Supplementary Material for "Privately Designed Correlated Equilibrium"

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Contents

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1 Introduction

This paper contains supplementary material to our paper "Privately Designed Correlated Equilibrium". It contains:

Section 2: Limits of perfect conditional ε-equilibria.

Section 3: Mixed information designs.

Section 4: Extension to the case of more than two players.

2 Limits of perfect conditional ε-equilibria

In this section we establish the following claim made in Section 4.5.

Theorem 2.1 *For each 2-player game G, $U^{\text{limit}}(G) = \mathcal{U}$.*

Proof. We have that $\mathcal{U} \subseteq U(G) \subseteq U^{\text{limit}}(G)$, where the first inclusion follows by Corollary 2. Thus, it remains to show that $U^{\text{limit}}(G) \subseteq \mathcal{U}$.

For each $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ and $i \in N$, let

$$C^i_\alpha = u(N(G)) \cap \{u \in \mathbb{R}^2 : u_i = \alpha_i \text{ and } u_j \leq \alpha_j\}, \text{ and}$$
$$C_\alpha = \sum_i \beta_i \text{co}(C^i_\alpha).$$

We then have that $\cup_\alpha C_\alpha \subseteq \mathcal{U}$;[1] hence, it is enough to show that $U^{\text{limit}}(G) \subseteq \cup_\alpha C_\alpha$.

Let $u \in U^{\text{limit}}(G)$. We have that

$$u = \lim_L \Big(\beta_1 \sum_{m \in \text{supp}(\phi^L_1)} \phi^L_1[m] u(\pi^L(m)) + \beta_2 \sum_{m \in \text{supp}(\phi^L_2)} \phi^L_2[m] u(\pi^L(m))\Big).$$

For each $L \in \mathbb{N}$ and $i \in N$, let $u^{L,i} = \sum_{m \in \text{supp}(\phi^L_i)} \phi^L_i[m] u(\pi^L(m))$. For each $k \in N$,

$$\begin{aligned} u^{L,i}_k &= \sum_{m_j} \phi^L_{i,M_j}[m_j] \sum_{m_i} \frac{\phi^L_i[m_i, m_j]}{\phi^L_{i,M_j}[m_j]} u_k(\pi^L_i(m_i), \pi^L_j(m_j)) \\ &= \sum_{m_j} \phi^L_{i,M_j}[m_j] u_k \left(\sum_{m_i} \frac{\phi^L_i[m_i, m_j]}{\phi^L_{i,M_j}[m_j]} \pi^L_i(m_i), \pi^L_j(m_j) \right). \end{aligned}$$

[1]Indeed, if $u \in C_\alpha$, then $u = \sum_i \beta_i u^i$, $u^1 = \sum_{k=1}^{K_1} \lambda_{1,k} u^{1,k}$ and $u^2 = \sum_{k=1}^{K_2} \lambda_{2,k} u^{2,k}$ where $\lambda_{i,k} \geq 0$, $\sum_k \lambda_{i,k} = 1$, $u^{i,k} \in u(N(G))$, $u^{i,k}_i = u^{i,k'}_i = \alpha_i \geq u^{j,k}_i$ for each $i \in N$ and k, k'. In fact, the converse also holds and, thus, $\cup_\alpha C_\alpha = \mathcal{U}$.

Thus,

$$u^{L,i} = \sum_{m_j} \phi^L_{i,M_j}[m_j] u \left(\sum_{m_i} \frac{\phi^L_i[m_i, m_j]}{\phi^L_{i,M_j}[m_j]} \pi^L_i(m_i), \pi^L_j(m_j) \right).$$

Taking a subsequence if necessary, we may assume that $\{u^{L,i}\}_{L=1}^\infty$ converges; let $u^i = \lim_L u^{L,i}$. Then $u = \beta_1 \lim_L u^{L,1} + \beta_2 \lim_L u^{L,2} = \beta_1 u^1 + \beta_2 u^2$.

Note that $\text{supp}(\phi^L_{M_j}) = \text{supp}(\phi^L_{1,M_j}) \cup \text{supp}(\phi^L_{2,M_j})$ for each $j \in N$.

Lemma 2.1 *For each $\eta > 0$, there exists $\bar{L} \in \mathbb{N}$ such that, for each $L \geq \bar{L}$ and $i \in N$,*

$$\phi^L_{i,M_j}\left(\left\{ m_j \in \text{supp}(\phi^L_{i,M_j}) : v_i(\pi^L_j(m_j)) \geq \max_{m'_j \in \text{supp}(\phi^L_{M_j})} v_i(\pi^L_j(m'_j)) - \eta \right\} \right) > 1 - \eta.$$

Proof. Suppose not; then there is $\eta > 0$, $i \in N$ and a subsequence $\{\pi^{L_k}\}_k$ such that $\phi^{L_k}_{i,M_j}(M_k) \geq \eta$, where

$$M_k = \left\{ m_j \in \text{supp}(\phi^{L_k}_{i,M_j}) : v_i(\pi^{L_k}_j(m_j)) < \max_{m'_j \in \text{supp}(\phi^{L_k}_{M_j})} v_i(\pi^{L_k}_j(m'_j)) - \eta \right\}.$$

Let $\varepsilon > 0$ be such that $2\varepsilon < \beta_i \eta^2$ and $k \in \mathbb{N}$ be such that π^{L_k} is a perfect conditional ε-equilibrium. Let $m^*_j \in \text{supp}(\phi^{L_k}_{M_j})$ be such that $v_i(\pi^{L_k}_j(m^*_j)) = \max_{m'_j \in \text{supp}(\phi^{L_k}_{M_j})} v_i(\pi^{L_k}_j(m'_j))$. Let $\bar{m}_i \notin \text{supp}(\phi^{L_k}_{M_i})$ and $\phi'_i = 1_{(\bar{m}_i, m^*_j)}$.

Condition 6(b) implies, in the limit, that

$$u_i(\pi^{L_k}_i(\bar{m}_i, \phi'_i), \pi^{L_k}_j(m^*_j)) \geq \max_{a_i} u_i(a_i, \pi^{L_k}_j(m^*_j)) - \varepsilon = v_i(\pi^{L_k}_j(m^*_j)) - \varepsilon.$$

It also implies that, for each $m_i \in \text{supp}(\phi^{L_k}_{j,M_i})$,

$$\sum_{m_j} \frac{\phi^{L_k}_j[m_i, m_j]}{\phi^{L_k}_{j,M_i}[m_i]} u_i(\pi^{L_k}_i(m_i, \phi'_i), \pi^{L_k}_j(m_j)) \geq \max_{a_i} \sum_{m_j} \frac{\phi^{L_k}_j[m_i, m_j]}{\phi^{L_k}_{j,M_i}[m_i]} u_i(a_i, \pi^{L_k}_j(m_j)) - \varepsilon.$$

Hence,

$$\sum_m \phi^{L_k}_j[m] u_i(\pi^{L_k}_i(m_i, \phi'_i), \pi^{L_k}_j(m_j)) \geq \sum_m \phi^{L_k}_j[m] u_i(\pi^{L_k}_i(m_i), \pi^{L_k}_j(m_j)) - \varepsilon.$$

Then

$$\sum_m (\phi'_i, \phi_j^{L_k})[m] u_i(\pi_i^{L_k}(m_i, \phi'_i), \pi_j^{L_k}(m_j)) - \sum_m \phi^{L_k}[m] u_i(\pi^{L_k}(m))$$
$$= \beta_i \sum_m \left(\phi'_i[m] u_i(\pi_i^{L_k}(m_i, \phi'_i), \pi_j^{L_k}(m_j)) - \phi_i^{L_k}[m] u_i(\pi_i^{L_k}(m_i), \pi_j^{L_k}(m_j))\right)$$
$$+ \beta_j \sum_m \phi_j^{L_k}[m] \big(u_i(\pi_i^{L_k}(m_i, \phi'_i), \pi_j^{L_k}(m_j)) - u_i(\pi_i^{L_k}(m_i), \pi_j^{L_k}(m_j))\big)$$
$$\geq \beta_i \Big(v_i(\pi_j^{L_k}(m_j^*)) - \varepsilon - \sum_m \phi_i^{L_k}[m] v_i(\pi_j^{L_k}(m_j))\Big) - \beta_j \varepsilon$$
$$\geq \beta_i \sum_{m_j \in M_k} \phi_{i,M_j}^{L_k}[m_j] \left(v_i(\pi_j^{L_k}(m_j^*)) - u_i(\pi_j^{L_k}(m_j))\right) - \varepsilon$$
$$\geq \beta_i \eta^2 - \varepsilon > \varepsilon.$$

But this contradicts condition 6(a). ■

For each $L \in \mathbb{N}$ and $i \in N$, let

$$M_j^{L,i,\eta} = \left\{ m_j \in \text{supp}(\phi_{i,M_j}^L) : v_i(\pi_j^L(m_j)) \geq \max_{m'_j \in \text{supp}(\phi_{M_j}^L)} v_i(\pi_j^L(m'_j)) - \eta \right\}.$$

For each $\eta > 0$ and $\delta \in \Delta(A_j)$, let

$$BR_i^\eta(\delta) = \{\delta' \in \Delta(A_i) : u_i(\delta', \delta) \geq \max_{a_i \in A_i} u_i(a_i, \delta) - \eta\}.$$

Lemma 2.2 *For each $\eta > 0$, there exists $\bar{L} \in \mathbb{N}$ such that, for each $L \geq \bar{L}$ and $i \in N$,*

$$\phi_i^L \left(\left\{ m \in \text{supp}(\phi_i^L) : m_j \in M_j^{L,i,\eta} \text{ and } \pi_i^L(m_i) \in BR_i^\eta(\pi_j^L(m_j)) \right\} \right) > 1 - \eta.$$

Proof. Suppose not; then there is $\eta > 0$, $i \in N$ and a subsequence $\{\pi^{L_k}\}_k$ such that $\phi_i^{L_k}(\hat{M}_k) \geq \eta$, where

$$\hat{M}_k = \left\{ m \in \text{supp}(\phi_i^{L_k}) : m_j \notin M_j^{L_k,i,\eta} \text{ or } \pi_i^{L_k}(m_i) \notin BR_i^\eta(\pi_j^{L_k}(m_j)) \right\}.$$

Let $K \in \mathbb{N}$ be such that, for each $k \geq K$, $\phi_{i,M_j}^{L_k}(M_j^{L_k,i,\eta}) \geq \phi_{i,M_j}^{L_k}(M_j^{L_k,i,\eta/2}) > 1 - \eta/2$. Fix $k \geq K$ and let

$$M_k = \left\{ m \in \text{supp}(\phi_i^{L_k}) : m_j \in M_j^{L_k,i,\eta} \text{ and } \pi_i^{L_k}(m_i) \notin BR_i^\eta(\pi_j^{L_k}(m_j)) \right\}.$$

Then,

$$\eta \le \phi_i^{L_k}(\hat{M}_k) \le \phi_i^{L_k}(M_k) + \phi_i^{L_k}(\mathrm{supp}(\phi_i^{L_k}) \setminus (M_i \times M_j^{L_k,i,\eta})) < \phi_i^{L_k}(M_k) + \frac{\eta}{2}.$$

Hence, $\phi_i^{L_k}(M_k) \ge \eta/2$.

Let $\varepsilon > 0$ be such that $\varepsilon < \beta_i \eta^2/4$ and $k \ge K$ be such that π^{L_k} is a perfect conditional ε-equilibrium. Let $m_j^* \in \mathrm{supp}(\phi_{M_j}^{L_k})$ be such that $v_i(\pi_j^{L_k}(m_j^*)) = \max_{m_j' \in \mathrm{supp}(\phi_{M_j}^{L_k})} v_i(\pi_j^{L_k}(m_j'))$, $\bar{m}_i \notin \mathrm{supp}(\phi_{M_i}^{L_k})$ and $\phi_i' = 1_{(\bar{m}_i, m_j^*)}$.

Condition 6(b) implies that

$$u_i(\pi_i^{L_k}(\bar{m}_i, \phi_i'), \pi_j^{L_k}(m_j^*)) \ge \max_{a_i} u_i(a_i, \pi_j^{L_k}(m_j^*)) - \varepsilon = v_i(\pi_j^{L_k}(m_j^*)) - \varepsilon.$$

Condition 6(b) also implies that, for each $m_i \in \mathrm{supp}(\phi_{j,M_i}^{L_k})$,

$$\sum_{m_j} \frac{\phi_j^{L_k}[m_i, m_j]}{\phi_{j,M_i}^{L_k}[m_i]} u_i(\pi_i^{L_k}(m_i, \phi_i'), \pi_j^{L_k}(m_j)) \ge \max_{a_i} \sum_{m_j} \frac{\phi_j^{L_k}[m_i, m_j]}{\phi_{j,M_i}^{L_k}[m_i]} u_i(a_i, \pi_j^{L_k}(m_j)) - \varepsilon.$$

Hence,

$$\sum_m \phi_j^{L_k}[m] u_i(\pi_i^{L_k}(m_i, \phi_i'), \pi_j^{L_k}(m_j)) \ge \sum_m \phi_j^{L_k}[m] u_i(\pi_i^{L_k}(m_i), \pi_j^{L_k}(m_j)) - \varepsilon.$$

Then

$$\begin{aligned}
&\sum_m (\phi_i', \phi_j^{L_k})[m] u_i(\pi_i^{L_k}(m_i, \phi_i'), \pi_j^{L_k}(m_j)) - \sum_m \phi^{L_k}[m] u_i(\pi^{L_k}(m)) \\
&\ge \beta_i u_i(\pi_i^{L_k}(\bar{m}_i, \phi_i'), \pi_j^{L_k}(m_j^*)) - \sum_m \beta_i \phi_i^{L_k}[m] u_i(\pi^{L_k}(m)) - \beta_j \varepsilon \\
&\ge \beta_i \Big(v_i(\pi_j^{L_k}(m_j^*)) - \varepsilon - \phi_i^{L_k}[M_k](v_i(\pi_j^{L_k}(m_j^*)) - \eta) - (1 - \phi_i^{L_k}[M_k]) v_i(\pi_j^{L_k}(m_j^*)) \Big) - \beta_j \varepsilon \\
&\ge \beta_i \frac{\eta^2}{2} - \varepsilon > \varepsilon.
\end{aligned}$$

But this contradicts condition 6(a). ■

Corollary 2.1 *For each $\eta > 0$, there exists $\bar{L} \in \mathbb{N}$ such that, for each $L \ge \bar{L}$ and $i \in N$, $\phi_{i,M_j}^L(M_j^{L,i,\eta}) > 1 - \eta$ and*

$$\phi_{i,M_j}^L \left(\left\{ m_j \in M_j^{L,i,\eta} : \frac{\sum_{m_i : \pi_i^L(m_i) \in B_i^\eta(\pi_j^L(m_j))} \phi_i^L[m_i, m_j]}{\phi_{i,M_j}^L[m_j]} > 1 - \eta \right\} \right) > 1 - \eta.$$

Proof. We may assume that $\eta < 1$. Let $\eta > 0$ and let $\bar{L} \in \mathbb{N}$ be such that

$$\phi_i^L\left(\left\{m \in \operatorname{supp}(\phi_i^L) : m_j \in M_j^{L,i,\eta^2} \text{ and } \pi_i^L(m_i) \in BR_i^{\eta^2}(\pi_j^L(m_j))\right\}\right) > 1 - \eta^2$$

for each $L \geq \bar{L}$. Fix $L \geq \bar{L}$ and note that $\phi_{i,M_j}^L(M_j^{L,i,\eta^2}) > 1 - \eta^2$ and, hence, $\phi_{i,M_j}^L(M_j^{L,i,\eta}) \geq \phi_{i,M_j}^L(M_j^{L,i,\eta^2}) > 1 - \eta^2$.

Let, for each $m_j \in M_j^{L,i,\eta}$,

$$E_{m_j} = \{m_i \in M_i : (m_i, m_j) \in \operatorname{supp}(\phi_i^L) \text{ and } \pi_i^L(m_i) \in BR_i^{\eta^2}(\pi_j^L(m_j))\}.$$

Then

$$\left\{m \in \operatorname{supp}(\phi_i^L) : m_j \in M_j^{L,i,\eta^2} \text{ and } \pi_i^L(m_i) \in BR_i^{\eta^2}(\pi_j^L(m_j))\right\} = \bigcup_{m_j \in M_j^{L,i,\eta^2}} (\{m_j\} \times E_{m_j})$$

and

$$1 - \eta^2 < \phi_i^L\left(\left\{m \in \operatorname{supp}(\phi_i^L) : m_j \in M_j^{L,i,\eta^2} \text{ and } \pi_i^L(m_i) \in BR_i^{\eta^2}(\pi_j^L(m_j))\right\}\right) = \sum_{m_j \in M_j^{L,i,\eta^2}} \phi_{i,M_j}^L[m_j] \frac{\sum_{m_i \in E_{m_j}} \phi_i^L[m_i, m_j]}{\phi_{i,M_j}^L[m_j]}.$$

If

$$\phi_{i,M_j}^L\left(\left\{m_j \in M_j^{L,i,\eta^2} : \frac{\sum_{m_i : \pi_i^L(m_i) \in B_i^{\eta^2}(\pi_j^L(m_j))} \phi_i^L[m_i, m_j]}{\phi_{i,M_j}^L[m_j]} > 1 - \eta\right\}\right) \leq 1 - \eta$$

then

$$1 - \eta^2 < \sum_{m_j \in M_j^{L,i,\eta^2}} \phi_{i,M_j}^L[m_j] \frac{\sum_{m_i \in E_{m_j}} \phi_i^L[m_i, m_j]}{\phi_{i,M_j}^L[m_j]} \leq 1 - \eta + \eta(1 - \eta) = 1 - \eta^2,$$

a contradiction. Hence,

$$\phi_{i,M_j}^L\left(\left\{m_j \in M_j^{L,i,\eta^2} : \frac{\sum_{m_i : \pi_i^L(m_i) \in B_i^{\eta^2}(\pi_j^L(m_j))} \phi_i^L[m_i, m_j]}{\phi_{i,M_j}^L[m_j]} > 1 - \eta\right\}\right) > 1 - \eta.$$

Since $M_j^{L,i,\eta^2} \subseteq M_j^{L,i,\eta}$ and $B_i^{\eta^2}(\pi_j^L(m_j)) \subseteq B_i^{\eta}(\pi_j^L(m_j))$ for each $m_j \in M_j$, it follows that

$$\phi_{i,M_j}^L \left(\left\{ m_j \in M_j^{L,i,\eta} : \frac{\sum_{m_i:\pi_i^L(m_i)\in B_i^\eta(\pi_j^L(m_j))} \phi_i^L[m_i,m_j]}{\phi_{i,M_j}^L[m_j]} > 1-\eta \right\} \right) > 1-\eta.$$

■

Lemma 2.3 *For each $\eta > 0$, there exists $\bar{L} \in \mathbb{N}$ such that, for each $L \geq \bar{L}$ and $i \in N$,*

$$\phi_{i,M_j}^L \left(\left\{ m_j \in \text{supp}(\phi_{i,M_j}^L) : \pi_j^L(m_j) \in B_j^\eta \left(\sum_{m_i} \frac{\phi_i^L[m_i,m_j]}{\phi_{i,M_j}^L[m_j]} \pi_i^L(m_i) \right) \right\} \right) > 1-\eta.$$

Proof. Suppose not; then there is $\eta > 0$, $i \in N$ and a subsequence $\{\pi^{L_k}\}_k$ such that $\phi_i^{L_k}(M_k) \geq \eta$, where

$$M_k = \left\{ m_j \in \text{supp}(\phi_{i,M_j}^{L_k}) : \pi_j^{L_k}(m_j) \notin B_j^\eta \left(\sum_{m_i} \frac{\phi_i^{L_k}[m_i,m_j]}{\phi_{i,M_j}^{L_k}[m_j]} \pi_i^{L_k}(m_i) \right) \right\}.$$

Let $\varepsilon > 0$ be such that $\varepsilon < \beta_i \eta^2/2$ and $k \in \mathbb{N}$ be such that π^{L_k} is a perfect conditional ε-equilibrium. Let $m_i^* \in \text{supp}(\phi_{M_i}^{L_k})$ be such that $v_j(\pi_i^{L_k}(m_i^*)) = \max_{m_i' \in \text{supp}(\phi_{M_i}^{L_k})} v_j(\pi_i^{L_k}(m_i'))$, $\bar{m}_j \notin \text{supp}(\phi_{M_j}^{L_k})$ and $\phi_j' = 1_{(\bar{m}_j, m_i^*)}$.

Condition 6(b) implies that

$$u_j(\pi_j^{L_k}(\bar{m}_j, \phi_j'), \pi_i^{L_k}(m_i^*)) \geq \max_{a_j} u_j(a_j, \pi_i^{L_k}(m_i^*)) - \varepsilon = v_j(\pi_i^{L_k}(m_i^*)) - \varepsilon.$$

Condition 6(b) also implies that, for each $m_j \in \text{supp}(\phi_{i,M_j}^{L_k})$,

$$\sum_{m_i} \frac{\phi_i^{L_k}[m_i,m_j]}{\phi_{i,M_j}^{L_k}[m_j]} u_j(\pi_j^{L_k}(m_j,\phi_j'), \pi_i^{L_k}(m_i)) \geq \max_{a_j} \sum_{m_i} \frac{\phi_i^{L_k}[m_i,m_j]}{\phi_{i,M_j}^{L_k}[m_j]} u_j(a_j, \pi_i^{L_k}(m_i)) - \varepsilon.$$

Hence,

$$u_j \left(\pi_j^{L_k}(m_j,\phi_j'), \sum_{m_i} \frac{\phi_i^{L_k}[m_i,m_j]}{\phi_{i,M_j}^{L_k}[m_j]} \pi_i^{L_k}(m_i) \right) \geq \max_{a_j} u_j \left(a_j, \sum_{m_i} \frac{\phi_i^{L_k}[m_i,m_j]}{\phi_{i,M_j}^{L_k}[m_j]} \pi_i^{L_k}(m_i) \right) - \varepsilon.$$

Then

$$
\begin{aligned}
&\sum_m (\phi'_j, \phi_i^{L_k})[m] u_j(\pi_j^{L_k}(m_j, \phi'_j), \pi_i^{L_k}(m_i)) - \sum_m \phi^{L_k}[m] u_j(\pi^{L_k}(m)) \\
&= \beta_j u_j(\pi_j^{L_k}(\bar{m}_j, \phi'_j), \pi_i^{L_k}(m_i^*)) - \sum_m \beta_j \phi_j^{L_k}[m] u_j(\pi^{L_k}(m)) \\
&+ \beta_i \sum_{m_j} \phi_{i,M_j}^{L_k}[m_j] \left(\sum_{m_i} \frac{\phi_i^{L_k}[m_i, m_j]}{\phi_{i,M_j}^{L_k}[m_j]} u_j(\pi_j^{L_k}(m_j, \phi'_j), \pi_i^{L_k}(m_i)) \right. \\
&\qquad\qquad \left. - \sum_{m_i} \frac{\phi_i^{L_k}[m_i, m_j]}{\phi_{i,M_j}^{L_k}[m_j]} u_j(\pi_j^{L_k}(m_j), \pi_i^{L_k}(m_i)) \right) \\
&\geq -\beta_j \varepsilon + \beta_i \sum_{m_j} \phi_{i,M_j}^{L_k}[m_j] \left(\max_{a_j} u_j \left(a_j, \sum_{m_i} \frac{\phi_i^{L_k}[m_i, m_j]}{\phi_{i,M_j}^{L_k}[m_j]} \pi_i^{L_k}(m_i) \right) - \varepsilon \right. \\
&\qquad\qquad \left. - u_j \left(\pi_j^{L_k}(m_j), \sum_{m_i} \frac{\phi_i^{L_k}[m_i, m_j]}{\phi_{i,M_j}^{L_k}[m_j]} \pi_i^{L_k}(m_i) \right) \right) \\
&\geq -\beta_j \varepsilon + \beta_i \Big(-\varepsilon + \phi_{i,M_j}^{L_k}[M_k] \eta \Big) \\
&\geq \beta_i \eta^2 - \varepsilon > \varepsilon.
\end{aligned}
$$

But this contradicts condition 6(a). ■

The following corollary follows from Corollary 2.1 and Lemma 2.3.

Corollary 2.2 *For each $\eta > 0$, there exists $\bar{L} \in \mathbb{N}$ such that, for each $L \geq \bar{L}$ and $i \in N$,*

$$
\begin{aligned}
\phi_{i,M_j}^L & \left(\left\{ m_j \in M_j^{L,i,\eta} : \frac{\sum_{m_i : \pi_i^L(m_i) \in B_i^\eta(\pi_j^L(m_j))} \phi_i^L[m_i, m_j]}{\phi_{i,M_j}^L[m_j]} > 1 - \eta \right\} \bigcap \right. \\
& \left. \left\{ m_j \in \text{supp}(\phi_{i,M_j}^L) : \pi_j^L(m_j) \in B_j^\eta \left(\sum_{m_i} \frac{\phi_i^L[m_i, m_j]}{\phi_{i,M_j}^L[m_j]} \pi_i^L(m_i) \right) \right\} \right) > 1 - \eta.
\end{aligned}
$$

Proof. Let $\eta > 0$, $\bar{L}_1$ be given by Corollary 2.1 and $\bar{L}_2$ be given by Lemma 2.3, both corresponding to $\eta/2$. Then let $\bar{L} = \max\{L_1, L_2\}$. ■

Let

$$\hat{M}_j^{L,i,\eta} = \left\{ m_j \in M_j^{L,i,\eta} : \frac{\sum_{m_i:\pi_i^L(m_i)\in B_i^\eta(\pi_j^L(m_j))} \phi_i^L[m_i,m_j]}{\phi_{i,M_j}^L[m_j]} > 1-\eta \right\} \bigcap$$
$$\left\{ m_j \in \text{supp}(\phi_{i,M_j}^L) : \pi_j^L(m_j) \in B_j^\eta \left(\sum_{m_i} \frac{\phi_i^L[m_i,m_j]}{\phi_{i,M_j}^L[m_j]} \pi_i^L(m_i) \right) \right\}$$

$$\hat{B} = \max_{i\in N} \max_{a\in A} |u_i(a)|,$$

$$B = \hat{B} + 1,$$

$$\gamma = \frac{1}{4B+2}, \text{ and}$$

$$\bar{M}_j^{L,i,\eta} = \hat{M}_j^{L,i,\gamma\eta}.$$

The following corollary follows by Corollary 2.2 and the definition of $\bar{M}_j^{L,i,\eta}$.

Corollary 2.3 *For each $\eta > 0$, there exists $\bar{L} \in \mathbb{N}$ such that, for each $L \geq \bar{L}$ and $i \in N$, $\phi_{i,M_j}^L(\bar{M}_j^{L,i,\eta}) > 1-\eta$ and, for each $m_j \in \bar{M}_j^{L,i,\eta}$,*

$$\pi_j^L(m_j) \in B_j^\eta \left(\sum_{m_i} \frac{\phi_i^L[m_i,m_j]}{\phi_{i,M_j}^L[m_j]} \pi_i^L(m_i) \right),$$

$$\sum_{m_i} \frac{\phi_i^L[m_i,m_j]}{\phi_{i,M_j}^L[m_j]} \pi_i^L(m_i) \in B_i^\eta(\pi_j^L(m_j)), \text{ and}$$

$$u_i \left(\sum_{m_i} \frac{\phi_i^L[m_i,m_j]}{\phi_{i,M_j}^L[m_j]} \pi_i^L(m_i), \pi_j^L(m_j) \right) \geq \max_{m_j' \in \text{supp}(\phi_{M_j}^L)} v_i(\pi_j^L(m_j')) - \eta.$$

Proof. Let $\eta > 0$ and let $\bar{L} \in \mathbb{N}$ be given by Corollary 2.2 and such that $\phi_{i,M_j}^L(\hat{M}_j^{L,i,\gamma\eta}) > 1-\gamma\eta$ for each $L \geq \bar{L}$ and $i \in N$. Fix $L \geq \bar{L}$ and $i \in N$. Then $\phi_{i,M_j}^L(\bar{M}_j^{L,i,\eta}) = \phi_{i,M_j}^L(\hat{M}_j^{L,i,\gamma\eta}) > 1-\gamma\eta > 1-\eta$.

Let $m_j \in \bar{M}_j^{L,i,\eta} = \hat{M}_j^{L,i,\gamma\eta}$. Then,

$$\pi_j^L(m_j) \in B_j^\eta \left(\sum_{m_i} \frac{\phi_i^L[m_i,m_j]}{\phi_{i,M_j}^L[m_j]} \pi_i^L(m_i) \right)$$

since $\gamma\eta < \eta$. Furthermore,

$$-B = -(\hat{B}+1) = -\hat{B} - 1 \leq \max_{a_i} u_i(a_i, \pi_j^L(m_j)) - 1 < \max_{a_i} u_i(a_i, \pi_j^L(m_j)) - \gamma\eta$$

and, hence,

$$u_i\left(\sum_{m_i}\frac{\phi_i^L[m_i,m_j]}{\phi_{i,M_j}^L[m_j]}\pi_i^L(m_i),\pi_j^L(m_j)\right)=$$

$$\sum_{m_i:\pi_i^L(m_i)\in B_i^{\gamma\eta}(\pi_j^L(m_j))}\frac{\phi_i^L[m_i,m_j]}{\phi_{i,M_j}^L[m_j]}u_i(\pi_i^L(m_i),\pi_j^L(m_j))+$$

$$\sum_{m_i:\pi_i^L(m_i)\notin B_i^{\gamma\eta}(\pi_j^L(m_j))}\frac{\phi_i^L[m_i,m_j]}{\phi_{i,M_j}^L[m_j]}u_i(\pi_i^L(m_i),\pi_j^L(m_j))>$$

$$(1-\gamma\eta)(\max_{a_i}u_i(a_i,\pi_j^L(m_j))-\gamma\eta)-\gamma\eta B=$$

$$\max_{a_i}u_i(a_i,\pi_j^L(m_j))-\gamma\eta-\gamma\eta\max_{a_i}u_i(a_i,\pi_j^L(m_j))+\gamma^2\eta^2-\gamma\eta B>$$

$$\max_{a_i}u_i(a_i,\pi_j^L(m_j))-\gamma\eta-\gamma\eta B-\gamma\eta B=$$

$$\max_{a_i}u_i(a_i,\pi_j^L(m_j))-\frac{\eta}{2}\gamma(2+4B)=$$

$$\max_{a_i}u_i(a_i,\pi_j^L(m_j))-\frac{\eta}{2}.$$

Since $\gamma\eta<\eta/2$ and $m_j\in M_j^{L,i,\gamma\eta}$, it follows by the above that

$$u_i\left(\sum_{m_i}\frac{\phi_i^L[m_i,m_j]}{\phi_{i,M_j}^L[m_j]}\pi_i^L(m_i),\pi_j^L(m_j)\right)>v_i(\pi_j^L(m_j))-\frac{\eta}{2}\geq\max_{m_j'\in\operatorname{supp}(\phi_{M_j}^L)}v_i(\pi_j^L(m_j'))-\eta.$$

■

Note that, for each $L\in\mathbb{N}$ and $i\in N$,

$$u^{L,i}=\sum_{m_j}\phi_{i,M_j}^L[m_j]u\left(\sum_{m_i}\frac{\phi_i^L[m_i,m_j]}{\phi_{i,M_j}^L[m_j]}\pi_i^L(m_i),\pi_j^L(m_j)\right)$$

$$=(1-\phi_{i,M_j}^L[\bar{M}_j^{L,i,\eta}])\sum_{m_j\notin\bar{M}_j^{L,i,\eta}}\frac{\phi_{i,M_j}^L[m_j]}{1-\phi_{i,M_j}^L[\bar{M}_j^{L,i,\eta}]}u\left(\sum_{m_i}\frac{\phi_i^L[m_i,m_j]}{\phi_{i,M_j}^L[m_j]}\pi_i^L(m_i),\pi_j^L(m_j)\right)$$

$$+\phi_{i,M_j}^L[\bar{M}_j^{L,i,\eta}]\sum_{m_j\in\bar{M}_j^{L,i,\eta}}\frac{\phi_{i,M_j}^L[m_j]}{\phi_{i,M_j}^L[\bar{M}_j^{L,i,\eta}]}u\left(\sum_{m_i}\frac{\phi_i^L[m_i,m_j]}{\phi_{i,M_j}^L[m_j]}\pi_i^L(m_i),\pi_j^L(m_j)\right).$$

Define

$$\bar{u}^{L,i}=\sum_{m_j\in\bar{M}_j^{L,i,\eta}}\frac{\phi_{i,M_j}^L[m_j]}{\phi_{i,M_j}^L[\bar{M}_j^{L,i,\eta}]}u\left(\sum_{m_i}\frac{\phi_i^L[m_i,m_j]}{\phi_{i,M_j}^L[m_j]}\pi_i^L(m_i),\pi_j^L(m_j)\right).$$

Then $u^i=\lim_L\bar{u}^{L,i}$.

Let $\eta > 0$ and let $\bar{L} \in \mathbb{N}$ be as in Corollary 2.2. For each $L \geq \bar{L}$, let $\alpha_L = (\alpha_{L,1}, \alpha_{L,2})$ be defined by setting, for each $i \in N$,

$$\alpha_{L,i} = \max_{m_j \in \text{supp}(\phi^L_{M_j})} v_i(\pi^L_j(m_j));$$

note that $\alpha_{L,i} \in \text{co}(u_i(A))$ and that $\text{co}(u_i(A))$ is compact. Let

$$N^\eta(G) = \{(\sigma_1, \sigma_2) \in \Delta(A_1) \times \Delta(A_2) : \sigma_i \in BR^\eta_i(\sigma_j) \text{ for each } i \in N\}, \text{ and}$$
$$C^{i,\eta}_{\alpha_L} = u(N^\eta(G)) \cap \{u \in \mathbb{R}^2 : \alpha_{L,i} - \eta \leq u_i \leq \alpha_{L,i} \text{ and } u_j \leq \alpha_{L,j}\}.$$

It then follows that

$$\bar{u}^{L,i} \in \text{co}(C^{i,\eta}_{\alpha_L}).$$

Indeed, for each $m_j \in \bar{M}^{L,i,\eta}_j$,

$$\pi^L_j(m_j) \in B^\eta_j\left(\sum_{m_i} \frac{\phi^L_i[m_i, m_j]}{\phi^L_{i,M_j}[m_j]} \pi^L_i(m_i)\right),$$
$$\sum_{m_i} \frac{\phi^L_i[m_i, m_j]}{\phi^L_{i,M_j}[m_j]} \pi^L_i(m_i) \in B^\eta_i(\pi^L_j(m_j)),$$
$$u_i\left(\sum_{m_i} \frac{\phi^L_i[m_i, m_j]}{\phi^L_{i,M_j}[m_j]} \pi^L_i(m_i), \pi^L_j(m_j)\right) \geq \max_{m'_j \in \text{supp}(\phi^L_{M_j})} v_i(\pi^L_j(m'_j)) - \eta = \alpha_{L,i} - \eta,$$
$$u_i\left(\sum_{m_i} \frac{\phi^L_i[m_i, m_j]}{\phi^L_{i,M_j}[m_j]} \pi^L_i(m_i), \pi^L_j(m_j)\right) \leq \max_{m'_j \in \text{supp}(\phi^L_{M_j})} v_i(\pi^L_j(m'_j)) = \alpha_{L,i} \text{ and}$$
$$u_j\left(\sum_{m_i} \frac{\phi^L_i[m_i, m_j]}{\phi^L_{i,M_j}[m_j]} \pi^L_i(m_i), \pi^L_j(m_j)\right) = \sum_{m_i} \frac{\phi^L_i[m_i, m_j]}{\phi^L_{i,M_j}[m_j]} u_j\left(\pi^L_i(m_i), \pi^L_j(m_j)\right)$$
$$\leq \max_{m'_i \in \text{supp}(\phi^L_{M_i})} v_j(\pi^L_i(m'_i)) = \alpha_{L,j}.$$

It then follows by Caratheodory's Theorem that $\bar{u}^{L,i} = \sum_{k=1}^3 \lambda_{L,i,k} u^{L,i,k}$ for some $\lambda_{L,i,1}, \lambda_{L,i,2}, \lambda_{L,i,3} \in [0,1]$ and $u^{L,i,1}, u^{L,i,2}, u^{L,i,3} \in C^{i,\eta}_{\alpha_L}$ such that $\sum_{k=1}^3 \lambda_{L,i,k} = 1$. Taking a subsequence if necessary, we may assume that $\{\alpha_L\}^\infty_{L=\bar{L}}$, $\{u^{L,i,k}\}^\infty_{L=\bar{L}}$ and $\{\lambda_{L,i,k}\}^\infty_{L=\bar{L}}$ converge for each $k = 1, 2, 3$; let $\alpha = \lim_L \alpha_L$, $u^{i,k} = \lim_L u^{L,i,k}$ and

$\lambda_{i,k} = \lim_L \lambda_{L,i,k}$ for each $k = 1, 2, 3$. Hence,

$$u^i = \lim_L \bar{u}^{L,i} = \sum_{k=1}^{3} \lambda_{i,k} u^{i,k},$$

$$\sum_{k=1}^{3} \lambda_{i,k} = 1,$$

and, for each $k = 1, 2, 3$,

$$\lambda_{i,k} \geq 0,$$

$$\alpha_i - \eta \leq u_i^{i,k} \leq \alpha_i,$$

$$u_j^{i,k} \leq \alpha_j \text{ and}$$

$$u^{i,k} \in u(N^\eta(G)).$$

Since this holds for each $\eta > 0$, it follows that, for each $k = 1, 2, 3$, $u_i^{i,k} = \alpha_i$ and $u^{i,k} \in u(N(G))$. Hence, $u^i \in \text{co}(C_\alpha^i)$ and $u \in C_\alpha \subseteq \mathcal{U}$. ■

3 Mixed information designs

In this section we establish the claims made in Section 4.6. The first one is that, for each 2-player game in $\mathcal{G}$, the sequential equilibrium payoffs of G_{id} are specific combinations of two Nash equilibria of G.

Theorem 3.1 *For each 2-player game $G \in \mathcal{G}$,*

$$U(G) \subseteq U^*(G) \subseteq \{\beta_1 u(\sigma) + \beta_2 u(\sigma') : \sigma, \sigma' \in N(G)\}.$$

The second claim is that $\beta_1(1,1) + \beta_2(2,2)$ is a sequential equilibrium payoff of G_{id} when G is the battle of the sexes.

Claim 1 *If G is the battle of the sexes, then $\beta_1(1,1) + \beta_2(2,2) \in U^*(G)$.*

The final claim is that, for each 2-player game G, the limit payoffs of perfect conditional ε-equilibria are combinations of two Nash equilibria.

Theorem 3.2 *For each 2-player game G,*

$$\{\beta_1 u(\sigma) + \beta_2 u(\sigma') : \sigma, \sigma' \in N(G)\} \subseteq U^{\text{limit}*}(G).$$

3.1 Proof of Theorem 3.1

Let $\pi \in \Pi^*$ be a sequential (or Nash) equilibrium of G_{id}. Then

$$\sum_{\phi} \pi^1[\phi] \sum_{m} \phi[m] u_i(\pi(m)) \geq \sum_{\phi_j} \pi_j^1[\phi_j] \sum_{m} (\phi_i', \phi_j)[m] u_i(\pi_i'(m_i, \phi_i'), \pi_j(m_j)), \quad (3.1)$$

for each $i, j \in N$, $j \neq i$, $\phi_i' \in S$ and $\pi_i' : M_i \times S \to \Delta(A_i)$.

For each $i \in N$ and $m_i \in M_i$, let $\pi_i(m_i) = \sum_{\phi_i} \pi_i^1[\phi_i]\pi_i(\phi_i, m_i)$. Then, for each $m \in M$, let $\pi(m) = (\pi_1(m_1), \pi_2(m_2))$.

Lemma 3.1 *For each $i, j \in N$, $j \neq i$, $\phi_i \in \operatorname{supp}(\pi_i^1)$ and $m \in \operatorname{supp}(\phi_i)$,*

$$u_i(\pi_i^2(\phi_i, m_i), \pi_j(m_j)) = \sup_{m' \in M} u_i(\pi_i^2(\phi_i, m_i'), \pi_j(m_j')).$$

Proof. Suppose not; then there is $i \in N$, $\phi_i^* \in \operatorname{supp}(\pi_i^1)$, $m' \in \operatorname{supp}(\phi_i^*)$ and $m^* \in M$ such that $u_i(\pi_i^2(\phi_i, m_i^*), \pi_j(m_j^*)) > u_i(\pi_i^2(\phi_i, m_i'), \pi_j(m_j'))$.

Define $\hat{\phi}_i$ by setting, for each $m \in \operatorname{supp}(\phi_i^*)$,

$$\hat{\phi}_i[m] = \begin{cases} \lambda \phi_i^*[m'] & \text{if } m = m', \\ \phi_i^*[m^*] + (1-\lambda)\phi_i^*[m'] & \text{if } m = m^*, \\ \phi_i^*[m] & \text{otherwise,} \end{cases}$$

where $\lambda \in (0,1)$ is such that $\hat{\phi}_i \notin \operatorname{supp}(\pi_i^1)$. Define $\hat{\pi}_i^1$ by setting, for each $\phi_i \in \operatorname{supp}(\pi_i^1)$,

$$\hat{\pi}_i^1[\phi_i] = \begin{cases} 0 & \text{if } \phi_i = \phi_i^*, \\ \pi_i^1[\phi_i^*] & \text{if } \phi_i = \hat{\phi}_i, \\ \pi_i^1[\phi_i] & \text{otherwise,} \end{cases}$$

and define $\hat{\pi}_i^2 : S \times M_i \to \Delta(A_i)$ by setting, for each $(\phi_i, m_i) \in S \times M_i$,

$$\hat{\pi}_i^2(\phi_i, m_i) = \begin{cases} \pi_i^2(\phi_i^*, m_i) & \text{if } \phi_i = \hat{\phi}_i, \\ \pi_i^2(\phi_i, m_i) & \text{otherwise.} \end{cases}$$

Then, letting $\hat{\pi}^1 = (\hat{\pi}_i^1, \pi_j^1)$ and $\hat{\pi}^2 = (\hat{\pi}_i^2, \pi_j^2)$,

$$\sum_\phi \hat{\pi}^1[\phi] \sum_m \phi[m] u_i(\hat{\pi}^2(\phi, m)) - \sum_\phi \pi^1[\phi] \sum_m \phi[m] u_i(\pi^2(\phi, m)) =$$
$$\pi_i^1[\phi_i^*] \Big(\sum_{\phi_j} \pi_j^1[\phi_j] \sum_m (\hat{\phi}_i, \phi_j)[m] u_i(\pi^2(\phi_i^*, \phi_j, m))$$
$$- \sum_{\phi_j} \pi_j^1[\phi_j] \sum_m (\phi_i^*, \phi_j)[m] u_i(\pi^2(\phi_i^*, \phi_j, m)) \Big) =$$
$$\pi_i^1[\phi_i^*] \beta_i (1-\lambda) \phi_i^*[m'] \Big(\sum_{\phi_j} \pi_j^1[\phi_j] u_i(\pi^2(\phi_i^*, \phi_j, m^*)) - \sum_{\phi_j} \pi_j^1[\phi_j] u_i(\pi^2(\phi_i^*, \phi_j, m')) \Big) =$$
$$\pi_i^1[\phi_i^*] \beta_i (1-\lambda) \phi_i^*[m'] \Big(u_i(\pi_i^2(\phi_i, m_i^*), \pi_j(m_j^*)) - u_i(\pi_i^2(\phi_i, m_i'), \pi_j(m_j')) \Big) > 0.$$

But this is a contradiction to (3.1). ■

Lemma 3.2 *For each $i, j \in N$, $i \neq j$, $\phi_i, \phi_i' \in \operatorname{supp}(\pi_i^1)$, $m \in \operatorname{supp}(\phi_i)$ and $m' \in \operatorname{supp}(\phi_i')$,*

$$u_i(\pi_i^2(\phi_i, m_i), \pi_j(m_j)) = u_i(\pi_i^2(\phi_i', m_i'), \pi_j(m_j')).$$

Proof. Condition (3.1) implies that

$$\sum_{\phi_j} \pi_j^1[\phi_j] \sum_m (\phi_i, \phi_j)[m] u_i(\pi^2(\phi_i, \phi_j, m)) =$$
$$\sum_{\phi_j} \pi_j^1[\phi_j] \sum_m (\phi_i', \phi_j)[m] u_i(\pi^2(\phi_i', \phi_j, m)).$$

Lemma 3.1 implies that $u_i(\pi_i^2(\phi_i, \hat{m}_i), \pi_j(\hat{m}_j)) = u_i(\pi_i^2(\phi_i, m_i), \pi_j(m_j))$ for each $\hat{m} \in \operatorname{supp}(\phi_i)$ and that $u_i(\pi_i^2(\phi_i', \hat{m}_i), \pi_j(\hat{m}_j)) = u_i(\pi_i^2(\phi_i', m_i'), \pi_j(m_j'))$ for each $\hat{m} \in \operatorname{supp}(\phi_i')$. Hence,

$$0 = \sum_{\phi_j} \pi_j^1[\phi_j] \sum_m (\phi_i, \phi_j)[m] u_i(\pi^2(\phi_i, \phi_j, m)) -$$
$$\sum_{\phi_j} \pi_j^1[\phi_j] \sum_m (\phi_i', \phi_j)[m] u_i(\pi^2(\phi_i', \phi_j, m)) =$$
$$\beta_i \Big(u_i(\pi_i^2(\phi_i, m_i), \pi_j(m_j)) - u_i(\pi_i^2(\phi_i', m_i'), \pi_j(m_j')) \Big).$$

■

Lemma 3.3 *For each $i, j \in N$, $i \neq j$ $\phi_i \in \operatorname{supp}(\pi_i^1)$ and $m \in \operatorname{supp}(\phi_i)$,*

$$\pi_i^2(\phi_i, m_i) \in BR_i(\pi_j^2(m_j)).$$

Proof. Suppose not; then there is $i, j \in N$, $j \neq i$, $\phi_i^* \in \text{supp}(\pi_i^1)$ and $m^* \in \text{supp}(\phi_i^*)$ such that $\pi_i^2(\phi_i^*, m_i^*) \notin BR_i(\pi_j^2(m_j^*))$. Let $a_i^* \in BR_i(\pi_j^2(m_j^*))$, $\bar{m}_i \notin \cup_{\phi \in \text{supp}(\pi^1)} \text{supp}(\phi_{M_i})$, $\hat{\phi}_i = 1_{(\bar{m}_i, m_j^*)}$, $\hat{\pi}_1^1 = 1_{\hat{\phi}_i}$ and $\hat{\pi}_i^2 : S \times M_i \to \Delta(A_i)$ be such that $\hat{\pi}_i^2(\phi_i, m_i) = a_i^*$ if $(\phi_i, m_i) = (\hat{\phi}_i, \bar{m}_i)$ and $\hat{\pi}_i^2(\phi_i, m_i) = \pi_i^2(\phi_i, m_i)$ otherwise. Then

$$u_i(\hat{\pi}_i, \pi_j) - u_i(\pi) = \beta_i \Big(u_i(a_i^*, \pi_j^2(m_j^*)) - u_i(\pi_i^2(\phi_i^*, m_i^*), \pi_j^2(m_j^*)) \Big) > 0.$$

But this contradicts (3.1). ∎

Lemma 3.4 *For each $i, j \in N$, $i \neq j$, $\phi_i \in \text{supp}(\pi_i^1)$ and $m \in \text{supp}(\phi_i)$ such that $m_i \in \cup_{\phi_j \in \text{supp}(\pi_j^1)} \text{supp}(\phi_{j,M_i})$, $\pi_i(\phi_i, m_i)$ solves*

$$\max_{\alpha_i \in \Delta(A_i)} \frac{\sum_{\phi_j} \pi_j^1[\phi_j] \sum_{m_j} \phi_j[m_i, m_j] u_i(\alpha_i, \pi_j(\phi_j, m_j))}{\sum_{\phi_j} \pi_j^1[\phi_j] \phi_{j,M_i}[m_i]}.$$

Proof. Suppose not; then there is $i, j \in N$, $i \neq j$, $\phi_i^* \in \text{supp}(\pi_i^1)$ and $m' \in \text{supp}(\phi_i^*)$ such that $m_i' \in \cup_{\phi_j \in \text{supp}(\pi_j^1)} \text{supp}(\phi_{j,M_i})$ and $\pi_i(\phi_i^*, m_i')$ does not solve

$$\max_{\alpha_i \in \Delta(A_i)} \frac{\sum_{\phi_j} \pi_j^1[\phi_j] \sum_{m_j} \phi_j[m_i', m_j] u_i(\alpha_i, \pi_j(\phi_j, m_j))}{\sum_{\phi_j} \pi_j^1[\phi_j] \phi_{j,M_i}[m_i']}. \quad (3.2)$$

Let a_i^* be a solution to problem (3.2), $\bar{m}_i \notin \cup_{\phi \in \text{supp}(\pi^1)} \text{supp}(\phi_{M_i})$, $\phi_i' = 1_{(\bar{m}_i, m_j')}$, $\hat{\pi}_i^1 = 1_{\phi_i'}$ and $\hat{\pi}_i : S \times M_i \to \Delta(A_i)$ be such that

$$\hat{\pi}_i^2(\phi_i', m_i) = \begin{cases} a_i^* & \text{if } m_i = m_i', \\ \pi_i(\phi_i^*, m_i') & \text{if } m_i = \bar{m}_i, \\ \pi_i(\phi_i^*, m_i) & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} u_i(\hat{\pi}_i, \pi_j) - u_i(\pi) &= \beta_i \Big(u_i(\pi_i^2(\phi_i^*, m_i'), \pi_j(m_j')) - u_i(\pi_i^2(\phi_i^*, m_i'), \pi_j(m_j')) \Big) \\ &+ \beta_j \sum_{\phi_j} \pi_j^1[\phi_j] \sum_{m_j} \phi_j[m_i', m_j] \Big(u_i(a_i^*, \pi_j(\phi_j, m_j)) - u_i(\pi_i(\phi_i^*, m_i'), \pi_j(\phi_j, m_j)) \Big) \\ &= \beta_j \sum_{\phi_j} \pi_j^1[\phi_j] \sum_{m_j} \phi_j[m_i', m_j] \Big(u_i(a_i^*, \pi_j(\phi_j, m_j)) - u_i(\pi_i(\phi_i^*, m_i'), \pi_j(\phi_j, m_j)) \Big). \end{aligned}$$

Since $\pi_i(\phi_i^*, m_i')$ does not solve problem (3.2) but a_i^* does, it follows that

$$\frac{\sum_{\phi_j} \pi_j^1[\phi_j] \sum_{m_j} \phi_j[m_i', m_j]}{\sum_{\phi_j} \pi_j^1[\phi_j] \phi_{j,M_i}[m_i']} \Big(u_i(a_i^*, \pi_j(\phi_j, m_j)) - u_i(\pi_i(\phi_i^*, m_i'), \pi_j(\phi_j, m_j)) \Big)$$

is strictly positive and, since $m_i' \in \cup_{\phi_j \in \text{supp}(\pi_j^1)} \text{supp}(\phi_{j,M_i})$,

$$\beta_j \sum_{\phi_j} \pi_j^1[\phi_j] \sum_{m_j} \phi_j[m_i', m_j]\Big(u_i(a_i^*, \pi_j(\phi_j, m_j)) - u_i(\pi_i(\phi_i^*, m_i'), \pi_j(\phi_j, m_j))\Big) > 0.$$

Hence, $u_i(\hat{\pi}_i, \pi_j) - u_i(\pi) > 0$. But this contradicts (3.1). ■

Lemma 3.5 *For each $i, j \in N$, $j \neq i$, $\phi_i \in \text{supp}(\pi_i^1)$ and $m_i \in \cup_{\phi_j \in \text{supp}(\pi_j^1)} \text{supp}(\phi_{j,M_i})$, $\pi_i(\phi_i, m_i)$ solves*

$$\max_{\alpha_i \in \Delta(A_i)} \frac{\sum_{\phi_j} \pi_j^1[\phi_j] \sum_{m_j} \phi_j[m_i, m_j] u_i(\alpha_i, \pi_j(\phi_j, m_j))}{\sum_{\phi_j} \pi_j^1[\phi_j] \phi_{j,M_i}[m_i]}.$$

Proof. If m_i is such that $(m_i, m_j) \in \text{supp}(\phi_i)$ for some m_j, then the conclusion follows by Lemma 3.4. Otherwise, it follows from (3.1). ■

Let $i, j \in N$ with $i \neq j$. We then have that, for each $m_j \in \cup_{\phi_i \in \text{supp}(\pi_i^1)} \text{supp}(\phi_{i,M_j})$,

$$\left(\frac{\sum_{\phi_i} \pi_i^1[\phi_i] \sum_{m_i} \phi_i[m_i, m_j] \pi_i^2(\phi_i, m_i)}{\sum_{\phi_i} \pi_i^1[\phi_i] \phi_{i,M_j}[m_j]}, \pi_j^2(m_j) \right) \text{ is a Nash equilibrium of } G.$$

Indeed, it follows by Lemma 3.5 that $\pi_j^2(\phi_j, m_j) \in BR_j \left(\frac{\sum_{\phi_i} \pi_i^1[\phi_i] \sum_{m_i} \phi_i[m_i, m_j] \pi_i^2(\phi_i, m_i)}{\sum_{\phi_i} \pi_i^1[\phi_i] \phi_{i,M_j}[m_j]} \right)$. Hence, $\pi_j^2(m_j) = \sum_{\phi_j} \pi_j^1[\phi_j] \pi_j^2(\phi_j, m_j) \in BR_j \left(\frac{\sum_{\phi_i} \pi_i^1[\phi_i] \sum_{m_i} \phi_i[m_i, m_j] \pi_i^2(\phi_i, m_i)}{\sum_{\phi_i} \pi_i^1[\phi_i] \phi_{i,M_j}[m_j]} \right)$.

Furthermore, for each $\phi_i \in \text{supp}(\pi_i^1)$ and $m_i \in M_i$ such that $(m_i, m_j) \in \text{supp}(\phi_i)$, $\pi_i^2(\phi_i, m_i) \in BR_i(\pi_j^2(m_j))$ by Lemma 3.3. Thus,

$$\frac{\sum_{\phi_i} \pi_i^1[\phi_i] \sum_{m_i} \phi_i[m_i, m_j] \pi_i^2(\phi_i, m_i)}{\sum_{\phi_i} \pi_i^1[\phi_i] \phi_{i,M_j}[m_j]} \in BR_i(\pi_j^2(m_j)).$$

We have that

$$\begin{aligned}
u(\pi) &= \beta_1 \sum_{\phi} \pi^1[\phi] \sum_{m \in \text{supp}(\phi_1)} \phi_1[m] u(\pi^2(\phi, m)) + \beta_2 \sum_{\phi} \pi^1[\phi] \sum_{m \in \text{supp}(\phi_2)} \phi_2[m] u(\pi^2(\phi, m)) \\
&= \beta_1 \sum_{\phi_1} \pi_1^1[\phi_1] \sum_{m \in \text{supp}(\phi_1)} \phi_1[m] u(\pi_1^2(\phi_1, m_1), \pi_2^2(m_2)) \\
&+ \beta_2 \sum_{\phi_2} \pi_2^1[\phi_2] \sum_{m \in \text{supp}(\phi_2)} \phi_2[m] u(\pi_1^2(m_1), \pi_2^2(\phi_2, m_2)).
\end{aligned}$$

Hence, we compute $u^i := \sum_{\phi_i} \pi_i^1[\phi_i] \sum_{m \in \text{supp}(\phi_i)} \phi_i[m] u(\pi_i^2(\phi_i, m_i), \pi_j^2(m_j))$ for each $i \in \text{supp}(\beta)$. Let $i, k \in N$. Then

$$\begin{aligned} u_k^i &= \sum_{m_j} \frac{\sum_{\phi_i} \pi_i^1[\phi_i]\phi_{i,M_j}[m_j]}{\sum_{\phi_i} \pi_i^1[\phi_i]\phi_{i,M_j}[m_j]} \sum_{\phi_i} \pi_i^1[\phi_i] \sum_{m_i} \phi_i[m_i, m_j] u_k(\pi_i^2(\phi_i, m_i), \pi_j(m_j)) \\ &= \sum_{m_j} \sum_{\phi_i} \pi_i^1[\phi_i]\phi_{i,M_j}[m_j] u_k \left(\frac{\sum_{\phi_i} \pi_i^1[\phi_i] \sum_{m_i} \phi_i[m_i, m_j] \pi_i(\phi_i, m_i)}{\sum_{\phi_i} \pi_i^1[\phi_i]\phi_{i,M_j}[m_j]}, \pi_j(m_j) \right). \end{aligned}$$

Thus,

$$u^i = \sum_{m_j} \sum_{\phi_i} \pi_i^1[\phi_i]\phi_{i,M_j}[m_j] u \left(\frac{\sum_{\phi_i} \pi_i^1[\phi_i] \sum_{m_i} \phi_i[m_i, m_j] \pi_i(\phi_i, m_i)}{\sum_{\phi_i} \pi_i^1[\phi_i]\phi_{i,M_j}[m_j]}, \pi_j(m_j) \right).$$

Hence, for each $m_j \in \text{supp}(\sum_{\phi_i} \pi_i^1[\phi_i]\phi_{i,M_j})$, there is a Nash equilibrium

$$\sigma^{i,m_j} = \left(\frac{\sum_{\phi_i} \pi_i^1[\phi_i] \sum_{m_i} \phi_i[m_i, m_j] \pi_i(\phi_i, m_i)}{\sum_{\phi_i} \pi_i^1[\phi_i]\phi_{i,M_j}[m_j]}, \pi_j(m_j) \right)$$

of G such that $u^i = \sum_{m_j} \alpha^{i,m_j} u(\sigma^{i,m_j})$ with $\alpha^{i,m_j} = \sum_{\phi_i} \pi_i^1[\phi_i]\phi_{i,M_j}[m_j]$. Then let $L_i = |\text{supp}(\sum_{\phi_i} \pi_i^1[\phi_i]\phi_{i,M_j})|$ and, writing $\text{supp}(\sum_{\phi_i} \pi_i^1[\phi_i]\phi_{i,M_j}) = \{m_j^1, \ldots, m_j^{L_i}\}$, let $\alpha^{i,l} = \sum_{\phi_i} \pi_i^1[\phi_i]\phi_{i,M_j}[m_j^l]$ and $\sigma^{i,l} = \sigma^{i,m_j^l}$ for each $l \in \{1, \ldots, L_i\}$.

For each $m_j \in \text{supp}(\sum_{\phi_i} \pi_i^1[\phi_i]\phi_{i,M_j})$, it follows by Lemmas 3.1 and 3.2 that

$$\begin{aligned} u_i(\sigma^{i,m_j}) &= \frac{\sum_{\phi_i} \pi_i^1[\phi_i] \sum_{m_i} \phi_i[m_i, m_j]}{\sum_{\phi_i} \pi_i^1[\phi_i]\phi_{i,M_j}[m_j]} u_i(\pi_i^2(\phi_i, m_i), \pi_j^2(m_j)) \\ &= \max_{\phi^* \in \text{supp}(\pi_i^1), m^* \in M} u_i(\pi_i^2(\phi^*, m_i^*), \pi_j^2(m_j^*)). \end{aligned}$$

Thus, $u_i(\sigma^{i,m_j}) = u_i(\sigma^{i,m_j'})$ for each $m_j' \in \text{supp}(\sum_{\phi_i} \pi_i^1[\phi_i]\phi_{i,M_j})$.

3.2 Proof of Claim 1

The proof uses a similar construction to the one used in the proof of Theorem 1.

Let $\sigma^1 = (B, B)$, $\sigma^2 = (A, A)$ and $\bar{\sigma}$ be the mixed Nash equilibrium of the battle of the sexes and pick $L \in \mathbb{N}$ such that

$$\left(1 - \frac{x}{L} - x'\right) \frac{2}{3} + \frac{x}{L} 2 + x' < 1 \tag{3.3}$$

for each $x, x' \in [0, 1]$ such that $x' < 1$ and $x + x' \leq 1$.

Let $i \in N$ and $j \neq i$. For each $1 \leq l \leq L$, let

$$\phi_1^l = 1_{(l,L+1)} \text{ and } \phi_2^l = 1_{(L+1,l)}.$$

Let

$$\pi_i^1 = \frac{1}{L}\sum_{l=1}^{L} 1_{\phi_i^l}$$

be the first period strategy.

The second period strategy is as follows. For each $1 \leq l \leq L$ and $m_i \in M_i$, let

$$\pi_i^2(m_i, \phi_i^l) = \begin{cases} \sigma_i^i & \text{if } m_i = l, \\ \sigma_i^j & \text{if } m_i = L+1, \\ \bar{\sigma}_i & \text{otherwise.} \end{cases}$$

We will specify the remaining values of π_i^2 as follows. For each $m_i \in M_i$ and $\phi_i \in S \setminus \{\phi_i^l : 1 \leq l \leq L\}$ such that $\beta_i \phi_{i,M_i}[m_i] + \beta_j \sum_{l=1}^{L} \frac{\phi_{j,M_i}^l[m_i]}{L} = 0$, let $\pi_i^2(m_i, \phi_i) = \bar{\sigma}_i$.

Note that, for each $1 \leq l \leq L$, $\beta_i \phi_{i,M_i}^l[m_i] + \beta_j \sum_{l=1}^{L} \frac{\phi_{j,M_i}^l[m_i]}{L} = 0$ for each $m_i \notin \{l, L+1\}$ and $\pi_i^2(m_i, \phi_i^l) = \bar{\sigma}_i$. Thus,

$$\pi_i^2(m_i, \phi_i) = \bar{\sigma}_i$$

for each $m_i \in M_i$ and $\phi_i \in S$ such that $\beta_i \phi_{i,M_i}[m_i] + \beta_j \sum_{l=1}^{L} \frac{\phi_{j,M_i}^l[m_i]}{L} = 0$.

For each $m_i \in M_i$ and $\phi_i \in S \setminus \{\phi_i^l : 1 \leq l \leq L\}$ such that $\beta_i \phi_{i,M_i}[m_i] + \beta_j \sum_{l=1}^{L} \frac{\phi_{j,M_i}^l[m_i]}{L} > 0$, let $\pi_i^2(m_i, \phi_i)$ be a best-reply against

$$\sum_{m_j} \frac{\beta_i \phi_i[m_i, m_j] + \beta_j \sum_{l=1}^{L} \frac{\phi_j^l[m_i, m_j]}{L}}{\beta_i \phi_{i,M_i}[m_i] + \beta_j \sum_{l=1}^{L} \frac{\phi_{j,M_i}^l[m_i]}{L}} \pi_j(m_j, \phi_j^l).$$

Note that, for each $1 \leq l \leq L$, $\beta_i \phi_{i,M_i}^l[L+1] + \beta_j \sum_{h=1}^{L} \frac{\phi_{j,M_i}^h[L+1]}{L} = \beta_j > 0$ and

$$\sum_{m_j} \frac{\beta_i \phi_i^l[L+1, m_j] + \beta_j \sum_{h=1}^{L} \frac{\phi_j^h[L+1, m_j]}{L}}{\beta_i \phi_{i,M_i}^l[L+1] + \beta_j \sum_{h=1}^{L} \frac{\phi_{j,M_i}^h[L+1]}{L}} \pi_j(m_j, \phi_j^h) =$$
$$\frac{1}{L}\sum_{h=1}^{L}\sum_{m_j} \phi_j^h[L+1, m_j]\pi_j(m_j, \phi_j^h) = \frac{1}{L}\sum_{h=1}^{L} \pi_j(h, \phi_j^h) = \sigma_j^j.$$

Thus, $\pi_i^2(L+1, \phi_i^l) = \sigma_i^j \in BR_i\left(\sum_{m_j} \frac{\beta_i\phi_i[m_i,m_j]+\beta_j\sum_{l=1}^L \frac{\phi_j^l[m_i,m_j]}{L}}{\beta_i\phi_{i,M_i}[m_i]+\beta_j\sum_{l=1}^L \frac{\phi_{j,M_i}^l[m_i]}{L}}\pi_j(m_j,\phi_j^l)\right)$. Furthermore, for each $1 \le l \le L$, $\beta_i\phi_{i,M_i}^l[l] + \beta_j\sum_{h=1}^L \frac{\phi_{j,M_i}^h[l]}{L} = \beta_i > 0$ and

$$\sum_{m_j} \frac{\beta_i\phi_i^l[L+1,m_j] + \beta_j\sum_{h=1}^L \frac{\phi_j^h[L+1,m_j]}{L}}{\beta_i\phi_{i,M_i}^l[L+1] + \beta_j\sum_{h=1}^L \frac{\phi_{j,M_i}^h[L+1]}{L}}\pi_j(m_j,\phi_j^h) =$$
$$\sum_{m_j} \phi_i^l[l,m_j]\frac{1}{L}\sum_{h=1}^L \pi_j(m_j,\phi_j^l) = \frac{1}{L}\sum_{h=1}^L \pi_j(L+1,\phi_j^h) = \sigma_j^i.$$

Thus, $\pi_i^2(l, \phi_i^l) = \sigma_i^i \in BR_i\left(\sum_{m_j} \frac{\beta_i\phi_i[m_i,m_j]+\beta_j\sum_{l=1}^L \frac{\phi_j^l[m_i,m_j]}{L}}{\beta_i\phi_{i,M_i}[m_i]+\beta_j\sum_{l=1}^L \frac{\phi_{j,M_i}^l[m_i]}{L}}\pi_j(m_j,\phi_j^l)\right)$. Hence,

$$\pi_i^2(m_i,\phi_i) \in BR_i\left(\sum_{m_j} \frac{\beta_i\phi_i[m_i,m_j] + \beta_j\sum_{l=1}^L \frac{\phi_j^l[m_i,m_j]}{L}}{\beta_i\phi_{i,M_i}[m_i] + \beta_j\sum_{l=1}^L \frac{\phi_{j,M_i}^l[m_i]}{L}}\pi_j(m_j,\phi_j^l)\right)$$

for each $m_i \in M_i$ and $\phi_i \in S$ such that $\beta_i\phi_{i,M_i}[m_i] + \beta_j\sum_{l=1}^L \frac{\phi_{j,M_i}^l[m_i]}{L} > 0$.

We may assume that $\pi_i : M_i \times S \to \Delta(A_i)$ is measurable as in the proof of Theorem 1.

We define $\{\pi^\alpha, p^\alpha\}_\alpha$ as follows. The index set consists of $\alpha = (k, F, \hat{F})$ such that $k \in \mathbb{N}$, F is a finite subset of $\mathbb{N}$ and $\hat{F}$ is a finite subset of S; this set is partially ordered by defining $(k', F', \hat{F}') \ge (k, F, \hat{F})$ if $k' \ge k$, $F \subseteq F'$ and $\hat{F} \subseteq \hat{F}'$. For each $\alpha = (k, F, \hat{F})$, let

$$\begin{aligned}
&\tau_i^\alpha = \frac{\sum_{l\in F\cup\left(\cup_{\phi\in\hat{F}}\mathrm{supp}(\phi_{M_i})\right)} 2^{-l}1_l}{\sum_{l\in F\cup\left(\cup_{\phi\in\hat{F}}\mathrm{supp}(\phi_{M_i})\right)} 2^{-l}},\\
&q_i^\alpha = \tau_i^\alpha \times \bar{q}_i,\\
&\tau^\alpha = \tau_1^\alpha \times \tau_2^\alpha,\\
&q^\alpha = k^{-1}\tau_1^\alpha \times 1_{L+2} + k^{-1}1_{L+2} \times \tau_2^\alpha + (1-2k^{-1})1_{(L+2,L+2)},\\
&\mu^\alpha = (1-k^{-2})q^\alpha + k^{-3}\tau^\alpha, \text{ and}\\
&p^\alpha(\phi) = (1-k^{-1})(\beta_1\phi_1 + \beta_2\phi_2) + k^{-1}\mu^\alpha.
\end{aligned}$$

Furthermore, let $\upsilon_X \in \Delta(X)$ be uniform on X whenever X is a finite set and let

$$\pi_i^{1,\alpha} = (1-k^{-3})\pi_i^1 + k^{-3}\upsilon_{\hat{F}} \text{ and } \pi_i^{2,\alpha}(m_i,\phi_i) = (1-k^{-1})\pi_i^2(m_i,\phi_i) + k^{-1}\upsilon_{A_i}$$

for each $(m_i, \phi_i) \in M_i \times S$.

Let $\varepsilon > 0$. We have that the conditions (i)–(v) in the definition of perfect conditional ε-equilibrium hold by construction. We will show that condition (vi) holds for some subnet of $\{\pi^\alpha, p^\alpha\}_\alpha$. Some technical details of this argument are simplified by our construction of $\{\pi^\alpha, p^\alpha\}_\alpha$ which is such that $\text{supp}(\pi^{1,\alpha})$ and $\text{supp}(p^\alpha)$ are finite for each α. We define

$$S_i(F, \hat{F}) = \Big(F \cup (\cup_{\phi\in\hat{F}} \text{supp}(\phi_{M_i})) \cup \{1, \ldots, L+2\}\Big) \times \Big(\hat{F} \cup \{\phi_i^l : 1 \leq l \leq L\}\Big)$$

which is the set of pairs (m_i, ϕ_i) that can occur with strictly positive probability. Indeed, if $(m, \phi) \in \mathbb{N}^2 \times S^2$ is such that $\pi^{1,\alpha}[\phi] > 0$ and $\sum_{\phi'\in\text{supp}(\pi^{1,\alpha})} p^\alpha(\phi')[m] > 0$, then $(m_i, \phi_i) \in S_i(F, \hat{F})$ for each $i \in N$.

Recall that $\alpha = (k, F, \hat{F})$. In what follows, we will often fix F and $\hat{F}$ and take limits as $k \to \infty$. Regarding condition (vi) (a), let $i, j \in N$, $j \neq i$ and $\phi_i' \in S$. We have that, for each finite subsets F and $\hat{F}$ of $\mathbb{N}$ and S, respectively,

$$\lim_k \sum_{\phi\in\text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] \left(\sum_m p^\alpha(\phi)[m] u_i(\pi^{2,\alpha}(m,\phi))\right) = \sum_\phi \pi^1[\phi] \sum_m \phi[m] u_i(\pi(m,\phi))$$
$$= \frac{1}{L} \sum_{\phi_i} \pi_i^1[\phi_i] \sum_{l=1}^{L} \sum_m (\phi_i, \phi_j^l)[m] u_i(\pi_i(m_i, \phi_i), \pi_j(m_j, \phi_j^l))$$

and that

$$\lim_k \sum_{\phi\in\text{supp}(1_{\phi_i'} \times \pi_j^{1,\alpha})} (1_{\phi_i'} \times \pi_j^{1,\alpha})[\phi] \left(\sum_m p^\alpha(\phi)[m] u_i(\pi^{2,\alpha}(m,\phi))\right)$$
$$= \frac{1}{L} \sum_{l=1}^{L} \sum_m (\phi_i', \phi_j^l)[m] u_i(\pi_i(m_i, \phi_i'), \pi_j(m_j, \phi_j^l)).$$

Hence, by considering α such that $k \geq k_0$ for some $k_0 \in \mathbb{N}$, it is enough to show that, for each $1 \leq h \leq L$,

$$\sum_{l=1}^{L} \sum_m (\phi_i^h, \phi_j^l)[m] u_i(\pi_i(m_i, \phi_i^h), \pi_j(m_j, \phi_j^l)) \geq \sum_{l=1}^{L} \sum_m (\phi_i', \phi_j^l)[m] u_i(\pi_i(m_i, \phi_i'), \pi_j(m_j, \phi_j^l)),$$

which is equivalent to

$$\frac{1}{L} \sum_{l=1}^{L} \sum_m \phi_i^h[m] u_i(\pi_i(m_i, \phi_i^h), \pi_j(m_j, \phi_j^l)) \geq \frac{1}{L} \sum_{l=1}^{L} \sum_m \phi_i'[m] u_i(\pi_i(m_i, \phi_i'), \pi_j(m_j, \phi_j^l)). \tag{3.4}$$

We have that $\frac{1}{L}\sum_{l=1}^{L}\sum_m \phi_i^h[m]u_i(\pi_i(m_i,\phi_i^h),\pi_j(m_j,\phi_j^l)) = v_i(\sigma_j^i)$ and that

$$\frac{1}{L}\sum_{l=1}^{L}\sum_m \phi_i'[m]u_i(\pi_i(m_i,\phi_i'),\pi_j(M_j,\phi_j^l)) =$$
$$\frac{1}{L}\sum_{l=1}^{L}\left(\phi_{i,M_j}'[l]v_i(\sigma_j^j)+\phi_{i,M_j}'[L+1]v_i(\sigma_j^i)+(1-\phi_{i,M_j}'[l]-\phi_{i,M_j}'[L+1])v_i(\bar\sigma_j)\right) =$$
$$\frac{\phi_{i,M_j}'[\{1,\ldots,L\}]}{L}v_i(\sigma_j^j)+\phi_{i,M_j}'[L+1]v_i(\sigma_j^i)+\left(1-\frac{\phi_{i,M_j}'[\{1,\ldots,L\}]}{L}-\phi_{i,M_j}'[L+1]\right)v_i(\bar\sigma_j) =$$
$$\frac{x}{L}2+x'+\left(1-\frac{x}{L}-x'\right)\frac{2}{3}$$

where $x=\phi_{i,M_j}'[\{1,\ldots,L\}]$ and $x'=\phi_{i,M_j}'[L+1]$. Thus, (3.4) holds if $x'=1$; it also holds when $x'<1$ by (3.3).

Consider next condition (vi) (b). For each $i,j\in N$, $i\neq j$, finite subset F of $\mathbb{N}$, finite subset $\hat F$ of S, $(m_i,\phi_i)\in S_i(F,\hat F)$ and $\gamma_i\in\Delta(A_i)$, we have that

$$\lim_k \frac{\sum_{\phi_j\in\text{supp}(\pi_j^{1,\alpha})}\pi_j^{1,\alpha}[\phi_j]\left(\sum_{m_j}p^\alpha(\phi_i,\phi_j)[m_i,m_j]u_i(\gamma_i,\pi_j^{2,\alpha}(m_j,\phi_j))\right)}{\sum_{\phi_j\in\text{supp}(\pi_j^{1,\alpha})}\pi_j^{1,\alpha}[\phi_j]p_{M_i}^\alpha(\phi_i,\phi_j)[m_i]}$$
$$= u_i(\gamma_i,\bar\sigma_j)$$

if $\beta_i\phi_{i,M_i}[m_i]+\beta_j\sum_{l=1}^{L}\frac{\phi_{j,M_i}^l[m_i]}{L}=0$, and

$$\lim_k \frac{\sum_{\phi_j\in\text{supp}(\pi_j^{1,\alpha})}\pi_j^{1,\alpha}[\phi_j]\left(\sum_{m_j}p^\alpha(\phi_i,\phi_j)[m_i,m_j]u_i(\gamma_i,\pi_j^{2,\alpha}(m_j,\phi_j))\right)}{\sum_{\phi_j\in\text{supp}(\pi_j^{1,\alpha})}\pi_j^{1,\alpha}[\phi_j]p_{M_i}^\alpha(\phi_i,\phi_j)[m_i]} =$$
$$\sum_{m_j}\frac{\beta_i\phi_i[m_i,m_j]+\beta_j\sum_{l=1}^{L}\frac{\phi_{j,M_i}^l[m_i,m_j]}{L}}{\beta_i\phi_{i,M_i}[m_i]+\beta_j\sum_{l=1}^{L}\frac{\phi_{j,M_i}^l[m_i]}{L}}u_i(\gamma_i,\pi_j(m_j,\phi_j^l))$$

if $\beta_i\phi_{i,M_i}[m_i]+\beta_j\sum_{l=1}^{L}\frac{\phi_{j,M_i}^l[m_i]}{L}>0$. The latter case is clear since all terms in the denominator of the fraction converge to zero except the one that converges to $\beta_i\phi_{i,M_i}[m_i]+\beta_j\sum_{l=1}^{L}\frac{\phi_{j,M_i}^l[m_i]}{L}$ and similarly regarding the numerator.

In the former case, both the numerator and the denominator converge to zero since $\beta_i\phi_{i,M_i}[m_i]+\beta_j\sum_{l=1}^{L}\frac{\phi_{j,M_i}^l[m_i]}{L}=0$. Multiplying each by k, it follows that all terms converge to zero except the ones corresponding to the case where $\pi_j^{1,\alpha}=\phi_j^1$

and $p^\alpha(\phi_i, \phi_j^*) = q^\alpha$. If $m_i \neq L+2$, then

$$q^\alpha[m_i, m_j] = \begin{cases} k^{-1}\tau_i^\alpha[m_i] & \text{if } m_j = L+2, \\ 0 & \text{otherwise,} \end{cases}$$

$q_{M_i}^\alpha[m_i] = k^{-1}\tau_i^\alpha[m_i]$ and

$$\frac{q^\alpha[m_i, L+2]}{q_{M_i}^\alpha[m_i]} = 1;$$

if $m_i = L+2$, then

$$q^\alpha[L+2, m_j] = \begin{cases} k^{-1}\tau_i^\alpha[L+2] + k^{-1}\tau_j^\alpha[L+2] + 1 - 2k^{-1} & \text{if } m_j = L+2, \\ k^{-1}\tau_j^\alpha[m_j] & \text{otherwise,} \end{cases}$$

$q_{M_i}^\alpha[L+2] = 1 - k^{-1} + k^{-1}\tau_i^\alpha[L+2]$ and

$$\lim_k \frac{q^\alpha[L+2, L+2]}{q_{M_i}^\alpha[L+2]} = 1.$$

Thus,

$$\lim_k \frac{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] \left(\sum_{m_j} p^\alpha(\phi_i, \phi_j)[m_i, m_j] u_i(\gamma_i, \pi_j^{2,\alpha}(m_j, \phi_j))\right)}{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] p_{M_i}^\alpha(\phi_i, \phi_j)[m_i]}$$

$$= \frac{1}{L}\sum_{l=1}^{L} u_i(\gamma_i, \pi_j(L+2, \phi_j^l)) = u_i(\gamma_i, \bar{\sigma}_j).$$

We will next show that $\pi_i(m_i, \phi_i)$ solves

$$\max_{\gamma_i \in \Delta(A_i)} \lim_k \frac{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] \left(\sum_{m_j} p^\alpha(\phi_i, \phi_j)[m_i, m_j] u_i(\gamma_i, \pi_j^{2,\alpha}(m_j, \phi_j))\right)}{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] p_{M_i}^\alpha(\phi_i, \phi_j)[m_i]} \quad (3.5)$$

for each $i \in N$ and $(m_i, \phi_i) \in S_i(F, \hat{F})$.

If $\beta_i \phi_{i,M_i}[m_i] + \beta_j \sum_{l=1}^{L} \frac{\phi_{j,M_i}^l[m_i]}{L} = 0$, then (3.5) follows because $\pi_i(m_i, \phi_i) = \bar{\sigma}_i$ and $\bar{\sigma}$ is a Nash equilibrium.

If $\beta_i \phi_{i,M_i}[m_i] + \sum_{l=1}^{L} \frac{\phi_{j,M_i}^l[m_i]}{L} > 0$, then

$$\lim_k \frac{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] \left(\sum_{m_j} p^\alpha(\phi_i, \phi_j)[m_i, m_j] u_i(\gamma_i, \pi_j^{2,\alpha}(m_j, \phi_j))\right)}{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] p_{M_i}^\alpha(\phi_i, \phi_j)[m_i]}$$

$$= \sum_{m_j} \frac{\beta_i \phi_i[m_i, m_j] + \beta_j \sum_{l=1}^{L} \frac{\phi_j^l[m_i, m_j]}{L}}{\beta_i \phi_{i,M_i}[m_i] + \beta_j \sum_{l=1}^{L} \frac{\phi_{j,M_i}^l[m_i]}{L}} u_i(\gamma_i, \pi_j(m_j, \phi_j^l))$$

$$= u_i\left(\gamma_i, \sum_{m_j} \frac{\beta_i \phi_i[m_i, m_j] + \beta_j \sum_{l=1}^{L} \frac{\phi_j^l[m_i, m_j]}{L}}{\beta_i \phi_{i,M_i}[m_i] + \beta_j \sum_{l=1}^{L} \frac{\phi_{j,M_i}^l[m_i]}{L}} \pi_j(m_j, \phi_j^l)\right).$$

Since $\pi_i(m_i, \phi_i) \in BR_i\left(\sum_{m_j} \frac{\beta_i \phi_i[m_i,m_j] + \beta_j \sum_{l=1}^{L} \frac{\phi_j^l[m_i,m_j]}{L}}{\beta_i \phi_{i,M_i}[m_i] + \beta_j \sum_{l=1}^{L} \frac{\phi_{j,M_i}^l[m_i]}{L}} \pi_j(m_j, \phi_j^l)\right)$, it follows that (3.5) holds in this case.

The above arguments show that, for each finite subsets F of $\mathbb{N}$ and $\hat{F}$ of S, condition 6 holds whenever k is sufficiently high. Specifically, condition 6 (a) holds for each $i \in N$ whenever $k \geq k_0$. For each $i \in N$ and $(m_i, \phi_i) \in S_i(F, \hat{F})$, there is $k(m_i, \phi_i)$ such that condition 6 (b) holds whenever $k \geq k(m_i, \phi_i)$. Thus, let

$$k(F, \hat{F}) = \max\left\{k_0, \max_{i \in N} \max_{(m_i,\phi_i) \in S_i(F,\hat{F})} k(m_i, \phi_i)\right\}.$$

Since condition 6 (b) is trivially satisfied when

$$\pi_i^{1,\alpha}[\phi_i] \sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] p_{M_i}^{\alpha}(\phi_i, \phi_j)[m_i] = 0,$$

i.e. when $i \in N$ and $(m_i, \phi_i) \notin S_i(F, \hat{F})$, it follows that condition 6 holds whenever $k \geq k(F, \hat{F})$. This allows us to define the following subnet $\{\pi^{\varphi(\eta)}, p^{\varphi(\eta)}\}_\eta$ of $\{\pi^\alpha, p^\alpha\}_\alpha$ such that condition 6 holds.

The index set of the subnet $\{\pi^{\varphi(\eta)}, p^{\varphi(\eta)}\}_\eta$ is the same as the one in the net $\{\pi^\alpha, p^\alpha\}_\alpha$. The function $\varphi : \eta \mapsto \alpha$ is defined by setting, for each $\eta = (k, F, \hat{F})$,

$$\varphi(\eta) = \left(\max\left\{k, k(F, \hat{F})\right\}, F, \hat{F}\right).$$

It is then clear that condition 6 holds and that, as required by the definition of a subnet, for each α_0, there exists η_0, e.g. $\eta_0 = \alpha_0$, such that $\varphi(\eta) \geq \alpha_0$ for each $\eta \geq \eta_0$.

3.3 Proof of Theorem 3.2

Let $i \in N$ and $j \neq i$. Let $\bar{\sigma}^i$ be such that $u_i(\bar{\sigma}^i) = \min_{\sigma \in N(G)} u_i(\sigma)$.

For each $i \in N$ and $1 \leq l \leq L$, let

$$\phi_1^l = 1_{(l,L+1)} \text{ and } \phi_2^l = 1_{(L+1,l)} \text{ and}$$

$$\pi_i^{L,1} = \frac{1}{L} \sum_{l=1}^{L} 1_{\phi_i^l}.$$

For each $i \in N$ and $1 \leq l \leq L$, let

$$\pi_i^{L,2}(m_i, \phi_i^l) = \begin{cases} \sigma_i^i & \text{if } m_i = l, \\ \sigma_i^j & \text{if } m_i = L+1, \\ \bar{\sigma}_i^j & \text{otherwise.} \end{cases}$$

For each $i, j \in N$, $j \neq i$ and $m_j \in M_j \setminus \{L+1\}$, let

$$\bar{\phi}_i^{m_j} = 1_{(L+2, m_j)}.$$

Then set

$$\pi_i^{L,2}(L+2, \bar{\phi}_i^{m_j}) = \bar{\sigma}_i^i.$$

For each $m_i \in M_i$ and $\phi_i \in S \setminus \{\phi_i^l : 1 \leq l \leq L\}$ such that $\beta_i \phi_{i,M_i}[m_i] + \beta_j \sum_{l=1}^L \frac{\phi_{j,M_i}^l[m_i]}{L} = 0$, let $\pi_i^{L,2}(m_i, \phi_i) = \bar{\sigma}_i^j$.

For each $(m_i, \phi_i) \in (M \times S \setminus \{\phi_i^l : 1 \leq l \leq L\}) \setminus \{(L+2, \bar{\phi}_i^{m_j}) : m_j \in M_j \setminus \{L+1\}\}$ such that $\beta_i \phi_{i,M_i}[m_i] + \beta_j \sum_{l=1}^L \frac{\phi_{j,M_i}^l[m_i]}{L} > 0$, let $\pi_i^{L,2}(m_i, \phi_i)$ be a best-reply against

$$\frac{1}{L} \sum_{l=1}^{L} \sum_{m_j} \frac{\beta_i \phi_i[m_i, m_j] + \beta_j \phi_j^l[m_i, m_j]}{\beta_i \phi_{i,M_i}[m_i] + \beta_j \sum_{l=1}^L \frac{\phi_{j,M_i}^l[m_i]}{L}} \pi_j^2(m_j, \phi_j^l).$$

We may assume that $\pi_i^{L,2} : M_i \times S \to \Delta(A_i)$ is measurable as in the proof of Theorem 1.

Note that $\pi_i^{L,2}(m_i, \phi_i) = \bar{\sigma}_i^j$ for each $m_i \in M_i$ and $\phi_i \in S$ such that $\beta_i \phi_{i,M_i}[m_i] + \beta_j \sum_{l=1}^L \frac{\phi_{j,M_i}^l[m_i]}{L} = 0$ and that $\pi_i^{L,2}(m_i, \phi_i)$ is a best-reply against

$$\frac{1}{L} \sum_{l=1}^{L} \sum_{m_j} \frac{\beta_i \phi_i[m_i, m_j] + \beta_j \phi_j^l[m_i, m_j]}{\beta_i \phi_{i,M_i}[m_i] + \beta_j \sum_{l=1}^L \frac{\phi_{j,M_i}^l[m_i]}{L}} \pi_j^2(m_j, \phi_j^l).$$

for each $(m_i, \phi_i) \in (M_i \times S) \setminus \{(L+2, \bar{\phi}_i^{m_j}) : m_j \in M_j \setminus \{L+1\}\}$ such that $\beta_i \phi_{i,M_i}[m_i] + \beta_j \sum_{l=1}^L \frac{\phi_{j,M_i}^l[m_i]}{L} > 0$.

Let $\varepsilon > 0$ and let $\bar{L} \in \mathbb{N}$ be such that, for each $L \geq \bar{L}$,

$$\bar{\sigma}_i^i \text{ is a } \frac{\varepsilon}{2}\text{-best-reply against } \frac{L-1}{L} \bar{\sigma}_j^i + \frac{1}{L} \sigma_j^j, \text{ and} \tag{3.6}$$

$$\left(1 - \frac{1}{L}\right) u_i(\bar{\sigma}^i) + \frac{1}{L} u_i(\sigma^j) - u_i(\sigma^i) < \frac{\varepsilon}{2}. \tag{3.7}$$

It follows from (3.7) that

$$\left(1-\frac{\alpha}{L}-\alpha'\right)v_i(\bar{\sigma}_j^i)+\frac{\alpha}{L}v_i(\sigma_j^j)+\alpha' v_i(\sigma_j^i)<v_i(\sigma_j^i)+\frac{\varepsilon}{2} \tag{3.8}$$

for each $\alpha,\alpha'\in[0,1]$ such that $\alpha+\alpha'=1$.

Let $L\geq\bar{L}$. We define perturbations $\{\pi^\alpha,p^\alpha\}_\alpha$ such that $\{\pi^\alpha\}_\alpha$ converges to π^L as follows. The index set consists of $\alpha=(k,F,\hat{F})$ such that $k\in\mathbb{N}$, F is a finite subset of $\mathbb{N}$ and $\hat{F}$ is a finite subset of S; this set is partially ordered by defining $(k',F',\hat{F}')\geq(k,F,\hat{F})$ if $k'\geq k$, $F\subseteq F'$ and $\hat{F}\subseteq\hat{F}'$. Let $\upsilon_X\in\Delta(X)$ be uniform on X whenever X is a finite set. For each $\alpha=(k,F,\hat{F})$, let

$$p^\alpha(\phi)=(1-k^{-2})(\beta_1\phi_1+\beta_2\phi_2)+k^{-2}\upsilon_{F^2}.$$

For each $\alpha=(k,F,\hat{F})$ and $i\in N$, let $T_i(F,\hat{F})=(F\cup(\cup_{\phi\in\hat{F}}\mathrm{supp}(\phi_{M_i})))\setminus\{L+1\}$,

$$\pi_i^{1,\alpha}=(1-k^{-1})\pi_i^1+k^{-1}(1-k^{-1})|T_j(F,\hat{F})|^{-1}\sum_{m_j\in T_j(F,\hat{F})}1_{\bar{\phi}_i^{m_j}}+k^{-2}\upsilon_{\hat{F}}.$$

For each $\alpha=(k,F,\hat{F})$, $i\in N$ and $(m_i,\phi_i)\in M_i\times S$, let

$$\pi_i^{2,\alpha}(m_i,\phi_i)=(1-k^{-1})\pi_i^2(m_i,\phi_i)+k^{-1}\upsilon_{A_i}.$$

We have that the conditions (i)–(v) hold by construction. We will show that condition (vi) holds for some subnet of $\{\pi^\alpha,p^\alpha\}_\alpha$. Recall that $\alpha=(k,F,\hat{F})$. In what follows, we often fix F and $\hat{F}$ and take limits as $k\to\infty$.

Regarding condition (vi) (a), let $i,j\in N$, $j\neq i$ and $\phi_i'\in S$. We have that, for each finite subsets F and $\hat{F}$ of $\mathbb{N}$ and S, respectively,

$$\lim_k\sum_{\phi\in\mathrm{supp}(\pi^{1,\alpha})}\pi^{1,\alpha}[\phi]\left(\sum_m p^\alpha(\phi)[m]u_i(\pi^{2,\alpha}(m,\phi))\right)=\sum_\phi\pi^{L,1}[\phi]\sum_m\phi[m]u_i(\pi^L(m,\phi))$$
$$=\frac{1}{L}\sum_{\phi_i}\pi_i^{L,1}[\phi_i]\sum_{l=1}^{L}\sum_m(\phi_i,\phi_j^l)[m]u_i(\pi_i^{L,2}(m_i,\phi_i),\pi_j^{L,2}(m_j,\phi_j^l))$$

and that

$$\lim_k\sum_{\phi\in\mathrm{supp}(1_{\phi_i'}\times\pi_j^{1,\alpha})}(1_{\phi_i'}\times\pi_j^{1,\alpha})[\phi]\left(\sum_m p^\alpha(\phi)[m]u_i(\pi^{2,\alpha}(m,\phi))\right)$$
$$=\frac{1}{L}\sum_{l=1}^{L}\sum_m(\phi_i',\phi_j^l)[m]u_i(\pi_i^{L,2}(m_i,\phi_i'),\pi_j^{L,2}(m_j,\phi_j^l)).$$

Hence, by considering α such that $k \geq k_0$ for some $k_0 \in \mathbb{N}$, it is enough to show that, for each $1 \leq h \leq L$,

$$\frac{1}{L}\sum_{l=1}^{L}\sum_{m}(\phi_i^h, \phi_j^l)[m]u_i(\pi_i^{L,2}(m_i, \phi_i^h), \pi_j^{L,2}(m_j, \phi_j^l)) \geq$$
$$\frac{1}{L}\sum_{l=1}^{L}\sum_{m}(\phi_i', \phi_j^l)[m]u_i(\pi_i^{L,2}(m_i, \phi_i'), \pi_j^{L,2}(m_j, \phi_j^l)) - \frac{\varepsilon}{2},$$

which is equivalent to

$$\begin{aligned}&\frac{1}{L}\sum_{l=1}^{L}\sum_{m}\phi_i^h[m]u_i(\pi_i^{L,2}(m_i, \phi_i^h), \pi_j^{L,2}(m_j, \phi_j^l)) \geq \\ &\frac{1}{L}\sum_{l=1}^{L}\sum_{m}\phi_i'[m]u_i(\pi_i^{L,2}(m_i, \phi_i'), \pi_j^{L,2}(m_j, \phi_j^l)) - \frac{\varepsilon}{2}.\end{aligned} \tag{3.9}$$

We have that $\frac{1}{L}\sum_{l=1}^{L}\sum_{m}\phi_i^h[m]u_i(\pi_i^{L,2}(m_i, \phi_i^h), \pi_j^{L,2}(m_j, \phi_j^l)) = v_i(\sigma_j^i)$ and that

$$\frac{1}{L}\sum_{l=1}^{L}\sum_{m}\phi_i'[m]u_i(\pi_i^{L,2}(m_i, \phi_i'), \pi_j^{L,2}(m_j, \phi_j^l)) =$$
$$\frac{1}{L}\sum_{l=1}^{L}\left(\phi'_{i,m_j}[l]v_i(\sigma_j^j) + \phi'_{i,m_j}[L+1]v_i(\sigma_j^i) + (1 - \phi'_{i,m_j}[l] - \phi'_{i,m_j}[L+1])v_i(\bar{\sigma}_j^i)\right) =$$
$$\frac{\phi'_{i,m_j}[\{1,\ldots,L\}]}{L}v_i(\sigma_j^j) + \phi'_{i,m_j}[L+1]v_i(\sigma_j^i) + \left(1 - \frac{\phi'_{i,m_j}[\{1,\ldots,L\}]}{L} - \phi'_{i,m_j}[L+1]\right)v_i(\bar{\sigma}_j^i).$$

Thus, (3.9) holds by (3.8).

Consider next condition (vi) (b). Let $i, j \in N$, $i \neq j$, F be a finite subset of $\mathbb{N}$, $\hat{F}$ be a finite subset of S, (m_i, ϕ_i) be such that $\pi_i^{1,\alpha}[\phi_i]\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})}\pi_j^{1,\alpha}[\phi_j]p_{M_i}^{\alpha}(\phi_i, \phi_j)[m_i] > 0$ and $\gamma_i \in \Delta(A_i)$ be given. We have

$$\lim_k \frac{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})}\pi_j^{1,\alpha}[\phi_j]\left(\sum_{m_j}p^{\alpha}(\phi_i, \phi_j)[m_i, m_j]u_i(\gamma_i, \pi_j^{2,\alpha}(m_j, \phi_j))\right)}{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})}\pi_j^{1,\alpha}[\phi_j]p_{M_i}^{\alpha}(\phi_i, \phi_j)[m_i]} =$$
$$u_i(\gamma_i, \bar{\sigma}_j^j)$$

if $\beta_i\phi_{i,M_i}[m_i] + \beta_j\sum_{l=1}^{L}\frac{\phi_{j,M_i}^l[m_i]}{L} = 0$, and

$$\lim_k \frac{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})}\pi_j^{1,\alpha}[\phi_j]\left(\sum_{m_j}p^{\alpha}(\phi_i, \phi_j)[m_i, m_j]u_i(\gamma_i, \pi_j^{2,\alpha}(m_j, \phi_j))\right)}{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})}\pi_j^{1,\alpha}[\phi_j]p_{M_i}^{\alpha}(\phi_i, \phi_j)[m_i]} =$$
$$\frac{1}{L}\sum_{l=1}^{L}\sum_{m_j}\frac{\beta_i\phi_i[m_i, m_j] + \beta_j\phi_j^l[m_i, m_j]}{\beta_i\phi_{i,M_i}[m_i] + \beta_j\sum_{l=1}^{L}\frac{\phi_{j,M_i}^l[m_i]}{L}}u_i(\gamma_i, \pi_j^{L,2}(m_j, \phi_j^l))$$

if $\beta_i \phi_{i,M_i}[m_i] + \beta_j \sum_{l=1}^{L} \frac{\phi_{j,M_i}^l[m_i]}{L} > 0$. The latter case is clear since all terms in the denominator of the fraction converge to zero except the one that converges to $\beta_i \phi_{i,M_i}[m_i] + \beta_j \sum_{l=1}^{L} \frac{\phi_{j,M_i}^l[m_i]}{L}$ and similarly regarding the numerator.

In the former case, both the numerator and the denominator converge to zero since $\beta_i \phi_{i,M_i}[m_i] + \beta_j \sum_{l=1}^{L} \frac{\phi_{j,M_i}^l[m_i]}{L} = 0$. Multiplying each by k, it follows that all terms converge to zero except the ones corresponding to the case where $\phi_j = \bar{\phi}_j^{m_i}$ and $p^\alpha(\phi_i, \bar{\phi}_j) = \beta_i \phi_i + \beta_j \phi_j$. Thus,

$$
\begin{aligned}
&\lim_k \frac{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] \left(\sum_{m_j} p^\alpha(\phi_i, \phi_j)[m_i, m_j] u_i(\gamma_i, \pi_j^{2,\alpha}(m_j, \phi_j)) \right)}{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] p_{M_i}^\alpha(\phi_i, \phi_j)[m_i]} \\
&= \frac{\sum_{m_j} \beta_j \bar{\phi}_j^{m_i}[m_i, m_j] u_i(\gamma_i, \pi_j^{L,2}(m_j, \bar{\phi}_j^{m_i}))}{\sum_{m_j} \beta_j \bar{\phi}_j^{m_i}[m_i, m_j]} \\
&= u_i(\gamma_i, \pi_j^{L,2}(L+2, \bar{\phi}_j^{m_i})) = u_i(\gamma_i, \bar{\sigma}_j^j).
\end{aligned}
$$

We will next show that $\pi_i^{L,2}(m_i, \phi_i)$ solves

$$
\max_{\gamma_i \in \Delta(A_i)} \lim_k \frac{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] \left(\sum_{m_j} p^\alpha(\phi_i, \phi_j)[m_i, m_j] u_i(\gamma_i, \pi_j^{2,\alpha}(m_j, \phi_j)) \right)}{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] p_{M_i}^\alpha(\phi_i, \phi_j)[m_i]} \tag{3.10}
$$

for each $i \in N$ and (m_i, ϕ_i) such that $\pi_i^{1,\alpha}[\phi_i] \sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] p_{M_i}^\alpha(\phi_i, \phi_j)[m_i] > 0$.

If $\beta_i \phi_{i,M_i}[m_i] + \beta_j \sum_{l=1}^{L} \frac{\phi_{j,M_i}^l[m_i]}{L} = 0$, then (3.10) follows because $\pi_i^{L,2}(m_i, \phi_i) = \bar{\sigma}_i^j$ and $\bar{\sigma}^j$ is a Nash equilibrium.

If $\beta_i \phi_{i,M_i}[m_i] + \sum_{l=1}^{L} \frac{\phi_{j,M_i}^l[m_i]}{L} > 0$ and $(m_i, \phi_i) \notin \{(L+2, \bar{\phi}_i^{m_j}) : m_j \in M_j \setminus \{L+1\}\}$, then

$$
\begin{aligned}
&\lim_k \frac{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] \left(\sum_{m_j} p^\alpha(\phi_i, \phi_j)[m_i, m_j] u_i(\gamma_i, \pi_j^{2,\alpha}(m_j, \phi_j)) \right)}{\sum_{\phi_j \in \text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] p_{M_i}^\alpha(\phi_i, \phi_j)[m_i]} \\
&= \frac{1}{L} \sum_{l=1}^{L} \sum_{m_j} \frac{\beta_i \phi_i[m_i, m_j] + \beta_j \phi_j^l[m_i, m_j]}{\beta_i \phi_{i,M_i}[m_i] + \beta_j \sum_{l=1}^{L} \frac{\phi_{j,M_i}^l[m_i]}{L}} u_i(\gamma_i, \pi_j^{L,2}(m_j, \phi_j^l)) \\
&= u_i \left(\gamma_i, \frac{1}{L} \sum_{l=1}^{L} \sum_{m_j} \frac{\beta_i \phi_i[m_i, m_j] + \beta_j \phi_j^l[m_i, m_j]}{\beta_i \phi_{i,M_i}[m_i] + \beta_j \sum_{l=1}^{L} \frac{\phi_{j,M_i}^l[m_i]}{L}} \pi_j^{L,2}(m_j, \phi_j^l) \right).
\end{aligned}
$$

Since $\pi_i^{L,2}(m_i, \phi_i) \in BR_i\left(\frac{1}{L}\sum_{l=1}^{L}\sum_{m_j}\frac{\beta_i\phi_i[m_i,m_j]+\beta_j\phi_j^l[m_i,m_j]}{\beta_i\phi_{i,M_i}[m_i]+\beta_j\sum_{l=1}^{L}\frac{\phi^l_{j,M_i}[m_i]}{L}}\pi_j^{L,2}(m_j,\phi_j^l)\right)$, it follows that (3.10) holds in this case.

Finally, for $(m_i, \phi_i) \in \{(L+2, \bar{\phi}_i^{m_j}) : m_j \in M_j \setminus \{L+1\}\}$, note that

$$\begin{aligned}
&\frac{1}{L}\sum_{l=1}^{L}\frac{\beta_i\bar{\phi}_i^{m_j}[L+2,m_j]+\beta_j\phi_j^l[L+2,m_j]}{\beta_i\bar{\phi}_{i,M_i}^{m_j}[L+2]+\beta_j\sum_{l=1}^{L}\frac{\phi^l_{j,M_i}[L+2]}{L}}\pi_j^{L,2}(m_j,\phi_j^l)\\
&=\frac{1}{L}\sum_{l=1}^{L}\pi_j^{L,2}(m_j,\phi_j^l)\\
&=\lambda\bar{\sigma}_j^i+(1-\lambda)\sigma_j^j,
\end{aligned}$$

where

$$\lambda = \begin{cases} 1 & \text{if } m_j \notin \{1,\dots,L\},\\ 1-1/L & \text{if } m_j \in \{1,\dots,L\}.\end{cases}$$

Thus, $\pi_i^{L,2}(L+2, \bar{\phi}_i^{m_j}) = \bar{\sigma}_i^i$ is an $\frac{\varepsilon}{2}$-best-reply against $\lambda\bar{\sigma}_j^i + (1-\lambda)\sigma_j^j$.

The above arguments show that, for each finite subsets F of $\mathbb{N}$ and $\hat{F}$ of S, condition (vi) holds whenever k is sufficiently high. Specifically, condition (vi) (a) holds for each $i \in N$ whenever $k \geq k_0$. For each $i \in N$ and (m_i, ϕ_i) such that

$$\pi_i^{1,\alpha}[\phi_i]\sum_{\phi_j\in\mathrm{supp}(\pi_j^{1,\alpha})}\pi_j^{1,\alpha}[\phi_j]p^\alpha_{M_i}(\phi_i,\phi_j)[m_i] > 0,$$

there is $k(m_i, \phi_i)$ such that condition (vi) (b) holds whenever $k \geq k(m_i, \phi_i)$. Thus, let

$$k(F,\hat{F}) = \max\left\{k_0, \max_{i\in N}\max_{(m_i,\phi_i)} k(m_i,\phi_i)\right\}.$$

Since condition (vi) (b) is trivially satisfied when

$$\pi_i^{1,\alpha}[\phi_i]\sum_{\phi_j\in\mathrm{supp}(\pi_j^{1,\alpha})}\pi_j^{1,\alpha}[\phi_j]p^\alpha_{M_i}(\phi_i,\phi_j)[m_i] = 0,$$

it follows that condition (vi) holds whenever $k \geq k(F, \hat{F})$. This allows us to define the following subnet $\{\pi^{\varphi(\eta)}, p^{\varphi(\eta)}\}_\eta$ of $\{\pi^\alpha, p^\alpha\}_\alpha$ such that condition (vi) holds.

The index set of the subnet $\{\pi^{\varphi(\eta)}, p^{\varphi(\eta)}\}_\eta$ is the same as the one in the net $\{\pi^\alpha, p^\alpha\}_\alpha$. The function $\varphi : \eta \mapsto \alpha$ is defined by setting, for each $\eta = (k, F, \hat{F})$,

$$\varphi(\eta) = \left(\max\left\{k, k(F,\hat{F})\right\}, F, \hat{F}\right).$$

It is then clear that condition (vi) holds and that, as required by the definition of a subnet, for each α_0, there exists η_0, e.g. $\eta_0 = \alpha_0$, such that $\varphi(\eta) \geq \alpha_0$ for each $\eta \geq \eta_0$.

4 More than two players

Consider a normal-form game $G = (A_i, u_i)_{i \in N}$ where the set N of players is finite. The number of players is $n = |N| \geq 2$. Let S be the set of finitely supported probability measures on $M = \prod_{i \in N} M_i = \mathbb{N}^n$.

We allow for $\beta_i = 0$ for some $i \in N$, in which case only the players in $\mathrm{supp}(\beta) = \{i \in N : \beta_i > 0\}$ choose an information design $\phi_i \in S$. The players' interaction is then described by the following extensive-form game G_{id}. At the beginning of the game, each player $i \in \mathrm{supp}(\beta)$ chooses an information design $\phi_i \in S$. After all players in $\mathrm{supp}(\beta)$ have chosen their information design, a profile of signals $m \in M$ is realized according to $\phi \in \Delta(M)$ defined by setting, for each $m \in M$,

$$\phi[m] = \sum_{i \in \mathrm{supp}(\beta)} \beta_i \phi_i[m].$$

Each player $i \in N$ observes $m_i \in M_i$ and, if $i \in \mathrm{supp}(\beta)$, his choice $\phi_i \in S$, and then chooses an action $a_i \in A_i$. Player i's payoff is then $u_i(a_1, \ldots, a_n)$.

A (behavioral) strategy for player $i \in \mathrm{supp}(\beta)$ is $\pi_i = (\pi_i^1, \pi_i^2)$ such that $\pi_i^1 \in \Delta(S)$ and $\pi_i^2 : M_i \times S \to \Delta(A_i)$ is measurable; and, for $i \in N \setminus \mathrm{supp}(\beta)$, it is $\pi_i = \pi_i^2$ with $\pi_i^2 : M_i \to \Delta(A_i)$. A strategy is $\pi = (\pi_1, \ldots, \pi_n)$. Let Π be the set of strategies π such that $\pi_i^1 \in S$ (i.e. π_i^1 is pure) for each $i \in \mathrm{supp}(\beta)$ and we focus on $\pi \in \Pi$.

In the statement of Theorem 4.1, we use the convention that $\mathrm{supp}(\phi_i^*) = \emptyset$ for each $i \notin \mathrm{supp}(\beta)$ and let, for each $i \in N$, $\mathrm{supp}(\beta_{-i}) = \mathrm{supp}(\beta) \setminus \{i\}$.

Theorem 4.1 *For each n-player game G, $\Big((\phi_i^*)_{i \in \mathrm{supp}(\beta)}, \Big((\pi_i(m_i))_{m_i \in \mathrm{supp}(\phi^*_{M_i})}\Big)_{i \in N}\Big)$ is the outcome of a sequential equilibrium of G_{id} if and only if, for each $i \in N$,*

$$v_i(\pi_{-i}(m_{-i})) = \max_{m'_{-i} \in M^*_{-i}} v_i(\pi_{-i}(m'_{-i})) \text{ and } \pi_i(m_i) \in BR_i(\pi_{-i}(m_{-i})) \tag{4.1}$$

for each $m \in \mathrm{supp}(\phi_i^*)$, and

$$\pi_i(m_i) \text{ solves } \max_{\alpha_i \in \Delta(A_i)} \sum_{m_{-i}} \frac{\sum_{j \in \mathrm{supp}(\beta_{-i})} \beta_j \phi_j^*[m_i, m_{-i}]}{\sum_{j \in \mathrm{supp}(\beta_{-i})} \beta_j \phi^*_{j,M_i}[m_i]} u_i(\alpha_i, \pi_{-i}(m_{-i})) \tag{4.2}$$

for each $m_i \in \cup_{j\in \text{supp}(\beta_{-i})}\text{supp}(\phi^*_{j,M_i})$.

We use Theorem 4.1 to show that, in the Example of Section 4.7, $(1-\beta_3)(2,2,2)+\beta_3(0,0,3)$ is a sequential equilibrium payoff when $\min\{2\beta_1, 2\beta_2\} \geq \beta_3$.

Let $i \in \{1,2\}$ and $m \in \text{supp}(\phi^*_i)$. Then $\pi_i(m_i) = A$ and $\pi_{-i}(m_{-i}) = (A, M)$ or $\pi_i(m_i) = B$ and $\pi_{-i}(m_{-i}) = (B, M)$. In either case, $\pi_i(m_i) \in BR_i(\pi_{-i}(m_{-i}))$ and $v_i(\pi_{-i}(m_{-i})) = 2 \geq v_i(\pi_{-i}(m'_{-i}))$ for each $m'_{-i} \in M^*_{-i}$.

Furthermore, for each $m_i \in \cup_{j\in \text{supp}(\beta_{-i})}\text{supp}(\phi^*_{j,M_i}) = \{m'_i, m''_i\}$, $\pi_i(m_i)$ solves

$$\max_{a_i\in A_i}\sum_{m_{-i}}\frac{\sum_{j\neq i}\beta_j\phi^*_j[m_i,m_{-i}]}{\sum_{j\neq i}\beta_j\phi^*_{j,M_i}[m_i]}u_i(a_i,\pi_{-i}(m_{-i})).$$

Indeed, if $m_i = m'_i$, then $\pi_i(m_i) = A$ and, letting $j \in \{1,2\}$ with $j \neq i$, the maximization problem is

$$\max_{a_i\in A_i}\frac{\beta_j u_i(a_i,(A,M)) + \beta_3 u_i(a_i,(A,L))}{\beta_j+\beta_3};$$

if $i = 1$, $a_i = A$ yields $\frac{2\beta_2}{\beta_2+\beta_3}$ whereas $a_i = B$ yields $\frac{\beta_3}{\beta_2+\beta_3}$; thus, $\pi_1(m'_1)$ solves the maximization problem since $2\beta_2 \geq \beta_3$; if $i = 2$, then $a_i = A$ yields $\frac{2\beta_1}{\beta_1+\beta_3}$ whereas $a_i = B$ yields 0; thus, $\pi_2(m'_2)$ solves the maximization problem. If $m_i = m''_i$, the maximization problem is

$$\max_{a_i\in A_i}\frac{\beta_j u_i(a_i,(B,M)) + \beta_3 u_i(a_i,(B,R))}{\beta_j+\beta_3};$$

if $i = 1$, $a_i = \pi_i(m''_i) = B$ yields $\frac{2\beta_2}{\beta_2+\beta_3}$ whereas $a_i = A$ yields 0; thus $\pi_i(m''_i)$ solves the maximization problem; if $i = 2$, then $a_i = \pi_i(m''_i) = B$ yields $\frac{2\beta_1}{\beta_1+\beta_3}$ whereas $a_i = A$ yields $\frac{\beta_3}{\beta_1+\beta_3}$; thus $\pi_i(m''_i)$ solves the maximization problem since $2\beta_1 \geq \beta_3$.

Consider next $m \in \text{supp}(\phi^*_3)$. Then $\pi_3(m_3) = L$ and $\pi_{-3}(m_{-3}) = (A, A)$ or $\pi_3(m_3) = R$ and $\pi_{-3}(m_{-3}) = (B, B)$. In either case, $\pi_3(m_3) \in BR_3(\pi_{-3}(m_{-3}))$ and $v_3(\pi_{-3}(m_{-3})) = 3 \geq v_3(\pi_{-3}(m'_{-3}))$ for each $m'_{-3} \in M^*_{-3}$. It follows that condition (4.1) in Theorem 4.1 is satisfied.

Finally, note that $\cup_{j\in \text{supp}(\beta_{-3})}\text{supp}(\phi^*_{j,M_3}) = \{\hat{m}_3\}$ and that $\pi_3(\hat{m}_3) = M$ solves

$$\begin{aligned}\max_{a_3\in A_3}\sum_{m_{-3}}\frac{\sum_{j\neq 3}\beta_j\phi^*_j[\hat{m}_3,m_{-3}]}{\sum_{j\neq 3}\beta_j\phi^*_{j,M_3}[\hat{m}_3]}u_3(a_3,\pi_{-3}(m_{-3}))\\ = \max_{a_3\in A_3}\frac{u_3(A,A,a_3)+u_3(B,B,a_3)}{2}.\end{aligned}$$

Thus, condition (2) in Theorem 4.1 is also satisfied. Hence, it follows by Theorem 4.1 that $(1-\beta_3)(2,2,2)+\beta_3(0,0,3)$ is a sequential equilibrium payoff when $\min\{2\beta_1, 2\beta_2\} \geq \beta_3$.

4.1 Proof of the necessity part of Theorem 4.1

We start by noting the properties that sequential equilibrium imposes on the equilibrium outcome. Namely, for each sequential equilibrium $\pi \in \Pi$,

$$\sum_m \phi^*[m] u_i(\pi(m)) \geq \sum_m (\phi_i', \phi_{-i}^*)[m] u_i(\pi_i'(m_i, \phi_i'), \pi_{-i}(m_{-i})), \tag{4.3}$$

for each $i \in \text{supp}(\beta)$, $\phi_i' \in S$ and $\pi_i' : M_i \times S \to \Delta(A_i)$, where $(\phi_i', \phi_{-i}^*) = \beta_i \phi_i' + \sum_{j \in \text{supp}(\beta) \setminus \{i\}} \beta_j \phi_j^*$, and

$$\sum_{m_{-i}} \frac{\phi^*[m]}{\phi_{M_i}^*[m_i]} u_i(\pi(m)) \geq \sum_{m_{-i}} \frac{\phi^*[m]}{\phi_{M_i}^*[m_i]} u_i(a_i, \pi_{-i}(m_{-i})) \tag{4.4}$$

for each $i \in N$, $m_i \in \text{supp}(\phi_{M_i}^*)$ and $a_i \in A_i$.

In each sequential equilibrium of G_{id}, any player $i \in \text{supp}(\beta)$ must send optimal messages m in the sense that they induce an action profile $\pi(m)$ that maximizes i's payoff function. This is stated in Lemma 4.1 which is a preliminary result for condition (4.1).

Lemma 4.1 *If G is an n-player game and π is a sequential equilibrium of G_{id}, then* $\text{supp}(\phi_i^*) \subseteq \{m \in M : u_i(\pi(m)) = \sup_{m' \in M} u_i(\pi(m'))\}$ *for each* $i \in \text{supp}(\beta)$.

Proof. Suppose not; then there is $i \in \text{supp}(\beta)$, $m' \in \text{supp}(\phi_i^*)$ and $m^* \in M$ such that $u_i(\pi(m^*)) > u_i(\pi(m'))$. Define ϕ_i' by setting, for each $m \in \text{supp}(\phi_i^*)$,

$$\phi_i'[m] = \begin{cases} 0 & \text{if } m = m', \\ \phi_i^*[m^*] + \phi_i^*[m'] & \text{if } m = m^*, \\ \phi_i^*[m] & \text{otherwise,} \end{cases}$$

and let $\pi'_i : M_i \times S \to \Delta(A_i)$ be such that $\pi'_i(m_i, \phi'_i) = \pi_i(m_i, \phi^*_i)$ for each $m_i \in M_i$. Then

$$\begin{aligned}
&\sum_m (\phi'_i, \phi^*_{-i})[m] u_i(\pi'_i(m_i, \phi'_i), \pi_{-i}(m_{-i})) - \sum_m \phi^*[m] u_i(\pi(m)) \\
&= \sum_m (\phi'_i, \phi^*_{-i})[m] u_i(\pi(m)) - \sum_m \phi^*[m] u_i(\pi(m)) \\
&= \sum_m \beta_i \left(\phi'_i[m] - \phi^*_i[m]\right) u_i(\pi(m)) \\
&= \beta_i \phi^*_i[m'] \Big(u_i(\pi(m^*)) - u_i(\pi(m')) \Big) > 0.
\end{aligned}$$

But this is a contradiction to (4.3) since π is a sequential equilibrium of G_{id}. ■

The conclusion of Lemma 4.1 can be strengthened: for a message m to be optimal, $u_i(\pi(m))$ must achieve $\max_{m'_{-i}} v_i(\pi_{-i}(m'_{-i}))$ and, thus, $\pi_i(m_i)$ be a best-reply to $\pi_{-i}(m_{-i})$.

Lemma 4.2 *If G is an n-player game and π is a sequential equilibrium of G_{id}, then*

$$\begin{aligned}
\operatorname{supp}(\phi^*_i) \subseteq \{m \in M : \ \ & v_i(\pi_{-i}(m_{-i})) = \sup_{m'_{-i} \in M_{-i}} v_i(\pi_{-i}(m'_{-i})) \\
& \textit{and } \pi_i(m_i) \in BR_i(\pi_{-i}(m_{-i}))\}
\end{aligned}$$

for each $i \in \operatorname{supp}(\beta)$.

Proof. Suppose not; then there is $i \in \operatorname{supp}(\beta)$, $m' \in \operatorname{supp}(\phi^*_i)$ and $m^* \in M$ such that (i) $v_i(\pi_{-i}(m^*_{-i})) > v_i(\pi_{-i}(m'_{-i}))$ or (ii) $v_i(\pi_{-i}(m'_{-i})) = \sup_{\hat{m}_{-i} \in M_{-i}} v_i(\pi_{-i}(\hat{m}_{-i}))$ and $\pi_i(m'_i) \notin BR_i(\pi_{-i}(m'_{-i}))$; in case (ii), let $m^* = m'$. Let $a^*_i \in BR_i(\pi_{-i}(m^*_{-i}))$, $\bar{m}_i \notin \operatorname{supp}(\phi^*_{M_i})$, $\phi'_i = 1_{(\bar{m}_i, m^*_{-i})}$ and $\pi'_i : M_i \times S \to \Delta(A_i)$ be such that $\pi'_i(\bar{m}_i, \phi'_i) = a^*_i$ and $\pi'_i(m_i, \phi'_i) = \pi_i(m_i, \phi^*_i)$ for each $m_i \neq \bar{m}_i$. Then

$$\begin{aligned}
&\sum_m (\phi'_i, \phi^*_{-i})[m] u_i(\pi'_i(m_i, \phi'_i), \pi_{-i}(m_{-i})) - \sum_m \phi^*[m] u_i(\pi(m)) \\
&= \sum_m \beta_i \phi'_i[m] u_i(\pi'_i(m_i, \phi'_i), \pi_{-i}(m_{-i})) - \sum_m \beta_i \phi^*_i[m] u_i(\pi(m)) \\
&= \beta_i \Big(u_i(a^*_i, \pi_{-i}(m^*_{-i})) - \sum_{m \in \operatorname{supp}(\phi^*_i)} \phi^*_i[m] u_i(\pi(m)) \Big) \\
&= \beta_i \Big(v_i(\pi_{-i}(m^*_{-i})) - u_i(\pi(m')) \Big)
\end{aligned}$$

because $u_i(\pi(m)) = u_i(\pi(m'))$ for each $m \in \text{supp}(\phi_i^*)$ by Lemma 4.1 as $m' \in \text{supp}(\phi_i^*)$. Thus, if $v_i(\pi_{-i}(m_{-i}^*)) > v_i(\pi_{-i}(m'_{-i}))$, then $v_i(\pi_{-i}(m_{-i}^*)) - u_i(\pi(m')) \geq v_i(\pi_{-i}(m_{-i}^*)) - v_i(\pi_{-i}(m'_{-i})) > 0$; if $v_i(\pi_{-i}(m_{-i}^*)) = v_i(\pi_{-i}(m'_{-i}))$, then $\pi_i(m'_i) \notin BR_i(\pi_{-i}(m'_{-i}))$ and $v_i(\pi_{-i}(m_{-i}^*)) - u_i(\pi(m')) > v_i(\pi_{-i}(m_{-i}^*)) - v_i(\pi_{-i}(m'_{-i})) \geq 0$. In either case, it follows that $\sum_m (\phi'_i, \phi_{-i}^*)[m] u_i(\pi'_i(m_i, \phi'_i), \pi_{-i}(m_{-i})) - \sum_m \phi^*[m] u_i(\pi(m)) > 0$. But this is a contradiction to (4.3) since π is a sequential equilibrium. ■

Lemma 4.2 implies that $\pi_i(m_i)$ is a best-reply against $\pi_{-i}(m_{-i})$ whenever $m \in \text{supp}(\phi_i^*)$ and $i \in \text{supp}(\beta)$. We will now show that if, in addition,

$$m_i \in \cup_{j \in \text{supp}(\beta_{-i})} \text{supp}(\phi_{j,M_i}^*),$$

then $\pi_i(m_i)$ solves

$$\max_{\alpha_i \in \Delta(A_i)} \sum_{m_{-i}} \frac{\sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_j^*[m_i, m_{-i}]}{\sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_{j,M_i}^*[m_i]} u_i(\alpha_i, \pi_{-i}(m_{-i})).$$

Thus, whenever $m_i \in \text{supp}(\phi_i^*) \cap (\cup_{j \in \text{supp}(\beta_{-i})} \text{supp}(\phi_{j,M_i}^*))$, $\pi_i(m_i)$ solves player i's expected payoff conditional on his information design ϕ_i^* being chosen and also conditional on it not being chosen. The reason for this is that player i can always differentiate the messages he receives from himself from those that he receives from the other players: if $m \in \text{supp}(\phi_i^*)$ is such that $\pi_i(m_i)$ does not maximize i's expected payoff conditional on his information design ϕ_i^* not being chosen, then player i would gain by deviating from ϕ_i^* by simply sending a message $(\bar{m}_i, m_{-i})$ with probability one for some $\bar{m}_i \notin \text{supp}(\phi_{M_i}^*)$. If he receives message m_i, then he can be sure that his information design has not been chosen and can choose a solution to that problem in response to m_i; if he receives message $\bar{m}_i$, then the can be sure that his information design has been chosen and choose $\pi_i(m_i)$, which is a best-reply against m_{-i}, in response to $\bar{m}_i$.

Lemma 4.3 *If G is an n-player game and π is a sequential equilibrium of G_{id}, then*

$$\text{supp}(\phi_i^*) \subseteq \Big\{ m \in M : m_i \notin \cup_{j \in \text{supp}(\beta_{-i})} \text{supp}(\phi_{j,M_i}^*) \text{ or } \pi_i(m_i) \text{ solves } \max_{\alpha_i \in \Delta(A_i)} \sum_{m_{-i}} \frac{\sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_j^*[m_i, m_{-i}]}{\sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_{j,M_i}^*[m_i]} u_i(\alpha_i, \pi_{-i}(m_{-i})) \Big\}$$

for each $i \in \text{supp}(\beta)$.

Proof. Suppose not; then there is $i \in \text{supp}(\beta)$ and $m' \in \text{supp}(\phi_i^*)$ such that $m_i' \in \cup_{j\in\text{supp}(\beta_{-i})}\text{supp}(\phi_{j,M_i}^*)$ and $\pi_i(m_i')$ does not solve

$$\max_{\alpha_i\in\Delta(A_i)} \sum_{m_{-i}} \frac{\sum_{j\in\text{supp}(\beta_{-i})} \beta_j \phi_j^*[m_i', m_{-i}]}{\sum_{j\in\text{supp}(\beta_{-i})} \beta_j \phi_{j,M_i}^*[m_i']} u_i(\alpha_i, \pi_{-i}(m_{-i})). \tag{4.5}$$

Let a_i^* be a solution to problem (4.5), $\bar{m}_i \notin \text{supp}(\phi_{M_i}^*)$, $\phi_i' = 1_{(\bar{m}_i, m_{-i}')}$ and $\pi_i' : M_i \times S \to \Delta(A_i)$ be such that

$$\pi_i'(m_i, \phi_i') = \begin{cases} a_i^* & \text{if } m_i = m_i', \\ \pi_i(m_i') & \text{if } m_i = \bar{m}_i, \\ \pi_i(m_i) & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}
&\sum_m (\phi_i', \phi_{-i}^*)[m] u_i(\pi_i'(m_i, \phi_i'), \pi_{-i}(m_{-i})) - \sum_m \phi^*[m] u_i(\pi(m)) \\
&= \beta_i \Big(u_i(\pi(m')) - \sum_{m\in\text{supp}(\phi_i^*)} \phi_i^*[m] u_i(\pi(m)) \Big) \\
&+ \sum_{j\in\text{supp}(\beta_{-i})} \beta_j \sum_{m_{-i}} \phi_j^*[m_i', m_{-i}] \Big(u_i(a_i^*, \pi_{-i}(m_{-i})) - u_i(\pi_i(m_i'), \pi_{-i}(m_{-i})) \Big) \\
&= \sum_{j\in\text{supp}(\beta_{-i})} \beta_j \sum_{m_{-i}} \phi_j^*[m_i', m_{-i}] \Big(u_i(a_i^*, \pi_{-i}(m_{-i})) - u_i(\pi_i(m_i'), \pi_{-i}(m_{-i})) \Big)
\end{aligned}$$

where the last equality follows by Lemma 4.1 since $m' \in \text{supp}(\phi_i^*)$. Since $\pi_i(m_i')$ does not solve problem (4.5) but a_i^* does, it follows that

$$\sum_{m_{-i}} \frac{\sum_{j\in\text{supp}(\beta_{-i})} \beta_j \phi_j^*[m_i', m_{-i}]}{\sum_{j\in\text{supp}(\beta_{-i})} \beta_j \phi_{j,M_i}^*[m_i']} \Big(u_i(a_i^*, \pi_{-i}(m_{-i})) - u_i(\pi_i(m_i'), \pi_{-i}(m_{-i})) \Big) > 0$$

and, since $m_i' \in \cup_{j\in\text{supp}(\beta_{-i})}\text{supp}(\phi_{j,M_i}^*)$,

$$\sum_{j\in\text{supp}(\beta_{-i})} \beta_j \sum_{m_{-i}} \phi_j^*[m_i', m_{-i}] \Big(u_i(a_i^*, \pi_{-i}(m_{-i})) - u_i(\pi_i(m_i'), \pi_{-i}(m_{-i})) \Big) > 0.$$

Hence, $\sum_m (\phi_i', \phi_{-i}^*)[m] u_i(\pi_i'(m_i, \phi_i'), \pi_{-i}(m_{-i})) - \sum_m \phi^*[m] u_i(\pi(m)) > 0$. But this is a contradiction to (4.3) since π is a sequential equilibrium of G_{id}. ■

It follows by Lemmas 4.2 and 4.3 that, for each sequential equilibrium outcome, $i \in N$ and $m \in \text{supp}(\phi_i^*)$, condition (4.1) in Theorem 4.1 holds and $\pi_i(m_i)$ solves

$$\max_{\alpha_i\in\Delta(A_i)} \sum_{m_{-i}} \frac{\sum_{j\in\text{supp}(\beta_{-i})} \beta_j \phi_j^*[m_i, m_{-i}]}{\sum_{j\in\text{supp}(\beta_{-i})} \beta_j \phi_{j,M_i}^*[m_i]} u_i(\alpha_i, \pi_{-i}(m_{-i}))$$

whenever $m_i \in \cup_{j\in \text{supp}(\beta_{-i})}\text{supp}(\phi^*_{j,M_i})$ and, hence,

$$m_i \in \text{supp}(\phi^*_i) \cap (\cup_{j\in \text{supp}(\beta_{-i})}\text{supp}(\phi^*_{j,M_i})).$$

In fact, regarding (4.1), note that if $i \in \text{supp}(\beta)$ and $m \in \text{supp}(\phi^*_i)$, then $m_j \in \text{supp}(\phi^*_{M_j})$ for each $j \in N$ and, thus, $m \in M^*$. Hence,

$$v_i(\pi_{-i}(m_{-i})) \leq \max_{m'_{-i}\in M^*_{-i}} v_i(\pi_{-i}(m'_{-i})) \leq \sup_{m'_{-i}\in M_{-i}} v_i(\pi_{-i}(m'_{-i})) = v_i(\pi_{-i}(m_{-i})).$$

Condition (4.4) implies that, for each $i \in N$, $\pi_i(m_i)$ solves

$$\max_{\alpha_i\in\Delta(A_i)} \sum_{m_{-i}} \frac{\sum_{j\in\text{supp}(\beta_{-i})} \beta_j \phi^*_j[m_i, m_{-i}]}{\sum_{j\in\text{supp}(\beta_{-i})} \beta_j \phi^*_{j,M_i}[m_i]} u_i(\alpha_i, \pi_{-i}(m_{-i}))$$

whenever $m_i \in \cup_{j\in\text{supp}(\beta_{-i})}\text{supp}(\phi^*_{j,M_i}) \setminus \text{supp}(\phi^*_i)$. This, together with what has been shown in the previous paragraph, shows that condition (4.2) in Theorem 4.1 holds.

4.2 Proof of the sufficiency part of Theorem 4.1

Let $\left((\phi^*_i)_{i\in\text{supp}(\beta)}, \left((\pi_i(m_i))_{m_i\in\text{supp}(\phi^*_{M_i})}\right)_{i\in N}\right)$ be such that conditions (4.1) and (4.2) in Theorem 4.1 hold; we will show that it is the outcome of a sequential equilibrium.

We will construct a sequential equilibrium π with the desired outcome. To this end, consider $\{\pi^\alpha, p^\alpha\}_\alpha$ defined as follows: The index set consists of $\alpha = (k, F, \hat{F})$ such that $k \in \mathbb{N}$, F is a finite subset of $\mathbb{N}$ and $\hat{F}$ is a finite subset of S; this set is partially ordered by defining $(k', F', \hat{F}') \geq (k, F, \hat{F})$ if $k' \geq k$, $F \subseteq F'$ and $\hat{F} \subseteq \hat{F}'$. If X is a finite set, let $\upsilon_X \in \Delta(X)$ be uniform on X. For each $i \in N$, let

$$\bar{m}_i \in \begin{cases} \text{supp}(\phi^*_{i,M_i}) & \text{if } i \in \text{supp}(\beta), \\ \text{supp}(\phi^*_{M_i}) & \text{if } i \notin \text{supp}(\beta), \end{cases}$$

$$\bar{q}_i[m_{-i}] = \begin{cases} \frac{\phi^*_i[\bar{m}_i, m_{-i}]}{\phi^*_{i,M_i}[\bar{m}_i]} & \text{if } i \in \text{supp}(\beta) \\ \frac{\sum_{j\in\text{supp}(\beta)} \beta_j \phi^*_j[\bar{m}_i, m_{-i}]}{\sum_{j\in\text{supp}(\beta)} \beta_j \phi^*_{j,M_i}[\bar{m}_i]} & \text{if } i \notin \text{supp}(\beta), \end{cases}$$

for each $m_{-i} \in M_{-i}$, and for each $\alpha = (k, F, \hat{F})$, let

$$\begin{aligned}
\tau_i^\alpha &= \frac{\sum_{l \in F \cup \left(\cup_{\phi \in \hat{F}} \mathrm{supp}(\phi_{M_i})\right)} 2^{-l} 1_l}{\sum_{l \in F \cup \left(\cup_{\phi \in \hat{F}} \mathrm{supp}(\phi_{M_i})\right)} 2^{-l}}, \\
q_i^\alpha &= \tau_i^\alpha \times \bar{q}_i, \\
\tau^\alpha &= \prod_{j \in N} \tau_j^\alpha, \\
q^\alpha &= (n')^{-1} \sum_{j \in \mathrm{supp}(\beta)} q_j^\alpha, \\
\hat{q}^\alpha &= n^{-1} \sum_{j \in N} q_j^\alpha, \\
\mu^\alpha &= (1 - k^{-1} - k^{-2}) q^\alpha + k^{-1} \hat{q}^\alpha + k^{-2} \tau^\alpha, \text{ and} \\
p^\alpha(\phi) &= (1 - k^{-1}) \sum_{j \in \mathrm{supp}(\beta)} \beta_j \phi_j + k^{-1} \mu^\alpha.
\end{aligned}$$

For each $m_i \notin \mathrm{supp}(\phi^*_{M_i})$, set $\pi_i(m_i, \phi_i^*) = \pi_i(\bar{m}_i)$ if $i \in \mathrm{supp}(\beta)$ and $\pi_i(m_i) = \pi_i(\bar{m}_i)$ if $i \notin \mathrm{supp}(\beta)$; hence, $\pi_i(m_i)$ is defined for each $i \in N$ and $m_i \in M_i$.

For each $i \in \mathrm{supp}(\beta)$, $m_i \in M_i$ and $\phi_i \neq \phi_i^*$ such that

$$\beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \mathrm{supp}(\beta_{-i})} \beta_j \phi^*_{j,M_i}[m_i] = 0,$$

let $\pi_i(m_i, \phi_i) = \pi_i(\bar{m}_i)$.

For each $i \in \mathrm{supp}(\beta)$, $m_i \in M_i$ and $\phi_i \neq \phi_i^*$ such that

$$\beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \mathrm{supp}(\beta_{-i})} \beta_j \phi^*_{j,M_i}[m_i] > 0,$$

let $\pi_i(m_i, \phi_i)$ be a best-reply against

$$\sum_{m_{-i}} \frac{\beta_i \phi_i[m_i, m_{-i}] + \sum_{j \in \mathrm{supp}(\beta_{-i})} \beta_j \phi_j^*[m_i, m_{-i}]}{\beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \mathrm{supp}(\beta_{-i})} \beta_j \phi^*_{j,M_i}[m_i]} \pi_{-i}(m_{-i}).$$

We may assume that $\pi_i : M_i \times S \to \Delta(A_i)$ is measurable. Note first that $M_i \times S = \cup_{r=1}^3 B_r$ with

$$\begin{aligned}
B_1 &= \{(m_i, \phi_i) : \phi_i = \phi_i^*\}, \\
B_2 &= \{(m_i, \phi_i) : \phi_i \neq \phi_i^* \text{ and } \beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \mathrm{supp}(\beta_{-i})} \beta_j \phi^*_{j,M_i}[m_i] = 0\} \text{ and} \\
B_3 &= \{(m_i, \phi_i) : \phi_i \neq \phi_i^* \text{ and } \beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \mathrm{supp}(\beta_{-i})} \beta_j \phi^*_{j,M_i}[m_i] > 0\}.
\end{aligned}$$

For each $r \in \{1,2,3\}$, B_r is measurable. Indeed, B_1 is closed, B_2 is the intersection of an open set, $\{(m_i,\phi_i) : \phi_i \neq \phi_i^*\}$, with a closed set, $\{(m_i,\phi_i) : \beta_i \phi_{i,M_i}[m_i] + \sum_{j\in \text{supp}(\beta_{-i})} \beta_j \phi^*_{j,M_i}[m_i] = 0\}$, and B_3 is open. Then, for each measurable $B \subseteq \Delta(A_i)$, $\pi_i^{-1}(B) \cap B_1$ is measurable since $\pi_i^{-1}(B) \cap B_1$ is countable. Regarding $\pi_i^{-1}(B) \cap B_3$: Let $f : M_i \times S \to \Delta(A_{-i})$ be defined by setting, for each $(m_i,\phi_i) \in B_3$, $f(m_i,\phi_i) = \sum_{m_{-i}} \frac{\beta_i\phi_i[m_i,m_{-i}] + \sum_{j\in \text{supp}(\beta_{-i})} \beta_j \phi^*_j[m_i,m_{-i}]}{\beta_i \phi_{i,M_i}[m_i] + \sum_{j\in \text{supp}(\beta_{-i})} \beta_j \phi^*_{j,M_i}[m_i]} \pi_{-i}(m_{-i})$. Letting $BR_i : \Delta(A_{-i}) \rightrightarrows \Delta(A_i)$ be player i's best-reply correspondence in G, define $\Psi : M_i \times S \rightrightarrows \Delta(A_i)$ by setting, for each $(m_i,\phi_i) \in B_3$, $\Psi(m_i,\phi_i) = BR_i(f(m_i,\phi_i))$. Since $\Delta(A_i)$ is compact, f is continuous and BR_i is upper hemicontinuous, it follows that Ψ is upper hemicontinuous and, hence, measurable (and, thus, weakly measurable). Hence, Ψ has a measurable selection by the Kuratowski-Ryll-Nardzewski Selection Theorem (e.g. Aliprantis and Border (2006, Theorem 18.13, p. 600)). Finally, for each measurable $B \subseteq \Delta(A_i)$, $\pi_i^{-1}(B) = B_2$ if $\pi_i(\bar{m}_i) \in B$ and $\pi_i^{-1}(B) = \emptyset$ otherwise; thus $\pi_i^{-1}(B) \cap B_2$ is measurable.

Furthermore, let

$$\pi_i^{1,\alpha} = (1-k^{-3}) 1_{\phi_i^*} + k^{-3} \upsilon_{\hat{F}} \text{ and } \pi_i^{2,\alpha}(m_i,\phi_i) = (1-k^{-1})\pi_i(m_i,\phi_i) + k^{-1}\upsilon_{A_i}$$

if $i \in \text{supp}(\beta)$. For each $i \notin \text{supp}(\beta)$, let

$$\pi_i^{2,\alpha}(m_i) = (1-k^{-1})\pi_i(m_i) + k^{-1}\upsilon_{A_i}.$$

Let $\varepsilon > 0$. We have that the following conditions in the definition of perfect conditional ε-equilibrium hold by construction:

1. For each α, π^α is a strategy and $p^\alpha : S^{n'} \to \Delta(M)$ is measurable,

2. For each $i \in \text{supp}(\beta)$, $\sup_{B\in\mathcal{B}(S)} |\pi_i^{1,\alpha}[B] - 1_{\phi_i^*}[B]| \to 0$ and

$$\sup_{(m_i,\phi_i)\in M_i\times S, a_i \in A_i} |\pi_i^{2,\alpha}(m_i,\phi_i)[a_i] - \pi_i(m_i,\phi_i)[a_i]| \to 0,$$ [2]

3. For each $i \in \text{supp}(\beta)$, $m_i \in M_i$, $\phi_i \in S$ and $a_i \in A_i$, there is $\bar{\alpha}$ such that $\pi_i^{1,\alpha}[\phi_i] > 0$ and $\pi_i^{2,\alpha}(m_i,\phi_i)[a_i] > 0$ for each $\alpha \geq \bar{\alpha}$,

[2] We let $\mathcal{B}(S)$ denote the class of Borel measurable subsets of S and, for each $\phi \in S$, 1_ϕ denote the probability measure on S degenerate at ϕ.

4. For each $i \in N \setminus \text{supp}(\beta)$, $\sup_{m_i \in M_i, a_i \in A_i} |\pi_i^{2,\alpha}(m_i)[a_i] - \pi_i(m_i)[a_i]| \to 0$,
5. For each $i \in N \setminus \text{supp}(\beta)$, $m_i \in M_i$ and $a_i \in A_i$, there is $\bar{\alpha}$ such that $\pi_i^{2,\alpha}(m_i)[a_i] > 0$ for each $\alpha \geq \bar{\alpha}$,
6. $\sup_{\phi \in S^{n'}, B \subseteq M} |p^\alpha(\phi)[B] - \sum_{i \in \text{supp}(\beta)} \beta_i \phi_i[B]| \to 0$, and
7. For each $\phi \in S^{n'}$ and $m \in M$, there is $\bar{\alpha}$ such that $p^\alpha(\phi)[m] > 0$ for each $\alpha \geq \bar{\alpha}$.

Note also that, for each α, $\text{supp}(\pi^{1,\alpha})$ and $\text{supp}(p^\alpha)$ are finite. We define

$$S_i(F, \hat{F}) = \Big(\Big(F \cup (\cup_{\phi \in \hat{F}} \text{supp}(\phi_{M_i})) \cup (\cup_{j \in \text{supp}(\beta_{-i})} \text{supp}(\phi^*_{j,M_i}))\Big) \times \hat{F}\Big)$$
$$\cup \Big(\Big(F \cup (\cup_{\phi \in \hat{F}} \text{supp}(\phi_{M_i})) \cup (\cup_{j \in \text{supp}(\beta)} \text{supp}(\phi^*_{j,M_i}))\Big) \times \{\phi_i^*\}\Big)$$

for each $i \in \text{supp}(\beta)$ and

$$S_i(F, \hat{F}) = F \cup (\cup_{\phi \in \hat{F}} \text{supp}(\phi_{M_i})) \cup (\cup_{j \in \text{supp}(\beta)} \text{supp}(\phi^*_{j,M_i}))$$

for each $i \in N \setminus \text{supp}(\beta)$. If $(m, \phi) \in \mathbb{N}^n \times S^{n'}$ is such that $\pi^{1,\alpha}[\phi] > 0$ and $\sum_{\phi' \in \text{supp}(\pi^{1,\alpha})} p^\alpha(\phi')[m] > 0$, then $(m_i, \phi_i) \in S_i(F, \hat{F})$ for each $i \in \text{supp}(\beta)$ and $m_i \in S_i(F, \hat{F})$ for each $i \in N \setminus \text{supp}(\beta)$.

Thus, to show that π is a perfect conditional ε-equilibrium, it remains to show that

8. for each α,

   (a) For each $i \in \text{supp}(\beta)$ and $\phi_i' \in S$,

   $$\sum_{\phi \in \text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] \left(\sum_m p^\alpha(\phi)[m] u_i(\pi^{2,\alpha}(m, \phi))\right) \geq$$
   $$\sum_{\phi \in \text{supp}(1_{\phi_i'} \times \pi_{-i}^{1,\alpha})} (1_{\phi_i'} \times \pi_{-i}^{1,\alpha})[\phi] \left(\sum_m p^\alpha(\phi)[m] u_i(\pi^{2,\alpha}(m, \phi))\right) - \varepsilon,$$

   where $\pi^{1,\alpha} = \prod_{i \in \text{supp}(\beta)} \pi_i^{1,\alpha}$ and $1_{\phi_i'} \times \pi_{-i}^{1,\alpha} = 1_{\phi_i'} \times \prod_{j \in \text{supp}(\beta) \setminus \{i\}} \pi_j^{1,\alpha}$,

   (b) For each $i \in \text{supp}(\beta)$, $(m_i, \phi_i) \in M_i \times S$ such that

   $$\pi_i^{1,\alpha}[\phi_i] \sum_{\phi_{-i} \in \text{supp}(\pi_{-i}^{1,\alpha})} \pi_{-i}^{1,\alpha}[\phi_{-i}] p_{M_i}^\alpha(\phi_i, \phi_{-i})[m_i] > 0$$

and $a_i \in A_i$,

$$\frac{\sum_{\phi_{-i}\in\text{supp}(\pi^{1,\alpha}_{-i})} \pi^{1,\alpha}_{-i}[\phi_{-i}] \left(\sum_{m_{-i}} p^\alpha(\phi_i,\phi_{-i})[m_i,m_{-i}]u_i(\pi^{2,\alpha}(m,\phi))\right)}{\sum_{\phi_{-i}\in\text{supp}(\pi^{1,\alpha}_{-i})} \pi^{1,\alpha}_{-i}[\phi_{-i}]p^\alpha_{M_i}(\phi_i,\phi_{-i})[m_i]} \geq$$

$$\frac{\sum_{\phi_{-i}\in\text{supp}(\pi^{1,\alpha}_{-i})} \pi^{1,\alpha}_{-i}[\phi_{-i}] \left(\sum_{m_{-i}} p^\alpha(\phi_i,\phi_{-i})[m_i,m_{-i}]u_i(a_i,\pi^{2,\alpha}_{-i}(m_{-i},\phi_{-i}))\right)}{\sum_{\phi_{-i}\in\text{supp}(\pi^{1,\alpha}_{-i})} \pi^{1,\alpha}_{-i}[\phi_{-i}]p^\alpha_{M_i}(\phi_i,\phi_{-i})[m_i]}$$

$$-\varepsilon,$$

(c) For each $i \in N \setminus \text{supp}(\beta)$, $m_i \in M_i$ such that

$$\sum_{\phi\in\text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi]p^\alpha_{M_i}(\phi)[m_i] > 0$$

and $a_i \in A_i$,

$$\frac{\sum_{\phi\in\text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] \left(\sum_{m_{-i}} p^\alpha(\phi)[m_i,m_{-i}]u_i(\pi^{2,\alpha}(m,\phi))\right)}{\sum_{\phi\in\text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi]p^\alpha_{M_i}(\phi)[m_i]} \geq$$

$$\frac{\sum_{\phi\in\text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] \left(\sum_{m_{-i}} p^\alpha(\phi)[m_i,m_{-i}]u_i(a_i,\pi^{2,\alpha}_{-i}(m_{-i},\phi_{-i}))\right)}{\sum_{\phi\in\text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi]p^\alpha_{M_i}(\phi)[m_i]} - \varepsilon.$$

We will show that condition 8 holds for some subnet of $\{\pi^\alpha, p^\alpha\}_\alpha$. Recall that $\alpha = (k, F, \hat{F})$. In what follows, we will often fix F and $\hat{F}$ and take limits as $k \to \infty$.

Regarding condition 8 (a), let $i \in \text{supp}(\beta)$ and $\phi'_i \in S$. We have that, for each finite subsets F and $\hat{F}$ of $\mathbb{N}$ and S, respectively,

$$\lim_k \sum_{\phi\in\text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] \left(\sum_m p^\alpha(\phi)[m]u_i(\pi^{2,\alpha}(m,\phi))\right) = \sum_m \phi^*[m]u_i(\pi(m))$$

and that

$$\lim_k \sum_{\phi\in\text{supp}(1_{\phi'_i}\times\pi^{1,\alpha}_{-i})} (1_{\phi'_i}\times\pi^{1,\alpha}_{-i})[\phi] \left(\sum_m p^\alpha(\phi)[m]u_i(\pi^{2,\alpha}(m,\phi))\right) =$$

$$\sum_m (\phi'_i,\phi^*_{-i})[m]u_i(\pi_i(m_i,\phi'_i),\pi_{-i}(m_{-i})).$$

Hence, by considering α such that $k \geq k_0$ for some $k_0 \in \mathbb{N}$, it is enough to show that

$$\sum_m \phi^*[m]u_i(\pi(m)) \geq \sum_m (\phi'_i,\phi^*_{-i})[m]u_i(\pi_i(m_i,\phi'_i),\pi_{-i}(m_{-i})),$$

which is equivalent to

$$\sum_m \phi_i^*[m]u_i(\pi(m)) \geq \sum_m \phi_i'[m]u_i(\pi_i(m_i,\phi_i'),\pi_{-i}(m_{-i})). \tag{4.6}$$

For each $j \in N$ and $m_j \in M_j$, $\pi_j(m_j) \in \{\pi_j(m_j') : m_j' \in \text{supp}(\phi^*_{M_j})\}$ since $\pi_j(m_j) = \pi_j(\bar{m}_j)$ whenever $m_j \notin \text{supp}(\phi^*_{M_j})$. Thus, by (4.1),

$$\sum_m \phi_i'[m]u_i(\pi_i(m_i,\phi_i'),\pi_{-i}(m_{-i})) \leq \sum_m \phi_i'[m]v_i(\pi_{-i}(m_{-i}))$$
$$\leq \max_{m_{-i}\in M^*_{-i}} v_i(\pi_{-i}(m_{-i})) = \sum_m \phi_i^*[m]u_i(\pi(m))$$

and, hence, (4.6) holds. It then follows that condition 8 (a) also holds.

Consider condition 8 (b) and (c). For each $i \in \text{supp}(\beta)$, finite subset F of $\mathbb{N}$, finite subset $\hat{F}$ of S, $(m_i,\phi_i) \in S_i(F,\hat{F})$ and $\gamma_i \in \Delta(A_i)$, we have that

$$\lim_k \frac{\sum_{\phi_{-i}\in\text{supp}(\pi^{1,\alpha}_{-i})} \pi^{1,\alpha}_{-i}[\phi_{-i}] \left(\sum_{m_{-i}} p^\alpha(\phi_i,\phi_{-i})[m_i,m_{-i}]u_i(\gamma_i,\pi^{2,\alpha}_{-i}(m_{-i},\phi_{-i}))\right)}{\sum_{\phi_{-i}\in\text{supp}(\pi^{1,\alpha}_{-i})} \pi^{1,\alpha}_{-i}[\phi_{-i}]p^\alpha_{M_i}(\phi_i,\phi_{-i})[m_i]}$$
$$= \sum_{m_{-i}} \frac{\phi_i^*[\bar{m}_i,m_{-i}]}{\phi^*_{i,M_i}[\bar{m}_i]}u_i(\gamma_i,\pi_{-i}(m_{-i}))$$

if $\beta_i\phi_{i,M_i}[m_i] + \sum_{j\in\text{supp}(\beta_{-i})}\beta_j\phi^*_{j,M_i}[m_i] = 0$, and

$$\lim_k \frac{\sum_{\phi_{-i}\in\text{supp}(\pi^{1,\alpha}_{-i})} \pi^{1,\alpha}_{-i}[\phi_{-i}] \left(\sum_{m_{-i}} p^\alpha(\phi_i,\phi_{-i})[m_i,m_{-i}]u_i(\gamma_i,\pi^{2,\alpha}_{-i}(m_{-i},\phi_{-i}))\right)}{\sum_{\phi_{-i}\in\text{supp}(\pi^{1,\alpha}_{-i})} \pi^{1,\alpha}_{-i}[\phi_{-i}]p^\alpha_{M_i}(\phi_i,\phi_{-i})[m_i]} =$$
$$\sum_{m_{-i}} \frac{\beta_i\phi_i[m_i,m_{-i}] + \sum_{j\in\text{supp}(\beta_{-i})}\beta_j\phi_j^*[m_i,m_{-i}]}{\beta_i\phi_{i,M_i}[m_i] + \sum_{j\in\text{supp}(\beta_{-i})}\beta_j\phi^*_{j,M_i}[m_i]}u_i(\gamma_i,\pi_{-i}(m_{-i}))$$

if $\beta_i\phi_{i,M_i}[m_i] + \sum_{j\in\text{supp}(\beta_{-i})}\beta_j\phi^*_{j,M_i}[m_i] > 0$. The latter case is clear since all terms in the denominator of the fraction converge to zero except the one that converges to $\beta_i\phi_{i,M_i}[m_i] + \sum_{j\in\text{supp}(\beta_{-i})}\beta_j\phi^*_{j,M_i}[m_i]$ and similarly regarding the numerator.

In the former case, both the numerator and the denominator converge to zero since $\beta_i\phi_{i,M_i}[m_i] + \sum_{j\in\text{supp}(\beta_{-i})}\beta_j\phi^*_{j,M_i}[m_i] = 0$. Multiplying each by k, it follows that all terms converge to zero except the ones corresponding to the case where $\pi_j^{1,\alpha} = \phi_j^*$ for each $j \neq i$ and $p^\alpha(\phi_i,\phi^*_{-i}) = q^\alpha$. Furthermore, for each $m_{-i} \in M_{-i}$,

$$q^\alpha[m_i,m_{-i}] = (n')^{-1}(q_i^\alpha[m_i,m_{-i}] + \sum_{j\in\text{supp}(\beta)\setminus\{i\}} q_j^\alpha[m_i,m_{-i}]),$$
$$q_i^\alpha[m_i,m_{-i}] = \tau_i^\alpha[m_i]\bar{q}_i[m_{-i}] \text{ and}$$
$$q_j^\alpha[m_i,m_{-i}] = 0 \text{ for each } j \in \text{supp}(\beta)\setminus\{i\},$$

the latter since $m_i \notin \text{supp}(\phi^*_{j,M_i})$. Hence, $q^\alpha[m_i, m_{-i}] = (n')^{-1}\tau^\alpha_i[m_i]\bar{q}_i[m_{-i}]$ and $q^\alpha_{M_i}[m_i] = (n')^{-1}\tau^\alpha_i[m_i]$. Thus,

$$\frac{q^\alpha[m_i, m_{-i}]}{q^\alpha_{M_i}[m_i]} = \bar{q}_i[m_{-i}] = \frac{\phi^*_i[\bar{m}_i, m_{-i}]}{\phi^*_{i,M_i}[\bar{m}_i]}.$$

Similarly, for each $i \notin \text{supp}(\beta)$, finite subset F of $\mathbb{N}$, finite subset $\hat{F}$ of S, $m_i \in S_i(F, \hat{F})$ and $\gamma_i \in \Delta(A_i)$, we have that

$$\lim_k \frac{\sum_{\phi\in\text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] \left(\sum_{m_{-i}} p^\alpha(\phi)[m_i, m_{-i}]u_i(\gamma_i, \pi^{2,\alpha}_{-i}(m_{-i}, \phi_{-i}))\right)}{\sum_{\phi\in\text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi]p^\alpha_{M_i}(\phi)[m_i]} =$$
$$\sum_{m_{-i}} \frac{\sum_{j\in\text{supp}(\beta)} \beta_j\phi^*_j[\bar{m}_i, m_{-i}]}{\sum_{j\in\text{supp}(\beta)} \beta_j\phi^*_{j,M_i}[\bar{m}_i]} u_i(\gamma_i, \pi_{-i}(m_{-i}))$$

if $\sum_{j\in\text{supp}(\beta)} \beta_j\phi^*_{j,M_i}[m_i] = 0$, and

$$\lim_k \frac{\sum_{\phi\in\text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] \left(\sum_{m_{-i}} p^\alpha(\phi)[m_i, m_{-i}]u_i(\gamma_i, \pi^{2,\alpha}_{-i}(m_{-i}, \phi_{-i}))\right)}{\sum_{\phi\in\text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi]p^\alpha_{M_i}(\phi)[m_i]} =$$
$$\sum_{m_{-i}} \frac{\sum_{j\in\text{supp}(\beta)} \beta_j\phi^*_j[m_i, m_{-i}]}{\sum_{j\in\text{supp}(\beta)} \beta_j\phi^*_{j,M_i}[m_i]} u_i(\gamma_i, \pi_{-i}(m_{-i}))$$

if $\sum_{j\in\text{supp}(\beta)} \beta_j\phi^*_{j,M_i}[m_i] > 0$. The latter case is as in the case $i \in \text{supp}(\beta)$. In the former case, both the numerator and the denominator converge to zero since $\sum_{j\in\text{supp}(\beta)} \beta_j\phi^*_{j,M_i}[m_i] = 0$; furthermore, $q^\alpha_{M_i}[m_i] = 0$ for the same reason. Multiplying each by k^2, it follows that all terms converge to zero except the ones corresponding to the case where $\pi^{1,\alpha}_j = \phi^*_j$ for each $j \neq i$ and $p^\alpha(\phi_i, \phi^*_{-i}) = \hat{q}^\alpha$. Furthermore, for each $m_{-i} \in M_{-i}$,

$$\hat{q}^\alpha[m_i, m_{-i}] = n^{-1}(q^\alpha_i[m_i, m_{-i}] + \sum_{j\in N} q^\alpha_j[m_i, m_{-i}]),$$
$$q^\alpha_i[m_i, m_{-i}] = \tau^\alpha_i[m_i]\bar{q}_i[m_{-i}] \text{ and}$$
$$q^\alpha_j[m_i, m_{-i}] = 0 \text{ for each } j \neq i,$$

the latter since $m_i \notin \text{supp}(\phi^*_{M_i})$. Thus,

$$\frac{\hat{q}^\alpha[m_i, m_{-i}]}{\hat{q}^\alpha_{M_i}[m_i]} = \bar{q}_i[m_{-i}] = \frac{\sum_{j\in\text{supp}(\beta)} \beta_j\phi^*_j[\bar{m}_i, m_{-i}]}{\sum_{j\in\text{supp}(\beta)} \beta_j\phi^*_{j,M_i}[\bar{m}_i]}.$$

We will next show that $\pi_i(m_i, \phi_i)$ solves

$$\max_{\gamma_i \in \Delta(A_i)} \lim_k \frac{\sum_{\phi_{-i} \in \text{supp}(\pi_{-i}^{1,\alpha})} \pi_{-i}^{1,\alpha}[\phi_{-i}] \left(\sum_{m_{-i}} p^{\alpha}(\phi_i, \phi_{-i})[m_i, m_{-i}] u_i(\gamma_i, \pi_{-i}^{2,\alpha}(m_{-i}, \phi_{-i})) \right)}{\sum_{\phi_{-i} \in \text{supp}(\pi_{-i}^{1,\alpha})} \pi_{-i}^{1,\alpha}[\phi_{-i}] p_{M_i}^{\alpha}(\phi_i, \phi_{-i})[m_i]} \tag{4.7}$$

for each $i \in \text{supp}(\beta)$, $m_i \in M_i$, $\phi_i \in S$, and $\pi_i(m_i)$ solves

$$\max_{\gamma_i \in \Delta(A_i)} \lim_k \frac{\sum_{\phi \in \text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] \left(\sum_{m_{-i}} p^{\alpha}(\phi)[m_i, m_{-i}] u_i(\gamma_i, \pi_{-i}^{2,\alpha}(m_{-i}, \phi_{-i})) \right)}{\sum_{\phi \in \text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] p_{M_i}^{\alpha}(\phi)[m_i]} \tag{4.8}$$

for each $i \notin \text{supp}(\beta)$ and $m_i \in M_i$.

We first establish (4.7). If $\beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_{j,M_i}^*[m_i] = 0$, then

$$\begin{aligned}
&\lim_k \frac{\sum_{\phi_{-i} \in \text{supp}(\pi_{-i}^{1,\alpha})} \pi_{-i}^{1,\alpha}[\phi_{-i}] \left(\sum_{m_{-i}} p^{\alpha}(\phi_i, \phi_{-i})[m_i, m_{-i}] u_i(\gamma_i, \pi_{-i}^{2,\alpha}(m_{-i}, \phi_{-i})) \right)}{\sum_{\phi_{-i} \in \text{supp}(\pi_{-i}^{1,\alpha})} \pi_{-i}^{1,\alpha}[\phi_{-i}] p_{M_i}^{\alpha}(\phi_i, \phi_{-i})[m_i]} \\
&= \sum_{m_{-i}} \frac{\phi_i^*[\bar{m}_i, m_{-i}]}{\phi_{i,M_i}^*[\bar{m}_i]} u_i(\gamma_i, \pi_{-i}(m_{-i})).
\end{aligned}$$

Since $\pi_i(m_i, \phi_i) = \pi_i(\bar{m}_i)$ and $\pi_i(\bar{m}_i) \in BR_i(\pi_{-i}(m_{-i}))$ for each $m_{-i} \in M_{-i}$ such that $(\bar{m}_i, m_{-i}) \in \text{supp}(\phi_i^*)$ by (4.1), it follows that (4.7) holds in this case.

If $\beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_{j,M_i}^*[m_i] > 0$ and $\phi_i \neq \phi_i^*$, then

$$\begin{aligned}
&\lim_k \frac{\sum_{\phi_{-i} \in \text{supp}(\pi_{-i}^{1,\alpha})} \pi_{-i}^{1,\alpha}[\phi_{-i}] \left(\sum_{m_{-i}} p^{\alpha}(\phi_i, \phi_{-i})[m_i, m_{-i}] u_i(\gamma_i, \pi_{-i}^{2,\alpha}(m_{-i}, \phi_{-i})) \right)}{\sum_{\phi_{-i} \in \text{supp}(\pi_{-i}^{1,\alpha})} \pi_{-i}^{1,\alpha}[\phi_{-i}] p_{M_i}^{\alpha}(\phi_i, \phi_{-i})[m_i]} \\
&= \sum_{m_{-i}} \frac{\beta_i \phi_i[m_i, m_{-i}] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_j^*[m_i, m_{-i}]}{\beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_{j,M_i}^*[m_i]} u_i(\gamma_i, \pi_{-i}(m_{-i})) \\
&= u_i \left(\gamma_i, \sum_{m_{-i}} \frac{\beta_i \phi_i[m_i, m_{-i}] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_j^*[m_i, m_{-i}]}{\beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_{j,M_i}^*[m_i]} \pi_{-i}(m_{-i}) \right).
\end{aligned}$$

Since $\pi_i(m_i, \phi_i)$ is optimal against $\sum_{m_{-i}} \frac{\beta_i \phi_i[m_i, m_{-i}] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_j^*[m_i, m_{-i}]}{\beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_{j,M_i}^*[m_i]} \pi_{-i}(m_{-i})$, it follows that (4.7) holds in this case.

Finally, consider the case where $\phi_i = \phi_i^*$ and

$$\beta_i \phi_{i,M_i}[m_i] + \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_{j,M_i}^*[m_i] > 0.$$

Note that it is enough to show that

$$\sum_{m_{-i}} \phi^*[m]\big(u_i(\pi(m)) - u_i(a_i, \pi_{-i}(m_{-i}))\big) \geq 0 \tag{4.9}$$

for each $a_i \in A_i$ and that

$$\begin{aligned}
&\sum_{m_{-i}} \phi^*[m]\big(u_i(\pi(m)) - u_i(a_i, \pi_{-i}(m_{-i}))\big) \\
&= \sum_{m_{-i}} \beta_i \phi_i^*[m]\big(u_i(\pi(m)) - u_i(a_i, \pi_{-i}(m_{-i}))\big) \\
&+ \sum_{m_{-i}} \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_j^*[m]\big(u_i(\pi(m)) - u_i(a_i, \pi_{-i}(m_{-i}))\big).
\end{aligned}$$

We have that $u_i(\pi(m)) \geq u_i(a_i, \pi_{-i}(m_{-i}))$ for each m_{-i} such that $\phi_i^*[m] > 0$ by (4.1); moreover, for each m_{-i} such that $\phi_j^*[m] > 0$ for some $j \in \text{supp}(\beta_{-i})$, then

$$m_i \in \cup_{j \in \text{supp}(\beta_{-i})} \text{supp}(\phi_{j,M_i}^*)$$

and, hence, $\sum_{m_{-i}} \sum_{j \in \text{supp}(\beta_{-i})} \beta_j \phi_j^*[m]\big(u_i(\pi(m)) - u_i(a_i, \pi_{-i}(m_{-i}))\big) \geq 0$ by (4.2). Thus, (4.9) holds and so does (4.7).

We next establish (4.8). If $\sum_{j \in \text{supp}(\beta)} \beta_j \phi_{j,M_i}^*[m_i] = 0$, then it follows that

$$\begin{aligned}
&\lim_k \frac{\sum_{\phi \in \text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] \left(\sum_{m_{-i}} p^\alpha(\phi)[m_i, m_{-i}] u_i(a_i, \pi_{-i}^{2,\alpha}(m_{-i}, \phi_{-i}))\right)}{\sum_{\phi \in \text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] p_{M_i}^\alpha(\phi)[m_i]} \\
&= \sum_{m_{-i}} \frac{\sum_{j \in \text{supp}(\beta)} \beta_j \phi_j^*[\bar{m}_i, m_{-i}]}{\sum_{j \in \text{supp}(\beta)} \beta_j \phi_{j,M_i}^*[\bar{m}_i]} u_i(a_i, \pi_{-i}(m_{-i})).
\end{aligned}$$

Since $\pi_i(m_i) = \pi_i(\bar{m}_i)$, it follows by (4.2) that (4.8) holds in this case.

If $\sum_{j \in \text{supp}(\beta)} \beta_j \phi_{j,M_i}^*[m_i] > 0$, then it is enough to establish (4.9). For each $a_i \in A_i$, we have that

$$\begin{aligned}
&\sum_{m_{-i}} \phi^*[m]\big(u_i(\pi(m)) - u_i(a_i, \pi_{-i}(m_{-i}))\big) \\
&= \sum_{m_{-i}} \sum_{j \in \text{supp}(\beta)} \beta_j \phi_j^*[m]\big(u_i(\pi(m)) - u_i(a_i, \pi_{-i}(m_{-i}))\big) \geq 0
\end{aligned}$$

by (4.2). Thus, (4.9) holds and so does (4.8).

The above arguments show that, for each finite subsets F of $\mathbb{N}$ and $\hat{F}$ of S, condition 8 holds whenever k is sufficiently high. Specifically, condition 8 (a) holds for each $i \in N$ whenever $k \geq k_0$. For each $i \in \text{supp}(\beta)$ and $(m_i, \phi_i) \in S_i(F, \hat{F})$, there is $k(m_i, \phi_i)$ such that condition 8 (b) holds whenever $k \geq k(m_i, \phi_i)$. For each $i \in N \setminus \text{supp}(\beta)$ and $m_i \in S_i(F, \hat{F})$, there is $k(m_i)$ such that condition 8 (c) holds whenever $k \geq k(m_i)$. Thus, let

$$k(F, \hat{F}) = \max \left\{ k_0, \max_{i \in \text{supp}(\beta)} \max_{(m_i, \phi_i) \in S_i(F, \hat{F})} k(m_i, \phi_i), \max_{i \in N \setminus \text{supp}(\beta)} \max_{m_i \in S_i(F, \hat{F})} k(m_i) \right\}.$$

Since condition 8 (b) is trivially satisfied when

$$\pi_i^{1,\alpha}[\phi_i] \sum_{\phi_{-i} \in \text{supp}(\pi_{-i}^{1,\alpha})} \pi_{-i}^{1,\alpha}[\phi_{-i}] p_{M_i}^{\alpha}(\phi_i, \phi_{-i})[m_i] = 0,$$

i.e. when $i \in \text{supp}(\beta)$ and $(m_i, \phi_i) \notin S_i(F, \hat{F})$, and that condition 8 (c) is trivially satisfied when $\sum_{\phi \in \text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] p_{M_i}^{\alpha}(\phi)[m_i] = 0$, i.e. when $i \in N \setminus \text{supp}(\beta)$ and $m_i \notin S_i(F, \hat{F})$, it follows that condition 8 holds whenever $k \geq k(F, \hat{F})$. This allows us to define the following subnet $\{\pi^{\varphi(\eta)}, p^{\varphi(\eta)}\}_\eta$ of $\{\pi^\alpha, p^\alpha\}_\alpha$ such that condition 8 holds.

The index set of the subnet $\{\pi^{\varphi(\eta)}, p^{\varphi(\eta)}\}_\eta$ is the same as the one in the net $\{\pi^\alpha, p^\alpha\}_\alpha$. The function $\varphi : \eta \mapsto \alpha$ is defined by setting, for each $\eta = (k, F, \hat{F})$,

$$\varphi(\eta) = \left(\max \left\{ k, k(F, \hat{F}) \right\}, F, \hat{F} \right).$$

It is then clear that condition 8 holds and that, as required by the definition of a subnet, for each α_0, there exists η_0, e.g. $\eta_0 = \alpha_0$, such that $\varphi(\eta) \geq \alpha_0$ for each $\eta \geq \eta_0$.

References

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