. This set is easiest to describe for generic 2-player simultaneous-move games, where the only achievable payoffs are convex combinations of two Nash equilibrium payoffs with weights β_1 and β_2 and the restriction that the payoff with weight β_i is no worse for player i than the payoff with weight β_j.[3] These results are in contrast with, e.g. Aumann and Hart (2003) or Ben-Porath (1998), and show that the details of what is allowed for players to choose in self-mediated interaction matter for the payoffs that can be achieved in equilibrium. In particular, this paper shows that when information is designed optimally by the individuals involved in a strategic situation, very few correlated equilibrium payoffs can be achieved and there is a push from correlated to Nash equilibria.

The paper is organized as follows. Section 2 introduces our model of privately designed correlated equilibrium and characterizes the equilibrium outcomes of the information design extensive-form game. Related literature is discussed in Section 3, along with extensions and concluding remarks. Proofs of our main results can be found in the Appendix. Some details of the extensions we consider in Section 3 are left to the supplementary material.[4]

2 Privately designed correlated equilibrium

This section presents our model (Section 2.1) and main results (Section 2.2).

2.1 Model

Consider a 2-player simultaneous-move game $G = (A_i, u_i)_{i \in N}$ where $N = \{1, 2\}$ is the set of players and, for each $i \in N$, A_i is a finite set of player i's actions and $u_i : A \to \mathbb{R}$ is player i's payoff function, where $A = \prod_{i \in N} A_i$. We extend the domain of u_i to the space of mixed actions in the usual way.[5] Let $N(G) \subseteq \Delta(A_1) \times \Delta(A_2)$ denote the set of Nash equilibria of G.[6]

[3] In the non-generic case, the payoff with weight β_i may itself be a convex combination of payoffs of Nash equilibria among which player i is indifferent.

[4] Available at https://klaohakunakorn.com/idsm.pdf

[5] I.e. for each $\alpha_1 \in \Delta(A_1)$ and $\alpha_2 \in \Delta(A_2)$, $u_i(\alpha_1, \alpha_2) = \sum_{a \in A} \alpha_1[a_1]\alpha_2[a_2]u_i(a_1, a_2)$.

[6] Given a metric space X, $\Delta(X)$ denotes the set of Borel probability measures on X. For each $\mu \in \Delta(X)$, $\text{supp}(\mu)$ denotes the support of μ. For each $x \in X$, the probability measure in $\Delta(X)$ that assigns probability one to x is denoted by 1_x. When $X = \prod_{j \in J} X_j$ for some finite set J, μ_{X_j} denotes the marginal of μ on X_j for each $j \in J$. Given $(\mu_1, \mu_2) \in \Delta(X_1) \times \Delta(X_2)$, $\mu_1 \times \mu_2 \in \Delta(X_1 \times X_2)$ denotes the product measure on $X_1 \times X_2$.

Before the game G is played, each player privately receives a message on which he can potentially condition his action in G. However, rather than assuming that the joint distribution of these private messages is exogenously given (as in the standard model of correlated equilibrium), we will allow each player to privately design this information structure. Our model of *privately designed correlated equilibrium* is formalized by the following extensive-form game G_{id}, which we refer to as the information design extension of G.

At the beginning of the game, each player $i \in N$ chooses simultaneously an information structure which is a probability distribution over message profiles. The set of messages each player $i \in N$ can potentially receive is $M_i = \mathbb{N}$. An *information structure* is a finitely supported probability measure on $M = M_1 \times M_2 = \mathbb{N}^2$. Let S be the set of information structures. Thus, each player i chooses an information structure $\phi_i \in S$. Given a profile of chosen information structures (ϕ_1, ϕ_2), a message profile $m \in M$ is drawn from the distribution $\phi \in \Delta(M)$ defined by setting, for each $m \in M$,

$$\phi[m] = \beta_1 \phi_1[m] + \beta_2 \phi_2[m],$$

where $\beta_1, \beta_2 > 0$ and $\beta_1 + \beta_2 = 1$; the probabilities β_1 and β_2 are exogenous and fixed throughout the paper, and one interpretation for them is that the information structure of each $i \in N$ is chosen by nature with probability β_i. Each player $i \in N$ observes his coordinate $m_i \in M_i$ of the realized message profile m and his choice $\phi_i \in S$ but not the other player's coordinate $m_j \in M_j$ of the realized message profile m or choice $\phi_j \in S$, where $j \neq i$. Then each player i chooses an action $a_i \in A_i$ of the underlying simultaneous-move game G conditional on the observed (m_i, ϕ_i). Player i's payoff is then $u_i(a_1, a_2)$.

A formal description of G_{id} is as follows: the set of terminal histories is the set of sequences $(\phi_1, \phi_2, m, a_1, a_2) \in S \times S \times M \times A_1 \times A_2$. Player 1 moves following the empty history, denoted by w, and histories of the form (ϕ_1, ϕ_2, m). Player 2 moves following histories of the form (ϕ_1) and (ϕ_1, ϕ_2, m, a_1). For each history $(\phi_1, \phi_2) \in S^2$, nature draws $m \in M$ from the distribution $\beta_1 \phi_1 + \beta_2 \phi_2 \in \Delta(M)$. Player 1's information sets are $\{w\}$ and $H(\phi_1, m_1) = \{(\phi_1, \phi_2, (m_1, m_2)) : \phi_2 \in S \text{ and } m_2 \in M_2\}$ for each

$\phi_1 \in S$ and $m_1 \in M_1$; player 2's information sets are $\{(\phi_1) : \phi_1 \in S\}$ and $H(\phi_2, m_2) = \{(\phi_1, \phi_2, (m_1, m_2), a_1) : \phi_1 \in S, m_1 \in M_1$ and $a_1 \in A_1\}$ for each $\phi_2 \in S$ and $m_2 \in M_2$. Finally, for each $i \in \{1, 2\}$, player i's payoff from terminal history $(\phi_1, \phi_2, m, a_1, a_2)$ is $u_i(a_1, a_2)$.

The design of information structures in G_{id} is private in the sense that (i) it is done by the players, (ii) each player's choice of information structure is his own private information and (iii) no player observes nature's choice of the aggregated information structure. Assuming that information structures have finite support implies that each player always has the choice of knowing whether his information structure is the one that was chosen by nature; indeed, the set of messages he can receive if his opponent's information structure is chosen is the finite subset $\text{supp}(\phi_{j,M_i})$ of $\mathbb{N}$ and, hence, he can choose ϕ_i such that $\text{supp}(\phi_{i,M_i})$ is contained in the complement of $\text{supp}(\phi_{j,M_i})$.

A (behavioral) strategy for player $i \in N$ is $\pi_i = (\pi_i^1, \pi_i^2)$ such that $\pi_i^1 \in \Delta(S)$ and $\pi_i^2 : M_i \times S \to \Delta(A_i)$ is measurable.[7] A strategy is $\pi = (\pi_1, \pi_2)$ and let Π^* be the set of strategies. We focus mostly on strategies where players do not mix over the choice of information structures.[8] Let Π be the set of strategies π such that $\pi_i^1 \in S$ (i.e. π_i^1 is pure) for each $i \in N$.

For each strategy $\pi \in \Pi$ and for each $i \in N$ and $m_i \in M_i$, we often write $\phi_i^* = \pi_i^1$ and $\sigma_i^*(m_i) = \pi_i^2(m_i, \phi_i^*)$. We define $\phi^* \in \Delta(M)$ such that, for each $m \in M$, $\phi^*[m] = \beta_1\phi_1^*[m] + \beta_2\phi_2^*[m]$. For each $m \in M$, we write $\sigma^*(m) = (\sigma_1^*(m_1), \sigma_2^*(m_2))$, and for each $\pi \in \Pi$, we write $u_i(\pi) = \sum_{m \in M} \phi^*[m]u_i(\sigma^*(m))$ for each $i \in N$.

We use Nash equilibrium and sequential equilibrium as solution concepts. Sequential equilibrium is defined analogously to Myerson and Reny (2020): a strategy $\pi \in \Pi^*$ is a *sequential equilibrium* if it is a perfect conditional ε-equilibrium for each $\varepsilon > 0$.[9] Informally, π is a *perfect conditional ε-equilibrium* if there exists a net of behavioral strategies converging to π and a net of nature's choices converging to $(\phi_1, \phi_2) \mapsto \beta_1\phi_1 + \beta_2\phi_2$ such that every profile of information structures $\phi = (\phi_1, \phi_2)$

[7]The set S is endowed with the topology of the weak convergence of probability measures.

[8]See Section 3.4 for an extension of our results to the case where players can mix over the information structure.

[9]See A.1 in the Appendix for the definition of perfect conditional ε-equilibrium in our setting.

is played with positive probability sufficiently far in the tails of the net, every message profile $m = (m_1, m_2)$ is realized with positive probability following any ϕ sufficiently far in the tails of the net, every action profile is played with positive probability following any (ϕ, m) sufficiently far in the tails of the net, and all strategies in the net are ε-optimal conditional on every positive probability information set. Note that each strategy in the net is also an ε-Nash equilibrium which only requires unconditional ε-optimality.

In finite games, π is a sequential equilibrium if and only if it is a perfect conditional ε-equilibrium for each $\varepsilon > 0$; thus, our definition of sequential equilibrium is the natural extension to infinite games. However, as argued by Myerson and Reny (2020), a drawback of this definition is that a sequential equilibrium may not exist in general. To circumvent this non-existence issue, they define a slightly weaker notion of equilibrium. However, in our setting, Theorem 1 implies that there exists a sequential equilibrium as we have defined it and in the supplementary material to this paper, we show that the two definitions lead to the same set of equilibrium payoffs.

2.2 Main results

Our setting is in contrast to the case where an impartial mediator sends messages according to some exogenously given information structure $\phi \in S$. In this case, the set of equilibrium action distributions that result from varying ϕ is exactly the set of correlated equilibria of G, as shown by Aumann (1987).[10] With privately designed information structures there will, in general, be a reduction in the set of equilibrium outcomes. The reason is that the messages $m \in \text{supp}(\phi_i^*)$ that each player i sends must be optimal for player i. This is established in Theorem 1 which fully characterizes the set of sequential equilibrium outcomes of G_{id}.

The following notation is used in the statement of Theorem 1. Given a strategy π, for each $i \in N$, let $M_i^* = \text{supp}(\phi_{M_i}^*)$ be the set of messages that player i receives with strictly positive probability. The *outcome* of a strategy $\pi \in \Pi$ is

[10]This result is also implied by Myerson (1982, Proposition 2).

$\left(\phi_i^*, (\sigma_i^*(m_i))_{m_i \in M_i^*}\right)_{i \in N}$; it consists of the information structure chosen by each player and, for each message that he receives with strictly positive probability, the mixed action he will choose in response. For each $i \in N$, $j \neq i$ and $\delta \in \Delta(A_j)$, let $v_i(\delta) = \max_{\alpha \in \Delta(A_i)} u_i(\alpha, \delta)$ and $BR_i(\delta) = \{\alpha \in \Delta(A_i) : u_i(\alpha, \delta) = v_i(\delta)\}$ be, respectively, player i's value function and best-reply correspondence.

Theorem 1. *For each 2-player game G, the following conditions are equivalent:*

1. $\left(\phi_i^*, (\sigma_i^*(m_i))_{m_i \in M_i^*}\right)_{i \in N}$ *is the outcome of a Nash equilibrium of G_{id}.*

2. $\left(\phi_i^*, (\sigma_i^*(m_i))_{m_i \in M_i^*}\right)_{i \in N}$ *is the outcome of a sequential equilibrium of G_{id}.*

3. $\left(\phi_i^*, (\sigma_i^*(m_i))_{m_i \in M_i^*}\right)_{i \in N}$ *is such that, for each $i, j \in N$ and $j \neq i$,*

$$v_i(\sigma_j^*(m_j)) = \max_{m_j' \in M_j^*} v_i(\sigma_j^*(m_j')) \text{ and } \sigma_i^*(m_i) \in BR_i(\sigma_j^*(m_j)) \tag{1}$$

for each $m \in \operatorname{supp}(\phi_i^*)$, and

$$\sigma_i^*(m_i) \text{ solves } \max_{\alpha_i \in \Delta(A_i)} \sum_{m_j} \frac{\phi_j^*[m_i, m_{-i}]}{\phi_{j,M_i}^*[m_i]} u_i(\alpha_i, \sigma_j^*(m_j)) \tag{2}$$

for each $m_i \in \operatorname{supp}(\phi_{j,M_i}^*)$.

Theorem 1 shows that Nash and sequential equilibrium outcomes of the information design extension of G coincide.[11] These are characterized by the optimality of the messages each player sends and of the actions he chooses: (1) requires that each message profile sent by a player i, i.e. each $m \in \operatorname{supp}(\phi_i^*)$, must be optimal in the sense that it induces a mixed action $\sigma_j^*(m_j)$ of the other player $j \neq i$ that maximizes player i's value function over $\{\sigma_j^*(m_j') : m_j' \in M_j^*\}$ and, moreover, $\sigma_i^*(m_i)$ must be a best response to $\sigma_j^*(m_j)$. To see why this condition is necessary, consider a deviation by player i to an information structure ϕ_i that sends message profile $\bar{m}$ with probability one, where $\bar{m}_i \notin M_i^*$ and $\bar{m}_j \in \arg\max_{m_j' \in M_j^*} v_i(\sigma_j^*(m_j'))$. Then following $\bar{m}_i$, player i can best respond to $\sigma_j^*(\bar{m}_j)$ and following $m_i \neq \bar{m}_i$, player i can choose an action to maximize his payoff conditional on the information structure being ϕ_j^*.

[11] This is in contrast to mediated communication games where this equivalence fails when priors lack full support (see e.g. Gerardi and Myerson (2007) and Sugaya and Wolitzky (2021)).

This deviation is profitable unless $\sigma_j^*(m_j)$ maximizes v_i over $\{\sigma_j^*(m_j') : m_j' \in M_j^*\}$ and $\sigma_i^*(m_i)$ is a best response to $\sigma_j^*(m_j)$, i.e. unless (1) is satisfied.

In addition, this deviation is profitable unless for each $m_i \in \text{supp}(\phi_{j,M_i}^*)$, $\sigma_i^*(m_i)$ maximizes player i's payoff conditional on ϕ_j^* being the information structure. Thus, for messages m_i that player i receives with strictly positive probability from the information structure ϕ_j^*, (2) requires that $\sigma_i^*(m_i)$ maximizes player i's payoff conditional on ϕ_j^* being the information structure.

We show that (1) and (2) are sufficient for the outcome to be supported in sequential equilibrium by constructing one where for each off-path message $m_i \notin M_i^*$, player i's belief is the same as his belief following some message $\bar{m}_i \in \text{supp}(\phi_{i,M_i}^*)$ which then makes $\sigma_i^*(m_i) = \sigma_i^*(\bar{m}_i)$ sequentially rational.[12] Since for each off-path message $m_i \notin M_i^*$, player i plays an on-path continuation action $\sigma_i^*(\bar{m}_i) \in \{\sigma_i^*(m_i') : m_i' \in M_i^*\}$, player j's choice of information structure is optimal since it induces continuation actions of player i that maximize player j's value function over this set.

The following corollary of Theorem 1 characterizes the action distributions of sequential equilibria of G_{id}. For each strategy $\pi \in \Pi$, the *action distribution* of π is $\sigma_\pi \in \Delta(A)$ such that $\sigma_\pi = \sum_{m \in M^*} \phi^*[m](\sigma_1^*(m) \times \sigma_2^*(m))$.[13] Let

$$A(G) = \{\sigma_\pi : \pi \in \Pi \text{ is a sequential equilibrium of } G_{id}\}$$

be the set of action distributions of the sequential equilibria of G_{id}. Corollary 1 characterizes each equilibrium action distribution as a specific convex combination of the action distributions of Nash equilibria of G by showing that $A(G)$ equals the

[12]Although our definition of sequential equilibrium does not make reference to systems of beliefs, the beliefs are implicitly specified through the net of perturbations and the required belief can easily be generated by the appropriate perturbation of nature's choices.

[13]Recall that for each $(\alpha_1, \alpha_2) \in \Delta A_1 \times \Delta A_2$, $\alpha_1 \times \alpha_2 \in \Delta(A)$ is the product distribution, i.e., for each $a \in A$, $(\alpha_1 \times \alpha_2)[a] = \alpha_1[a_1]\alpha_2[a_2]$.

following set:

$$\mathcal{A} = \Bigg\{ \sigma \in \Delta(A) : \forall i \in N, \text{ there exists } L_i, (\eta^{i,l})_{l=1}^{L_i}, (\sigma^{i,l})_{l=1}^{L_i} \text{ such that}$$
$$\sigma = \beta_1 \sum_{l=1}^{L_1} \eta^{1,l}(\sigma_1^{1,l} \times \sigma_2^{1,l}) + \beta_2 \sum_{l=1}^{L_2} \eta^{2,l}(\sigma_1^{2,l} \times \sigma_2^{2,l}),$$
$$\eta^i \geq 0, \sum_{l=1}^{L_i} \eta^{i,l} = 1, \sigma^{i,l} \in N(G) \text{ and } u_i(\sigma^{i,k}) = u_i(\sigma^{i,l}) \geq u_i(\sigma^{j,r})$$
$$\forall k, l \in \{1, \ldots, L_i\}, j \in N \text{ and } r \in \{1, \ldots, L_j\} \Bigg\}.$$

Corollary 1. *For each 2-player game G, $A(G) = \mathcal{A}$.*

Corollary 1 characterizes the equilibrium action distributions of G_{id} for 2-player games. It shows that when player i's information structure is chosen, there is a resulting distribution over Nash equilibria of G, all of which give the same payoff to player i. Furthermore, this common payoff is no less than the payoff player i obtains in each of the Nash equilibria that result when player j's information structure is chosen.

The characterization of equilibrium action distributions in Corollary 1 implies an analogous characterization of the set of equilibrium payoffs of G_{id}. Let

$$U(G) = \{u(\pi) : \pi \in \Pi \text{ is a sequential equilibrium of } G_{id}\}$$

be the set of sequential equilibrium payoffs of G_{id}. Corollary 2 shows that $U(G)$ equals the following set:

$$\mathcal{U} = \Bigg\{ \beta_1 u^1 + \beta_2 u^2 : \forall i \in N, \text{ there exists } L_i, (\eta^{i,l})_{l=1}^{L_i}, (\sigma^{i,l})_{l=1}^{L_i} \text{ such that}$$
$$u^i = \sum_{l=1}^{L_i} \eta^{i,l} u(\sigma^{i,l}), \eta^i \geq 0, \sum_{l=1}^{L_i} \eta^{i,l} = 1,$$
$$\sigma^{i,l} \in N(G) \text{ and } u_i(\sigma^{i,k}) = u_i(\sigma^{i,l}) \geq u_i(\sigma^{j,r})$$
$$\forall k, l \in \{1, \ldots, L_i\}, j \in N \text{ and } r \in \{1, \ldots, L_j\} \Bigg\}.$$

Corollary 2. *For each 2-player game G, $U(G) = \mathcal{U}$.*

$1\backslash 2$	A	B
A	$2,1$	$0,0$
B	$0,0$	$1,2$

Figure 1: The battle of the sexes.

$1\backslash 2$	A	B
A	$6,6$	$1,7$
B	$7,1$	$0,0$

Figure 2: The game of chicken.

Thus, in general, not all correlated equilibrium payoffs of G can be achieved in the information design extensive-form game G_{id}. Indeed, equilibrium payoffs of G_{id} form a particular subset of the convex hull of the Nash equilibrium payoffs of G.

When G is the battle of the sexes in Figure 1, Corollary 2 implies that $U(G) = u(N(G)) \cup \{\beta_1(2,1) + \beta_2(1,2)\}$. The payoff profile $\beta_1(2,1) + \beta_2(1,2)$, for example, can be generated by $\phi_i^* = 1_{(i,i)}$, $\sigma_i^*(1) = A$ and $\sigma_i^*(2) = B$ for each i, which satisfies conditions (1) and (2) and hence is the outcome of a sequential equilibrium of G_{id}.

When G is the game of chicken in Figure 2, the Nash equilibria are (A, B), (B, A) and $(\frac{1}{2}1_A + \frac{1}{2}1_B, \frac{1}{2}1_A + \frac{1}{2}1_B)$, with payoffs $(7,1)$, $(1,7)$ and $(\frac{7}{2}, \frac{7}{2})$ respectively, and it is well-known that there are correlated equilibria with payoffs outside the convex hull of the Nash equilibrium payoffs. Corollary 2 implies that

$$U(G) = u(N(G)) \cup \left\{\beta_1(7,1) + \beta_2(1,7), \beta_1(7,1) + \beta_2(\tfrac{7}{2}, \tfrac{7}{2}), \beta_1(\tfrac{7}{2}, \tfrac{7}{2}) + \beta_2(1,7)\right\}.$$

In particular, $(4\frac{2}{3}, 4\frac{2}{3})$ is not in $U(G)$ and the action distribution $\frac{1}{3}1_{(A,A)} + \frac{1}{3}1_{(A,B)} + \frac{1}{3}1_{(B,A)}$ is not the action distribution of a Nash equilibrium of the information design extensive-form game. This payoff profile and action distribution could be obtained with $\phi_1 = \phi_2 = \frac{1}{3}1_{(1,1)} + \frac{1}{3}1_{(1,2)} + \frac{1}{3}1_{(2,1)}$ and $\pi_i^2(1, \phi_i) = A$ and $\pi_i^2(2, \phi_i) = B$ for each i. But then player 1 would gain by deviating to $\phi_1' = 1_{(2,1)}$ thereby increasing the probability that his preferred action profile, (B, A), is played.

The characterization of $U(G)$ is simpler in generic games, such as the battle of the sexes and the game of chicken, since then the payoff resulting after each information structure is chosen is that of a Nash equilibrium. Let $\mathcal{G}$ be the set of games such that, for each Nash equilibria σ and σ' of G, if $u_i(\sigma) = u_i(\sigma')$ for some $i \in N$, then $u_j(\sigma) = u_j(\sigma')$ for $j \neq i$ (equivalently, if $u_i(\sigma) \neq u_i(\sigma')$ for some $i \in N$ then $u_j(\sigma) \neq u_j(\sigma')$ for $j \neq i$). We regard $\mathcal{G}$ as a subset of $\mathbb{R}^{2|A|}$. A subset of a Euclidean space is *generic* if the closure of its complement has Lebesgue measure zero.

Corollary 3. *The set $\mathcal{G}$ is generic and, for each 2-player game $G \in \mathcal{G}$,*

$$U(G) = \{\beta_1 u(\sigma) + \beta_2 u(\sigma') : \sigma, \sigma' \in N(G), u_1(\sigma) \geq u_1(\sigma'), u_2(\sigma') \geq u_2(\sigma)\}.$$

The proof of Corollary 3 actually shows that the set of games such that $u_i(\sigma) \neq u_i(\sigma')$ for each $i \in N$ and $\sigma, \sigma' \in N(G)$ such that $\sigma \neq \sigma'$ is generic. This set is contained in $\mathcal{G}$ and contains all games with a unique equilibrium as well as the battle of the sexes. It is clear from Corollary 3 that $U(G) = u(N(G))$ for each 2-player game G with a unique Nash equilibrium.

3 Related literature and discussion

Many papers have considered whether correlated equilibrium payoffs can be sustained as the outcome of an extended game where players can take "cheap" pre-play actions that determine the distribution of their information. The distinguishing feature of our model is that we allow each player to choose any information structure he desires, and with some probability the information structure he chooses is the one that actually determines the joint distribution of the messages of all players. This section provides a discussion of how our model relates to alternative formalizations in the literature and possible extensions.

3.1 Cheap talk

For 2-player games, Aumann and Hart's (2003) results imply that any payoff in the convex hull of the Nash equilibrium payoffs can be achieved as the outcome of an extended game where players engage in cheap talk for as long as they like before playing the original game. In Aumann and Hart (2003), messages are common knowledge so there is no possibility of getting payoffs outside of the convex hull, but cheap talk is enough for players to reach any outcome achievable using publicly observed lotteries. On the other hand, in our model, there are privately observed lotteries but nevertheless players can only get payoffs in $\mathrm{co}(u(N(G)))$ and not even all of those (even if we were to vary β). The key difference between our specification and the setting of Aumann and Hart (2003) is that in the latter, each player fully determines the message received by the other player. On the other hand, according to our specification, there is always a possibility that each player gets to determine the messages of both players. For example, if player 2 benefits from player 1 sending some message m_1, then player 2 may want to take certain (unmodelled) actions that increase the likelihood that player 1 will send message m_1.

It is possible to unify the two cases through an abstract aggregation function $\alpha : S^2 \to S$ such that if player 1 chooses information structure $\phi_1 \in S$ and player 2 chooses information structure $\phi_2 \in S$, then the realized information structure is $\alpha(\phi_1, \phi_2) \in S$. Our formalization corresponds to $\alpha(\phi_1, \phi_2) = \beta_1\phi_1 + \beta_2\phi_2$ and it is possible to specify an alternative aggregation function to capture Aumann and Hart's (2003) model when communication is restricted to one period only. Indeed, assume that each player i chooses $\phi_i \in S$ concentrated on $\{m^i \in \mathbb{N}^2 : m^i_1 = m^i_2\}$, let $\psi : \mathbb{N}^2 \times \mathbb{N}^2 \to \mathbb{N}^2$ be a bijection and let $\alpha(\phi_1, \phi_2) = (\phi_1 \times \phi_2) \circ \psi^{-1}$. Note that each message received by player i is in bijective correspondence with (m^1, m^2), where m^j is drawn from ϕ_j and can be identified with the message sent by player j in Aumann and Hart (2003).

Other papers (e.g. Bárány (1992), Gerardi (2004), and Āzacis, Laclau, and Vida (2025)) attempt to achieve the entire set of correlated equilibrium payoffs via cheap talk, which requires more than two players.[14] Although our focus is primarily on

[14] See Forges (2020), Section 4, for a detailed survey. For pure unmediated interaction (no balls and urns or verification) and sequential rationality, Āzacis, Laclau, and Vida (2025) requires at least four players, improving on Gerardi's (2004) result which required five players. Bárány (1992) established

2-player games, we discuss the extension of our results to more than two players in Section 3.5, which implies that, in general, payoffs from privately designed correlated equilibrium are a strict subset of the set of correlated equilibrium payoffs.

3.2 Communication protocols and manipulability

Another strand of the literature has focused on whether players can communicate in a more sophisticated manner to achieve correlated equilibrium payoffs. For instance, Ben-Porath (1998) shows that each correlated equilibrium can be approximated by the action distribution of a sequential equilibrium in a specific information design extensive-form game that includes the possibility of credibly revealing messages and (in the case of two players) ball and urns.[15]

However, the specification of such extensive form games rules out the possibility of certain manipulations by assumption. In the chicken game, Ben-Porath's (1998) result implies that the correlated equilibrium $\phi = \frac{1}{3}1_{(A,A)} + \frac{1}{3}1_{(A,B)} + \frac{1}{3}1_{(B,A)}$ is close to the action distribution of a sequential equilibrium of his information design extensive-form game. This sequential equilibrium involves, in particular, player 1 drawing from a ball from an urn which contains $\frac{2}{3}$ green balls and $\frac{1}{3}$ red balls and playing A if and only if the ball is green. This urn is provided by player 2 and, by assumption, player 1 cannot manipulate its contents before choosing from it. In contrast, such a deviation corresponds to an alternative choice of information structure by player 1 and has a positive probability of being successful in the extensive-form game we consider.

3.3 Privacy

An alternative to our assumption that the information structures are chosen privately is to assume that aggregated information structure is public, i.e. that each player observes the information structure $\beta_1\phi_1 + \beta_2\phi_2$ chosen by nature. Under this assumption and for the chicken game, the payoff $(4\frac{2}{3}, 4\frac{2}{3})$ can be achieved. Intuitively, both players choose the information structure inducing this payoff (e.g. $\frac{1}{3}1_{(1,1)} + \frac{1}{3}1_{(1,2)} + \frac{1}{3}1_{(2,1)}$)

a result for four players but without sequential rationality and assuming public verification.

[15] See also Urbano and Vila (2002) for 2-player games where players are boundedly rational.

and deviations from it can be deterred by the threat of reverting to the mixed strategy Nash equilibrium whenever some alternative information structure is realized. Thus, when the choice of information is observed, certain information structures can be sustained by the threat of punishment. Our aim is instead to ask which outcomes can arise abstracting away from the possibility of such threats.

3.4 Mixing over information structures

We have focused so far on the case where players are not allowed to mix in their choice of an information structure. As we argue in this section, allowing randomization in the choice of information structures does not significantly change our results.[16]

We focus on Corollary 3 and let

$$U^*(G) = \{u(\pi) : \pi \in \Pi^* \text{ is a sequential equilibrium of } G_{id}\},$$

where, recall, Π^* is the set of mixed strategies of G_{id}. We then have that, for each 2-player game $G \in \mathcal{G}$, $U(G) \subseteq U^*(G) \subseteq \{\beta_1 u(\sigma) + \beta_2 u(\sigma') : \sigma, \sigma' \in N(G)\}$. Thus, in generic 2-player games, sequential equilibrium payoffs of G_{id} when mixing over information structures is allowed continue to be specific convex combinations of two Nash equilibrium payoffs.

In games with a unique Nash equilibrium, it then follows that $U(G) = U^*(G) = u(N(G))$. If G is a 2-player game that has more than one Nash equilibrium, then mixing over information structures can expand the set of equilibrium payoffs. We illustrate this claim in the battle of the sexes by showing in the supplementary material to this paper that $\beta_1 u(B, B) + \beta_2 u(A, A) \in U^*(G) \setminus U(G)$. The reason why this payoff profile does not belong to $U(G)$ is that each player obtains a lower payoff following his own information structure being chosen by nature than following the information structure of his opponent being chosen. Thus, e.g. player 2 could deviate in his choice of information structure by sending a message to player 1 from player 1's information structure which triggers player 1 to choose B. When mixing over information structures is allowed, player 1 can prevent this deviation from being profitable by, for

[16] See the supplementary material to this paper for the details of this section.

1\2	A	B
A	$0,0,3$	$0,0,0$
B	$1,0,0$	$0,0,0$

1\2	A	B
A	$2,2,2$	$0,0,0$
B	$0,0,0$	$2,2,2$

1\2	A	B
A	$0,0,0$	$0,0,0$
B	$0,1,0$	$0,0,3$

Figure 3: Game from Aumann (1974).

instance, uniformly randomizing over L information structures, $\phi_1^1, \ldots, \phi_1^L$, with ϕ_1^l sending message l to himself and choosing B if and only if she receives message l and had chosen ϕ_1^l. In this way, if player 2 sends message $l \in \{1, \ldots, L\}$ to player 1, this will trigger B only with probability $1/L$ (note that player 2 does not observe which ϕ_1^l realizes and cannot condition the message he sends on it).

3.5 More than two players

The extension of our setting to the case of more than two players is straightforward.[17] Theorem 1 extends, but corollaries 1, 2 and 3 do not. Consider the game in Figure 3, which is Example 2.5 in Aumann (1974), where player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix ($A_3 = \{L, M, R\}$). In this game, $u_i \leq 1$ for each $u \in u(N(G))$ and $i \in \{1, 2, 3\}$ but $(2, 2, 2)$ is a correlated equilibrium payoff.

Using the extension of Theorem 1, we show in the supplementary material to this paper that if $\min\{2\beta_1, 2\beta_2\} \geq \beta_3$, then $(1 - \beta_3)(2, 2, 2) + \beta_3(0, 0, 3)$ is a sequential equilibrium payoff of the information design extensive-form game. This payoff can be obtained by setting $\phi_1^* = \phi_2^* = \frac{1}{2}1_{(m_1', m_2', \hat{m}_3)} + \frac{1}{2}1_{(m_1'', m_2'', \hat{m}_3)}$, $\phi_3^* = \frac{1}{2}1_{(m_1', m_2', \hat{m}_3')} + \frac{1}{2}1_{(m_1'', m_2'', \hat{m}_3'')}$, $\sigma_1^*(m_1') = A$, $\sigma_1^*(m_1'') = B$, $\sigma_2^*(m_2') = A$, $\sigma_2^*(m_2'') = B$, $\sigma_3^*(\hat{m}_3) = M$, $\sigma_3^*(\hat{m}_3') = L$ and $\sigma_3^*(\hat{m}_3'') = R$. Thus, the four message profiles that can occur induce the following action profiles: $\sigma^*(m_1', m_2', \hat{m}_3) = (A, A, M)$, $\sigma^*(m_1'', m_2'', \hat{m}_3) = (B, B, M)$, $\sigma^*(m_1', m_2', \hat{m}_3') = (A, A, L)$ and $\sigma^*(m_1'', m_2'', \hat{m}_3'') = (B, B, R)$. This shows that correlation of players' actions through privately designed information structures can achieve payoffs outside the convex hull of the Nash equi-

[17] See the supplementary material to this paper for the details for this section.

librium payoffs when there are more than two players.

Nevertheless, it is also clear that, with privately designed information structures, not all correlated equilibrium payoffs can be achieved. For example, if $(2,2,2) \in U(G)$, then, for some sequential equilibrium $\pi \in \Pi$,

$$(2,2,2) = \sum_{m \in \text{supp}(\phi^*)} \phi^*[m] u(\sigma^*(m))$$

and, thus, $\sigma^*(m) = (A, A, M)$ or $\sigma^*(m) = (B, B, M)$ for each $m \in \text{supp}(\phi^*)$. But then, for each $m \in \text{supp}(\phi_3^*)$, $\sigma_3^*(m_3)$ is not a best-reply against $\sigma_{-3}^*(m_{-3})$, contradicting (the extension of) Theorem 1.

Note that the correlated equilibrium payoff $(2,2,2)$ cannot be approximated by $u \in U(G)$. Indeed, to get close to $(2,2,2)$, ϕ^* must put small probability on m such that $\sigma^*(m) \notin \{(A, A, M), (B, B, M)\}$. Thus, ϕ_3^* must also put small probability on such m. But then there exists $m' \in \text{supp}(\phi_3^*)$ such that $\sigma^*(m') \in \{(A, A, M), (B, B, M)\}$, which contradicts (the extension of) Theorem 1.

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A Appendix

A.1 Definition of perfect conditional ε-equilibrium

A sequential equilibrium $\pi \in \Pi^*$ is, by definition, a perfect conditional ε-equilibrium for each $\varepsilon > 0$. For each $\varepsilon > 0$, $\pi \in \Pi^*$ is a *perfect conditional ε-equilibrium* if there exists a net $\{\pi^\alpha, p^\alpha\}_\alpha$ such that the following properties hold. The first five require that $\{\pi^\alpha\}_\alpha$ is a net of strategies converging to π that assign strictly positive probability to each action and information structure beyond a certain order, and that $\{p^\alpha\}_\alpha$ is a net of nature's choices regarding the probability distribution of message profiles for each profile of information structures (ϕ_1, ϕ_2) that converges to $\beta_1\phi_1 + \beta_2\phi_2$ and assigns strictly positive probability to each message profile beyond a certain order:

(i) For each α, π^α is a strategy and $p^\alpha : S^2 \to \Delta(M)$ is measurable,

(ii) For each $i \in N$, $\sup_{B \in \mathcal{B}(S)} |\pi_i^{1,\alpha}[B] - \pi_i^1[B]| \to 0$ and

$$\sup_{(m_i,\phi_i) \in M_i \times S, a_i \in A_i} |\pi_i^{2,\alpha}(m_i, \phi_i)[a_i] - \pi_i^2(m_i, \phi_i)[a_i]| \to 0,$$ [18]

(iii) For each $i \in N$, $m_i \in M_i$, $\phi_i \in S$ and $a_i \in A_i$, there is $\bar{\alpha}$ such that $\pi_i^{1,\alpha}[\phi_i] > 0$ and $\pi_i^{2,\alpha}(m_i, \phi_i)[a_i] > 0$ for each $\alpha \geq \bar{\alpha}$,

(iv) $\sup_{\phi \in S^2, B \subseteq M} |p^\alpha(\phi)[B] - \sum_{i \in N} \beta_i \phi_i[B]| \to 0$, and

(v) For each $\phi \in S^2$ and $m \in M$, there is $\bar{\alpha}$ such that $p^\alpha(\phi)[m] > 0$ for each $\alpha \geq \bar{\alpha}$.

A final condition requires that, for each α, π^α is such that the payoff that each player obtains by following it at each information set which is reached with strictly positive probability is within ε of his maximum payoff conditional on that information set:

(vi) for each α and $i, j \in N$, with $j \neq i$,

[18] We let $\mathcal{B}(S)$ denote the class of Borel measurable subsets of S and, for each $\phi \in S$, 1_ϕ denote the probability measure on S degenerate at ϕ.

(a) For each $\phi_i' \in S$,

$$\sum_{\phi \in \text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] \left(\sum_m p^\alpha(\phi)[m] u_i(\pi^{2,\alpha}(m,\phi)) \right) \geq$$
$$\sum_{\phi \in \text{supp}(1_{\phi_i'} \times \pi_j^{1,\alpha})} (1_{\phi_i'} \times \pi_j^{1,\alpha})[\phi] \left(\sum_m p^\alpha(\phi)[m] u_i(\pi^{2,\alpha}(m,\phi)) \right) - \varepsilon,$$

where $\pi^{1,\alpha} = \pi_1^{1,\alpha} \times \pi_2^{1,\alpha}$ and $\pi^{2,\alpha}(m,\phi) = (\pi_1^{2,\alpha}(m_1,\phi_1), \pi_2^{2,\alpha}(m_2,\phi_2))$, and

(b) For each $i \in N$, $(m_i, \phi_i) \in M_i \times S$ such that

$$\pi_i^{1,\alpha}[\phi_i] \sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] p^\alpha_{M_i}(\phi_i,\phi_j)[m_i] > 0$$

and $a_i \in A_i$,

$$\frac{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] \left(\sum_{m_j} p^\alpha(\phi_i,\phi_j)[m_i,m_j] u_i(\pi^{2,\alpha}(m,\phi)) \right)}{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] p^\alpha_{M_i}(\phi_i,\phi_j)[m_i]} \geq$$
$$\frac{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] \left(\sum_{m_j} p^\alpha(\phi_i,\phi_j)[m_i,m_j] u_i(a_i, \pi_j^{2,\alpha}(m_j,\phi_j)) \right)}{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] p^\alpha_{M_i}(\phi_i,\phi_j)[m_i]} - \varepsilon.$$

A.2 Proof of Theorem 1

Every sequential equilibrium is a Nash equilibrium, hence condition 2 implies condition 1. Thus, it suffices to show that condition 1 implies condition 3 and that condition 3 implies condition 2.

A.2.1 Proof that condition 1 implies condition 3

Let $\pi \in \Pi$ be a Nash equilibrium of G_{id}. Then

$$\sum_m \phi^*[m] u_i(\sigma^*(m)) \geq \sum_m (\phi_i', \phi_j^*)[m] u_i(\zeta(m_i), \sigma_j^*(m_j)), \tag{3}$$

for each $i, j \in N$, $j \neq i$, $\phi_i' \in S$ and $\zeta : M_i \to \Delta(A_i)$, where $(\phi_i', \phi_j^*) = \beta_i \phi_i' + \beta_j \phi_j^*$. It follows from (3) that

$$\sum_{m_j} \frac{\phi^*[m]}{\phi^*_{M_i}[m_i]} u_i(\sigma^*(m)) \geq \sum_{m_j} \frac{\phi^*[m]}{\phi^*_{M_i}[m_i]} u_i(a_i, \sigma_j^*(m_j)) \tag{4}$$

for each $i, j \in N$, $j \neq i$, $m_i \in \text{supp}(\phi^*_{M_i})$ and $a_i \in A_i$.

In each Nash equilibrium of G_{id}, any player $i \in N$ must send optimal messages m in the sense that they induce an action profile $\sigma^*(m)$ that maximizes i's payoff function. This is stated in Lemma 1 which is a preliminary result for condition (1).

Lemma 1. *For each* $i \in N$*,* $\text{supp}(\phi^*_i) \subseteq \{m \in M : u_i(\sigma^*(m)) = \sup_{m' \in M} u_i(\sigma^*(m'))\}$.

Proof. Suppose not; then there is $i \in N$, $m' \in \text{supp}(\phi^*_i)$ and $m^* \in M$ such that $u_i(\sigma^*(m^*)) > u_i(\sigma^*(m'))$. Define ϕ'_i by setting, for each $m \in \text{supp}(\phi^*_i)$,

$$\phi'_i[m] = \begin{cases} 0 & \text{if } m = m', \\ \phi^*_i[m^*] + \phi^*_i[m'] & \text{if } m = m^*, \\ \phi^*_i[m] & \text{otherwise,} \end{cases}$$

and let $\zeta : M_i \to \Delta(A_i)$ be such that $\zeta(m_i) = \sigma^*_i(m_i)$ for each $m_i \in M_i$. Then

$$\begin{aligned} &\sum_m (\phi'_i, \phi^*_j)[m] u_i(\zeta(m_i), \sigma^*_j(m_j)) - \sum_m \phi^*[m] u_i(\sigma^*(m)) \\ &= \sum_m (\phi'_i, \phi^*_j)[m] u_i(\sigma^*(m)) - \sum_m \phi^*[m] u_i(\sigma^*(m)) \\ &= \sum_m \beta_i \left(\phi'_i[m] - \phi^*_i[m]\right) u_i(\sigma^*(m)) \\ &= \beta_i \phi^*_i[m'] \Big(u_i(\sigma^*(m^*)) - u_i(\sigma^*(m')) \Big) > 0. \end{aligned}$$

But this contradicts (3). ■

The conclusion of Lemma 1 can be strengthened: for a message m to be optimal, $u_i(\sigma^*(m))$ must achieve $\max_{m'_j} v_i(\sigma^*_j(m'_j))$ and, thus, $\sigma^*_i(m_i)$ must be a best-reply to $\sigma^*_j(m_j)$.

Lemma 2. *For each* $i, j \in N$ *with* $i \neq j$*,*

$$\text{supp}(\phi^*_i) \subseteq \{m \in M : v_i(\sigma^*_j(m_j)) = \sup_{m'_j \in M_j} v_i(\sigma^*_j(m'_j)) \text{ and } \sigma^*_i(m_i) \in BR_i(\sigma^*_j(m_j))\}.$$

Proof. Suppose not; then there is $i \in N$, $j \neq i$, $m' \in \text{supp}(\phi^*_i)$ and $m^* \in M$ such that (i) $v_i(\sigma^*_j(m^*_j)) > v_i(\sigma^*_j(m'_j))$ or (ii) $v_i(\sigma^*_j(m'_j)) = \sup_{\hat{m}_j \in M_j} v_i(\sigma^*_j(\hat{m}_j))$ and $\sigma^*_i(m'_i) \notin BR_i(\sigma^*_j(m'_j))$; in case (ii), let $m^* = m'$. Let $a^*_i \in BR_i(\sigma^*_j(m^*_j))$,

$\bar{m}_i \notin \text{supp}(\phi^*_{M_i})$, $\phi'_i = 1_{(\bar{m}_i, m^*_j)}$ and $\zeta : M_i \to \Delta(A_i)$ be such that $\zeta(\bar{m}_i) = a^*_i$ and $\zeta(m_i) = \sigma^*_i(m_i)$ for each $m_i \neq \bar{m}_i$. Then

$$\begin{aligned}
&\sum_m (\phi'_i, \phi^*_j)[m] u_i(\zeta(m_i), \sigma^*_j(m_j)) - \sum_m \phi^*[m] u_i(\sigma^*(m)) \\
&= \sum_m \beta_i \phi'_i[m] u_i(\zeta(m_i), \sigma^*_j(m_j)) - \sum_m \beta_i \phi^*_i[m] u_i(\sigma^*(m)) \\
&= \beta_i \Big(u_i(a^*_i, \sigma^*_j(m^*_j)) - \sum_{m \in \text{supp}(\phi^*_i)} \phi^*_i[m] u_i(\sigma^*(m)) \Big) \\
&= \beta_i \Big(v_i(\sigma^*_j(m^*_j)) - u_i(\sigma^*(m')) \Big)
\end{aligned}$$

because $u_i(\sigma^*(m)) = u_i(\sigma^*(m'))$ for each $m \in \text{supp}(\phi^*_i)$ by Lemma 1 as $m' \in \text{supp}(\phi^*_i)$. Thus, if $v_i(\sigma^*_j(m^*_j)) > v_i(\sigma^*_j(m'_j))$, then

$$v_i(\sigma^*_j(m^*_j)) - u_i(\sigma^*(m')) \geq v_i(\sigma^*_j(m^*_j)) - v_i(\sigma^*_j(m'_j)) > 0;$$

if $v_i(\sigma^*_j(m^*_j)) = v_i(\sigma^*_j(m'_j))$, then $\sigma^*_i(m'_i) \notin BR_i(\sigma^*_j(m'_j))$ and

$$v_i(\sigma^*_j(m^*_j)) - u_i(\sigma^*(m')) > v_i(\sigma^*_j(m^*_j)) - v_i(\sigma^*_j(m'_j)) \geq 0.$$

It then follows that $\sum_m (\phi'_i, \phi^*_j)[m] u_i(\zeta(m_i), \sigma^*_j(m_j)) - \sum_m \phi^*[m] u_i(\sigma^*(m)) > 0$ in either case. But this contradicts (3). ∎

Lemma 2 implies that $\sigma^*_i(m_i)$ is a best-reply against $\sigma^*_j(m_j)$ whenever $m \in \text{supp}(\phi^*_i)$ and $i, j \in N$ with $i \neq j$. We will now show that if, in addition, $m_i \in \text{supp}(\phi^*_{j,M_i})$, then $\sigma^*_i(m_i)$ solves

$$\max_{\alpha_i \in \Delta(A_i)} \sum_{m_j} \frac{\phi^*_j[m_i, m_j]}{\phi^*_{j,M_i}[m_i]} u_i(\alpha_i, \sigma^*_j(m_j)).$$

Thus, whenever $m_i \in \text{supp}(\phi^*_{i,M_i}) \cap \text{supp}(\phi^*_{j,M_i})$, $\sigma^*_i(m_i)$ solves player i's expected payoff conditional on his information structure ϕ^*_i being chosen and also conditional on it not being chosen.

Lemma 3. *For each $i, j \in N$ with $i \neq j$,*

$$\text{supp}(\phi^*_i) \subseteq \Big\{ m \in M : m_i \notin \text{supp}(\phi^*_{j,M_i}) \text{ or } \sigma^*_i(m_i) \text{ solves } \max_{\alpha_i \in \Delta(A_i)} \sum_{m_j} \frac{\phi^*_j[m_i, m_j]}{\phi^*_{j,M_i}[m_i]} u_i(\alpha_i, \sigma^*_j(m_j)) \Big\}.$$

Proof. Suppose not; then there is $i \in N$ and $m' \in \text{supp}(\phi_i^*)$ such that $m'_i \in \text{supp}(\phi_{j,M_i}^*)$, $j \neq i$, and $\sigma_i^*(m'_i)$ does not solve

$$\max_{\alpha_i \in \Delta(A_i)} \sum_{m_j} \frac{\phi_j^*[m'_i, m_j]}{\phi_{j,M_i}^*[m'_i]} u_i(\alpha_i, \sigma_j^*(m_j)). \tag{5}$$

Let a_i^* be a solution to problem (5), $\bar{m}_i \notin \text{supp}(\phi_{M_i}^*)$, $\phi'_i = 1_{(\bar{m}_i, m'_j)}$ and $\zeta : M_i \to \Delta(A_i)$ be such that

$$\zeta(m_i) = \begin{cases} a_i^* & \text{if } m_i = m'_i, \\ \sigma_i^*(m'_i) & \text{if } m_i = \bar{m}_i, \\ \sigma_i^*(m_i) & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} &\sum_m (\phi'_i, \phi_j^*)[m] u_i(\zeta(m_i), \sigma_j^*(m_j)) - \sum_m \phi^*[m] u_i(\sigma^*(m)) \\ &= \beta_i \Big(u_i(\sigma^*(m')) - \sum_{m \in \text{supp}(\phi_i^*)} \phi_i^*[m] u_i(\sigma^*(m)) \Big) \\ &+ \beta_j \sum_{m_j} \phi_j^*[m'_i, m_j] \Big(u_i(a_i^*, \sigma_j^*(m_j)) - u_i(\sigma_i^*(m'_i), \sigma_j^*(m_j)) \Big) \\ &= \beta_j \sum_{m_j} \phi_j^*[m'_i, m_j] \Big(u_i(a_i^*, \sigma_j^*(m_j)) - u_i(\sigma_i^*(m'_i), \sigma_j^*(m_j)) \Big) \end{aligned}$$

where the last equality follows by Lemma 1 since $m' \in \text{supp}(\phi_i^*)$. Since $\sigma_i^*(m'_i)$ does not solve problem (5) but a_i^* does, it follows that

$$\sum_{m_j} \frac{\phi_j^*[m'_i, m_j]}{\phi_{j,M_i}^*[m'_i]} \Big(u_i(a_i^*, \sigma_j^*(m_j)) - u_i(\sigma_i^*(m'_i), \sigma_j^*(m_j)) \Big) > 0$$

and, since $m'_i \in \text{supp}(\phi_{j,M_i}^*)$,

$$\sum_{m_j} \phi_j^*[m'_i, m_j] \Big(u_i(a_i^*, \sigma_j^*(m_j)) - u_i(\sigma_i^*(m'_i), \sigma_j^*(m_j)) \Big) > 0.$$

Hence, $\sum_m (\phi'_i, \phi_j^*)[m] u_i(\zeta(m_i), \sigma_j^*(m_j)) - \sum_m \phi^*[m] u_i(\sigma^*(m)) > 0$. But this contradicts (3). ■

It follows by Lemmas 2 and 3 that, for each Nash equilibrium outcome, $i, j \in N$, $i \neq j$, and $m \in \text{supp}(\phi_i^*)$, condition (1) in Theorem 1 holds and $\sigma_i^*(m_i)$ solves

$$\max_{\alpha_i \in \Delta(A_i)} \sum_{m_j} \frac{\phi_j^*[m_i, m_j]}{\phi_{j,M_i}^*[m_i]} u_i(\alpha_i, \sigma_j^*(m_j))$$

whenever $m_i \in \text{supp}(\phi^*_{j,M_i})$ and, hence, $m_i \in \text{supp}(\phi^*_{i,M_i}) \cap \text{supp}(\phi^*_{j,M_i})$. In fact, regarding (1), note that $m \in \text{supp}(\phi^*_i)$ implies that $m_j \in \text{supp}(\phi^*_{M_j}) = M^*_j$. Hence,

$$v_i(\sigma^*_j(m_j)) \leq \max_{m'_j \in M^*_j} v_i(\sigma^*_j(m'_j)) \leq \sup_{m'_j \in M_j} v_i(\sigma^*_j(m'_j)) = v_i(\sigma^*_j(m_j)).$$

Condition (4) implies that, for each $i \in N$, $\sigma^*_i(m_i)$ solves

$$\max_{\alpha_i \in \Delta(A_i)} \sum_{m_j} \frac{\phi^*_j[m_i, m_j]}{\phi^*_{j,M_i}[m_i]} u_i(\alpha_i, \sigma^*_j(m_j))$$

whenever $m_i \in \text{supp}(\phi^*_{j,M_i}) \setminus \text{supp}(\phi^*_{i,M_i})$. This, together with what has been shown in the previous paragraph, shows that condition (2) in Theorem 1 holds.

A.2.2 Proof that condition 3 implies condition 2

Let $\left(\phi^*_i, (\sigma^*_i(m_i))_{m_i \in M^*_i}\right)_{i \in N}$ be such that conditions (1) and (2) in Theorem 1 hold; we will show that it is the outcome of a sequential equilibrium.

We will construct a sequential equilibrium π with the desired outcome. Let $i \in N$ and $j \neq i$. Set $\pi^1_i = \phi^*_i$ and $\pi^2_i(m_i, \phi^*_i) = \sigma^*_i(m_i)$ for each $m_i \in M^*_i$ since the goal is to define a strategy with outcome $\left(\phi^*_i, (\sigma^*_i(m_i))_{m_i \in M^*_i}\right)_{i \in N}$.

We will specify the remaining values of π^2_i as follows. Let

$$\bar{m}_i \in \text{supp}(\phi^*_{i,M_i}).$$

Informally, we will define $\{\pi^\alpha, p^\alpha\}_\alpha$ such that player i, after choosing ϕ_i and receiving m_i, believes that $\phi_j = \phi^*_j$ and that m_j occurs with probability $\frac{\phi^*_i[\bar{m}_i, m_j]}{\phi^*_{i,M_i}[\bar{m}_i]}$. In this case, we set player i's action to be $\sigma^*_i(\bar{m}_i)$, which is a best-reply against the action $\sigma^*_j(m_j)$ of player j for each m_j such that $\phi^*_i[\bar{m}_i, m_j] > 0$.

The above belief is only possible when $\beta_i \phi_{i,M_i}[m_i] + \beta_i \phi^*_{j,M_i}[m_i] = 0$ since otherwise, player i has to assign probability $\frac{\beta_i \phi_i[m_i, m_j] + \beta_j \phi^*_j[m_i, m_j]}{\beta_i \phi_{i,M_i}[m_i] + \beta_j \phi^*_{j,M_i}[m_i]}$ to (m_j, ϕ^*_j). In this case, we specify player i's action to be a best-reply against the expected action of player j.

The formal details are as follows. For each $m_i \in M_i$ and $\phi_i \in S$ such that $\beta_i \phi_{i,M_i}[m_i] + \beta_j \phi^*_{j,M_i}[m_i] = 0$, let $\pi^2_i(m_i, \phi_i) = \sigma^*_i(\bar{m}_i)$. In particular, $\pi^2_i(m_i, \phi^*_i) = \sigma^*_i(\bar{m}_i)$ if $m_i \notin M^*_i$ since then $\beta_i \phi^*_{i,M_i}[m_i] + \beta_j \phi^*_{j,M_i}[m_i] = 0$. Thus, $\sigma^*_i(m_i) = \pi^2_i(m_i, \phi^*_i)$ is defined for each $i \in N$ and $m_i \in M_i$.

For each $m_i \in M_i$ and $\phi_i \neq \phi_i^*$ such that $\beta_i \phi_{i,M_i}[m_i] + \beta_j \phi_{j,M_i}^*[m_i] > 0$, let $\pi_i^2(m_i, \phi_i)$ be a best-reply against

$$\sum_{m_j} \frac{\beta_i \phi_i[m_i, m_j] + \beta_j \phi_j^*[m_i, m_j]}{\beta_i \phi_{i,M_i}[m_i] + \beta_j \phi_{j,M_i}^*[m_i]} \sigma_j^*(m_j).$$

We may assume that $\pi_i^2 : M_i \times S \to \Delta(A_i)$ is measurable. Note first that $M_i \times S = \cup_{r=1}^3 B_r$ with

$$B_1 = \{(m_i, \phi_i) : \phi_i = \phi_i^*\},$$

$$B_2 = \{(m_i, \phi_i) : \phi_i \neq \phi_i^* \text{ and } \beta_i \phi_{i,M_i}[m_i] + \beta_j \phi_{j,M_i}^*[m_i] = 0\} \text{ and}$$

$$B_3 = \{(m_i, \phi_i) : \phi_i \neq \phi_i^* \text{ and } \beta_i \phi_{i,M_i}[m_i] + \beta_j \phi_{j,M_i}^*[m_i] > 0\}.$$

For each $r \in \{1, 2, 3\}$, B_r is measurable. Indeed, B_1 is closed, B_2 is the intersection of an open set, $\{(m_i, \phi_i) : \phi_i \neq \phi_i^*\}$, with a closed set, $\{(m_i, \phi_i) : \beta_i \phi_{i,M_i}[m_i] + \beta_j \phi_{j,M_i}^*[m_i] = 0\}$, and B_3 is open. Then, for each measurable $B \subseteq \Delta(A_i)$, $(\pi_i^2)^{-1}(B) \cap B_1$ is measurable since $(\pi_i^2)^{-1}(B) \cap B_1$ is countable. Regarding $(\pi_i^2)^{-1}(B) \cap B_3$: Let $f : M_i \times S \to \Delta(A_j)$ be defined by setting, for each $(m_i, \phi_i) \in B_3$, $f(m_i, \phi_i) = \sum_{m_j} \frac{\beta_i \phi_i[m_i, m_j] + \beta_j \phi_j^*[m_i, m_j]}{\beta_i \phi_{i,M_i}[m_i] + \beta_j \phi_{j,M_i}^*[m_i]} \sigma_j^*(m_j)$. Letting $BR_i : \Delta(A_j) \rightrightarrows \Delta(A_i)$ be player i's best-reply correspondence in G, define $\Psi : M_i \times S \rightrightarrows \Delta(A_i)$ by setting, for each $(m_i, \phi_i) \in B_3$, $\Psi(m_i, \phi_i) = BR_i(f(m_i, \phi_i))$. Since $\Delta(A_i)$ is compact, f is continuous and BR_i is upper hemicontinuous, it follows that Ψ is upper hemicontinuous and, hence, measurable (and, thus, weakly measurable). Hence, Ψ has a measurable selection by the Kuratowski-Ryll-Nardzewski Selection Theorem (e.g. Aliprantis and Border (2006, Theorem 18.13, p. 600)). Finally, for each measurable $B \subseteq \Delta(A_i)$, $(\pi_i^2)^{-1}(B) = B_2$ if $\sigma_i^*(\bar{m}_i) \in B$ and $(\pi_i^2)^{-1}(B) \cap B_2 = \emptyset$ otherwise; thus $(\pi_i^2)^{-1}(B) \cap B_2$ is measurable.

We define $\{\pi^\alpha, p^\alpha\}_\alpha$ as follows. The index set consists of $\alpha = (k, F, \hat{F})$ such that $k \in \mathbb{N}$, F is a finite subset of $\mathbb{N}$ and $\hat{F}$ is a finite subset of S; this set is partially ordered by defining $(k', F', \hat{F}') \geq (k, F, \hat{F})$ if $k' \geq k$, $F \subseteq F'$ and $\hat{F} \subseteq \hat{F}'$. For each $m_j \in M_j$, let

$$\bar{q}_i[m_j] = \frac{\phi_i^*[\bar{m}_i, m_j]}{\phi_{i,M_i}^*[\bar{m}_i]}$$

be the ϕ_i^*-probability of m_j conditional on $\bar{m}_i$. For each $\alpha = (k, F, \hat{F})$, we define $p^\alpha(\phi)$ such that the probability distribution of message profiles is $\beta_1 \phi_1 + \beta_2 \phi_2$ with probability $1 - k^{-1}$; with probability k^{-3}, it equals a probability distribution τ^α that assigns strictly positive probability to each message in F and in the support of information structures in $\hat{F}$; and, with the remaining probability of $k^{-1}(1 - k^{-2})$, it equals a probability distribution q^α such that the probability of m_j conditional on m_i equals $\bar{q}_i[m_j]$. Let

$$\begin{aligned}
\tau_i^\alpha &= \frac{\sum_{l \in F \cup \left(\cup_{\phi \in \hat{F}} \operatorname{supp}(\phi_{M_i})\right)} 2^{-l} 1_l}{\sum_{l \in F \cup \left(\cup_{\phi \in \hat{F}} \operatorname{supp}(\phi_{M_i})\right)} 2^{-l}}, \\
q_i^\alpha &= \tau_i^\alpha \times \bar{q}_i, \\
\tau^\alpha &= \tau_1^\alpha \times \tau_2^\alpha, \\
q^\alpha &= (q_1^\alpha + q_2^\alpha)/2, \\
\mu^\alpha &= (1 - k^{-2}) q^\alpha + k^{-2} \tau^\alpha, \text{ and} \\
p^\alpha(\phi) &= (1 - k^{-1})(\beta_1 \phi_1 + \beta_2 \phi_2) + k^{-1} \mu^\alpha.
\end{aligned}$$

Furthermore, let $\upsilon_X \in \Delta(X)$ be uniform on X whenever X is a finite set and let

$$\pi_i^{1,\alpha} = (1 - k^{-3}) 1_{\phi_i^*} + k^{-3} \upsilon_{\hat{F}} \text{ and } \pi_i^{2,\alpha}(m_i, \phi_i) = (1 - k^{-1}) \pi_i^2(m_i, \phi_i) + k^{-1} \upsilon_{A_i}$$

for each $(m_i, \phi_i) \in M_i \times S$. Thus, all information structures in $\hat{F}$ and all actions are chosen with strictly positive probability. Furthermore, the probability of an information structure different from ϕ_i^* is much smaller than the probability of q^α (i.e. their ratio is $k^{-3}/k^{-1}(1 - k^{-2})$ and goes to zero), which implies that if player i receives a message that is neither in the support of the information structure he chose nor in the support of ϕ_j^*, then player i believes that this happened because the message was drawn from q^α and not because player j chose an information structure different from ϕ_j^*.

Let $\varepsilon > 0$. We have that the conditions (i)–(v) in the definition of perfect conditional ε-equilibrium hold by construction. We will show that condition (vi) holds for some subnet of $\{\pi^\alpha, p^\alpha\}_\alpha$. Some technical details of this argument are simplified by our construction of $\{\pi^\alpha, p^\alpha\}_\alpha$ which is such that $\operatorname{supp}(\pi^{1,\alpha})$ and $\operatorname{supp}(p^\alpha)$ are finite for each α. We define

$$S_i(F,\hat{F}) = \Big(\Big(F \cup (\cup_{\phi\in\hat{F}}\text{supp}(\phi_{M_i})) \cup (\text{supp}(\phi^*_{j,M_i}))\Big) \times \hat{F}\Big)$$
$$\cup \Big(\Big(F \cup (\cup_{\phi\in\hat{F}}\text{supp}(\phi_{M_i})) \cup (\text{supp}(\phi^*_{M_i}))\Big) \times \{\phi^*_i\}\Big)$$

which is the set of pairs (m_i,ϕ_i) that can occur with strictly positive probability. Indeed, if $(m,\phi) \in \mathbb{N}^2 \times S^2$ is such that $\pi^{1,\alpha}[\phi] > 0$ and $\sum_{\phi'\in\text{supp}(\pi^{1,\alpha})} p^\alpha(\phi')[m] > 0$, then $(m_i,\phi_i) \in S_i(F,\hat{F})$ for each $i \in N$.

Recall that $\alpha = (k,F,\hat{F})$. In what follows, we will often fix F and $\hat{F}$ and take limits as $k \to \infty$. Regarding condition (vi) (a), let $i,j \in N$, $j \neq i$ and $\phi'_i \in S$. We have that, for each finite subsets F and $\hat{F}$ of $\mathbb{N}$ and S, respectively,

$$\lim_k \sum_{\phi\in\text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi]\left(\sum_m p^\alpha(\phi)[m]u_i(\pi^{2,\alpha}(m,\phi))\right) = \sum_m \phi^*[m]u_i(\sigma^*(m))$$

and that

$$\lim_k \sum_{\phi\in\text{supp}(1_{\phi'_i}\times\pi_j^{1,\alpha})} (1_{\phi'_i}\times\pi_j^{1,\alpha})[\phi]\left(\sum_m p^\alpha(\phi)[m]u_i(\pi^{2,\alpha}(m,\phi))\right) =$$
$$\sum_m (\phi'_i,\phi^*_j)[m]u_i(\pi^2_i(m_i,\phi'_i),\sigma^*_j(m_j)).$$

Hence, by considering α such that $k \geq k_0$ for some $k_0 \in \mathbb{N}$, it is enough to show that

$$\sum_m \phi^*[m]u_i(\sigma^*(m)) \geq \sum_m (\phi'_i,\phi^*_j)[m]u_i(\pi^2_i(m_i,\phi'_i),\sigma^*_j(m_j)),$$

which is equivalent to

$$\sum_m \phi^*_i[m]u_i(\sigma^*(m)) \geq \sum_m \phi'_i[m]u_i(\pi^2_i(m_i,\phi'_i),\sigma^*_j(m_j)). \tag{6}$$

For each $m_j \in M_j$, $\sigma^*_j(m_j) \in \{\sigma^*_j(m'_j) : m'_j \in M^*_j\}$ since $\sigma^*_j(m_j) = \sigma^*_j(\bar{m}_j)$ whenever $m_j \notin M^*_j$. Thus, by (1),

$$\sum_m \phi'_i[m]u_i(\pi^2_i(m_i,\phi'_i),\sigma^*_j(m_j)) \leq \sum_m \phi'_i[m]v_i(\sigma^*_j(m_j))$$
$$\leq \max_{m_j\in M^*_j} v_i(\sigma^*_j(m_j)) = \sum_m \phi^*_i[m]u_i(\sigma^*(m))$$

and, hence, (6) holds. It then follows that condition (vi) (a) also holds.

Consider next condition (vi) (b). For each $i, j \in N$, $i \neq j$, finite subset F of $\mathbb{N}$, finite subset $\hat{F}$ of S, $(m_i, \phi_i) \in S_i(F, \hat{F})$ and $\gamma_i \in \Delta(A_i)$, we have that

$$\lim_k \frac{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] \left(\sum_{m_j} p^\alpha(\phi_i, \phi_j)[m_i, m_j] u_i(\gamma_i, \pi_j^{2,\alpha}(m_j, \phi_j)) \right)}{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] p^\alpha_{M_i}(\phi_i, \phi_j)[m_i]}$$
$$= \sum_{m_j} \frac{\phi_i^*[\bar{m}_i, m_j]}{\phi^*_{i,M_i}[\bar{m}_i]} u_i(\gamma_i, \sigma_j^*(m_j))$$

if $\beta_i \phi_{i,M_i}[m_i] + \beta_j \phi^*_{j,M_i}[m_i] = 0$, and

$$\lim_k \frac{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] \left(\sum_{m_j} p^\alpha(\phi_i, \phi_j)[m_i, m_j] u_i(\gamma_i, \pi_j^{2,\alpha}(m_j, \phi_j)) \right)}{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] p^\alpha_{M_i}(\phi_i, \phi_j)[m_i]} =$$
$$\sum_{m_j} \frac{\beta_i \phi_i[m_i, m_j] + \beta_j \phi_j^*[m_i, m_j]}{\beta_i \phi_{i,M_i}[m_i] + \beta_j \phi^*_{j,M_i}[m_i]} u_i(\gamma_i, \sigma_j^*(m_j))$$

if $\beta_i \phi_{i,M_i}[m_i] + \beta_j \phi^*_{j,M_i}[m_i] > 0$. The latter case is clear since all terms in the denominator of the fraction converge to zero except the one that converges to $\beta_i \phi_{i,M_i}[m_i] + \beta_j \phi^*_{j,M_i}[m_i]$ and similarly regarding the numerator.

In the former case, both the numerator and the denominator converge to zero since $\beta_i \phi_{i,M_i}[m_i] + \beta_j \phi^*_{j,M_i}[m_i] = 0$. Multiplying each by k, it follows that all terms converge to zero except the ones corresponding to the case where $\pi_j^{1,\alpha} = \phi_j^*$ and $p^\alpha(\phi_i, \phi_j^*) = q^\alpha$. Furthermore, for each $m_j \in M_j$,

$$q^\alpha[m_i, m_j] = 2^{-1}(q_i^\alpha[m_i, m_j] + q_j^\alpha[m_i, m_j]),$$
$$q_i^\alpha[m_i, m_j] = \tau_i^\alpha[m_i] \bar{q}_i[m_j] \text{ and}$$
$$q_j^\alpha[m_i, m_j] = 0,$$

the latter since $m_i \notin \text{supp}(\phi^*_{j,M_i})$. Hence, $q^\alpha[m_i, m_j] = 2^{-1} \tau_i^\alpha[m_i] \bar{q}_i[m_j]$ and $q^\alpha_{M_i}[m_i] = 2^{-1} \tau_i^\alpha[m_i]$. Thus,

$$\frac{q^\alpha[m_i, m_j]}{q^\alpha_{M_i}[m_i]} = \bar{q}_i[m_j] = \frac{\phi_i^*[\bar{m}_i, m_j]}{\phi^*_{i,M_i}[\bar{m}_i]}.$$

We will next show that $\pi_i^2(m_i, \phi_i)$ solves

$$\max_{\gamma_i \in \Delta(A_i)} \lim_k \frac{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] \left(\sum_{m_j} p^\alpha(\phi_i, \phi_j)[m_i, m_j] u_i(\gamma_i, \pi_j^{2,\alpha}(m_j, \phi_j)) \right)}{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] p^\alpha_{M_i}(\phi_i, \phi_j)[m_i]} \quad (7)$$

for each $i, j \in N$, $j \neq i$ and $(m_i, \phi_i) \in S_i(F, \hat{F})$.

If $\beta_i \phi_{i,M_i}[m_i] + \beta_j \phi^*_{j,M_i}[m_i] = 0$, then

$$\lim_k \frac{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] \left(\sum_{m_j} p^\alpha(\phi_i, \phi_j)[m_i, m_j] u_i(\gamma_i, \pi_j^{2,\alpha}(m_j, \phi_j)) \right)}{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] p^\alpha_{M_i}(\phi_i, \phi_j)[m_i]}$$
$$= \sum_{m_j} \frac{\phi_i^*[\bar{m}_i, m_j]}{\phi^*_{i,M_i}[\bar{m}_i]} u_i(\gamma_i, \sigma_j^*(m_j)).$$

Since $\pi_i^2(m_i, \phi_i) = \sigma_i^*(\bar{m}_i)$ and $\sigma_i^*(\bar{m}_i) \in BR_i(\sigma_j^*(m_j))$ for each $m_j \in M_j$ such that $(\bar{m}_i, m_j) \in \text{supp}(\phi_i^*)$ by (1), it follows that (7) holds in this case.

If $\beta_i \phi_{i,M_i}[m_i] + \beta_j \phi^*_{j,M_i}[m_i] > 0$ and $\phi_i \neq \phi_i^*$, then

$$\lim_k \frac{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] \left(\sum_{m_j} p^\alpha(\phi_i, \phi_j)[m_i, m_j] u_i(\gamma_i, \pi_j^{2,\alpha}(m_j, \phi_j)) \right)}{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] p^\alpha_{M_i}(\phi_i, \phi_j)[m_i]}$$
$$= \sum_{m_j} \frac{\beta_i \phi_i[m_i, m_j] + \beta_j \phi_j^*[m_i, m_j]}{\beta_i \phi_{i,M_i}[m_i] + \beta_j \phi^*_{j,M_i}[m_i]} u_i(\gamma_i, \sigma_j^*(m_j))$$
$$= u_i \left(\gamma_i, \sum_{m_j} \frac{\beta_i \phi_i[m_i, m_j] + \beta_j \phi_j^*[m_i, m_j]}{\beta_i \phi_{i,M_i}[m_i] + \beta_j \phi^*_{j,M_i}[m_i]} \sigma_j^*(m_j) \right).$$

Since $\pi_i^2(m_i, \phi_i)$ is optimal against $\sum_{m_j} \frac{\beta_i \phi_i[m_i, m_j] + \beta_j \phi_j^*[m_i, m_j]}{\beta_i \phi_{i,M_i}[m_i] + \beta_j \phi^*_{j,M_i}[m_i]} \sigma_j^*(m_j)$, it follows that (7) holds in this case.

Finally, consider the case where $\phi_i = \phi_i^*$ and $\beta_i \phi_{i,M_i}[m_i] + \beta_j \phi^*_{j,M_i}[m_i] > 0$. Note that it is enough to show that

$$\sum_{m_j} \phi^*[m] \big(u_i(\sigma^*(m)) - u_i(a_i, \sigma_j^*(m_j)) \big) \geq 0 \tag{8}$$

for each $a_i \in A_i$ and that

$$\sum_{m_j} \phi^*[m] \big(u_i(\sigma^*(m)) - u_i(a_i, \sigma_j^*(m_j)) \big) =$$
$$\sum_{m_j} \beta_i \phi_i^*[m] \big(u_i(\sigma^*(m)) - u_i(a_i, \sigma_j^*(m_j)) \big) + \sum_{m_j} \beta_j \phi_j^*[m] \big(u_i(\sigma^*(m)) - u_i(a_i, \sigma_j^*(m_j)) \big).$$

We have that $u_i(\sigma^*(m)) \geq u_i(a_i, \sigma_j^*(m_j))$ for each m_j such that $\phi_i^*[m] > 0$ by (1); moreover, for each m_j such that $\phi_j^*[m] > 0$, then $m_i \in \text{supp}(\phi^*_{j,M_i})$ and, hence,

$\sum_{m_j} \beta_j \phi_j^*[m]\big(u_i(\sigma^*(m)) - u_i(a_i, \sigma_j^*(m_j))\big) \geq 0$ by (2). Thus, (8), and hence (7), holds.

The above arguments show that, for each finite subsets F of $\mathbb{N}$ and $\hat{F}$ of S, condition (vi) holds whenever k is sufficiently large. Specifically, condition (vi) (a) holds for each $i \in N$ whenever $k \geq k_0$. For each $i \in N$ and $(m_i, \phi_i) \in S_i(F, \hat{F})$, there is $k(m_i, \phi_i)$ such that condition (vi) (b) holds whenever $k \geq k(m_i, \phi_i)$. Thus, let

$$k(F, \hat{F}) = \max \left\{ k_0, \max_{i \in N} \max_{(m_i, \phi_i) \in S_i(F, \hat{F})} k(m_i, \phi_i) \right\}.$$

Since condition (vi) (b) is trivially satisfied when

$$\pi_i^{1,\alpha}[\phi_i] \sum_{\phi_j \in \mathrm{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] p^\alpha_{M_i}(\phi_i, \phi_j)[m_i] = 0,$$

i.e. when $i \in N$ and $(m_i, \phi_i) \notin S_i(F, \hat{F})$, it follows that condition (vi) holds whenever $k \geq k(F, \hat{F})$. This allows us to define the following subnet $\{\pi^{\varphi(\eta)}, p^{\varphi(\eta)}\}_\eta$ of $\{\pi^\alpha, p^\alpha\}_\alpha$ such that condition (vi) holds.

The index set of the subnet $\{\pi^{\varphi(\eta)}, p^{\varphi(\eta)}\}_\eta$ is the same as the one in the net $\{\pi^\alpha, p^\alpha\}_\alpha$. The function $\varphi : \eta \mapsto \alpha$ is defined by setting, for each $\eta = (k, F, \hat{F})$,

$$\varphi(\eta) = \left(\max\left\{k, k(F, \hat{F})\right\}, F, \hat{F}\right).$$

It is then clear that condition (vi) holds and that, as required by the definition of a subnet, for each α_0, there exists η_0, e.g. $\eta_0 = \alpha_0$, such that $\varphi(\eta) \geq \alpha_0$ for each $\eta \geq \eta_0$.

A.3 Proof of Corollary 1

We first show that $A(G) \subseteq \mathcal{A}$. Let $\pi \in \Pi$ be a sequential equilibrium of G_{id} and $\sigma = \sigma_\pi$ be its action distribution. Then $\big(\phi_i^*, (\sigma_i^*(m_i))_{m_i \in M_i^*}\big)_{i \in N}$ satisfies the conditions of Theorem 1. Define $\phi_i \in \Delta(\Delta(A_1) \times \Delta(A_2))$ such that $\phi_i[\alpha] = \sum_{m : \sigma^*(m) = \alpha} \phi_i^*[m]$ and note that ϕ_i has finite support. Then

$$\sigma = \sum_m \phi^*[m](\sigma_1^*(m) \times \sigma_2^*(m)) = \sum_{i,\alpha} \beta_i \phi_i[\alpha](\alpha_1 \times \alpha_2).$$

Claim 1. *For each $i \in N$ and $\alpha_j \in \mathrm{supp}(\phi_{i,\Delta(A_j)})$,*

$$\left(\sum_{\alpha_i \in \mathrm{supp}(\phi_{i,\Delta(A_i)})} \frac{\phi_i[\alpha_i, \alpha_j]}{\phi_{i,\Delta(A_j)}[\alpha_j]} \alpha_i, \alpha_j \right)$$

is a Nash equilibrium of G and

$$u_i(\alpha_i, \alpha_j) = \max_{\alpha' \in \cup_l \mathrm{supp}(\phi_l)} u_i(\alpha')$$

for each $\alpha_i \in \Delta(A_i)$ such that $\phi_i[\alpha_i, \alpha_j] > 0$.

Proof. Since $\alpha_j \in \mathrm{supp}(\phi_{i,\Delta(A_j)})$, there exists $m_j \in \mathrm{supp}(\phi^*_{i,M_j})$ such that $\alpha_j = \sigma^*_j(m_j)$. Let $\hat{M}_j$ be the set of such m_j and note that, for each $a_i \in A_i$,

$$\sum_{m_i} \sum_{m_j \in \hat{M}_j} \phi^*_i[m_i, m_j] \sigma^*_i(m_i)[a_i] = \sum_{\alpha_i} \phi_i[\alpha_i, \alpha_j] \alpha_i[a_i].$$

Thus, it follows that $\sum_{m_i} \frac{\phi^*_i[\{m_i\} \times \hat{M}_j]}{\phi^*_{i,M_j}[\hat{M}_j]} \sigma^*_i(m_i)[a_i] = \sum_{\alpha_i} \frac{\phi_i[\alpha_i, \alpha_j]}{\phi_{i,\Delta(A_j)}[\alpha_j]} \alpha_i[a_i]$. Since, for each $m_j \in \hat{M}_j$ and $\hat{\alpha}_j \in \Delta(A_j)$,

$$\sum_{m_i} \phi^*_i[m_i, m_j] u_j(\alpha_j, \sigma^*_i(m_i)) \geq \sum_{m_i} \phi^*_i[m_i, m_j] u_j(\hat{\alpha}_j, \sigma^*_i(m_i))$$

by (2), it follows that

$$\sum_{m_i} \sum_{m_j \in \hat{M}_j} \phi^*_i[m_i, m_j] u_j(\alpha_j, \sigma^*_i(m_i)) \geq \sum_{m_i} \sum_{m_j \in \hat{M}_j} \phi^*_i[m_i, m_j] u_j(\hat{\alpha}_j, \sigma^*_i(m_i))$$

and that α_j maximises $u_j(\cdot, \sum_{\alpha_i} \frac{\phi_i[\alpha_i, \alpha_j]}{\phi_{i,\Delta(A_j)}[\alpha_j]} \alpha_i)$.

For each $\alpha_i \in \Delta(A_i)$ such that $\phi_i[\alpha_i, \alpha_j] > 0$, there exists $m \in \mathrm{supp}(\phi^*_i)$ such that $\sigma^*(m) = (\alpha_i, \alpha_j)$. Furthermore, for each $\alpha' \in \cup_l \mathrm{supp}(\phi_l)$, there exists $m' \in M^*$ such that $\sigma^*(m') = \alpha'$. It then follows by (1) that $\alpha_i = \sigma^*_i(m_i) \in BR_i(\sigma^*_j(m_j)) = BR_i(\alpha_j)$ and

$$u_i(\alpha_i, \alpha_j) = v_i(\alpha_j) = v_i(\sigma^*_j(m_j)) \geq v_i(\sigma^*_j(m'_j)) \geq u_i(\sigma^*(m')) = u_i(\alpha').$$

Thus, $\sum_{\alpha_i \in \mathrm{supp}(\phi_{i,\Delta(A_i)})} \frac{\phi_i[\alpha_i, \alpha_j]}{\phi_{i,\Delta(A_j)}[\alpha_j]} \alpha_i \in BR_i(\alpha_j)$. □

Note that ϕ_i can be written as:

$$\phi_i = \sum_{\alpha_j} \phi_{i,\Delta(A_j)}[\alpha_j] \left(\left(\sum_{\alpha_i} \frac{\phi_i[\alpha_i, \alpha_j]}{\phi_{i,\Delta(A_j)}[\alpha_j]} 1_{\alpha_i} \right) \times 1_{\alpha_j} \right).$$

Hence,

$$\sigma = \sum_{i,\alpha} \beta_i \phi_i[\alpha](\alpha_1 \times \alpha_2) = \sum_i \beta_i \sum_{\alpha_j \in \text{supp}(\phi_{i,\Delta(A_j)})} \phi_{i,\Delta(A_j)}[\alpha_j](\sigma_1^{i,\alpha_j} \times \sigma_2^{i,\alpha_j}),$$

where, for each $\alpha_j \in \text{supp}(\phi_{i,\Delta(A_j)})$,

$$\sigma^{i,\alpha_j} = \left(\sum_{\alpha_i} \frac{\phi_i[\alpha_i, \alpha_j]}{\phi_{i,\Delta(A_j)}[\alpha_j]} \alpha_i, \alpha_j \right)$$

is a Nash equilibrium of G by Claim 1.

Then let $L_i = |\text{supp}(\phi_{i,\Delta(A_j)})|$ and, writing $\text{supp}(\phi_{i,\Delta(A_j)}) = \{\alpha_j^1, \ldots, \alpha_j^{L_i}\}$, let $\eta^{i,l} = \phi_{i,\Delta(A_j)}[\alpha_j^l]$ and $\sigma^{i,l} = \sigma^{i,\alpha_j^l}$ for each $l \in \{1, \ldots, L_i\}$.

For each $\alpha_j \in \text{supp}(\phi_{i,\Delta(A_j)})$, it follows by Claim 1 that

$$u_i(\sigma^{i,\alpha_j}) = \sum_{\alpha_i} \frac{\phi_i[\alpha_i, \alpha_j]}{\phi_{i,\Delta(A_j)}[\alpha_j]} u_i(\alpha_i, \alpha_j) = \max_{\hat{\alpha} \in \cup_l \text{supp}(\phi_l)} u_i(\hat{\alpha}).$$

Thus, $u_i(\sigma^{i,\alpha_j}) = u_i(\sigma^{i,\alpha'_j})$ for each $\alpha'_j \in \text{supp}(\phi_{i,\Delta(A_j)})$. Furthermore, for each $\alpha_i \in \text{supp}(\phi_{j,\Delta(A_i)})$,

$$u_i(\sigma^{j,\alpha_i}) = \sum_{\alpha_j} \frac{\phi_j[\alpha_i, \alpha_j]}{\phi_{j,\Delta(A_i)}[\alpha_i]} u_i(\alpha_i, \alpha_j) \leq \max_{\hat{\alpha} \in \cup_l \text{supp}(\phi_l)} u_i(\hat{\alpha});$$

thus, $u_i(\sigma^{i,\alpha_j}) \geq u_i(\sigma^{j,\alpha_i})$. This completes the proof that $A(G) \subseteq \mathcal{A}$.

We now show that $\mathcal{A} \subseteq A(G)$. Let $\sigma \in \mathcal{A}$, i.e. $\sigma = \sum_i \beta_i \sum_{l=1}^{L_i} \eta^{i,l}(\sigma_1^{i,l} \times \sigma_2^{i,l})$, $\sum_{l=1}^{L_i} \eta^{i,l} = 1$, $\eta^{i,l} \geq 0$, $\sigma^{i,l} \in N(G)$, $u_i(\sigma^{i,l}) = u_i(\sigma^{i,k}) \geq u_j(\sigma^{j,r})$ for each $i, j \in N$ with $i \neq j$, $l, k \in \{1, \ldots, L_i\}$ and $r \in \{1, \ldots, L_j\}$.

For each $i \in N$, $l \in \{1, \ldots, L_1\}$ and $k \in \{1, \ldots, L_2\}$, pick $m_i^{1,l}$ and $m_i^{2,k}$ in M_i such that $m_i^{1,l} \neq m_i^{1,r}$, $m_i^{2,k} \neq m_i^{2,s}$ and $m_i^{1,l} \neq m_i^{2,k}$ for each $l, r \in \{1, \ldots, L_1\}$ and $k, s \in \{1, \ldots, L_2\}$. Set $\phi_1^* = \sum_{l=1}^{L_1} \eta^{1,l} 1_{m^{1,l}}$, $\phi_2^* = \sum_{l=1}^{L_2} \eta^{2,l} 1_{m^{2,l}}$ and, for each $i \in N$, $j \in N$ and $l \in \{1, \ldots, L_j\}$, $\sigma_i^*(m_i^{j,l}) = \sigma_i^{j,l}$. It is then clear that σ is the action distribution of the outcome $\left(\phi_i^*, (\sigma_i^*(m_i))_{m_i \in M_i^*}\right)_{i \in N}$, i.e. $\sigma = \sum_{m \in M^*} \phi^*[m](\sigma_1^*(m) \times \sigma_2^*(m))$.

Fix $i \in N$ and let $j \neq i$. For each $m_j \in M_j^*$, $\sigma_j^*(m_j) = \sigma_j^{k,l}$ for some $k \in \{i, j\}$ and $l \in \{1, \ldots, L_k\}$. Since $v_i(\sigma_j^{i,l}) = u_i(\sigma^{i,l}) \geq u_i(\sigma^{j,r}) = v_i(\sigma_j^{j,r})$ for each $l \in \{1, \ldots, L_i\}$ and $r \in \{1, \ldots, L_j\}$, it follows that $\max_{m_j \in M_j^*} v_i(\sigma_j^*(m_j)) = u_i(\sigma^{i,l})$ for each $l \in \{1, \ldots, L_i\}$. Since, for each $m \in \text{supp}(\phi_i^*)$, there exists $l \in \{1, \ldots, L_i\}$ such that $\sigma_i^*(m_i) = \sigma_i^{i,l}$ and $\sigma_j^*(m_j) = \sigma_j^{i,l}$, it follows that $\text{supp}(\phi_i^*) \subseteq \{m \in M : v_i(\sigma_j^*(m_j)) = \max_{m_j' \in M_j^*} v_i(\sigma_j^*(m_j'))$ and $\sigma_i^*(m_i) \in BR_i(\sigma_j^*(m_j))\}$. Thus, (1) holds.

Moreover, for each $m_i \in \text{supp}(\phi_{j,M_i}^*)$, $m_i = m_i^{j,l}$ for some $l \in \{1, \ldots, L_j\}$ and hence $\sigma_i^*(m_i) = \sigma_i^{j,l}$ solves

$$\max_{\alpha_i \in \Delta(A_i)} \sum_{m_j} \frac{\phi_j^*[m_i, m_j]}{\phi_{j,M_i}^*[m_i]} u_i(\alpha_i, \sigma_j^*(m_j)) = \max_{\alpha_i \in \Delta(A_i)} u_i(\alpha_i, \sigma_j^{j,l}).$$

Thus, (2) holds. It then follows by Theorem 1 that $(\phi_i^*, (\sigma_i^*(m_i))_{m_i \in M_i^*})_{i \in N}$ is the outcome of a sequential equilibrium of G_{id} and, thus, that $\sigma \in A(G)$.

A.4 Proof of Corollary 3

The characterization of $U(G)$ follows from the definition of $\mathcal{G}$ and Corollary 2.

Standard results (e.g. Theorems 2.5.5 and 2.6.1 in van Damme (1991) and their proofs) imply that there is an open set O of $\mathbb{R}^{2|A|}$ such that its complement has Lebesgue measure zero and, for each $u \in O$, there is an open neighborhood V_u of u and $|N(u)|$ continuous functions, $f_k : V_u \to \Delta(A_1) \times \Delta(A_2)$ with $k \in \{1, \ldots, |N(u)|\}$ such that, for each $u' \in V_u$, $N(u') = \{f_k(u') : k \in \{1, \ldots, |N(u)|\}\}$ and $f_k(u) \neq f_l(u)$ for each $k, l \in \{1, \ldots, |N(u)|\}$ with $k \neq l$.[19] Shrinking V_u if needed, we may assume that, for each $a \in A$, $k, l \in \{1, \ldots, |N(u)|\}$ and $u' \in V_u$, $f_k(u')[a] \neq f_l(u')[a]$ if $f_k(u)[a] \neq f_l(u)[a]$.

We have that $\mathbb{R}^{2|A|}$ is separable, hence, there is a countable collection $\{V_{u_j}\}_{j=1}^{\infty}$ such that $O = \cup_{j=1}^{\infty} V_{u_j}$. Define, for each $j \in \mathbb{N}$, $I_j = \{1, \ldots, |N(u_j)|\}$ and

$$O_j = \cap_{(i,k,l) \in N \times I_j^2 : k \neq l} \{u \in V_{u_j} : u_i(f_k(u)) \neq u_i(f_l(u))\}.$$

Then O_j is open and $\cup_{j=1}^{\infty} O_j \subseteq \mathcal{G}$. It thus suffices to show that $C_{j,i,k,l} = \{u \in V_{u_j} :$

[19]The set $N(u)$ denotes the set of Nash equilibria of the game with payoff function u.

$u_i(f_k(u)) = u_i(f_l(u))\}$ has Lebesgue measure zero for each $j \in \mathbb{N}$ and $(i, k, l) \in N \times I_j^2$ such that $k \neq l$.

Let $j \in \mathbb{N}$ and $(i, k, l) \in N \times I_j^2$ be such that $k \neq l$. Since $f_k(u_j) \neq f_l(u_j)$, let $a \in A$ be such that $f_k(u)[a] \neq f_l(u)[a]$ for each $u \in V_{u_j}$. Then

$$C_{j,i,k,l} \subseteq \left\{ u \in V_{u_j} : u_i(a) = \frac{\sum_{a' \neq a} u_i(a')\big(f_l(u)[a'] - f_k(u)[a']\big)}{f_k(u)[a] - f_l(u)[a]} \right\}.$$

It then follows by Tonelli's Theorem (e.g. Wheeden and Zygmund (1977, Theorem 6.10, p. 92)) that $C_{j,i,k,l}$ has Lesbegue measure zero.