Monopoly Pricing with Optimal Information*

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October 24, 2024

Abstract

We analyze a monopoly pricing model where information about the buyer's valuation is endogenous. Before the seller sets a price, both the buyer and the seller receive private signals that may be informative about the buyer's valuation. The joint distribution of these signals, as a function of the valuation, is optimally chosen by the players. In general, players have conflicting incentives over the provision of information. As a modelling device, we assume that an aggregation function determines the information structure from the choices of the players, and we characterize the equilibrium payoffs for a natural class of aggregation functions. Every equilibrium payoff can be achieved by an information structure that is the result of the seller trying to make both players uninformed while the buyer tries to learn about his valuation. The seller can only price discriminate (set prices that depend on his belief about the buyer's valuation) to the extent that the buyer is indifferent between these prices.

*We wish to thank Peter Achim, Stephen Morris, Olivier Tercieux, Nikhil Vellodi and seminar participants at the University of York, PSE/Sciences Po, the 2023 SAET conference (Paris) and the Southeast Theory Festival (UCL) for helpful comments. Any remaining errors are, of course, ours.

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1 Introduction

Recent advances in the economic analysis of monopoly, such as Bergemann, Brooks, and Morris (2015) and Roesler and Szentes (2017), show that the impact of discriminatory pricing on consumer and producer surplus critically depends on the information available to both the seller and the buyer(s). Consequently, *both* parties may desire to influence or manipulate this information, and moreover there may be a conflict of interest regarding what information should be available. The idea that the information structure arises endogenously through the potentially conflicting actions of multiple parties raises several important questions. For example, what information does each party want to have, and what information do they want the other to have? In case they disagree, what information structure will result from their conflict? And given that the information structure arises endogenously, to what extent can the seller price discriminate? Our aim in this paper is to address these questions in a simple and tractable framework.

Answering these questions is not straightforward because it is infeasible to model all the possible ways each party can influence every piece of information provided. Moreover, when the buyer and the seller have different incentives over the information they wish to be provided, it is unclear how this conflict of interest will be resolved. The recent information design literature has generated many insights about the information structures that are likely to arise by carefully studying the incentives of some (metaphorical or literal) information designer who can choose from all possible information structures. However, with a few exceptions discussed in Section 2, only the case of a single information designer has been considered. The conceptual challenge of considering multiple designers is that ultimately they must decide on a single information structure, and any model of how a single information structure arises from the decisions of multiple designers requires non-obvious modelling choices. In short, there is a need for a general framework for modelling situations where the information structure is the result of the actions of *multiple* interested parties; specific instances of this framework must determine how conflicts between the parties are resolved.

We consider such framework to study how optimally chosen information affects monopoly pricing, the latter modelled in the standard way: the seller of a good, produced with zero marginal cost, makes a take-it-or-leave-it price offer to a buyer whose valuation is unknown and drawn from a finite set. Our point of departure from the standard model is that before the seller makes a price offer, both the buyer and the seller can take actions that determine the information they receive. In our model, an information structure is a function from the set of unknown valuations to a set of distributions over message profiles, consisting of one message for each player which he receives privately.[1] The information choices of the players will combine to produce some information structure, but as a tractable *reduced form representation* of the various actions players may take to influence this information structure, we assume that (i) each player (covertly) chooses the information structure directly, (ii) the set of possible messages is sufficiently rich as to not rule out any kind of information by assumption, and (iii) the true information structure (that determines the information that each player actually receives) is determined by an *aggregation function* that combines their choices.[2] An example of an aggregation function is one that maps the information structures chosen by the players into their convex combination for given strictly positive weights; it can be interpreted as each player trying to implement some information structure and nature choosing who is successful, with each player being successful with some fixed probability. We consider a class of aggregation functions

[1]For example, consider a seller who lists an object for sale on an online marketplace. Here each player's private message corresponds to the combination of the information provided by the listing, all the messages they receive from the other player and all other information that is relevant about either the value of the object or the beliefs of the other player.

[2]The seller's choice of information structure represents, in reduced form, all the actions that he takes to influence or manipulate the information that he and the buyer obtain, and likewise for the buyer, and an interpretation of the aggregation function is that it is a model of how the conflict between the buyer and the seller over the information they wish to be provided is resolved. The assumption that information choices are made covertly reflects our interpretation that the messages represent all the information the players receive. Thus, deviations can only be detected through their effect on the distribution of messages, but which deviations are detected is an endogenous feature of the model.

that includes the "convex-combination" one and characterize the equilibrium payoffs of a monopoly pricing game where the players choose their information in the way we have described before the seller makes a price offer.[3]

Our results generate a number of lessons about the implications of optimally chosen information structures for monopoly pricing: (i) All equilibrium payoffs can be obtained using a specific class of information structures where (ii) price discrimination is severely limited but (iii) multiple prices can be supported in equilibrium. Isolating the effect of requiring information to be optimally chosen leads to (iv) a unique equilibrium payoff and the conclusion that (v) the buyer is harmed by the seller's ability to price discriminate. We now discuss these lessons in turn.

Our main result shows that all pure strategy equilibrium payoffs can be achieved using an information structure where the buyer sometimes becomes informed about his valuation, and the seller knows only whether or not the buyer is informed; in particular, the seller is always uninformed about the buyer's valuation. The seller will set one price when the buyer is uninformed and another (possibly different) price when the buyer is informed. The buyer accepts when he is uninformed but accepts only if his valuation is at least as large as the price offer when he is informed. Thus, as far as the seller is concerned, it is better for both players to know nothing so he extracts surplus by charging the expected value; this is easier than first degree price discrimination which requires not only learning about the buyer's valuation but also credibly transmitting this information to the buyer. Thus, summing up:

(a) The buyer wants to learn about his valuation and wants the seller to know when he is successful in doing so, but the buyer does not transmit any information to the seller about his valuation.

(b) The seller does not want either player to learn about the buyer's valuation, but wants to know when the buyer is successful in doing so.

[3]As a solution concept we rely on Myerson and Reny (2020) who define sequential equilibrium for infinite games – our game is infinite because the set of information structures each player chooses from is a set of functions from valuations to distributions, and also because the seller can make any price offer from an interval of real numbers.

That the seller wants no information to be transmitted (or is unable to credibly transmit information) is often seen in reality. For example, many sellers list items on eBay with only very limited information;[4] wine producers in Bordeaux sell their product *en primeur* to négociants, before the wine is bottled and when the quality is still uncertain; and restaurant menus are often short and uninformative.[5] In each case, the buyer would prefer to have more information, and sometimes may be successful in acquiring such information (for example, a diner may ask the waiter to fully explain the menu).

A striking feature of our main result is that there is only one instance of price discrimination: the price may be lower for informed buyers than for uninformed ones but each is the same across valuations. This is a realistic feature: most service providers offer better deals for (in some cases, all) customers who call them to negotiate, which can be interpreted as the customer revealing that they are well informed about their valuation.

That the seller sets the same price for all valuations is due to the difficulty of credibly transmitting information when the information structure is designed by the players themselves (without commitment) as in our framework. For example, if the information structure is designed by the buyer, he will pretend to have whichever valuation gets the lowest price, thus rendering the message that the seller receives uninformative. Similarly, if the information structure is designed by the seller, he will try to make the buyer believe that his valuation is greater than the price. In reality, we often see, for example, marketing information provided by the seller that is essentially uninformative as in our model.

[4]Moreover, sellers on eBay are allowed to make a single time-limited offer to any buyer who views their listing; many sellers set these offers to be sent out automatically before any information is exchanged.

[5]In an article entitled "Why a minimalist menu can ruin my meal", published in the Financial Times on 10 August 2023, the writer laments the trend towards "obfuscating and withholding all but the most minimal information in the menu" which prevents him from making an informed choice. In our model, information is withheld precisely so that the item can be sold to those who would not have accepted if they were fully informed.

When mixed strategies are allowed, some price discrimination (across valuations) is possible since the argument that the buyer will pretend to have whichever valuation gets the lowest price breaks down: if the seller sets different *distributions* of prices following different messages, the buyer may be indifferent between them and can transmit information to the seller if different types randomize over messages with different probabilities. This information then rationalizes the different distributions of prices set by the seller. In other words, when mixed strategies are allowed, the buyer may induce certain distributions of posterior beliefs for the seller, who can then engage in third degree price discrimination. An interesting feature of this construction is that it is only possible when the messages are partially informative, i.e. the seller does not become fully informed. Thus, the requirement that information is provided optimally by the players themselves rules out first degree price discrimination (something rarely observed in real life). Moreover, although third degree price discrimination is possible, it is limited by the requirement that the buyer must find it optimal to provide the seller with the information required. For example, a seller may offer random discounts to those who are part of a loyalty scheme, but the decision to sign up is the buyer's.[6]

With the exception that the buyer may partially inform the seller about his valuation, properties (a) and (b) underlying the optimal information structure continue to hold with mixed strategies: the seller wants both parties to know nothing, the buyer wants to know whether her valuation is greater than the price, and both of them want the seller to know whether the buyer knows this.

Focusing again on the case of pure strategies, several pairs of prices, for informed and uninformed buyers, are possible in equilibrium.[7] For instance, when the buyer is uninformed, the price is at most the expected valuation but there are equilibria where it is lower; in fact it can be anything between the expected valuation and the lowest valuation of the buyer. What prevents the seller from raising the price in this case is that the buyer may rationally reject price changes by reasonably attributing it to

[6] Of course, this does not apply to (price) discrimination based on observable characteristics, such as gender-based price disparities.

[7] With mixed strategies, we can have several sets of distributions of prices.

sellers who know that the valuation is low. In other words, an instance of the classic lemons problem prevents the seller from adjusting the price in a way that would be profitable if the buyer's beliefs were fixed. That prices may depend on convention or social norms is realistic.[8] This creates a multiplicity of equilibrium payoffs, which is nevertheless smaller than what several recent papers, discussed in Section 2, have found when there is no requirement that the information is optimally chosen.[9] The multiplicity of payoffs arises because of the possibility that an uncertain buyer may become pessimistic when faced with an unexpected price change. This is a feature of any model where the seller has some information that the buyer does not; it is not a consequence of our requirement that the information structure is optimally chosen.[10]

We can rule out the possibility that the price offer affects the buyer's belief about his valuation – and isolate the effect of the requirement that the information structure is optimally chosen – by considering an alternative model where the buyer knows his own valuation, i.e. he is fully informed before engaging in the information game. For this model, we show that as the set of possible messages becomes large, there is a unique limit equilibrium payoff, which results from an information structure where the seller tries to become fully informed,[11] and the buyer wants the seller to learn nothing. The former property is intuitive, but the latter is not obvious: for example,

[8]It is natural, for example, to think that a buyer may reject price increases that he perceives as unfair. Our story implies that such a rejection can be made rationally, in the absence of any considerations of fairness.

[9]With pure strategies, we find that the set of equilibrium payoffs is significantly smaller (has lower dimension) than the set of feasible payoffs, but with mixed strategies, the set of equilibrium payoffs is larger and approaches the set of feasible payoffs in the case where the seller has almost perfect control over the information.

[10]Indeed, that there is an equilibrium where the seller sets the lowest possible price because an uninformed buyer rejects all other prices even though they are less than that his expected valuation is already established by, for example, Makris and Renou (2023) in a model with exogenous information. However, this feature is absent from Bergemann, Brooks, and Morris (2015) (where the buyer is fully informed) and Roesler and Szentes (2017) (where the seller is uninformed).

[11]To achieve this, the seller must play mixed strategies and randomize over the sets of messages he uses otherwise the buyer can send him the message corresponding to the lowest valuation and destroy the credibility of his information.

Bergemann, Brooks, and Morris (2015) show that there are information structures where both the buyer and seller do better than in the case where the seller learns nothing. In contrast, our result implies that such information structures cannot be optimally chosen by the buyer, and the best information structure the buyer can credibly choose is for the seller to be uninformed. When the seller is uninformed, he charges the same (uniform monopoly) price independently of the buyer's valuation; there is price discrimination only when the information structure is as the seller desires, which makes the buyer worse off than when the seller has no information.

The version of our model with an informed buyer allows a direct comparison with Bergemann, Brooks, and Morris (2015), since in their model the buyer is also fully informed. The key difference is that in our model the information of the seller is the result of optimal choices by both the buyer and the seller. In contrast to Bergemann, Brooks, and Morris (2015), who find that every payoff such that the seller gets at least the uniform monopoly profit can be supported in equilibrium, we find that there is a unique equilibrium payoff. In contrast to their result that the welfare implications of price discrimination depend on what information the seller has, we find that price discrimination (given optimally chosen information) is always bad for the buyer.

Proofs of our main results are in the Appendix. Supplementary material to this paper contains details for the alternative model with an informed buyer.[12]

2 Related literature

Our paper is inspired by the information design literature but in contrast to, e.g., Kamenica and Gentzkow (2011) or Bergemann and Morris (2016), we decentralize the role of the designer and relax the commitment assumption. In addition to monopoly pricing, our model can be applied to other strategic situations where multiple players design the information structure in a noncooperative way. For example, we have applied our model in Carmona and Laohakunakorn (2023) to study a repeated game where the monitoring structure is optimally chosen by the players themselves, and

[12] The supplementary material is available at https://klaohakunakorn.com/imsm.pdf.

in Carmona and Laohakunakorn (2024) to study correlated equilibrium where the correlation structure is optimally chosen by the players themselves. Besides these two papers, Gentzkow and Kamenica (2017) also consider the case of multiple information designers. In their setting, multiple senders choose what information to communicate to a single receiver who observes the realization of all information structures. In contrast, in our model, each player is both a sender and a receiver simultaneously and each observes only the realization of one information structure that aggregates their information choices.

Several recent papers have considered information design in a monopoly pricing setting. Bergemann, Brooks, and Morris (2015) consider a model where the buyer is fully informed and show that any feasible payoff such that the seller gets at least the uniform monopoly profit can be supported in equilibrium for *some* information provided to the seller. Makris and Renou (2023) consider all possible information structures (i.e. both the buyer and the seller can become informed) and show that any feasible payoff such that the seller gets at least the lowest valuation of the buyer can be supported in equilibrium. Kartik and Zhong (2023) allow the seller's cost also to be uncertain and characterize the payoffs from all information structures, as well payoffs under different restrictions on information structures. In contrast to these papers, we allow the players to choose their information structure optimally.

Many papers have considered information structures that are optimal for either the seller or the buyer(s). For example, Roesler and Szentes (2017) consider a model where the seller is uninformed and find that under the buyer-optimal information structure, the seller's payoff is less than the uniform monopoly profit.[13] Bergemann, Heumann, Morris, Sorokin, and Winter (2022) consider the revenue-maximizing information structure in a second price auction, and Bergemann, Heumann, and Morris (2023) consider the bidder-optimal information structure in an optimal auction. Bobkova (2024) compares the efficiency of different auctions when bidders can choose to learn about different components of their value. In contrast, we consider a setting where

[13]Moreover any feasible payoff such that the seller receives at least this amount can be supported in equilibrium for *some* information provided to the buyer.

both the buyer and the seller receive information, the information structure is the result of noncooperative optimal choices by both players, and the players can learn about each other's information as well as about the buyer's valuation.

In our model, information is free; several papers have considered related models with costly information. For example, Ravid, Roesler, and Szentes (2022) consider a model where the buyer can a purchase a signal about his valuation, and they show that as the cost of information goes to zero, equilibria converge to the worst free-learning equilibrium. Pernoud and Gleyze (2023) allow agents to acquire costly information and find that agents will typically choose to learn about others' preferences even when they are not directly payoff relevant. Achim and Li (2024) study monopoly pricing where the buyer can pay for expert advice and show that when the cost of advice is low, the seller "prices in" the expert which makes the buyer worse off. We abstract away from the cost of information since our goal is to demonstrate that certain information structures may be difficult to support in equilibrium, even when they can be freely chosen.

3 Model and main result

3.1 Model

A monopolist seller of a good makes a take-it-or-leave-it price offer to a buyer whose valuation is unknown and who chooses to buy the good at that price or not. In addition, before the seller makes a price offer, both the buyer and the seller choose an information structure.

The set of players is represented by $N = \{b, s\}$ with player b being the buyer and player s being the seller. The buyer's valuation of the seller's good belongs to the set $V = \{v_1, \ldots, v_K\}$ with $0 < v_1 < \cdots < v_K$; it is unknown to both players, and its prior distribution is $\nu \in \Delta(V)$ which is fully supported.

Each player chooses an information design. In the information design problems we consider, the designer sends messages to both players. The set of messages each player $i \in N$ can potentially receive is $M_i = \mathbb{N}$. This avoids imposing a bound on the number of different messages that the designer can send; to avoid unnecessary technical complications, we focus on (arbitrary) finite subsets of messages. Letting F be the set of finitely supported probability measures on $M = \prod_{i \in N} M_i = \mathbb{N}^2$, an *information design* consists of a function $\phi : V \to F$. Let Φ be the set of such functions.

The players' interaction is then described by the following extensive-form game G. At the beginning of the game, each player $i \in N$ chooses an information design $\phi_i \in \Phi$. After all players have chosen their information designs, a profile of buyer's valuations and messages $(v, m) \in V \times M$ is realized according to $\phi \in \Delta(V \times M)$ defined by setting, for each $(v, m) \in V \times M$,

$$\phi[v, m] = \nu[v]\beta(\phi_b(v), \phi_s(v))[m],$$

where $\beta : F^2 \to F$ is a function that aggregates the information choices of the players. That is, if the buyer chooses information structure ϕ_b, the seller chooses information structure ϕ_s and the buyer's valuation is v, the message profile m is drawn from $\beta(\phi_b(v), \phi_s(v))$.

We make the following assumptions on β:[14]

1. For each $m \in M$ and $(\gamma, \gamma') \in F^2$, if $\gamma[m] = \gamma'[m] = 0$, then $\beta(\gamma, \gamma')[m] = 0$.
2. For each $m \in M$, $\gamma \neq 1_m$ and $\tilde{\gamma} \in F$:
   (a) $\beta(1_m, \tilde{\gamma})[m] > \beta(\gamma, \tilde{\gamma})[m]$ and $\beta(\tilde{\gamma}, 1_m)[m] > \beta(\tilde{\gamma}, \gamma)[m]$,
   (b) $\beta(1_m, \tilde{\gamma})[m'] \leq \beta(\gamma, \tilde{\gamma})[m']$ and $\beta(\tilde{\gamma}, 1_m)[m'] \leq \beta(\tilde{\gamma}, \gamma)[m']$ for all $m' \neq m$, with strict inequality if $\gamma[m'] > 0$.
3. There exists $\beta_b, \beta_s \in (0, 1)$ with $\beta_b + \beta_s = 1$ such that for each $(\gamma_b, \gamma_s) \in F^2$, $\beta(\gamma_b, \gamma_s)[\text{supp}(\gamma_b)] \geq \beta_b$ and $\beta(\gamma_b, \gamma_s)[\text{supp}(\gamma_s)] \geq \beta_s$.

[14] For each $m \in M$, $1_m \in F$ denotes the probability measure degenerate on m and, for each $\gamma \in F$, $\text{supp}(\gamma)$ denotes the support of γ.

Property 1 requires that if both players agree that some message profile should arise with zero probability, then that message profile indeed arises with zero probability. Property 2 is similar to an independence of irrelevant alternatives condition: if a player chooses to send a message profile with probability 1, then the probability of that message profile should go up, and the probability of all other message profiles should go down independently of the choice of the other player. Property 3 requires that the realized message profile comes from the seller's information structure with probability at least β_s and from the buyer's information structure with probability at least β_b, for some $\beta_b, \beta_s \in (0, 1)$ with $\beta_b + \beta_s = 1$; in particular, if the players choose information structures with disjoint supports, then β_s (resp. β_b) is the probability that the realized message profile comes from the seller's (resp. buyer's) information structure. The parameter β_i can be interpreted as the amount of control player i has over the true information structure.

An example of an aggregation function that satisfies our assumptions is $\beta(\gamma, \gamma') = \beta_b\gamma + \beta_s\gamma'$, for some $\beta_b, \beta_s \in (0, 1)$ with $\beta_b + \beta_s = 1$. It is actually the only example if β extends (multi) linearly from degenerate distributions, i.e. if it satisfies $\beta(\gamma, \gamma') = \sum_{m\in\text{supp}(\gamma)} \sum_{m'\in\text{supp}(\gamma')} \beta(1_m, 1_{m'})$ for each $(\gamma, \gamma') \in F^2$. See Section 6 for a discussion of the model and aggregation function, as well as further examples.

Each player $i \in N$ observes $m_i \in M_i$ and his choice $\phi_i \in \Phi$ but neither m_j nor ϕ_j where $j \neq i$. The seller then makes a price offer $p \in [v_1, v_K]$ to the buyer, and the buyer chooses whether to accept ($a = 1$) or reject the offer ($a = 0$). Let $V^* = [v_1, v_K]$ and $A = \{0, 1\}$; payoffs are as follows: For each $(v, p, a) \in V \times V^* \times A$,

$$u_s(p, a) = pa,$$

$$u_b(v, p, a) = (v - p)a.$$

A pure strategy for the seller is $\pi_s = (\pi_s^1, \pi_s^2)$ such that $\pi_s^1 \in \Phi$ and $\pi_s^2 : \mathbb{N}\times\Phi \to V^*$ is measurable.[15] A pure strategy for the buyer is $\pi_b = (\pi_b^1, \pi_b^2)$ such that $\pi_b^1 \in \Phi$ and $\pi_b^2 : \mathbb{N} \times \Phi \times V^* \to A$ is measurable. A pure strategy is $\pi = (\pi_b, \pi_s)$ and let Π be the

[15]The set F is endowed with the topology of the weak convergence of probability measures and the corresponding Borel σ-algebra.

set of pure strategies. We use sequential equilibrium, defined in Myerson and Reny (2020), as our solution concept.[16]

Often, when we focus on pure strategies,[17] we write $\phi_i^* = \pi_i^1$, $p(m_s) = \pi_s^2(m_s, \phi_s^*)$, $a(m_b, p) = \pi_b^2(m_b, \phi_b^*, p)$ and $a(m_b, m_s) = a(m_b, p(m_s))$, where $(m_b, m_s, p) \in \mathbb{N}^2 \times V^*$. Let Π^* be the set of $\pi \in \Pi$ such that $a(m_b, v_1) = 1$ for each $m_b \in M_b$ and we focus on $\pi \in \Pi^*$. This is a mild refinement since, upon receiving any message m_b, the buyer is certain that his valuation is at least v_1 and thus is, at the very least, not worse off by buying at price v_1 than not buying.[18]

3.2 Examples

We present some examples of information structures and ask if they are optimal for the players under specific assumptions about behavior in the resulting monopoly pricing game. The examples feature $V = \{1, 2, 3, 4, 5\}$ with ν uniform (hence, the expected valuation is 3), and $\beta(\gamma, \gamma') = 0.5\gamma + 0.5\gamma'$ for each $(\gamma, \gamma') \in F^2$.

Example 1 *The information structure $\phi : V \to F$ such that*

$$\phi(v) = 1_{(v,v)} \text{ for each } v \in V$$

corresponds to full information. Suppose that, for each v, the seller makes the price offer v which the buyer accepts. Then ϕ cannot be the information structure in equilibrium, i.e. it is not optimal for both players to choose ϕ since the seller has a profitable deviation to choose an information structure $\phi'(v) = 1_{(5,5)}$ for each v. Then the distribution of messages is $(0.5)1_{(v,v)} + (0.5)1_{(5,5)}$ for each v. Thus, with probability 0.5, the seller will receive $m_s = v$ and get payoff $\sum_v \nu[v]v = 3$ as in the proposed equilibrium; however, with probability 0.5, the seller will get payoff 5 instead of 3.

[16] More precisely, π is a sequential equilibrium if it is a perfect conditional ε-equilibrium for every $\varepsilon > 0$.

[17] See Section 4 for an analysis of mixed strategies.

[18] If we allow the seller to make price offers less than v_1, then it can easily be shown that this refinement must be satisfied in every equilibrium.

Example 2 *The information structure* $\hat{\phi} : V \to F$ *such that*

$$\hat{\phi}(v) = \begin{cases} 1_{(1,1)} & \text{if } v \in \{1,2\} \\ 1_{(2,1)} & \text{if } v \in \{3,4,5\} \end{cases}$$

corresponds to a partially informed buyer and an uninformed seller. Here the buyer learns whether his value is less than or at least 3*, and the seller learns nothing. Suppose that the seller makes a price offer that is accepted if* $m_b = 2$ *and rejected if* $m_b = 1$*. Then* $\hat{\phi}$ *cannot be the information structure in equilibrium, since the seller has a profitable deviation to choose an information structure* $\phi'(v) = 1_{(2,1)}$ *for each* v*. In this case, with probability* 0.5*, the seller's price offer will be accepted with probability* 1 *instead of* $\frac{3}{5}$*.*

Example 3 *The information structure* $\bar{\phi} : V \to F$ *such that*

$$\bar{\phi}(v) = 1_{(3,3)} \text{ for each } v \in V$$

corresponds to no information. Suppose that the seller makes a price offer $p \in (1,3]$ *which the buyer accepts. Then* $\bar{\phi}$ *cannot be the information structure in equilibrium since the buyer has a profitable deviation to choose an information structure* $\phi(v) = 1_{(v,3)}$ *for each* v *and to accept only if* $p \leq v$*.*

We now argue that there is an equilibrium where the buyer chooses the information structure $\hat{\phi}$ and the seller chooses the information structure $\bar{\phi}$. First, we specify what happens on the equilibrium path: suppose that $\phi_b^* = \hat{\phi}$, $\phi_s^* = \bar{\phi}$, $p(1) = 3$, $p(3) = 3$, $a(1,p) = 0$ if $p = 3$, $a(2,p) = 1$ if $p = 3$ and $a(3,p) = 1$ if $p = 3$. Note that on the equilibrium path, the seller receives messages 1 and 3 and the buyer receives messages 1, 2 and 3. Also, since the seller sets price 3 after messages 1 and 3, the buyer will only see price 3 on the equilibrium path. Thus, the above is a complete description of the strategy for on-path histories.

We now argue that the price offer 3 is optimal for the seller. Crucially, note that any other price offer is off the equilibrium path and thus the belief following such price offer cannot be determined by Bayes' rule. In fact, it is possible to construct perturbations such that the buyer believes that he has valuation 1 after any unexpected price offer. Given such belief, we can specify that the buyer will only accept 1 (by assumption) and the equilibrium price offer, making the equilibrium price offer optimal.

Similarly, to ensure that the information structures are optimally chosen, we can construct perturbations such that following any zero probability message, the buyer believes that his valuation is 1 and the seller believes that the buyer would accept 5 (and hence makes price offer 5). Thus, we only have to ensure that the players do not want to deviate by sending different on-path messages to the other player. If the buyer sends message 3 instead of 1 to the seller, the price is the same, and he is making the correct decision ex post conditional on $\hat{\phi}$ being chosen, so $\hat{\phi}$ is optimal. For the seller, conditional on $\bar{\phi} = 1_{(3,3)}$ being chosen, he gets profit 3, which is the same profit he can get by sending the buyer message 2 (after which the buyer also accepts price 3) and higher than the profit he can get by sending the buyer message 1 (after which the buyer rejects all prices other than 1). Thus, $\bar{\phi}$ is optimal for the seller.

Note that $\bar{\phi}$ and $\hat{\phi}$ send different messages to each player, so in this equilibrium, the players know which information structure has been chosen. When the realized message profile is sent by the seller, the price is 3 and the buyer accepts. When the realized message profile is sent by the buyer, the price is 3 and the buyer accepts if and only if his valuation is at least 3. In the next subsection, Theorem 1 will imply that any equilibrium payoff can be achieved using a generalization of the above strategy, with the price following the seller's message being replaced by $p_s \in [1, 3]$ and the price following the buyer's message being replaced by $p_b \in [1, p_s]$.

3.3 Main result

Let $\Delta^0 = \{(\beta_b, \beta_s) \in (0,1)^2 : \beta_b + \beta_s = 1\}$ and, for each $(\beta_b, \beta_s) \in \Delta^0$, $\mathcal{B}(\beta_b, \beta_s)$ denote the set of aggregation functions β satisfying Properties 1–3 and $U^*(\beta_b, \beta_s)$ be the set of payoffs of the sequential equilibria $\pi \in \Pi^*$ of the game with aggregation function β for some $\beta \in \mathcal{B}(\beta_b, \beta_s)$.

Theorem 1 *For each* $(\beta_b, \beta_s) \in \Delta^0$, (u_b, u_s) *belongs to* $U^*(\beta_b, \beta_s)$ *if and only if there exists* $(p_b, p_s) \in (V^*)^2$ *and* $\lambda \in [0, 1]$ *such that*

$$u_b = \beta_s \left(\sum_v \nu[v]v - p_s \right) + \beta_b \left(\sum_{v \geq p_b} \nu[v](v - p_b) \right) \tag{1}$$

$$u_s = \beta_s p_s + \beta_b \left(p_b \sum_{v > p_b} \nu[v] + p_b \nu[p_b]\lambda \right), \tag{2}$$

$$p_b \leq p_s \leq \sum_v \nu[v]v \text{ and} \tag{3}$$

$$v_1 \leq p_b \sum_{v > p_b} \nu[v] + p_b \nu[p_b]\lambda. \tag{4}$$

Theorem 1 implies that any pure strategy equilibrium payoff can be achieved by an equilibrium with at most two prices, p_s and p_b. The price is p_s when the message comes from the seller's information structure, which happens with probability β_s, the price is p_b when the message comes from the buyer's information structure, which happens with probability β_b, and these prices do not contain any information about the buyer's valuation.

Conditions (1) and (2) describe the payoffs from such equilibrium, given that the buyer accepts p_s, accepts p_b whenever his valuation is greater than p_b and rejects p_b whenever his valuation is less than p_b. If the buyer's valuation is exactly p_b, he can accept with any probability λ.[19] Condition (3) requires that $p_b \leq p_s$, otherwise the buyer could deviate by sending the seller the message that results in p_s, and that $p_s \leq \sum_v \nu[v]v$, otherwise the buyer would not accept p_s. Condition (4) requires that the seller's payoff following each message must be at least v_1, since he can always offer v_1 which will be accepted.

The focus in Theorem 1 is on payoffs which has the advantage of abstracting from details of equilibrium strategies that are not relevant to the players' welfare. To illustrate this point, note that (u_b, u_s) such that $u_b = \sum_v \nu[v]v - v_1$ and $u_s = v_1$ is an equilibrium payoff (let $p_b = p_s = v_1$ and $\lambda = 1$), which can be obtained

[19] Although the buyer is playing pure strategies, he can choose an (nondegenerate) information structure that randomizes between sending himself two messages when his valuation is p_b: one where he accepts p_b and another where he rejects p_b.

with an equilibrium strategy in which the seller makes price offer v_1 regardless of his information. This then implies that any information structure can be optimally chosen by the players since then any price offer $p > v_1$ is off the equilibrium path and it is possible to construct perturbations such that the buyer optimally rejects any $p > v_1$, making v_1 optimal for the seller and any information structure optimal to each player. This multiplicity of equilibria is however irrelevant to players' welfare as all of them have the same payoff.

We thus focus on payoffs and on the properties of equilibrium strategies that support them. Focusing on the case $\lambda = 1$ for simplicity, Theorem 1 makes it clear that there is a set of strategies, which differ only in the prices p_b and p_s that the seller offers depending on which information design occurs, namely: (a) the buyer wants to learn about his valuation (in particular, whether it is less than p_b or at least p_b) and wants the seller to know when he is successful in doing so, but the buyer does not transmit any information to the seller about his valuation, e.g. the buyer's design ϕ_b^* is such that $\phi_b^*(v) = 1_{(1,1)}$ for each $v < p_b$ and $\phi_b^*(v) = 1_{(2,1)}$ for each $v \geq p_b$; (b) the seller does not want either player to learn about the buyer's valuation, but wants to know when the buyer is successful in doing so, e.g. the seller's design ϕ_s^* is such that $\phi_s^*(v) = 1_{(3,3)}$ for each $v \in V$; (c) the seller sets price p_b if he knows that the buyer has been successful in learning about his valuation and p_s otherwise; and (d) the buyer accepts p_s with probability one, and accepts p_b with probability one if his valuation is at least p_b and zero if his valuation is less than p_b. The strategy changes with p_b and p_s but the resulting information structure is always the same; since $\cup_v \text{supp}(\phi^*_{b,M_i}(v))$ and $\cup_v \text{supp}(\phi^*_{s,M_i}(v))$ are disjoint, each player i can find out which information design has occurred; each can conclude that the seller's (resp. buyer's) design has occurred if he receives message $m_i = 3$ (resp. $m_i < 3$); in addition, upon receiving message $m_b = 1$ (resp. $m_b = 2$), the buyer learns that his valuation is less than (resp. at least) p_b.

Theorem 1 and the above discussion make clear that prices do not depend on the valuation and that they depend only on which information structure occurs. Since the buyer is informed about his valuation if and only if his information structure realizes, it follows that price discrimination is limited to informed vs uninformed buyers; in particular, it does not extend to high vs low valuation buyers.

As already pointed out, it is possible to construct perturbations such that the buyer optimally rejects price offers that are off the equilibrium path. This, in particular, allows for prices p_s lower than the buyer's expected valuation to be offered when the seller's information design occurs. This accounts for the multiplicity of equilibrium payoffs.

To better understand the extent of payoff multiplicity, we provide a further characterization the set $U^*(\beta_b, \beta_s)$ of equilibrium payoffs. It uses the following notation: Let $E = \sum_{v \in V} \nu[v]v$, $E(p) = \sum_{v \geq p} \nu[v]v$ for each $v_1 \leq p \leq E$, $\nu(p) = \sum_{v \geq p} \nu[v]$ for each $v_1 \leq p \leq E$,

$$C_k = \{p \in (v_{k-1}, v_k] : p\nu(v_k) \geq v_1 \text{ and } p \leq E\} \text{ for each } k \in \{2, \dots, K\},$$

$$C_1 = \{v_1\},$$

$$\kappa = \{k \in \{1, \dots, K\} : C_k \neq \emptyset\},$$

$\underline{v}_k = \inf C_k$ and $\bar{v}_k = \max C_k$ for each $k \in \kappa$, and $v_0 = 0$. Furthermore, let p^* be a solution of $\max_{p \in [v_1, E]} p\nu(p)$. We say that $(u_b, u_s) \in U^*(\beta_b, \beta_s)$ is *represented by* (p_b, p_s, λ) if (1) and (2) hold for some $(p_b, p_s, \lambda) \in (V^*)^2 \times [0, 1]$ satisfying (3) and (4). Consider the set $U^{**}(\beta_b, \beta_s)$ of $(u_b, u_s) \in U^*(\beta_b, \beta_s)$ represented by (p_b, p_s, λ) with $\lambda = 1$. We then obtain the following corollary of Theorem 1.[20]

Corollary 1 *For each $(\beta_b, \beta_s) \in \Delta^0$, $U^{**}(\beta_b, \beta_s) = \cup_{k \in \kappa} U_k$ where, for each $k \in \kappa$ such that $\underline{v}_k > v_{k-1}$,*

$$\begin{aligned} U_k = &\{(u_b, u_s) \in \mathbb{R}^2 : u_b + u_s = \beta_s E + \beta_b E(v_k), \\ &\quad \text{and } \underline{v}_k(\beta_s + \beta_b \nu(v_k)) \leq u_s \leq \beta_s E + \beta_b \bar{v}_k \nu(v_k)\} \end{aligned}$$

and, for each $k \in \kappa$ such that $\underline{v}_k = v_{k-1}$,

$$\begin{aligned} U_k = &\{(u_b, u_s) \in \mathbb{R}^2 : u_b + u_s = \beta_s E + \beta_b E(v_k), \\ &\quad \text{and } \underline{v}_k(\beta_s + \beta_b \nu(v_k)) < u_s \leq \beta_s E + \beta_b \bar{v}_k \nu(v_k)\}. \end{aligned}$$

[20]See Section A.4 for an equivalent description of the element used in Corollary 1, which is useful to actually draw the set of equilibrium payoffs.

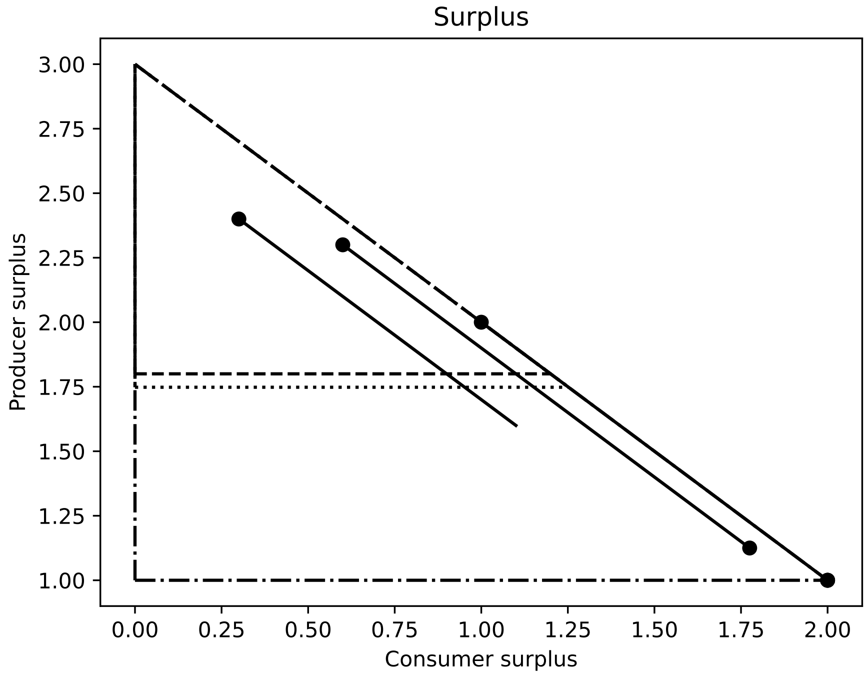

Figure 1: Equilibrium payoffs when $V = \{1, \ldots, 5\}$, ν is uniform and $\beta_b = 1/2$.

Furthermore, for each $k \in \kappa$*,*

$$\beta_s E + \beta_b E(v_K) \leq \beta_s E + \beta_b E(v_k) \leq \beta_s E + \beta_b E(v_1),$$

$$\beta_s E + \beta_b v_1 \leq \beta_s E + \beta_b \bar{v}_k \nu(v_k) \leq \beta_s E + \beta_b p^* \nu(p^*) \text{ and}$$

$$v_1 \leq \underline{v}_k(\beta_s + \beta_b \nu(v_k)) \leq \beta_s E + \beta_b p^* \nu(p^*).$$

Hence, $U^{**}(\beta_b, \beta_s)$ is the union of parallel lines with slope -1 with upper endpoints for u_s being minimized when $p_b = v_1$ and maximized when $p_b = p^*$ and the lower endpoints for u_s being minimized when $p_b = v_1$.[21] Figure 1 shows $U^{**}(\beta_b, \beta_s)$ in the case where $V = \{1, \ldots, 5\}$, ν is uniform and $\beta_b = 1/2$ (the three solid diagonal lines), and contrasts it with the surplus triangle in Bergemann, Brooks, and Morris (2015) (dashed triangle), Roesler and Szentes (2017) (dotted triangle) and Makris and Renou (2023) (dashdot triangle).

[21]See Figure 2 in Section A.4 for an example where the lower endpoints for u_s are not maximized when $p_b = p^*$.

The surplus triangle in Makris and Renou (2023) shows all the payoffs that can be obtained for some information structure; it is the largest one since they consider all possible information structures (Bergemann, Brooks, and Morris (2015) consider those where the buyer is fully informed and Roesler and Szentes (2017) consider those where the seller is uninformed).[22] In our setting, by contrast, the information structure must be chosen optimally by the players and this implies that only a small subset of the payoffs identified by Makris and Renou (2023) can be sustained in a pure strategy equilibrium.

4 Mixed strategies

In this section, we allow the players to mix, i.e. a strategy for the seller is $\pi_s = (\pi_s^1, \pi_s^2)$ such that $\pi_s^1 \in \Delta(\Phi)$ and $\pi_s^2 : M_s \times \Phi \to \Delta(V^*)$ is measurable, a strategy for the buyer is $\pi_b = (\pi_b^1, \pi_b^2)$ such that $\pi_b^1 \in \Delta(\Phi)$ and $\pi_b^2 : M_b \times \Phi \times V^* \to \Delta(A)$ is measurable, and a strategy is $\pi = (\pi_b, \pi_s)$. We identify $\Delta(A)$ with the one-dimensional simplex and write $\pi_b^2(m_b, \phi_b, p)$ to mean the probability that the buyer accepts following (m_b, ϕ_b, p), i.e. $\pi_b^2(m_b, \phi_b, p) = \pi_b^2(m_b, \phi_b, p)[1]$.

Let $\bar{\Pi}$ be the set of strategies, let $\bar{\Pi}^*$ be the set of $\pi \in \bar{\Pi}$ such that $\pi_b^2(m_b, \phi_b, v_1) = 1$ for each $(m_b, \phi_b) \in M_b \times \Phi$, and let $\bar{U}^*(\beta_b, \beta_s)$ be the set of payoffs of sequential equilibria $\pi \in \bar{\Pi}^*$ of the game with aggregation function β such that $\beta(\gamma, \gamma') = \beta_b \gamma + \beta_s \gamma'$ for each $\gamma, \gamma' \in F$. Recall that $E = \sum_v \nu[v]v$.

Mixed strategies are useful in our model for at least the following reasons: First, if an uninformed buyer accepts the price offer E with some probability $p \in [v_1/E, 1)$, then the seller can randomize between offering E and pE (which is accepted with

[22] In Makris and Renou's (2023) setting, payoffs where the seller gets v_1 can be achieved using, for example, an information structure where the seller but not the buyer knows the valuation. As in our model, there are perturbations such that the buyer optimally rejects any off-path price offer greater than v_1. Thus, to achieve $(E - v_1, v_1)$, the seller can set price v_1 which the buyer accepts. To achieve $(0, v_1)$, the seller can set price E which the buyer accepts with probability v_1/E. The first degree price discrimination payoff can be achieved with full information as usual. Then all other payoffs can be achieved by taking convex combinations of these information structures.

probability one). Note that the buyer's payoff is zero when the price is E and $(1-p)E$ when the price is pE. By choosing appropriate weights on the randomization and the probability of acceptance p, it turns out that following the seller's information structure being chosen, *any* feasible payoff can be supported in equilibrium.

Second, mixed strategies relax the requirements that the optimality of information impose on the informativeness of the messages sent from the buyer's information structure. Without any optimality requirement, the buyer can induce any Bayes's plausible distribution of posterior beliefs (about the buyer's valuation) for the seller. As we have seen in Section 3, with the restriction to pure strategies, the optimality requirements are so stringent that the only possible posterior belief that the buyer can induce is the prior. With mixed strategies, however, some nondegenerate distributions of posteriors will be possible. If the seller sets different distributions of prices following different messages, buyers with different valuations may be indifferent between these distributions, and hence willing to induce the posterior beliefs for the seller that in turn rationalize the distributions of prices. Thus, following the buyer's information structure being chosen, mixed strategies allow the buyer to optimally induce certain distributions of posterior beliefs for the seller, who must then set optimal prices given each belief. This is consistent with real-life instances of price discrimination where the information that the seller uses to price discriminate is provided by the buyer; for example, Amazon Prime customers face different distributions of prices compared with non-Prime customers.

Finally, by randomizing over the set of messages that a player sends to himself, it becomes more difficult for his opponent to mimic his message (for example, the seller cannot reliably send the buyer the message that corresponds to a high valuation if the buyer randomizes over which message he uses to mean this), which further relaxes some constraints that the equilibrium prices must satisfy.

For each sequential equilibrium, M_s^v in the statement of the following theorem will correspond to the set of messages that buyer type v sends to the seller with positive probability, $\gamma(m_s)$ and $\mu(m_s)$ to the price distribution and the seller's posterior belief after all such messages, $\xi(v, m_s)$ to the probability the buyer accepts v when indifferent after sending message m_s to the seller, and τ to the distribution of posterior beliefs of the seller induced by the buyer.

Theorem 2 *For each* $(\beta_b, \beta_s) \in \Delta^0$, $u \in \bar{U}^*(\beta_b, \beta_s)$ *if and only if there exists:*

(i) For each $v \in V$*:* $M_s^v \subseteq M_s$,

(ii) For each $m_s \in \cup_v M_s^v$*:* $\gamma(m_s) \in \Delta(V^*)$ *and* $\mu(m_s) \in \Delta(\{v \in V : m_s \in M_s^v\})$

(iii) For each $v \in V$ *and* $m_s \in M_s^v$*:* $\xi(v, m_s) \in [0,1]$, *and*

(iv) $\tau \in \Delta(\cup_v M_s^v)$

such that:

$$u = \beta_s(u_b^s, u_s^s) + \beta_b(u_b^b, u_s^b), \tag{5}$$

$$(u_b^s, u_s^s) \in \{(u_b, u_s) : v_1 \leq u_s \leq E, 0 \leq u_b \leq E - v_1, u_s + u_b \leq E\}, \tag{6}$$

$$u_b^b = \sum_v \nu[v] \sum_{p<v} \gamma(m_s^v)[p](v-p) \geq \sum_v \nu[v] \sum_{p<v} \gamma(h(v))[p](v-p) \tag{7}$$

for each $v \mapsto m_s^v$ *such that* $m_s^v \in M_s^v$ *for each* $v \in V$ *and* $h : V \to \cup_v M_s^v$,

$$u_s^b = \sum_{m_s \in \cup_v M_s^v} \tau(m_s) p_{m_s} \Big(\sum_{v > p_{m_s}} \mu(m_s)[v] + \mu(m_s)[p_{m_s}]\xi(p_{m_s}, m_s) \Big) \tag{8}$$

for each $m_s \mapsto p_{m_s}$ *such that* $p_{m_s} \in \mathrm{supp}(\gamma(m_s))$ *for each* $m_s \in \cup_v M_s^v$,

$$\sum_{m_s \in M_s^v} \tau(m_s)\mu(m_s)[v] = \nu[v] \text{ for each } v \in V, \tag{9}$$

$$p\Big(\sum_{v>p} \mu(m_s)[v] + \mu(m_s)[p]\xi(p, m_s)\Big) \geq v_1 \text{ if } m_s \in \cup_v M_s^v \text{ and } p \in \mathrm{supp}(\gamma(m_s)) \tag{10}$$

$$u_s^s \geq v \text{ for each } v \leq \min \cup_{m_s} \mathrm{supp}(\gamma(m_s)) \text{ if } u_b^s > 0, \tag{11}$$

$$E \geq v \text{ for each } v \leq \min \cup_{m_s} \mathrm{supp}(\gamma(m_s)) \text{ if } u_b^s = 0. \tag{12}$$

Since we focus in this section on the "convex-combination" aggregation function, we can interpret each payoff $u = \beta_s u^s + \beta_b u^b$ as consisting of u^s that arises when the seller's information structure is chosen and u^b that arises when the buyer's information structure is chosen.

Conditional on the seller's information structure being chosen, any feasible payoff where the seller gets at least v_1 can be supported in mixed strategy equilibrium – this is condition (6). For example, suppose that we wish to achieve some payoff $(\bar{u}_b, \bar{u}_s)$, with $\bar{u}_s \geq v_1$. This can be achieved by the seller sending an uninformative message to both players, making price offer $\bar{u}_s$ with probability $\frac{\bar{u}_b}{E-\bar{u}_s}$ and price offer E with remaining probability. Feasibility implies that $\frac{\bar{u}_b}{E-\bar{u}_s} \in [0,1]$. If the buyer accepts E with probability $\frac{\bar{u}_s}{E}$, accepts $\bar{u}_s$ with probability 1, and rejects all other price offers except v_1, then the seller gets $\bar{u}_s$, both prices are optimal for the seller, and the buyer gets $\frac{\bar{u}_b}{E-\bar{u}_s}(E-\bar{u}_s) = \bar{u}_b$.[23]

In contrast, when the buyer's information structure is chosen, the requirement that such information structure is chosen optimally imposes restrictions on the payoffs that can be achieved. In particular, after receiving a message m_s from the buyer's information structure, if the seller sets a distribution of prices $\gamma(m_s)$, then he must be indifferent between each price in the support of $\gamma(m_s)$ given his belief $\mu(m_s)$ following m_s – this is condition (8). In general, the buyer can induce any distribution of beliefs for the seller satisfying the Bayes' plausibility constraint (9). However, each type of buyer in the support of $\mu(m_s)$ must also be willing to send message m_s to the seller; thus, supposing that each type v sends messages in M_s^v with positive probability, $\text{supp}(\mu(m_s)) = \{v \in V : m_s \in M_s^v\}$ and all $m_s \in M_s^v$ must give the same payoff to type v as required by condition (7).

Condition (10) is the analogue of condition (4) from Theorem 1. Finally, to understand conditions (11) and (12), note that buyer types $v \leq \min \cup_{m_s} \text{supp}(\gamma(m_s))$ are getting zero payoff conditional on the buyer's information being chosen. Such types could deviate by sending the seller a message from $\cup_{v \in V, \phi_s \in \pi_s^1} \text{supp}(\phi_{s,M_s}(v))$ instead, i.e. by attempting to mimic the seller's message. Conditions (11) and (12) ensure that the lowest price following each of the seller's messages is greater than all such v so that these "mimicking deviations" are not profitable. Note that we only need to be concerned with "mimicking deviations" from buyer types that get zero payoff conditional on the buyer's information being chosen. This is because with

[23]That rejecting all prices except E, $\bar{u}_s$ and v_1 is optimal for the buyer relies on off-path beliefs.

mixed strategies, the seller can randomize over the messages that he sends himself so that these "mimicking deviations" are detected with high probability, making their payoff arbitrarily close to zero.[24]

Remark 1 *Theorem 2 implies that for each* $\bar{u} \in P = \text{co}(\{(0, v_1), (0, E), (E - v_1, v_1)\})$ *and* $\varepsilon > 0$*, there exists* $\bar{\beta}_s < 1$ *such that for* $\beta_s > \bar{\beta}_s$*, there exists a mixed strategy equilibrium payoff* u *such that* $\|\bar{u} - u\| < \varepsilon$*, i.e. for* β_s *close 1, every payoff in* P *can be approximated arbitrarily well in a mixed strategy equilibrium.*

5 Informed buyer

We consider in this section the same setting as in Section 4 except that the buyer is informed about his valuation, i.e. the buyer can condition his acceptance decision on v.[25] As in Bergemann, Brooks, and Morris (2015), we focus on strategies such that the buyer buys if indifferent,[26] which pins down the buyer's second period strategy, i.e.

$$\pi_b^2(p, v) = \begin{cases} 1 & \text{if } v \geq p, \\ 0 & \text{otherwise} \end{cases}$$

for each $(v, p) \in V \times V^*$. Note that the message of the buyer plays no role and, for simplicity, we assume that only the seller receives a message, i.e. $M = M_s$. It is also without loss of generality to assume that the seller will set prices in V. A strategy for the buyer is then $\pi_b = \pi_b^1$ such that $\pi_b^1 \in \Delta(\Phi)$. A strategy for the seller is $\pi_s = (\pi_s^1, \pi_s^2)$ such that $\pi_s^1 \in \Delta(\Phi)$ and $\pi_s^2 : M_s \times \Phi \to V$ is measurable. To avoid technical difficulties, we focus on the case where both π_b^1 and π_s^1 have finite support. Let $\hat{\Pi}$ be the set of such strategies and let $\hat{U}^*(\beta_b, \beta_s)$ be the set of payoffs of the sequential equilibria $\pi \in \hat{\Pi}$ when $M_s = \mathbb{N}$ of the game with aggregation function β such that $\beta(\gamma, \gamma') = \beta_b \gamma + \beta_s \gamma'$.

[24] Thus, conditions (11) and (12) are weaker than the requirement that $p_b \leq p_s$ in Theorem 1.

[25] Proofs and some omitted details for this section can be found in the supplementary material.

[26] This is without loss by the standard argument that, for each $\varepsilon > 0$, the seller could offer $v - \varepsilon$, which is accepted with probability 1 by type v. Note that this argument did not apply in previous sections, since offering $v - \varepsilon$ instead of v affects the buyer's belief when he does not observe v.

Recall that $\nu(p) = \sum_{v \geq p} \nu[v]$. It turns out that unless the uniform monopoly profit is v_1, there is no sequential equilibrium when $M_s = \mathbb{N}$.

Theorem 3 *If* $\max_{v \in V} v\nu(v) > v_1$*, then* $\hat{U}^*(\beta_b, \beta_s) = \emptyset$ *for each* $(\beta_b, \beta_s) \in \Delta^0$*.*

The reason why there is no equilibrium when $M_s = \mathbb{N}$ is roughly as follows. In any such equilibrium, the seller must get the first degree price discrimination payoff whenever his information structure is chosen and must set the optimal price conditional on the message being drawn from the buyer's information structure whenever the buyer's information structure is chosen.[27] It turns out that these requirements cannot be satisfied at the same time, since first degree price discrimination implies that there are some messages after which the seller sets price v_1, but best responding conditional on the buyer's information structure being chosen means that some types with valuation strictly greater than v_1 must be getting zero payoff in equilibrium.[28] Such types can profitably deviate by sending any message that leads to v_1 with strictly positive probability.

When $|M_s|$ is large but finite, a similar argument implies that there is no equilibrium in pure strategies. Now the issue is that when $|M_s|$ is sufficiently large, the seller can guarantee a payoff close to the one identified in the previous paragraph, and thus there must be some message $\hat{m}$ following which the seller sets price v_1 with probability 1 (because of the pure strategy assumption). But since the buyer can always send $\hat{m}$ to the seller, this implies that the price following every message from the buyer's information structure must be v_1, which contradicts the seller's (almost) best responding to each message drawn from the buyer's information structure.

[27]For instance, he can guarantee this payoff by choosing $K = |V|$ messages $m^1, \ldots, m^K$ that are not being used by the buyer — which is possible since $\cup_v \text{supp}(\phi_b^*(v))$ is finite and $M_s = \mathbb{N}$, — sending message m^k when the buyer's valuation is v_k (i.e. $\phi_s(v_k) = 1_{m^k}$), charging v_k when he receives message m^k, and for any other message (which must come from the buyer's information structure) he can charge the best response conditional on the buyer's information structure being chosen.

[28]Since $\max_{v \in V} v\nu(v) > v_1$, the best response cannot be v_1 for every message from the buyer's information structure. But if the best response is $\hat{v} > v_1$ following some message m, then $\hat{v}$ must have sent message m which implies that $\hat{v}$ gets zero payoff in equilibrium.

The above argument suggests that the seller will want to prevent the buyer from mimicking him; this can be done by encoding the messages of his information design so that only he knows which valuation corresponds to which messages. This may explain why real-world companies set prices based on certain consumer characteristics without revealing what those characteristics are.

To explore the above incentive to obfuscate the meaning of messages and since there does not exist an equilibrium when M_s is countably infinite, we focus instead on the case where M_s is finite and $|M_s|$ converges to infinity. For convenience, we also focus on the generic case where $v \mapsto v\nu(v)$ has a unique maximizer, v^*; this is then the uniform monopoly price.

For each $(\beta_b, \beta_s) \in \Delta^0$ and $n \in \mathbb{N}$, let $G_n(\beta_b, \beta_s)$ and $\hat{\Pi}_n$ be, respectively, the information-design game and the set of strategies $\pi \in \hat{\Pi}$ when $M_s = \{1, \ldots, n\}$ and $\beta(\gamma, \gamma') = \beta_b \gamma + \beta_s \gamma'$ for each $\gamma, \gamma' \in \Delta(M_s)$. Let $U_n(\beta_b, \beta_s)$ be the set of payoffs of sequential equilibria $\pi \in \hat{\Pi}_n$ of $G_n(\beta_b, \beta_s)$ and let $\hat{U}(\beta_b, \beta_s)$ be the limit of the sequence $\{U_n(\beta_b, \beta_s)\}_{n=1}^{\infty}$.[29] Recall that $E = \sum_v \nu[v]v$.

Theorem 4 *For each $(\beta_b, \beta_s) \in \Delta^0$, (u_b, u_s) belongs to $\hat{U}(\beta_b, \beta_s)$ if and only if*

$$u_b = \beta_b \sum_{v \geq v^*} \nu[v](v - v^*) \text{ and } \tag{13}$$

$$u_s = \beta_s E + \beta_b v^* \nu(v^*). \tag{14}$$

Theorem 4 demonstrates that in a model where the buyer knows his valuation, the limit equilibrium payoff is (generically) unique: with probability β_s, the seller becomes perfectly informed and first degree price discriminates; with probability β_b, the seller remains completely uninformed and sets the uniform monopoly price.

Thus, either there is no price discrimination – the price is the uniform monopoly price regardless of the buyer's valuation – or there is first degree price discrimination; since the latter is worse for the buyer than the former, price discrimination (given optimally chosen information) is always bad for the buyer. This contrasts with Bergemann, Brooks, and Morris's (2015) result, according to which the buyer can obtain,

[29] See the supplementary material for more details on the definition of $\hat{U}(\beta_b, \beta_s)$.

for some information structure, a higher welfare than what he gets when the seller sets the uniform monopoly price, i.e. price discrimination (without the optimality requirement) can be good for the buyer.

The information structure needed to support the unique limit equilibrium payoff has similar properties as when the buyer is uninformed, namely, the seller wants to be as informed as the buyer, but the buyer wants the seller to know nothing. Randomization is needed to obfuscate the messages: suppose that, e.g., the buyer chooses $\phi_b(v_k) = 1_{K+1}$ and the seller chooses $\phi_s(v_k) = 1_k$ for each $k \in \{1, \ldots, K\}$ and sets a price equal to v^* except when his message is $k \in \{1, \ldots, K\}$, in which case he sets a price equal to v_k. Then the buyer would gain by choosing instead $\phi_b(v) = 1_1$ for each v which then leads to a price of v_1.

The seller can obfuscate the meaning of his messages easily by considering different sets of K numbers in the message set $M_n = \{1, \ldots, n\}$. This can be done by taking a bijection $\psi : \{1, \ldots, n\} \to \{1, \ldots, n\}$, in which case $\psi(k)$ indicates that the buyer's valuation is v_k. Thus, the seller can choose $\phi_{s,\psi}(v_k) = 1_{\psi(k)}$ for each $k \in \{1, \ldots, K\}$ and randomize in an uniform way over the set of such ψ. Since the seller knows his information design, he knows the function ψ used and can decode the message, i.e. to charge $v_{\psi^{-1}(m)}$ when receiving message $m \in \psi(\{1, \ldots, K\})$; moreover, the seller knows that any message $m \notin \psi(\{1, \ldots, K\})$ must come from the buyer's information structure.

It is also important that the buyer obfuscates the meaning of the seller's message. To see this, suppose that, e.g., the buyer chooses $\phi_b(v) = 1_1$ for each $v \in V$ and the seller chooses the above mixed strategy and, if $\phi_{s,\psi}$ realizes, sets a price equal to v^* except when his message is $m \in \psi(\{1, \ldots, K\})$, in which case he sets a price equal to $v_{\psi^{-1}(m)}$. When the buyer's information design occurs, the seller gets a message $m = 1$ and should optimally set a price of v^* but whenever $1 \in \psi(\{1, \ldots, K\})$ and $v_{\psi^{-1}(1)} \neq v^*$, thus with strictly positive probability, will set a non-optimal price. Thus, the seller would gain by randomizing only over ψ such that $1 \notin \psi(\{1, \ldots, K\})$, which then gives him a perfectly informative message whenever his information structure realizes. The buyer can prevent the seller from becoming perfectly informed by randomizing uniformly over all messages, i.e. for each $m \in M_n = \{1, \ldots, n\}$, to choose $\phi_{b,m}(v) = 1_m$ for each $v \in V$ with probability $1/n$.

When n is large, the seller will be almost sure that the buyer's valuation is v_k after receiving $\psi(k)$; thus setting v_k is optimal. Given the seller's strategy, the buyer is indifferent between all messages and thus uniform randomization is optimal. Given the buyer's strategy, the seller is indifferent between using any set of K messages and uniform randomization over ψ is optimal.

Obfuscation by the seller prevents the buyer from manipulating the information the seller uses to first degree price discriminate and, as already mentioned, may correspond to the opacity of the set of buyer's characteristics that real-world sellers use to set prices. Obfuscation by the buyer prevents the seller from being certain of the buyer's valuation and may correspond to the reluctance of real-world buyers to provide information to the seller.

6 Discussion and concluding remarks

In this paper we propose a model of the conflicting goals that monopolists and their buyers have regarding the information available to them. In a variety of settings, we show that the seller will attempt to become as informed as the buyer (but no more), whereas the buyer will try to conceal his valuation from the seller. When the buyer is initially uninformed, he will try to become informed about his valuation; furthermore, both of them want it to be common knowledge whether or not the buyer succeeded in doing so.

Price discrimination is limited by our requirement that the information must be provided optimally by the buyer and the seller themselves. When the buyer is initially uninformed, prices may depend on whether or not the buyer becomes informed about his valuation but further price discrimination is limited by the requirement that the buyer must find it optimal to provide the seller with the information required. In the particular case of pure strategies, no further price discrimination is possible, i.e. price discrimination is exactly limited to informed vs uninformed buyers. When the buyer is initially informed, the conflict between the seller and the buyer is stark: the seller wants to become fully informed and the buyer wants the seller to know nothing. In this case, price discrimination is always bad for the buyer.

There are sharper conclusions when the buyer is initially informed because an informed buyer does not update his belief in response to different price offers. Our conclusions also depend on how the conflict of interest between the buyer and the seller over their information is resolved. The latter is captured in our setting by the aggregation function and the former concerns the (lack of) restrictions imposed by sequential equilibrium on how an initially uninformed buyer should update his belief after unexpected price offers. In what follows, we provide a brief discussion of these conceptual issues as well as some possible extensions of our framework.

(a) *Aggregation function.* The aggregation function determines the information acquisition possibilities in our model. For example, if one wishes to assume that the players are able to learn about the valuation, but not directly about each other's beliefs about the valuation, then one can assume that the aggregation function satisfies, for each $\gamma, \gamma' \in F$, $\beta(\gamma, \gamma') = \gamma_{M_b} \otimes \gamma'_{M_s}$. Under this specification, each player fully controls the distribution of his own private message but is not able to directly learn about the private message of his opponent since the messages are conditionally independent. On the other hand, the properties we impose on the aggregation function are meant to capture the possibility that players can influence, manipulate and learn about each other's information.

Aggregation functions satisfying properties 1–3 can be interpreted as nature choosing a player i with probability β_i and then the message being drawn from the support of ϕ_i but not necessarily according to ϕ_i (satisfying some monotonicity requirement as ϕ_i varies). In particular, properties 1–3 allow for some aggregation functions other than the convex combination one.[30] That Theorem 1 holds for any aggregation func-

[30] For example, let $\prec$ be the lexicographic order on $\mathbb{N}^2$ defined by $m \prec m'$ if (i) $m_1 < m'_1$ or (ii) $m_1 = m'_1$ and $m_2 < m'_2$. For each $\gamma \in F$, let $m(\gamma) \in \mathbb{N}^2$ be such that $\gamma[m(\gamma)] = \max_m \gamma[m]$ and there is no $\hat{m} \in \mathbb{N}^2$ such that $\gamma[\hat{m}] = \max_m \gamma[m]$ and $\hat{m} \prec m(\gamma)$. Let $0 < \varepsilon < 1$ and define, for each $\gamma, \gamma' \in F$, $\beta(\gamma, \gamma') = \frac{1}{2}\left((1-\varepsilon)1_{m(\gamma)} + \varepsilon\gamma\right) + \frac{1}{2}\left((1-\varepsilon)1_{m(\gamma')} + \varepsilon\gamma'\right)$.

tion in this class demonstrates that nothing depends on the true information structure being exactly equal to the ones chosen by the players as in the convex combination aggregation function. We assume the convex combination aggregation function in Sections 4 and 5 for convenience, but it is possible that, as in the case of Theorem 1, these results also hold for any aggregation function satisfying properties 1–3.

That nature picks each player i with probability β_i is of course an extreme assumption and it would be interesting to study the implications of alternative properties of the aggregation function (representing different models of information acquisition) for optimal price discrimination. Nevertheless, we view this as a simple and tractable way of representing the control each player has on the information structure using a single parameter.

(b) *Possible refinements.* Multiplicity in our model comes from the possibility that the price offer can affect the buyer's belief. In Section 5, we shut down this possibility in the most extreme way, by assuming that the buyer is fully informed. Alternatively, one could consider equilibrium refinements that put restrictions on the buyer's belief updating process as a result of price changes. Such restrictions may be empirically motivated by how buyers react to price changes in real life.

(c) *Other mechanisms.* Our aim is to understand the impact of optimally chosen information on classic mechanisms. Thus, we focus on a simple take-it-or-leave-it monopoly pricing mechanism. An alternative approach would be to consider the seller as a mechanism designer, who commits to a general mechanism that recommends a distribution of information designs to the buyer and implements an allocation as a function of the buyer's reported realized information design and private message and the seller's own realized information design and private message. We leave this interesting question for future work.

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Appendix: Proofs

A Proofs for Section 3

A.1 Preliminary Lemmas

Any sequential equilibrium $\pi \in \Pi$ satisfies the following conditions on the equilibrium path:

$$\begin{aligned}&\sum_v \nu[v] \sum_m \beta(\phi_b^*(v), \phi_s^*(v))[m] u_s(\pi^2(m, \phi^*)) \geq \\ &\sum_v \nu[v] \sum_m \beta(\phi_b^*(v), \phi_s(v))[m] u_s(\hat{\pi}^2(m, \phi_b^*, \phi_s)),\end{aligned} \tag{A.1}$$

for each $\phi_s \in \Phi$ and $\hat{\pi}_s^2 : M_s \times \Phi \to V^*$, where

$$\pi^2(m, \phi^*) = (\pi_s^2(m_s, \phi_s^*), \pi_b^2(m_b, \phi_b^*, \pi_s^2(m_s, \phi_s^*))) \text{ and}$$
$$\hat{\pi}^2(m, \phi_b^*, \phi_s) = (\hat{\pi}_s^2(m_s, \phi_s), \pi_b^2(m_b, \phi_b^*, \hat{\pi}_s^2(m_s, \phi_s))),$$

$$\begin{aligned}&\sum_v \nu[v] \sum_m \beta(\phi_b^*(v), \phi_s^*(v))[m] u_b(v, \pi^2(m, \phi^*)) \geq \\ &\sum_v \nu[v] \sum_m \beta(\phi_b(v), \phi_s^*(v))[m] u_b(v, \hat{\pi}^2(m, \phi_b, \phi_s^*)),\end{aligned} \tag{A.2}$$

for each $\phi_b \in \Phi$ and $\hat{\pi}_b^2 : M_b \times \Phi \times V^* \to A$, where

$$\hat{\pi}^2(m, \phi_b, \phi_s^*) = (\pi_s^2(m_s, \phi_s^*), \hat{\pi}_b^2(m_b, \phi_b, \pi_s^2(m_s, \phi_s^*))),$$

$$\begin{aligned}&\sum_{v,m_b} \frac{\phi^*[v, m_b, m_s]}{\sum_{\hat{v},\hat{m}_b} \phi^*[\hat{v}, \hat{m}_b, m_s]} u_s(\pi^2(m, \phi^*)) \geq \\ &\sum_{v,m_b} \frac{\phi^*[v, m_b, m_s]}{\sum_{\hat{v},\hat{m}_b} \phi^*[\hat{v}, \hat{m}_b, m_s]} u_s(p, \pi_b^2(m_b, \phi_b^*, p))\end{aligned} \tag{A.3}$$

for each $m_s \in \mathbb{N}$ such that $\sum_{v,m_b} \phi^*[v, m_b, m_s] > 0$ and $p \in V^*$, and

$$\begin{aligned}&\sum_{v,m_s} \frac{\phi^*[v, m_b, m_s]\pi_s^2(m_s, \phi_s^*)[p]}{\sum_{\hat{v},\hat{m}_s} \phi^*[\hat{v}, m_b, \hat{m}_s]\pi_s^2(\hat{m}_s, \phi_s^*)[p]} u_b(v, p, \pi_b^2(m_b, \phi_b^*, p)) \geq \\ &\sum_{v,m_s} \frac{\phi^*[v, m_b, m_s]\pi_s^2(m_s, \phi_s^*)[p]}{\sum_{\hat{v},\hat{m}_s} \phi^*[\hat{v}, m_b, \hat{m}_s]\pi_s^2(\hat{m}_s, \phi_s^*)[p]} u_b(v, p, a)\end{aligned} \tag{A.4}$$

for each $m_b \in \mathbb{N}$ and $p \in V^*$ such that $\sum_{v,m_s} \phi^*[v, m_b, m_s]\pi_s^2(m_s, \phi_s^*)[p] > 0$ and $a \in A$.

We will use the following notation: For each $(\phi_b, \phi_s) \in \Phi^2$ and $v \in V$, $\phi(v) = \beta(\phi_b(v), \phi_s(v))$. Furthermore, for each $i, j \in N$ with $i \neq j$, $m_i \in M_i$ and $m_j \in M_j$, $S_i(v) = \text{supp}(\phi_i(v))$, $S_{i,M_j}(v, m_i) = \{m_j : (m_i, m_j) \in S_i(v)\}$, $S_{i,M_i}(v, m_j) = \{m_i : (m_i, m_j) \in S_i(v)\}$, $S_{i,M_j}(v) = \cup_{m_i \in M_i} S_{i,M_j}(v, m_i)$, $S_{i,M_i}(v) = \cup_{m_j \in M_j} S_{i,M_i}(v, m_j)$ and $S(v) = S_b(v) \cup S_s(v)$. In particular, $\phi^*(v) = \beta(\phi_b^*(v), \phi_s^*(v))$, $S_i^*(v) = \text{supp}(\phi_i^*(v))$, $S_{i,M_j}^*(v, m_i) = \{m_j : (m_i, m_j) \in S_i^*(v)\}$, $S_{i,M_i}^*(v, m_j) = \{m_i : (m_i, m_j) \in S_i^*(v)\}$, $S_{i,M_j}^*(v) = \cup_{m_i \in M_i} S_{i,M_j}^*(v, m_i)$, $S_{i,M_i}^*(v) = \cup_{m_j \in M_j} S_{i,M_i}^*(v, m_j)$ and $S^*(v) = S_b^*(v) \cup S_s^*(v)$.

Lemma A.1 *If π is a sequential equilibrium of G, then* $\text{supp}(\phi_i^*(v)) \subseteq \{m \in M : u_i(v, \pi^2(m, \phi^*)) = \sup_{m' \in M} u_i(v, \pi^2(m', \phi^*))\}$ *for each $i \in N$ and $v \in V$.*

Proof. Suppose not; then there is $i \in N$, $v' \in V$, $m' \in \text{supp}(\phi_i^*(v'))$ and $m^* \in M$ such that $u_i(v', \pi^2(m^*, \phi^*)) > u_i(v', \pi^2(m', \phi^*))$. We may assume in addition that $u_i(v', \pi^2(m^*, \phi^*)) \geq u_i(v', \pi^2(m, \phi^*))$ for all $m \in S^*(v')$ (it is always possible to choose m^* satisfying this condition since $S^*(v')$ is finite).

Consider first the case where $i = s$. Define ϕ_s by setting, for each $v \in V$ and $m \in M$,

$$\phi_s(v)[m] = \begin{cases} 1 & \text{if } v = v' \text{ and } m = m^*, \\ 0 & \text{if } v = v' \text{ and } m \neq m^*, \\ \phi_s^*(v)[m] & \text{otherwise,} \end{cases}$$

and let $\hat{\pi}_s^2 : M_s \times \Phi \to V^*$ be such that $\hat{\pi}_s^2(m_s, \phi_s) = \pi_s^2(m_s, \phi_s^*)$ for each $m_s \in M_s$. Then $\hat{\pi}^2(m, \phi_b^*, \phi_s) = \pi^2(m, \phi^*)$ for each $m \in M$, $\beta(\phi_b^*(v), \phi_s^*(v)) = \beta(\phi_b^*(v), \phi_s(v))$ for each $v \neq v'$, $\beta(\phi_b^*(v'), \phi_s^*(v'))[m] = \beta(\phi_b^*(v'), 1_{m^*})[m] = 0$ for each $m \notin S^*(v') \cup \{m^*\}$

(by Property 1) and

$$\sum_v \nu[v] \sum_m \Big(\beta(\phi_b^*(v), \phi_s(v))[m] u_s(\hat{\pi}^2(m, \phi_b^*, \phi_s)) - \beta(\phi_b^*(v), \phi_s^*(v))[m] u_s(\pi^2(m, \phi^*))\Big)$$

$$= \sum_v \nu[v] \sum_m u_s(\pi^2(m, \phi^*))(\beta(\phi_b^*(v), \phi_s(v))[m] - \beta(\phi_b^*(v), \phi_s^*(v))[m])$$

$$= \nu[v']\Big((\beta(\phi_b^*(v'), 1_{m^*})[m^*] - \beta(\phi_b^*(v'), \phi_s^*(v'))[m^*]) u_s(\pi^2(m^*, \phi^*))$$

$$- \sum_{m \in S^*(v') \setminus \{m^*, m'\}} (\beta(\phi_b^*(v'), \phi_s^*(v'))[m] - \beta(\phi_b^*(v'), 1_{m^*})[m]) u_s(\pi^2(m, \phi^*))$$

$$- (\beta(\phi_b^*(v'), \phi_s^*(v'))[m'] - \beta(\phi_b^*(v'), 1_{m^*})[m']) u_s(\pi^2(m', \phi^*))\Big)$$

$$\geq \nu[v']\Big(\Big(\beta(\phi_b^*(v'), 1_{m^*})[m^*] - \beta(\phi_b^*(v'), \phi_s^*(v'))[m^*]$$

$$- \sum_{m \in S^*(v') \setminus \{m^*, m'\}} (\beta(\phi_b^*(v'), \phi_s^*(v'))[m] - \beta(\phi_b^*(v'), 1_{m^*})[m])\Big) u_s(\pi^2(m^*, \phi^*))$$

$$- (\beta(\phi_b^*(v'), \phi_s^*(v'))[m'] - \beta(\phi_b^*(v'), 1_{m^*})[m']) u_s(\pi^2(m', \phi^*))\Big)$$

$$= \nu[v']\Big(\beta(\phi_b^*(v'), \phi_s^*(v'))[m'] - \beta(\phi_b^*(v'), 1_{m^*})[m']\Big)\Big(u_s(\pi^2(m^*, \phi^*)) - u_s(\pi^2(m', \phi^*))\Big)$$

$$> 0$$

where the weak inequality follows because for all $m \in S^*(v') \setminus \{m^*, m'\}$,

$$u_s(\pi^2(m^*, \phi^*)) \geq u_s(\pi^2(m, \phi^*)) \text{ and}$$

$$\beta(\phi_b^*(v'), \phi_s^*(v'))[m] - \beta(\phi_b^*(v'), 1_{m^*})[m] \geq 0$$

(the latter by Property 2), the last equality follows because

$$\sum_{m \in S^*(v') \setminus \{m^*\}} (\beta(\phi_b^*(v'), \phi_s^*(v'))[m] - \beta(\phi_b^*(v'), 1_{m^*})[m]) =$$

$$\beta(\phi_b^*(v'), 1_{m^*})[m^*] - \beta(\phi_b^*(v'), \phi_s^*(v'))[m^*]$$

and hence

$$\beta(\phi_b^*(v'), 1_{m^*})[m^*] - \beta(\phi_b^*(v'), \phi_s^*(v'))[m^*]$$

$$- \sum_{m \in S^*(v') \setminus \{m^*, m'\}} (\beta(\phi_b^*(v'), \phi_s^*(v'))[m] - \beta(\phi_b^*(v'), 1_{m^*})[m]) =$$

$$\beta(\phi_b^*(v'), \phi_s^*(v'))[m'] - \beta(\phi_b^*(v'), 1_{m^*})[m'],$$

and the last inequality follows because

$$u_s(\pi^2(m^*, \phi^*)) > u_s(\pi^2(m', \phi^*)) \text{ and}$$
$$\beta(\phi_b^*(v'), \phi_s^*(v'))[m'] - \beta(\phi_b^*(v'), 1_{m^*})[m'] > 0$$

by Property 2. But this is a contradiction since π is a sequential equilibrium of G.

The proof for the case $i = b$ is analogous. Indeed, define ϕ_b by setting, for each $v \in V$ and $m \in M$,

$$\phi_b(v)[m] = \begin{cases} 1 & \text{if } v = v' \text{ and } m = m^*, \\ 0 & \text{if } v = v' \text{ and } m \neq m^*, \\ \phi_b^*(v)[m] & \text{otherwise,} \end{cases}$$

and let $\hat{\pi}_b^2 : M_b \times \Phi \times V^* \to A$ be such that $\hat{\pi}_b^2(m_b, \phi_b, p) = \pi_b^2(m_b, \phi_b^*, p)$ for each $(m_b, p) \in M_b \times V^*$. The remainder of the argument is as in the case $i = s$. ■

For each $v \in V$ and $m_s \in M_s$, let $w_b(v, m_s) = \max_{a \in A} u_b(v, \pi_s^2(m_s, \phi_s^*), a)$ and $BR_b(v, m_s) = \{a \in A : u_b(v, \pi_s^2(m_s, \phi_s^*), a) = w_b(v, m_s)\}$ be, respectively, the buyer's value function and best-reply correspondence. Analogously, for each $m_b \in M_b$, let $w_s(m_b) = \sup_{p \in V^*} u_s(p, \pi_b^2(m_b, \phi_b^*, p))$ and $BR_s(m_b) = \{p \in V^* : u_s(p, \pi_b^2(m_b, \phi_b^*, p)) = w_s(m_b)\}$. Furthermore, for each $v \in V$ and $m_b \in M_b$, let $w_s(v, m_b) = w_s(m_b)$ and $BR_s(v, m_b) = BR_s(m_b)$.

Lemma A.2 *If π is a sequential equilibrium of G, then*

$$\text{supp}(\phi_i^*(v)) \subseteq \{m \in M : \ w_i(v, m_{-i}) = \sup_{m'_{-i} \in M_{-i}} w_i(v, m'_{-i})$$
$$\text{and } \pi_i^2(m_i, \phi_i^*) \in BR_i(v, m_{-i})\}$$

for each $i \in N$ and $v \in V$, where $\pi_b^2(m_b, \phi_b^*) = \pi_b^2(m_b, \phi_b^*, \pi_s^2(m_s, \phi_s^*))$ for each $m \in M$.

Proof. Suppose not; then there is $i \in N$, $v' \in V$, $m' \in \text{supp}(\phi_i^*(v'))$ and $m^* \in M$ such that (i) $w_i(v', m_{-i}^*) > w_i(v', m'_{-i})$ or (ii) $w_i(v', m'_{-i}) = \sup_{\hat{m}_{-i} \in M_{-i}} w_i(v', \hat{m}_{-i})$ and $\pi_i^2(m'_i, \phi_{-i}^*) \notin BR_i(v', m'_{-i})$; in case (ii), let $m^* = m'$. In addition, we may assume that $w_i(v', m_{-i}^*) \geq w_i(v', m_{-i})$ for all $m \in S^*(v')$.

Consider the case where $i = b$. Let $a^* \in BR_b(v', m_s^*)$, $\bar{m}_b \in M_b$ be such that $(\bar{m}_b, m_s^*) \notin S^*(v')$,

$$\phi_b(v) = \begin{cases} 1_{(\bar{m}_b, m_s^*)} & \text{if } v = v', \\ \phi_b^*(v) & \text{otherwise,} \end{cases}$$

and $\hat{\pi}_b^2 : M_b \times \Phi \times V^* \to A$ be such that $\hat{\pi}_b^2(\bar{m}_b, \phi_b, \pi_s^2(m_s^*, \phi_s^*)) = a^*$ and $\hat{\pi}_b^2(m_b, \phi_b, p) = \pi_b^2(m_b, \phi_b^*, p)$ for each $(m_b, p) \neq (\bar{m}_b, \pi_s^2(m_s^*, \phi_s^*))$. Then $\hat{\pi}^2(m, \phi_b, \phi_s^*) = \pi^2(m, \phi^*)$ for each $m \in M$ such that $m_b \neq \bar{m}_b$, $\beta(\phi_b^*(v), \phi_s^*(v)) = \beta(\phi_b(v), \phi_s^*(v))$ for each $v \neq v'$, $\beta(\phi_b^*(v'), \phi_s^*(v'))[m] = 0$ for each $m \notin S^*(v')$ and $\beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[m] = 0$ for each $m \notin S^*(v') \cup \{(\bar{m}_b, m_s^*)\}$ (by Property 1), $(\bar{m}_b, m_s^*) \notin S^*(v')$ and

$$\begin{aligned}
&\sum_v \nu[v] \sum_m \Big(\beta(\phi_b(v), \phi_s^*(v))[m] u_b(v, \hat{\pi}^2(m, \phi_b, \phi_s^*)) \\
&-\beta(\phi_b^*(v), \phi_s^*(v))[m] u_b(v, \pi^2(m, \phi^*))\Big) \\
&= \nu[v'] \Bigg(\sum_m \Big(\beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[m] u_b(v', \hat{\pi}^2(m, \phi_b, \phi_s^*)) \\
&-\beta(\phi_b^*(v'), \phi_s^*(v'))[m] u_b(v', \pi^2(m, \phi^*))\Big)\Bigg) \\
&= \nu[v'] \Big(\beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[\bar{m}_b, m_s^*] w_b(v', m_s^*) \\
&- \sum_{m \in S^*(v') \setminus \{m'\}} (\beta(\phi_b^*(v'), \phi_s^*(v'))[m] - \beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[m]) u_b(v', \pi^2(m, \phi^*)) \\
&-(\beta(\phi_b^*(v'), \phi_s^*(v'))[m'] - \beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[m']) u_b(v', \pi^2(m', \phi^*))\Big) \\
&\geq \nu[v'] \Big(\Big(\beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[\bar{m}_b, m_s^*] \\
&- \sum_{m \in S^*(v') \setminus \{m'\}} (\beta(\phi_b^*(v'), \phi_s^*(v'))[m] - \beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[m])\Big) w_b(v', m_s^*) \\
&-(\beta(\phi_b^*(v'), \phi_s^*(v'))[m'] - \beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[m']) u_b(v', \pi^2(m', \phi^*))\Big) \\
&= \nu[v'] \Big(\beta(\phi_b^*(v'), \phi_s^*(v'))[m'] - \beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[m']\Big) \\
&\times \Big(w_b(v', m_s^*) - u_b(v', \pi^2(m', \phi^*))\Big)
\end{aligned}$$

where the weak inequality follows because for all $m \in S^*(v') \setminus \{m'\}$,

$$w_b(v', m_s^*) \geq w_b(v', m_s) \geq u_b(v', \pi^2(m, \phi^*)) \text{ and}$$

$$\beta(\phi_b^*(v'), \phi_s^*(v'))[m] - \beta(\phi_b^*(v'), 1_{(\bar{m}_b, m_s^*)})[m] \geq 0$$

(the latter by Property 2), and the last equality follows because

$$\beta(1_{(\bar{m}_b,m_s^*)},\phi_s^*(v'))[\bar{m}_b,m_s^*]-\sum_{m\in S^*(v')\setminus\{m'\}}(\beta(\phi_b^*(v'),\phi_s^*(v'))[m]-\beta(1_{(\bar{m}_b,m_s^*)},\phi_s^*(v'))[m])$$
$$=\beta(\phi_b^*(v'),\phi_s^*(v'))[m']-\beta(1_{(\bar{m}_b,m_s^*)},\phi_s^*(v'))[m'].$$

By Property 2, $\beta(\phi_b^*(v'),\phi_s^*(v'))[m']-\beta(1_{(\bar{m}_b,m_s^*)},\phi_s^*(v'))[m']>0$. If $w_b(v',m_s^*)>w_b(v',m_s')$, then

$$w_b(v',m_s^*)-u_b(v',\pi^2(m',\phi^*))\geq w_b(v',m_s^*)-w_b(v',m_s')>0;$$

if $w_b(v',m_s^*)=w_b(v',m_s')$, then $\pi_b^2(m_b',\phi_b^*)\notin BR_b(v',m_s')$ and

$$w_b(v',m_s^*)-u_b(v',\pi^2(m',\phi^*))>w_b(v',m_s^*)-w_b(v',m_s')\geq 0.$$

In either case, it follows that

$$\sum_v\nu[v]\sum_m(\beta(\phi_b(v),\phi_s^*(v))[m]u_b(v,\hat{\pi}^2(m,\phi_b,\phi_s^*))>$$
$$\sum_v\nu[v]\sum_m(\beta(\phi_b^*(v),\phi_s^*(v))[m]u_b(v,\pi^2(m,\phi^*)).$$

But this is a contradiction since π is a sequential equilibrium.

The proof for the case $i=s$ is analogous. Let $\bar{m}_s\in M_s$ be such that $(m_b^*,\bar{m}_s)\notin S^*(v')$ and, for each $k\in\mathbb{N}$, $p_k\in V^*$ be such that $u_s(p_k,\pi_b^2(m_b^*,\phi_b^*,p_k))>w_s(m_b^*)-1/k$. Then let

$$\phi_s(v)=\begin{cases}1_{(m_b^*,\bar{m}_s)} & \text{if } v=v',\\ \phi_s^*(v) & \text{otherwise,}\end{cases}$$

and $\hat{\pi}_s^2:M_s\times\Phi\to V^*$ be such that $\hat{\pi}_s^2(\bar{m}_s,\phi_s)=p_k$ and $\hat{\pi}_s^2(m_s,\phi_s)=\pi_s^2(m_s,\phi_s^*)$ for each $m_s\neq\bar{m}_s$. An argument analogous to the one for the case $i=b$ then shows that, for each $k\in\mathbb{N}$,

$$\sum_v\nu[v]\sum_m\Big(\beta(\phi_b(v),\phi_s^*(v))[m]u_b(v,\hat{\pi}^2(m,\phi_b,\phi_s^*))-\beta(\phi_b^*(v),\phi_s^*(v))[m]u_b(v,\pi^2(m,\phi^*))\Big)$$
$$\geq\nu[v']\Big(\beta(\phi_b^*(v'),\phi_s^*(v'))[m']-\beta(1_{(\bar{m}_b,m_s^*)},\phi_s^*(v'))[m']\Big)\Big(w_b(v',m_s^*)-u_b(v',\pi^2(m',\phi^*))\Big)$$
$$-\frac{1}{k}\nu[v']\beta(\phi_b^*(v'),1_{(m_b^*,\bar{m}_s)})[m_b^*,\bar{m}_s].$$

Since

$$\Big(\beta(\phi_b^*(v'),\phi_s^*(v'))[m'] - \beta(1_{(\bar{m}_b,m_s^*)},\phi_s^*(v'))[m']\Big)\Big(w_b(v',m_s^*) - u_b(v',\pi^2(m',\phi^*))\Big) > 0,$$

it follows that, for each k sufficiently large,

$$\sum_v \nu[v]\sum_m (\beta(\phi_b^*(v),\phi_s(v))[m]u_b(v,\hat{\pi}^2(m,\phi_b^*,\phi_s)) >$$
$$\sum_v \nu[v]\sum_m (\beta(\phi_b^*(v),\phi_s^*(v))[m]u_b(v,\pi^2(m,\phi^*)).$$

But this is a contradiction since π is a sequential equilibrium. ∎

A.2 Proof of Theorem 1

(Necessity) Let $\pi \in \Pi^*$ be a sequential equilibrium.

Lemma A.3 *$p(m_s) = p(m_s')$ for each $m_s, m_s' \in \cup_v S^*_{s,M_s}(v)$ and $a(m_b,p_s) = 1$ for each $m_b \in \cup_v S^*_{s,M_b}(v)$, where p_s is the common value of $p(m_s)$ for $m_s \in \cup_v S^*_{s,M_s}(v)$.*

Proof. Note first that $\max_{m^*\in\cup_v S^*(v)} u_s(\pi(m^*)) > 0$. Indeed, $u_s(\pi(m^*)) \geq 0$ for each $m^* \in \cup_v S^*(v)$ and if $\max_{m^*\in\cup_v S^*(v)} u_s(\pi(m^*)) = 0$, then $u_s(\pi) = 0$ by property 1. But then, letting $\hat{\pi}_s^1 = (\phi_s^*, \hat{\pi}_s^2)$ with $\hat{\pi}_s^2(m_s,\phi) = v_1$ for each $(m_s,\phi) \in M_s \times \Phi$, we have that

$$u_s(\pi_b,\hat{\pi}_s) \geq \sum_v \nu[v] \sum_{m\in S_s^*(v)} \phi^*(v)[m]v_1 \geq \beta_s v_1 > 0 = u_s(\pi)$$

by property 3. But this is a contradiction to the assumption that π is a sequential equilibrium.

Let $m_s, m_s' \in \cup_v S^*_{s,M_s}(v)$ and let $v, v' \in V$ and $m_b, m_b' \in M_b$ be such that $(m_s,m_b) \in S_s^*(v)$ and $(m_s',m_b') \in S_s^*(v')$. Then $a(m_b,p(m_s)) = a(m_b',p(m_s')) = 1$ since otherwise $\max_{m^*\in\cup_v S^*(v)} u_s(\pi(m^*)) = 0$ by Lemma A.1. Hence, by Lemma A.1,

$$p(m_s) = u_s(p(m_s),a(m_b,p(m_s))) = \max_{m^*\in\cup_v S^*(v)} u_s(\pi(m^*))$$
$$= u_s(p(m_s'),a(m_b',p(m_s'))) = p(m_s')$$

and, since $\max_{m^*\in\cup_v S^*(v)} u_s(\pi(m^*)) > 0$, $p_s > 0$. Thus, for each $\hat{m}_b \in \cup_v S^*_{s,M_b}(v)$, $p_s = \max_{m^*\in\cup_v S^*(v)} u_s(\pi(m^*)) = p_s a(\hat{m}_b,p_s)$ and, hence, $a(\hat{m}_b,p_s) = 1$. ∎

Lemma A.4 *There exist $v \in V$ and $m \in S_b^*(v)$ such that $a(m_b, p(m_s)) = 1$ and $\phi^*[v, m] > 0$.*

Proof. We will show that $a(m_b, p(m_s)) = 1$ for each $m \in S_b^*(v_K)$. This, together with $\nu[v_K] > 0$ and $\phi^*(v_K)[S_b^*(v_K)] > 0$ by property 3, implies the conclusion of the lemma.

We have that $p(m_s) = p_s$ and $a(m_b, p(m_s)) = 1$ for each $m \in \cup_v S_s^*(v)$ by Lemma A.3 and $\phi^*(v)[S_s^*(v)] \geq \beta_s > 0$ for each $v \in V$ by property 3. Thus, for each $v \in V$, there exists $m^v \in S_s^*(v)$ such that $p(m_s^v) = p_s$, $a(m_b^v, p(m_s^v)) = 1$ and $\phi^*[v, m_b^v, m_s^v] > 0$.

We have that $m_b^{v_1}$ such that $\sum_{(v,m_s):p(m_s)=p_s} \phi^*[\hat{v}, m_b, \hat{m}_s] \geq \phi^*[v_1, m_b^{v_1}, m_s^{v_1}] > 0$. Thus, by (A.4),

$$0 \leq \sum_{v,m_s:p(\hat{m}_s)=p_s} \phi^*[v, m_b^{v_1}, m_s](v - p_s) \Leftrightarrow$$
$$p_s \leq \frac{\sum_v v\nu[v] \sum_{m_s:p(\hat{m}_s)=p_s} \phi^*(v)[m_b^{v_1}, m_s]}{\sum_v \nu[v] \sum_{m_s:p(\hat{m}_s)=p_s} \phi^*(v)[m_b^{v_1}, m_s]}.$$

Hence, $p_s < v_K$ since

$$\nu[v_1] \sum_{m_s:p(\hat{m}_s)=p_s} \phi^*(v_1)[m_b^{v_1}, m_s] \geq \phi^*(v_1)[m_b^{v_1}, m_s^{v_1}] = \phi^*[v_1, m_b^{v_1}, m_s^{v_1}] > 0.$$

It follows from $p_s < v_K$ that $0 < v_K - p_s = u_b(v_K, m^{v_K}) \leq u_b(v_K, m)$ for each $m \in S_b^*(v_K)$ by Lemma A.1. Thus, $a(m_b, p(m_s)) = 1$ for each $m \in S_b^*(v_K)$. ∎

Lemma A.5 *$p(m_s) = p(m'_s)$ for each $m, m' \in \cup_v \{\tilde{m} \in S_b^*(v) : a(\tilde{m}_b, p(\tilde{m}_s)) = 1\}$. Furthermore, letting p_b be the common value of $p(m_s)$ for $m \in \cup_v \{\tilde{m} \in S_b^*(v) : a(\tilde{m}_b, p(\tilde{m}_s)) = 1\}$,*

$$a(m_b, p_b) = \begin{cases} 1 & \text{if } v > p_b, \\ 0 & \text{if } v < p_b \end{cases}$$

for each $v \in V$ and $m_b \in S_{b,M_b}^*(v)$.

Proof. Let $m, m' \in \cup_v \{\tilde{m} \in S_b^*(v) : a(\tilde{m}_b, p(\tilde{m}_s)) = 1\}$ be such that $p(m_s) > p(m'_s)$. Then let $v, v' \in V$ be such that $(m_b, m_s) \in S_b^*(v)$, $a(m_b, p(m_s)) = 1$, $(m'_b, m'_s) \in S_b^*(v')$ and $a(m'_b, p(m'_s)) = 1$.

Consider a deviation by b to a strategy $\hat{\pi}_b = (\phi_b, \hat{\pi}_b^2)$ such that $\phi_b(v) = 1_{(m_b', m_s')}$, $\phi_b(\hat{v}) = \phi_b^*(\hat{v})$ for each $\hat{v} \in V \setminus \{v\}$ and $\hat{\pi}_b^2(\hat{m}_b, \phi_b, p) = \pi_b^2(\hat{m}_b, \phi_b^*, p)$ for each $(\hat{m}_b, p) \in \mathbb{N} \times V^*$. This deviation is profitable since $u_b(\hat{\pi}_b, \pi_s) - u_b(\pi)$ equals

$$\begin{aligned}
&\nu[v] \sum_{\tilde{m} \notin \{m', m\}} \big(\beta(\phi_b(v), \phi_s^*(v))[\tilde{m}] - \beta(\phi_b^*(v), \phi_s^*(v))[\tilde{m}]\big) u_b(v, \pi^2(\tilde{m})) \\
&+ \nu[v]\big(\beta(\phi_b(v), \phi_s^*(v))[m] - \beta(\phi_b^*(v), \phi_s^*(v))[m]\big) u_b(v, \pi^2(m)) \\
&+ \nu[v]\big(\beta(\phi_b(v), \phi_s^*(v))[m'] - \beta(\phi_b^*(v), \phi_s^*(v))[m']\big) u_b(v, \pi^2(m')).
\end{aligned}$$

Lemma A.1 implies that $u_b(\pi^2(\tilde{m})) \leq u_b(\pi^2(m))$ for each $\tilde{m} \notin \{m', m\}$ since $m \in S_b^*(v)$. Furthermore, $u_b(v, \pi^2(m)) = v - p(m_s) < v - p(m_s') = u_b(v, \pi^2(m'))$. Thus,

$$\begin{aligned}
&u_b(\hat{\pi}_b, \pi_s) - u_b(\pi) \geq \\
&\nu[v]\big(\beta(\phi_b(v), \phi_s^*(v))[m'] - \beta(\phi_b^*(v), \phi_s^*(v))[m']\big)\big(u_b(v, \pi^2(m')) - u_b(v, \pi^2(m))\big) = \\
&\nu[v]\big(\beta(\phi_b(v), \phi_s^*(v))[m'] - \beta(\phi_b^*(v), \phi_s^*(v))[m']\big)\big(p(m_s) - p(m_s')\big) > 0
\end{aligned}$$

since $\beta(\phi_b(v), \phi_s^*(v))[m'] > \beta(\phi_b^*(v), \phi_s^*(v))[m']$ by property 2. But this contradicts the assumption that π is a sequential equilibrium.

Finally, let $v \in V$ and $m_b \in S_{b,M_b}^*(v)$. Then $a(m_b, p_b) \in BR_b(v, m_s)$ for each $m_s \in \mathbb{N}$ such that $(m_b, m_s) \in S_b^*(v)$ by Lemma A.2, hence $a(m_b, p_b) = 1$ if $v > p_b$ and $a(m_b, p_b) = 0$ if $v < p_b$. ■

For each $v \in V$, let $\beta[v] = \sum_{m \in S_s^*(v)} \phi^*(v)[m]$. Then

$$\sum_{m \in S_b^*(v) \setminus S_s^*(v)} \phi^*(v)[m] = 1 - \beta[v]$$

by property 1. If $p_b \in V$, let

$$\begin{aligned}
&\Lambda = \{m \in S_b^*(p_b) \setminus S_s^*(p_b) : a(m_b, p_b) = 1\}, \text{ and} \\
&\lambda(1 - \beta[p_b]) = \sum_{m \in \Lambda} \phi^*(p_b)[m];
\end{aligned}$$

if $p_b \notin V$, then let $\lambda = 0$. It follows by Lemmas A.3 and A.5 that

$$\begin{aligned}
u_b &= \sum_v \nu[v]\beta[v](v - p_s) + \sum_{v \geq p_b} \nu[v](1 - \beta[v])(v - p_b) \text{ and} \\
u_s &= p_s \sum_v \nu[v]\beta[v] + p_b \nu[p_b](1 - \beta[p_b])\lambda + p_b \sum_{v > p_b} \nu[v](1 - \beta[v]).
\end{aligned}$$

In the case where $p_b \neq p_s$, it follows that, for each $v \in V$, $S_s^*(v) \cap S_b^*(v) = \emptyset$ by Lemmas A.3 and A.5. Indeed, if $m \in S_s^*(v) \cap S_b^*(v)$, then $p(m_s) = p_s$ and $a(m_b, p(m_s)) = 1$ by Lemma A.3. Hence, Lemma A.5 implies that $p(m_s) = p_b \neq p_s = p(m_s)$, a contradiction.

It then follows by property 3 that $\beta[v] = \beta_s$ and $1 - \beta[v] = \beta_b$ and, thus,

$$u_b = \beta_s \left(\sum_v \nu[v]v - p_s \right) + \beta_b \left(\sum_{v \geq p_b} \nu[v](v - p_b) \right) \text{ and}$$
$$u_s = \beta_s p_s + \beta_b (p_b \sum_{v > p_b} \nu[v] + p_b \nu[p_b]\lambda).$$

Consider next the case $p_b = p_s$ and let $p = p_b = p_s$. Then

$$\begin{aligned} u_b &= \sum_{v>p} \nu[v](v-p) + \sum_{v \leq p} \nu[v]\beta[v](v-p) \leq \sum_{v>p} \nu[v](v-p) + \sum_{v \leq p} \nu[v]\beta_s(v-p) \\ &= \beta_s \left(\sum_v \nu[v]v - p_s \right) + \beta_b \left(\sum_{v \geq p_b} \nu[v](v-p_b) \right) \text{ and} \\ u_s &= p \sum_{v>p} \nu[v] + p\nu[p](\beta[p] + (1-\beta[p])\lambda) + p \sum_{v<p} \nu[v]\beta[v] \\ &\geq p \sum_{v>p} \nu[v] + p\nu[p](\beta_s + \beta_b\lambda) + p \sum_{v<p} \nu[v]\beta_s \\ &= \beta_s p_s + \beta_b (p_b \sum_{v>p_b} \nu[v] + p_b \nu[p_b]\lambda). \end{aligned}$$

For each $v \in V$, let $\hat{\beta}[v] = \sum_{m \in S_b^*(v)} \phi^*(v)[m]$. Then

$$\sum_{m \in S_s^*(v) \setminus S_b^*(v)} \phi^*(v)[m] = 1 - \hat{\beta}[v]$$

by property 1. If $p_b \in V$, let

$$\hat{\Lambda} = \{m \in S_b^*(p_b) : a(m_b, p_b) = 1\}, \text{ and} \tag{A.5}$$
$$\hat{\lambda}\hat{\beta}[p_b] = \sum_{m \in \hat{\Lambda}} \phi^*(p_b)[m]; \tag{A.6}$$

if $p_b \notin V$, then let $\hat{\lambda} = 0$. It follows by Lemmas A.3 and A.5 that

$$u_b = \sum_v \nu[v](1 - \hat{\beta}[v])(v - p_s) + \sum_{v \geq p_b} \nu[v]\hat{\beta}[v](v - p_b) \text{ and}$$
$$u_s = p_s \sum_v \nu[v](1 - \hat{\beta}[v]) + p_b \nu[p_b]\hat{\beta}[p_b]\hat{\lambda} + p_b \sum_{v > p_b} \nu[v]\hat{\beta}[v].$$

Then

$$
\begin{aligned}
u_b &= \sum_{v>p} \nu[v](v-p) + \sum_{v\leq p} \nu[v](1-\hat{\beta}[v])(v-p) \\
&\geq \sum_{v>p} \nu[v](v-p) + \sum_{v\leq p} \nu[v](1-\beta_b)(v-p) \\
&= \beta_s \left(\sum_{v} \nu[v]v - p_s \right) + \beta_b \left(\sum_{v\geq p_b} \nu[v](v-p_b) \right) \text{ and} \\
u_s &= p\sum_{v>p} \nu[v] + p\nu[p](\hat{\beta}[p]\hat{\lambda} + (1-\hat{\beta}[p])) + p\sum_{v<p} \nu[v](1-\hat{\beta}[v]) \\
&\leq p\sum_{v>p} \nu[v] + p\nu[p](\beta_b\hat{\lambda} + 1 - \beta_b) + p\sum_{v<p} \nu[v](1-\beta_b) \\
&= \beta_s p_s + \beta_b(p_b \sum_{v>p_b} \nu[v] + p_b\nu[p_b]\hat{\lambda}).
\end{aligned}
$$

It follows that

$$
u_b = \beta_s \left(\sum_{v} \nu[v]v - p_s \right) + \beta_b \left(\sum_{v\geq p_b} \nu[v](v-p_b) \right). \tag{A.7}
$$

Since $\lambda \geq 0$ and $\hat{\lambda} \leq 1$, we have that

$$
\beta_s p_s + \beta_b(p_b \sum_{v>p_b} \nu[v]) \leq u_s \leq \beta_s p_s + \beta_b(p_b \sum_{v>p_b} \nu[v] + p_b\nu[p_b]).
$$

Thus, for some $\lambda^* \in [0,1]$,

$$
u_s = \beta_s p_s + \beta_b(p_b \sum_{v>p_b} \nu[v] + \lambda^* p_b \nu[p_b]). \tag{A.8}
$$

Lemma A.6 *$p_s \geq p_b$, $p_s \leq \sum_v \nu[v]v$ and*

$$
v_1 \leq p_b \sum_{v>p_b} \nu[v] + p_b\nu[p_b]\lambda^*.
$$

Proof. Let, by Lemma A.4, $(v,m) \in V \times M$ be such that $m \in S_b^*(v)$ and $a(m_b, p(m_s)) = 1$. Thus, $u_s(\pi(m)) = p_b$ by Lemma A.5. Let $m' \in \cup_v S_s^*(v)$; then $u_s(\pi(m')) = p_s$ by Lemma A.3. Hence, it follows by Lemma A.1 that $p_s \geq p_b$.

We next show that $p_s \leq \sum_v \nu[v]v$. Suppose not; then $p_s > \sum_v \nu[v]v$. Let $\bar{m}_b \in M_b$ be such that $(\bar{m}_b, m_s) \notin \cup_v S^*(v)$ for each $m_s \in M_s$ and $\bar{m}_s \in M_s$ be such that, for some $m_b \in M_b$, $(m_b, \bar{m}_s) \in \cup_v S_b^*(v)$ and $a(m_b, p(\bar{m}_s)) = 1$. We have that $\bar{m}_b$ exists since $\cup_v S^*(v)$ is finite, $\bar{m}_s$ exists by Lemma A.4 and $p(\bar{m}_s) = p_b$ by Lemma A.5. Let ϕ_b be defined by setting, for each $v \in V$,

$$\phi_b(v) = \begin{cases} \phi_b^*(v) & \text{if } v < p_b, \\ 1_{(\bar{m}_b, \bar{m}_s)} & \text{if } v \geq p_b. \end{cases}$$

Let $\phi(v) = \beta(\phi_b(v), \phi_s^*(v))$, $S_s(v) = S_s^*(v)$ and $S_b(v) = \text{supp}(\phi_b(v))$. Then $S_b(v) \cap S_s(v) = \emptyset$ for each $v \in V$. This is clear if $v \geq p_b$ by the choice of $\bar{m}_b$. If $v < p_b$ and $m \in S_b(v) \cap S_s(v) = S_b^*(v) \cap S_s^*(v)$, then $p(m_s) = p_s$ and $a(m_b, p_s) = 1$ by Lemma A.3. Thus, by Lemma A.5, $p(m_s) = p_b$ and, therefore, $p_s = p_b$. Furthermore, $a(m_b, p_b) = 0$ implying that $1 = a(m_b, p_s) = a(m_b, p_b) = 0$, a contradiction.

It then follows that, for each $v \in V$,

$$1 = \phi(v)[S_b] + \phi(v)[S_s] \geq \beta_b + \beta_s = 1$$

by properties 1 and 3. Thus, $\phi(v)[S_b] = \beta_b$ and $\phi(v)[S_s] = \beta_s$ for each $v \in V$.

Consider $\hat{\pi}_b^2$ defined by setting, for each $(m_b, \hat{\phi}, p) \in \mathbb{N} \times \Phi \times V^*$,

$$\hat{\pi}_b^2(m_b, \hat{\phi}, p) = \begin{cases} 1 & \text{if } m_b = \bar{m}_b, \\ 0 & \text{otherwise.} \end{cases}$$

Letting $\hat{\pi}_b = (\phi, \hat{\pi}_b^2)$ and $\hat{u}_b = u_b(\hat{\pi}_b, \pi_s)$, it follows that

$$\hat{u}_b = \sum_v \nu[v] \left(\sum_{m \in S_b(v)} \phi(v)[m] u_b(v, \hat{\pi}^2(m)) + \sum_{m \in S_s(v)} \phi(v)[m] u_b(v, \hat{\pi}^2(m)) \right).$$

We have that $u_b(v, \hat{\pi}^2(m)) = 0$ for each $v \in V$ and $m \in S_s(v)$ since $m \in S_s(v) = S_s^*(v)$ implies that $m_b \neq \bar{m}_b$ and, hence, $\hat{\pi}_b^2(m_b, \phi_b, p_s) = 0$. Similarly, $u_b(v, \hat{\pi}^2(m)) = 0$ for each $v < p_b$ and $m \in S_b(v)$ since $m \in S_b(v) = S_b^*(v)$ implies that $m_b \neq \bar{m}_b$ and, hence, $\hat{\pi}_b^2(m_b, \phi_b, p_s) = 0$. Furthermore, $u_b(v, \hat{\pi}^2(m)) = v - p_b$ for each $v \geq p_b$ and $m \in S_b(v)$ since $S_b(v) = (\bar{m}_b, \bar{m}_s)$ implies that $p(\bar{m}_s) = p_b$ and $\hat{\pi}_b^2(m_b, \phi_b, p_b) = 1$. Thus,

$$\hat{u}_b = \sum_{v \geq p_b} \nu[v] \phi(v)[S_b(v)](v - p_b) = \beta_b \left(\sum_{v \geq p_b} \nu[v](v - p_b) \right)$$
$$> \beta_s \left(\sum_v \nu[v] v - p_s \right) + \beta_b \left(\sum_{v \geq p_b} \nu[v](v - p_b) \right) = u_b$$

since $p_s > \sum_v \nu[v]v$ and $\beta_s > 0$. But this is a contradiction since π is a sequential equilibrium.

Finally, we show that $v_1 \leq p_b \sum_{v>p_b} \nu[v] + p_b \nu[p_b]\lambda^*$. Suppose not; then $v_1 > p_b \sum_{v>p_b} \nu[v] + p_b \nu[p_b]\lambda^*$. Let $\bar{m}_s \in M_s$ be such that $(m_b, \bar{m}_s) \notin \cup_v S^*(v)$ for each $m_b \in M_b$ and $\bar{m}_b \in M_b$ be such that, for some $m_s \in M_s$, $(\bar{m}_b, m_s) \in \cup_v S^*_s(v)$. We have that $\bar{m}_s$ exists since $\cup_v S^*(v)$ is finite, $\bar{m}_b$ exists since $\cup_v S^*_s(v) \neq \emptyset$ and $a(\bar{m}_b, p_s) = 1$ by Lemma A.3. Let ϕ_s be defined by setting, for each $v \in V$,

$$\phi_s(v) = 1_{(\bar{m}_b, \bar{m}_s)}.$$

Let $\phi(v) = \beta(\phi^*_b(v), \phi_s(v))$, $S_b(v) = S^*_b(v)$ and $S_s(v) = \text{supp}(\phi_s(v))$. Then $S_b(v) \cap S_s(v) = \emptyset$ for each $v \in V$. It then follows that, for each $v \in V$,

$$1 = \phi(v)[S_b(v)] + \phi(v)[S_s(v)] \geq \beta_b + \beta_s = 1$$

by properties 1 and 3. Thus, $\phi(v)[S_b(v)] = \beta_b$ and $\phi(v)[S_s(v)] = \beta_s$ for each $v \in V$.

Consider $\hat{\pi}^2_s$ defined by setting, for each $(m_s, \hat{\phi}) \in \mathbb{N} \times \Phi \times V^*$,

$$\hat{\pi}^2_s(m_s, \hat{\phi}) = \begin{cases} p_s & \text{if } m_s = \bar{m}_s, \\ v_1 & \text{otherwise.} \end{cases}$$

Letting $\hat{\pi}_s = (\phi_s, \hat{\pi}^2_s)$ and $\hat{u}_s = u_s(\pi_b, \hat{\pi}_s)$, it follows that

$$\hat{u}_s = \sum_v \nu[v] \left(\sum_{m \in S_b(v)} \phi(v)[m] u_s(\hat{\pi}^2(m)) + \sum_{m \in S_s(v)} \phi(v)[m] u_s(\hat{\pi}^2(m)) \right).$$

We have that $u_s(\hat{\pi}^2(m)) = p_s$ for each $v \in V$ and $m \in S_s(v)$ since then $m = (\bar{m}_b, \bar{m}_s)$, $\hat{\pi}^2_s(\bar{m}_s, \phi_s) = p_s$ and $a(\bar{m}_b, p_s) = 1$. Furthermore, $u_s(\hat{\pi}^2(m)) = v_1$ for each $v \in V$ and $m \in S_b(v)$ since then $m_s \neq \bar{m}_s$, $\hat{\pi}^2_s(m_s, \phi_s) = v_1$ and $a(m_b, v_1) = 1$. Thus,

$$\hat{u}_s = \beta_s p_s + \beta_b v_1 > \beta_s p_s + \beta_b (p_b \sum_{v>p_b} \nu[v] + p_b \nu[p_b]\lambda^*) = u_b$$

since $v_1 > p_b \sum_{v>p_b} \nu[v] + p_b \nu[p_b]\lambda^*$ and $\beta_b > 0$. But this is a contradiction since π is a sequential equilibrium. ■

The necessity part of the theorem then follows from (A.7), (A.8) and Lemma A.6.

(Sufficiency) Let $(p_b, p_s) \in (V^*)^2$ and $\lambda \in [0,1]$ be as in the statement of the theorem, u_b defined by (1) and u_s defined by (2). We will show that $(u_b, u_s) \in U^*(\beta_b, \beta_s)$ by showing that there is a sequential equilibrium $\pi \in \Pi^*$ when the aggregation function β is such that $\beta(\gamma, \gamma') = \beta_b\gamma + \beta_s\gamma'$ for each $\gamma, \gamma' \in F$. It is clear that $\beta \in \mathcal{B}(\beta_b, \beta_s)$.

Let $\bar{m}^b, \bar{m}^s \in M$ with $\bar{m}^b_j \neq \bar{m}^s_j$ for each $j \in \{s, b\}$ and $\tilde{m}^b_b \in M_b \setminus \{\bar{m}^b_b, \bar{m}^s_b\}$. For each $v \in V$, define

$$\phi^*_s(v) = 1_{\bar{m}^s}$$

and

$$\phi^*_b(v) = \begin{cases} 1_{\bar{m}^b} & \text{if } v > p_b, \\ \lambda 1_{\bar{m}^b} + (1-\lambda)1_{(\tilde{m}^b_b, \bar{m}^b_s)} & \text{if } v = p_b, \\ 1_{(\tilde{m}^b_b, \bar{m}^b_s)} & \text{if } v < p_b. \end{cases}$$

For each $(m_b, m_s, p) \in \mathbb{N}^2 \times V^*$, let

$$\pi^2_s(m_s, \phi^*_s) = \begin{cases} p_s & \text{if } m_s = \bar{m}^s_s, \\ p_b & \text{if } m_s = \bar{m}^b_s, \\ v_K & \text{otherwise} \end{cases}$$

and

$$\pi^2_b(m_b, \phi^*_b, p) = \begin{cases} 1 & \text{if } m_b = \bar{m}^s_b \text{ and } p \in \{p_s, p_b\} \\ 1 & \text{if } m_b = \bar{m}^b_b \text{ and } p = p_b, \\ 1 & \text{if } p \leq v_1, \\ 0 & \text{otherwise.} \end{cases}$$

We will define perturbations such that whenever the buyer receives any price offer other than p_b following message $\bar{m}^b_b$, he believes that his value is v_1. In addition, whenever the buyer receives a zero-probability message following ϕ^*_b, he believes that his value is v_1 and whenever the seller receives a zero-probability message, he believes that the buyer knows that his value is v_K.

For each $m_s \in M_s$ and $\phi_s \neq \phi^*_s$ such that $\sum_v \nu[v](\beta_s\phi_s(v) + \beta_b\phi^*_b(v))_{M_s}[m_s] = 0$, let $\pi^2_s(m_s, \phi_s) = v_K$.

For each $m_s \in M_s$ and $\phi_s \neq \phi_s^*$ such that $\sum_v \nu[v](\beta_s\phi_s(v) + \beta_b\phi_b^*(v))_{M_s}[m_s] > 0$, let $\pi_s^2(m_s, \phi_s)$ maximize

$$p \sum_{m_b} \frac{\sum_v \nu[v](\beta_s\phi_s(v) + \beta_b\phi_b^*(v))[m_s, m_b]}{\sum_v \nu[v](\beta_s\phi_s(v) + \beta_b\phi_b^*(v))_{M_s}[m_s]} \pi_b^2(m_b, \phi_b^*, p).$$

For each $(m_b, p) \in M_b \times V^*$ and $\phi_b \neq \phi_b^*$ such that $\sum_{\{m_s:\pi_s^2(m_s,\phi_s^*)=p\}} \sum_v \nu[v](\beta_s\phi_s^*(v) + \beta_b\phi_b(v))[m_b, m_s] > 0$, let $\pi_b^2(m_b, \phi_b, p) = 1$ if and only if:

$$\frac{\sum_{\{m_s:\pi_s^2(m_s,\phi_s^*)=p\}} \sum_v \nu[v](\beta_s\phi_s^*(v) + \beta_b\phi_b(v))[m_s, m_b]v}{\sum_{\{m_s:\pi_s^2(m_s,\phi_s^*)=p\}} \sum_v \nu[v](\beta_s\phi_s^*(v) + \beta_b\phi_b(v))[m_b, m_s]} \geq p.$$

For each $(m_b, p) \in M_b \times V^*$ and $\phi_b \neq \phi_b^*$ such that $\sum_{\{m_s:\pi_s^2(m_s,\phi_s^*)=p\}} \sum_v \nu[v](\beta_s\phi_s^*(v) + \beta_b\phi_b(v))[m_b, m_s] = 0$, we will define $\pi_b^2(m_b, \phi_b, p)$ after the following net $\{\pi^\alpha, p^\alpha\}_\alpha$ has been defined, where,

1. for each α, $p^\alpha : \Phi^2 \to \Delta(V \times M)$ is measurable and π^α is a behavioral strategy,

i.e. $\pi_s^\alpha = (\pi_s^{1,\alpha}, \pi_s^{2,\alpha})$ is such that $\pi_s^{1,\alpha} \in \Delta(\Phi)$ and $\pi_s^{2,\alpha} : \mathbb{N}\times\Phi \to \Delta(V^*)$ is measurable, and $\pi_b^\alpha = (\pi_b^{1,\alpha}, \pi_b^{2,\alpha})$ such that $\pi_b^{1,\alpha} \in \Delta(\Phi)$ and $\pi_b^{2,\alpha} : \mathbb{N} \times \Phi \times V^* \to \Delta(A)$ is measurable.

Consider $\{\pi^\alpha, p^\alpha\}_\alpha$ defined as follows: The index set consists of $(k, F, \hat{F}, \tilde{F})$ such that $k \in \mathbb{N}$, F is a finite subset of $\mathbb{N}$, $\hat{F}$ is a finite subset of Φ and $\tilde{F}$ is a finite subset of V^*; this set is partially ordered by defining $(k', F', \hat{F}', \tilde{F}') \geq (k, F, \hat{F}, \tilde{F})$ if $k' \geq k$, $F \subseteq F'$, $\hat{F} \subseteq \hat{F}'$ and $\tilde{F} \subseteq \tilde{F}'$. If X is a finite set, let $\mathcal{U}_X \in \Delta(X)$ be uniform on X. For each $(F, \hat{F}, \tilde{F})$, define:

$$\Phi(F, \hat{F}) = \{\phi \in \hat{F} : \text{supp}(\phi) \subseteq F^2\} \text{ and}$$
$$P(F, \hat{F}, \tilde{F}) = \tilde{F} \cup \{p_s, p_b, v_K\} \cup \{\pi_s^2(m_s, \phi_s) : m_s \in F \cup \{\bar{m}_s^b\}, \phi_s \in \Phi(F, \hat{F})\}.$$

For each $v \in V$, define $m_s^v \in M_s \setminus \{\bar{m}_s^b\}$ such that $v \mapsto m_s^v$ is one-to-one. For each $m_b \in M_b$, let $\phi_s^{m_b}$ be such that $\phi_s^{m_b}(v_1) = 1_{(m_b, m_s^{v_1})}$ and $\phi_s^{m_b}(v) = 1_{(\bar{m}_b^b, m_s^v)}$ for $v \neq v_1$ (i.e. $\phi_s^{m_b}$ sends the seller message m_s^v when the valuation is v and sends the buyer message m_b only if the valuation is v_1; otherwise it sends the buyer message $\bar{m}_b^b$). Let $\pi_s^{m_b,\alpha}$ be such that $\pi_s^{m_b,1,\alpha} = \phi_s^{m_b}$ and $\pi_s^{m_b,2,\alpha}$ be such that $\pi_s^{m_b,2,\alpha}(m_s^{v_1}, \phi_s^{m_b}) = \mathcal{U}_{P(F,\hat{F},\tilde{F})}$, $\pi_s^{m_b,2,\alpha}(m_s, \phi_s^{m_b}) = p_b$ for $m_s \neq m_s^{v_1}$ and $\pi_s^{m_b,2,\alpha}(m_s, \phi_s) = p_b$ for all $m_s \in M_s$

and $\phi_s \neq \phi_s^{m_b}$. Let $\hat{\pi}_s^\alpha$ be such that $\hat{\pi}_s^{1,\alpha} = \mathcal{U}_{\Phi(F,\hat{F})}$ and $\hat{\pi}_s^{2,\alpha}(m_s,\phi_s) = \mathcal{U}_{P(F,\hat{F},\tilde{F})}$ for all m_s, ϕ_s. Let, for each $t = 1, 2$,

$$\pi_s^{t,\alpha} = (1-j^{-1})\pi_s^t + j^{-1}(1-j^{-j})|F|^{-1}\sum_{m_b \in F}\pi_s^{m_b,t,\alpha} + j^{-1}j^{-j}\hat{\pi}_s^{t,\alpha},$$

where $j = \max\{k, |F|, |\hat{F}|, |\tilde{F}|\}$.

For each $v \in V$, define $m_b^v \in M_b$ such that $v \mapsto m_b^v$ is one-to-one. For each $m_s \in M_s$, let $\phi_b^{m_s}$ be such that $\phi_b^{m_s}(v_K) = 1_{(m_b^{v_K}, m_s)}$ and $\phi_b^{m_s}(v) = 1_{(m_b^v, \bar{m}_s^s)}$ for $v \neq v_K$. Let $\pi_b^{m_s,1,\alpha} = \phi_b^{m_s}$. Let $\hat{\pi}_b^{1,\alpha} = \mathcal{U}_{\Phi(F,\hat{F})}$. Let:

$$\pi_b^{1,\alpha} = (1-j^{-1})\phi_b^* + j^{-1}(1-j^{-j})|F|^{-1}\sum_{m_s \in F}\pi_b^{m_s,1,\alpha} + j^{-1}j^{-j}\hat{\pi}_b^{1,\alpha}.$$

Let

$$p^\alpha(\phi)[v,m] = (1-j^{-j})\nu[v](\beta_s\phi_s(v) + \beta_b\phi_b(v))[m] + j^{-j}\mathcal{U}_{V\times F^2}[v,m].$$

For each $(m_b, p) \in M_b \times V^*$ and $\phi_b \neq \phi_b^*$ such that $\sum_{\{m_s:\pi_s^2(m_s,\phi_s^*)=p\}}\sum_v \nu[v](\beta_s\phi_s^*(v) + \beta_b\phi_b(v))[m_b, m_s] = 0$, let $\pi_b^2(m_b,\phi_b,p) = 1$ if and only if

$$\lim_\alpha \frac{\int_\Phi \left(\sum_{(v,m_s)} p^\alpha(\phi_b,\phi_s)[v,m_b,m_s]\pi_s^{2,\alpha}(m_s,\phi_s)[p]v\right)\mathrm{d}\pi_s^{1,\alpha}[\phi_s]}{\int_\Phi \sum_{(v,m_s)} p^\alpha(\phi_b,\phi_s)[v,m_b,m_s]\pi_s^{2,\alpha}(m_s,\phi_s)[p]\mathrm{d}\pi_s^{1,\alpha}[\phi_s]} \geq p.$$

Finally, let $\hat{\pi}_b^{2,\alpha}(m_b,\phi_b,p) = \mathcal{U}_A$ and $\pi_b^{2,\alpha}(m_b,\phi_b,p) = (1-j^{-1})\pi_b^2(m_b,\phi_b,p) + j^{-1}\hat{\pi}_b^{2,\alpha}(m_b,\phi_b,p)$ for each m_b, ϕ_b, p.

It is clear that the following conditions hold:

2. For each $i \in N$, $\sup_{B \in \mathcal{B}(\Phi)} |\pi_i^{1,\alpha}[B] - 1_{\phi_i^*}[B]| \to 0$,[31]

$$\sup_{(m,\phi)\in\mathbb{N}\times\Phi, B\in\mathcal{B}(V^*)} |\pi_s^{2,\alpha}(m,\phi)[B] - \pi_s^2(m,\phi)[B]| \to 0, \text{ and}$$

$$\sup_{(m,\phi,p)\in\mathbb{N}\times\Phi\times V^*, a\in A} |\pi_b^{2,\alpha}(m,\phi,p)[a] - \pi_b^2(m,\phi,p)[a]| \to 0,$$

3. For each $i \in N$, $m \in \mathbb{N}$, $\phi \in \Phi$, $p \in V^*$ and $a \in A$, there is $\bar{\alpha}$ such that $\pi_i^{1,\alpha}[\phi] > 0$, $\pi_s^{2,\alpha}(m,\phi)[p] > 0$ and $\pi_b^{2,\alpha}(m,\phi,p)[a] > 0$ for each $\alpha \geq \bar{\alpha}$,

[31] We let $\mathcal{B}(\Phi)$ denote the class of Borel measurable subsets of Φ and, for each $\phi \in \Phi$, 1_ϕ denote the probability measure on Φ degenerate at ϕ. Analogous definitions apply when Φ is replaced with V^*.

4. $\sup_{\phi\in\Phi^2, v\in V, B\subseteq M} |p^\alpha(\phi)[\{v\}\times B] - \nu[v]\sum_{i\in N}\beta_i\phi_i(v)[B]| \to 0$,

5. For each $\phi\in\Phi^2$, $v\in V$ and $m\in M$, there is $\bar\alpha$ such that $p^\alpha(\phi)[v,m]>0$ for each $\alpha\geq\bar\alpha$.

Note that, for each α and $(m_s,\phi_b,\phi_s)\in\mathbb{N}\times\Phi^2$, $\text{supp}(p^\alpha(\phi_b,\phi_s))$, $\text{supp}(\pi_b^{1,\alpha})$, $\text{supp}(\pi_s^{1,\alpha})$ and $\text{supp}(\pi_s^{2,\alpha}(m_s,\phi_s))$ are all finite. Moreover, if $\pi_b^{1,\alpha}[\phi_b]>0$ and $\pi_s^{1,\alpha}[\phi_s]>0$, then

$$\begin{aligned}\phi_b &\in \Phi_b^\alpha := \{\phi_b^*\}\cup\{\phi_b^{m_s} : m_s\in F\}\cup\Phi(F,\hat F) \text{ and}\\ \phi_s &\in \Phi_s^\alpha := \{\phi_s^*\}\cup\{\phi_s^{m_b} : m_b\in F\}\cup\Phi(F,\hat F).\end{aligned}$$

If $(m_s,\phi_s)\in\mathbb{N}\times\Phi$ is such that $\sum_{\phi_b\in\Phi}\pi_b^{1,\alpha}[\phi_b]p^\alpha_{M_s}(\phi_b,\phi_s)[m_s]>0$, then $m_s\in\cup_v\text{supp}(\phi_s(v)_{M_s})\cup\{\bar m_s^b\}\cup F$, and if $(m_b,\phi_b,p)\in\mathbb{N}\times\Phi\times V^*$ is such that

$$\sum_{\phi_s\in\text{supp}(\pi_s^{1,\alpha})}\pi_s^{1,\alpha}[\phi_s]\sum_{(v,m_s)\in\text{supp}(p^\alpha(\phi_b,\phi_s))}p^\alpha(\phi_b,\phi_s)[v,m_b,m_s]\pi_s^{2,\alpha}(m_s,\phi_s)[p]>0,$$

then $m_b\in\cup_v\text{supp}(\phi_b(v)_{M_b})\cup\{\bar m_b^s\}\cup F$ and $p\in P(F,\hat F,\tilde F)$.

Thus, to show that π is a sequential equilibrium, it suffices to show that the following conditions hold for each $\varepsilon>0$ and α:

6.(a) For each $i\in N$ and $\phi_i'\in\Phi$,

$$\begin{aligned}&\sum_{\phi\in\text{supp}(\pi^{1,\alpha})}\pi^{1,\alpha}[\phi]\sum_{(v,m)\in V\times\mathbb{N}^2}p^\alpha(\phi)[v,m]u_i(v,\pi^{2,\alpha}(m,\phi))\geq\\ &\sum_{\phi_j\in\text{supp}(\pi_j^{1,\alpha})}\pi_j^{1,\alpha}[\phi_j]\sum_{(v,m)\in V\times\mathbb{N}^2}p^\alpha(\phi_i',\phi_j)[v,m]u_i(v,\pi^{2,\alpha}(m,\phi_i',\phi_j))-\varepsilon,\end{aligned}$$

where $\pi^{1,\alpha}=\prod_{i\in N}\pi_i^{1,\alpha}$, $j\neq i$ and, for each $\phi\in\Phi^2$ and $m\in\mathbb{N}^2$, $\pi^{2,\alpha}(m,\phi)\in\Delta(V^*\times A)$ is defined by setting, for each $(p,a)\in V^*\times A$, $\pi^{2,\alpha}(m,\phi)[p,a]=\pi_s^{2,\alpha}(m_s,\phi_s)[p]\pi_b^{2,\alpha}(m_b,\phi_b,p)[a]$,

6.(b) For each $(m_s,\phi_s)\in\mathbb{N}\times\Phi$ such that $\pi_s^{1,\alpha}[\phi_s]\sum_{\phi_b\in\Phi}\pi_b^{1,\alpha}[\phi_b]p^\alpha_{M_s}(\phi_b,\phi_s)[m_s]>0$ and $p\in V^*$,

$$\begin{aligned}&\frac{\sum_{\phi_b\in\text{supp}(\pi_b^{1,\alpha})}\pi_b^{1,\alpha}[\phi_b]\left(\sum_{(v,m_b)}p^\alpha(\phi_b,\phi_s)[v,m]u_s(\pi^{2,\alpha}(m,\phi))\right)}{\sum_{\phi_b\in\text{supp}(\pi_b^{1,\alpha})}\pi_b^{1,\alpha}[\phi_b]p^\alpha_{M_s}(\phi_b,\phi_s)[m_s]}\geq\\ &\frac{\sum_{\phi_b\in\text{supp}(\pi_b^{1,\alpha})}\pi_b^{1,\alpha}[\phi_b]\left(\sum_{(v,m_b)}p^\alpha(\phi_b,\phi_s)[v,m]u_s(p,\pi_b^{2,\alpha}(m_b,\phi_b,p))\right)}{\sum_{\phi_b\in\text{supp}(\pi_b^{1,\alpha})}\pi_b^{1,\alpha}[\phi_b]p^\alpha_{M_s}(\phi_b,\phi_s)[m_s]}-\varepsilon.\end{aligned}$$

6.(c) For each $(m_b, \phi_b, p) \in \mathbb{N} \times \Phi \times V^*$ such that

$$\pi_b^{1,\alpha}[\phi_b] \sum_{\phi_s \in \text{supp}(\pi_s^{1,\alpha})} \pi_s^{1,\alpha}[\phi_s] \sum_{(v,m_s)} p^\alpha(\phi_s, \phi_b)[v, m_b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s)[p] > 0$$

and $a \in A$,

$$\frac{\sum_{\phi_s \in \text{supp}(\pi_s^{1,\alpha})} \pi_s^{1,\alpha}[\phi_s] \left(\sum_{(v,m_s)} p^\alpha(\phi_b, \phi_s)[v, m]\pi_s^{2,\alpha}(m_s, \phi_s)[p]u_b(v, p, \pi_b^{2,\alpha}(m_b, \phi_b, p)) \right)}{\sum_{\phi_s \in \text{supp}(\pi_s^{1,\alpha})} \pi_s^{1,\alpha}[\phi_s] \sum_{(v,m_s)} p^\alpha(\phi_s, \phi_b)[v, m]\pi_s^{2,\alpha}(m_s, \phi_s)[p]} \geq$$
$$\frac{\sum_{\phi_s \in \text{supp}(\pi_s^{1,\alpha})} \pi_s^{1,\alpha}[\phi_s] \left(\sum_{(v,m_s)} p^\alpha(\phi_b, \phi_s)[v, m]\pi_s^{2,\alpha}(m_s, \phi_s)[p]u_b(v, p, a) \right)}{\sum_{\phi_s \in \text{supp}(\pi_s^{1,\alpha})} \pi_s^{1,\alpha}[\phi_s] \sum_{(v,m_s)} p^\alpha(\phi_s, \phi_b)[v, m]\pi_s^{2,\alpha}(m_s, \phi_s)[p]} - \varepsilon.$$

Let $\varepsilon > 0$. We will show that these conditions holds for some subnet of $\{\pi^\alpha, p^\alpha\}_\alpha$. In particular, for each $(F, \hat{F}, \tilde{F})$, we will show that there exists a $k(F, \hat{F}, \tilde{F})$ such that for each $\alpha = (k, F, \hat{F}, \tilde{F})$ with $k \geq k(F, \hat{F}, \tilde{F})$, condition 6 is satisfied.

Consider condition 6.(a) with $i = s$ and $\phi_i' \in \Phi$. The left-hand side converges to $u_s = \beta_b p_b(\sum_{v>p_b} \nu[v] + \nu[p_b]\lambda) + \beta_s p_s$ and, when $\varepsilon = 0$, the right-hand side, for any $\phi_s' \in \Phi$, is at most:

$$\begin{aligned}
&(1 - j^{-1})^3(1 - j^{-j})\Big(\beta_b p_b(\sum_{v>p_b} \nu[v] + \nu[p_b]\lambda) \\
&+\beta_s \sum_{v,m} \nu[v]\phi_s'(v)[m_b, m_s]\pi_s^2(m_s, \phi_s')\pi_b^2(m_b, \phi_b^*, \pi_s^2(m_s, \phi_s'))\Big) \\
&+(1 - (1 - j^{-1})^3(1 - j^{-j}))v_K \\
&\leq (1 - j^{-1})^3(1 - j^{-j}) \left(\beta_b p_b(\sum_{v>p_b} \nu[v] + \nu[p_b]\lambda) + \beta_s \sum_{v,m} \nu[v]\phi_s'(v)[m_b, m_s]p_s \right) \\
&+(1 - (1 - j^{-1})^3(1 - j^{-j}))v_K
\end{aligned}$$

since v_K is the maximum payoff for the seller and $\pi_b^2(m_b, \phi_b^*, \pi_s^2(m_s, \phi_s')) = 0$ if $\pi_s^2(m_s, \phi_s') > p_s$. Thus, the inequality holds (uniformly across $\phi_i' \in \Phi$) for each α such that k (and hence j) is sufficiently large.

Consider next condition 6.(a) with $i = b$. The left-hand side converges to $u_b = \beta_b \sum_{v \geq p_b} \nu[v](v - p_b) + \beta_s(\sum_v \nu[v]v - p_s)$ and, when $\varepsilon = 0$, the right-hand side, for any $\phi_b' \in \Phi$, is at most:

$$(1-j^{-1})^3(1-j^{-j})\Big(\beta_b \sum_{v,m} \nu[v]\phi_b'(v)[m_b,m_s](v-\pi_s^2(m_s,\phi_s^*))\pi_b^2(m_b,\phi_b',\pi_s^2(m_s,\phi_s^*))$$
$$+\beta_s(\sum_v \nu[v]v-p_s)\Big)+(1-(1-j^{-1})^3(1-j^{-j}))v_K$$
$$\leq (1-j^{-1})^3(1-j^{-j})\Big(\beta_b \sum_{v,m} \nu[v]\phi_b'(v)[m_b,m_s](v-p_b)\pi_b^2(m_b,\phi_b',\pi_s^2(m_s,\phi_s^*))$$
$$+\beta_s(\sum_v \nu[v]v-p_s)\Big)+(1-(1-j^{-1})^3(1-j^{-j}))v_K$$
$$\leq (1-j^{-1})^3(1-j^{-j})\left(\beta_b \sum_{v\geq p_b} \nu[v](v-p_b)+\beta_s(\sum_v \nu[v]v-p_s)\right)$$
$$+(1-(1-j^{-1})^3(1-j^{-j}))v_K$$

since v_K is an upper bound on the buyer's payoff and $\pi_s^2(m_s,\phi_s^*) \geq p_b$ for each $m_s \in M_s$. Thus, the inequality holds (uniformly across $\phi_i' \in \Phi$) for each α such that k (and hence j) is sufficiently large.

Let k_a be such that condition 6.(a) holds for each α such that $k \geq k_a$.

Consider next condition 6.(b). We establish it by considering several cases.

Case 1: $\phi_s = \phi_s^*$ and $m_s = \bar{m}_s^s$. In the limit and when $\varepsilon = 0$, the inequality is $p_s \geq p\pi_b^2(\bar{m}_b^s,\phi_b^*,p)$. It holds since $p_b \leq p_s$ and

$$p\pi_b^2(\bar{m}_b^s,\phi_b^*,p) = \begin{cases} p & \text{if } p = p_b, \\ 0 & \text{if } p \neq p_b. \end{cases}$$

By similar arguments as for condition 6.(a), for sufficiently large k (and hence j), the inequality in fact holds uniformly across all $p \in V^*$. Let k_{b1} be such that condition 6.(b) holds for $(m_s,\phi_s) = (\bar{m}_s^s,\phi_s^*)$ for α such that $k \geq k_{b1}$.

Case 2: $\phi_s = \phi_s^*$ and $m_s = \bar{m}_s^b$. In the limit and when $\varepsilon = 0$, the inequality is

$$p_b(\sum_{v>p_b}\nu[v]+\nu[p_b]\lambda) \geq p\Big(\sum_{v<p_b}\nu[v]\pi_b^2(\tilde{m}_b^b,\phi_b^*,p)$$
$$+\nu[p_b](\lambda\pi_b^2(\bar{m}_b^b,\phi_b^*,p)+(1-\lambda)\pi_b^2(\tilde{m}_b^b,\phi_b^*,p))+\sum_{v>p_b}\nu[v]\pi_b^2(\bar{m}_b^b,\phi_b^*,p)\Big).$$

It holds since $p_b(\sum_{v>p_b}\nu[v]+\nu[p_b]\lambda) \geq v_1$ and its right-hand side is equal to v_1 if $p = v_1$ and zero if $p > v_1$ and $p \neq p_b$. Thus, the inequality holds for k sufficiently large (uniformly across $p \in V^*$). Let k_{b2} be such that condition 6.(b) holds for $(m_s, \phi_s) = (\bar{m}_s^b, \phi_s^*)$ for α such that $k \geq k_{b2}$

Case 3: $\phi_s = \phi_s^*$ and $m_s \notin \{\bar{m}_s^s, \bar{m}_s^b\}$. Note that we only need to consider $m_s \in F$ in this case since otherwise $\sum_{\phi_b \in \Phi} \pi_b^{1,\alpha}[\phi_b] p_{M_s}^{\alpha}(\phi_b, \phi_s)[m_s] = 0$. Given that $m_s \in F$, in the limit (as $k \to \infty$, i.e. we can keep F fixed) and when $\varepsilon = 0$, the inequality is

$$v_K \pi_b^2(m_b^{v_K}, \phi_b^{m_s}, v_K) \geq p\pi_b^2(m_b^{v_K}, \phi_b^{m_s}, p).$$

We have that $\pi_b^2(m_b^{v_K}, \phi_b^{m_s}, v_K) = 1$ since $\phi_b^{m_s}(v_K)[m_b^{v_K}, m_s] > 0$ and

$$\frac{\sum_{v, \hat{m}_s} v\nu[v](\beta_s \phi_s^*(v) + \beta_b \phi_b^{m_s}(v))[m_b^{v_K}, \hat{m}_s]}{\sum_{v', m_s'} \nu[v'](\beta_s \phi_s^*(v') + \beta_b \phi_b^{m_s}(v'))[m_b^{v_K}, m_s']} = v_K.$$

Hence, the inequality holds in the limit and, thus, for k sufficiently large (uniformly across $p \in V^*$). For each $m_s \in F \setminus \{\bar{m}_s^s, \bar{m}_s^b\}$, let $k_{b3}(m_s)$ be such that condition 6.(b) holds for (m_s, ϕ_s^*), for each α such that $k \geq k_{b3}(m_s)$, and let $k_{b3}(F) = \max_{m_s \in F \setminus \{\bar{m}_s^s, \bar{m}_s^b\}} k_{b3}(m_s)$. Note that for all $\alpha = (k, F, \hat{F}, \tilde{F})$ such that $k \geq k_{b3}(F)$, condition 6.(b) holds for all $(m_s, \phi_s) \in \{(m_s, \phi_s) : m_s \in F \setminus \{\bar{m}_s^s, \bar{m}_s^b\}, \phi_s = \phi_s^*\}$.

Case 4: $\phi_s \neq \phi_s^*$ and $m_s \in M_s$ such that $\sum_{v, m_b} \nu[v](\beta_b \phi_b^*(v) + \beta_s \phi_s(v))[m_b, m_s] > 0$. Note that we only have to consider $\phi_s \in \Phi_s^{\alpha} \setminus \{\phi_s^*\}$ and $\Phi_s^{\alpha} \setminus \{\phi_s^*\}$ is finite. In the limit and with $\varepsilon = 0$, the inequality is

$$\begin{aligned}
&\pi_s^2(\phi_s, m_s) \sum_{m_b} \frac{\sum_v \nu[v](\beta_s \phi_s(v) + \beta_b \phi_b^*(v))[m_s, m_b]}{\sum_v \nu[v](\beta_s \phi_s(v) + \beta_b \phi_b^*(v))_{M_s}[m_s]} \pi_b^2(m_b, \phi_b^*, \pi_s^2(\phi_s, m_s)) \\
&\geq p \sum_{m_b} \frac{\sum_v \nu[v](\beta_s \phi_s(v) + \beta_b \phi_b^*(v))[m_s, m_b]}{\sum_v \nu[v](\beta_s \phi_s(v) + \beta_b \phi_b^*(v))_{M_s}[m_s]} \pi_b^2(m_b, \phi_b^*, p),
\end{aligned}$$

which holds by definition. For each $(F, \hat{F})$, let $k_{b4}(F, \hat{F})$ be such that condition 6.(b) holds for all $\phi_s \in \Phi_s^{\alpha} \setminus \{\phi_s^*\}$ and $m_s \in M_s$ such that $\sum_{v, m_b} \nu[v](\beta_b \phi_b^*(v) + \beta_s \phi_s(v))[m_b, m_s] > 0$, for $\alpha = (k, F, \hat{F}, \tilde{F})$ such that $k \geq k_{b4}(F, \hat{F})$.

Case 5: $\phi_s \neq \phi_s^*$ and $m_s \in M_s$ such that $\sum_{v, m_b} \nu[v](\beta_b \phi_b^*(v) + \beta_s \phi_s(v))[m_b, m_s] = 0$. This is as in case 3. For each $(F, \hat{F})$, let $k_{b5}(F, \hat{F})$ be such that condition 6.(b) holds for all such (m_s, ϕ_s), for $\alpha = (k, F, \hat{F}, \tilde{F})$ such that $k \geq k_{b5}(F, \hat{F})$.

For each $(F, \hat{F})$, let $k_b(F, \hat{F}) = \max\{k_{b1}, k_{b2}, k_{b3}(F), k_{b4}(F, \hat{F}), k_{b5}(F, \hat{F})\}$.

Consider next condition 6.(c). We establish this condition by considering several cases.

Case 1: $\phi_b = \phi_b^*$, $p = p_s$ and $m_b = \bar{m}_b^s$. Since $\pi_b^2(\bar{m}_b^s, \phi_b^*, p_s) = 1$, we may consider $a = 0$. Thus, in the limit and with $\varepsilon = 0$, the inequality is $\sum_v \nu[v]v - p_s \geq 0$, which holds. Let k_{c1} be such that condition 6.(c) holds for $(m_b, \phi_b, p) = (\bar{m}_b^s, \phi_b^*, p_s)$, for α such that $k \geq k_{c1}$.

Case 2: $\phi_b = \phi_b^*$, $p = p_b$ and $m_b = \bar{m}_b^b$. Since $\pi_b^2(\bar{m}_b^b, \phi_b^*, p_b) = 1$, we may consider $a = 0$. Thus, in the limit and with $\varepsilon = 0$, the inequality is

$$\frac{\sum_{v>p_b} \nu[v](v - p_b) + \nu[p_b]\lambda(p_b - p_b)}{\sum_{v>p_b} \nu[v] + \nu[p_b]\lambda} \geq 0,$$

which holds. Let k_{c2} be such that condition 6.(c) holds for $(m_b, \phi_b, p) = (\bar{m}_b^b, \phi_b^*, p_b)$, for α such that $k \geq k_{c2}$.

Case 3: $\phi_b = \phi_b^*$, $p = p_b$ and $m_b = \tilde{m}_b^b$. Since $\pi_b^2(\tilde{m}_b^b, \phi_b^*, p_b) = 0$, we may consider $a = 1$. Thus, in the limit and with $\varepsilon = 0$, the inequality is

$$0 \geq \frac{\sum_{v<p_b} \nu[v](v - p_b) + \nu[p_b](1-\lambda)(p_b - p_b)}{\sum_{v<p_b} \nu[v] + \nu[p_b](1-\lambda)},$$

which holds. Let k_{c3} be such that condition 6.(c) holds for $(m_b, \phi_b, p) = (\tilde{m}_b^b, \phi_b^*, p_b)$, for α such that $k \geq k_{c3}$.

Case 4: $\phi_b = \phi_b^*$, $p \notin \{p_s, p_b\}$ and $m_b = \bar{m}_b^s$. Note that we only have to consider $p \in P(F, \hat{F}, \tilde{F})$ in this case. The strategy for the buyer is

$$\pi_b^2(\bar{m}_b^s, \phi_b^*, p) = \begin{cases} 1 & \text{if } p = v_1, \\ 0 & \text{if } p > v_1. \end{cases}$$

We have $p^\alpha(\phi_b^*, \phi_s^*)[v, \bar{m}_b^s, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^*)[p] \leq j^{-j}$ for each $v \in V$ and $m_s \in M_s$ since $(\beta_b\phi_b^*(v) + \beta_s\phi_s^*(v))[\bar{m}_b^s, m_s] = 0$ for $m_s \neq \bar{m}_s^s$ and $\pi_s^2(\bar{m}_s^s, \phi_s^*)[p] = 0$ implies:

$$\nu[v](\beta_b\phi_b^*(v) + \beta_s\phi_s^*(v))[\bar{m}_b^s, m_s]\pi_s^2(m_s, \phi_s^*)[p] = 0$$

and $\pi_s^{\bar{m}_b^s,2,\alpha}(\bar{m}_s^s, \phi_s^*)[p] = 0$ implies:

$$\nu[v](\beta_b\phi_b^*(v) + \beta_s\phi_s^*(v))[\bar{m}_b^s, m_s]\pi_s^{\bar{m}_b^s,2,\alpha}(m_s, \phi_s^*)[p] = 0.$$

If $v \neq v_1$, $m_b \neq \bar{m}_b^s$, or $m_s \neq m_s^{v_1}$, $p^\alpha(\phi_b^*, \phi_s^{m_b})[v, \bar{m}_b^s, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^{m_b})[p] \leq j^{-j}$. This is as follows: (1) if $m_b \neq \bar{m}_b^s$, then $(\beta_b\phi_b^*(v) + \beta_s\phi_s^{m_b}(v))[\bar{m}_b^s, m_s] = 0$ for each

$v \in V$ and $m_s \in M_s$; (2) if $m_b = \bar{m}_b^s$, $m_s = m_s^{v_1}$ and $v \neq v_1$, then $(\beta_b \phi_b^*(v) + \beta_s \phi_s^{\bar{m}_b^s}(v))[\bar{m}_b^s, m_s^{v_1}] = 0$; and (3) if $m_b = \bar{m}_b^s$, $m_s \neq m_s^{v_1}$ and $v \in V$, then (i) $(\beta_b \phi_b^*(v) + \beta_s \phi_s^{\bar{m}_b^s}(v))[\bar{m}_b^s, m_s] = 0$ for each $m_s \notin \{m_s^{v'} : v' \in V\}$, (ii) $\pi_s^2(m_s^{v'}, \phi_s^{\bar{m}_b^s})[p] = 0$ for each $v' \in V$ (since $\pi_b^2(\bar{m}_b^s, \phi_b^*, p) = 1$ if and only if $p \in \{p_b, p_s\}$ or $p \leq v_1$, and so $\pi_s^2(m_s^{v'}, \phi_s^{\bar{m}_b^s}) = p_s$ is optimal), and (iii) $\pi_s^{\bar{m}_b^s,2,\alpha}(m_s, \phi_s^{\bar{m}_b^s})[p] = 0$ for each $m_s \neq m_s^{v_1}$ and $\pi_s^{m_b,2,\alpha}(m_s, \phi_s^{\bar{m}_b^s})[p] = 0$ for each $m_b \neq \bar{m}_b^s$ and $m_s \in M_s$.

Finally, note that

$$\begin{aligned}\pi_s^{2,\alpha}(m_s^{v_1}, \phi_s^{\bar{m}_b^s})[p] &= j^{-1}(1-j^{-j})|F|^{-1} \sum_{m_b \in F} \pi_s^{m_b,2,\alpha}(m_s^{v_1}, \phi_s^{\bar{m}_b^s})[p] + O(j^{-j}) \\ &= j^{-1}(1-j^{-j})|F|^{-1}|P(F,\hat{F},\tilde{F})|^{-1} + O(j^{-j})\end{aligned}$$

since $\pi_s^{m_b,2,\alpha}(m_s^{v_1}, \phi_s^{\bar{m}_b^s}) = p_b$ for all $m_b \neq \bar{m}_b^s$.

Thus, the denominator of the inequality is (ignoring terms that are $O(j^{-j})$):

$$\begin{aligned}&(1-j^{-1}) \sum_{v,m_s} p^\alpha(\phi_b^*, \phi_s^*)[v, \bar{m}_b^s, m_s] \pi_s^{2,\alpha}(m_s, \phi_s^*)[p] \\ &+ j^{-1}(1-j^{-j})|F|^{-1} \sum_{m_b \in F} \sum_{v,m_s} p^\alpha(\phi_b^*, \phi_s^{m_b})[v, \bar{m}_b^s, m_s] \pi_s^{2,\alpha}(m_s, \phi_s^{m_b})[p] \\ &+ j^{-1} j^{-j} |\Phi(F,\hat{F})|^{-1} \sum_{\phi \in \Phi(F,\hat{F})} \sum_{v,m_s} p^\alpha(\phi_b^*, \phi)[v, \bar{m}_b^s, m_s] \pi_s^{2,\alpha}(m_s, \phi)[p] \\ &= j^{-1}(1-j^{-j})|F|^{-1}(1-j^{-j})\nu[v_1]\beta_s j^{-1}(1-j^{-j})|F|^{-1}|P(F,\hat{F},\tilde{F})|^{-1}\end{aligned}$$

Likewise, also ignoring terms that are $O(j^{-j})$, the numerator of the right-hand (resp. left-hand) side inequality is

$$j^{-1}(1-j^{-j})|F|^{-1}(1-j^{-j})\nu[v_1]\beta_s j^{-1}(1-j^{-j})|F|^{-1}|P(F,\hat{F},\tilde{F})|^{-1}(v_1 - p)$$

when $p > v_1$ (resp. $p = v_1$).

Thus, when $p > v_1$, the limit inequality (with $a = 1$ and $\varepsilon = 0$) is $0 \geq v_1 - p$. When $p = v_1$, the limit inequality (with $a = 0$ and $\varepsilon = 0$) is $v_1 - v_1 \geq 0$.

For each $p \in P(F,\hat{F},\tilde{F}) \setminus \{p_s, p_b\}$, let $k_{c4}(p)$ be such that condition 6.(c) holds for $(\bar{m}_b^s, \phi_b^*, p)$, for each $\alpha = (k, F, \hat{F}, \tilde{F})$ such that $k \geq k_{c4}(p)$, and let $k_{c4}(F,\hat{F},\tilde{F}) = \max_{p \in P(F,\hat{F},\tilde{F})} k_{c4}(p)$.

Case 5: $\phi_b = \phi_b^*$, $p = p_b < p_s$ and $m_b = \bar{m}_b^s$. Since $\pi_b^2(\bar{m}_b^s, \phi_b^*, p_b) = 1$, we may consider $a = 0$.

We have that $p^{\alpha}(\phi_b^*, \phi_s^*)[v, \bar{m}_b^s, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^*)[p_b] \leq j^{-j}$ for each $v \in V$ and $m_s \neq \bar{m}_s^s$ since $(\beta_b\phi_b^*(v) + \beta_s\phi_s^*(v))[\bar{m}_b^s, m_s] = 0$ for all $m_s \neq \bar{m}_s^s$. For $m_s = \bar{m}_s^s$, we have $\pi_s^2(\bar{m}_s^s, \phi_s^*)[p_b] = 0$ but $\pi_s^{m_b,2,\alpha}(\bar{m}_s^s, \phi_s^*)[p_b] = 1$ for each $m_b \in M_b$. Thus, $|F^{-1}|\sum_{m_b} \pi_s^{m_b,2,\alpha}(\bar{m}_s^s, \phi_s^*)[p_b] = 1$ and therefore $\pi_s^{2,\alpha}(\bar{m}_s^s, \phi_s^*)[p_b] = j^{-1}(1 - j^{-j})$.

Also, $(\beta_b\phi_b^*(v) + \beta_s\phi_s^{m_b}(v))[\bar{m}_b^s, m_s] > 0$ only if $m_b = \bar{m}_b^s$ and $m_s \in \{m_s^v : v \in V\}$, and $\pi_s^2(m_s^v, \phi_s^{\bar{m}_b^s})[p_b] = 0$ for each $v \in V$ (since $\pi_b^2(\bar{m}_b^s, \phi_b^*, p_s) = 1$). Thus, $\sum_{m_b\in F}\sum_{v,m_s} p^{\alpha}(\phi_b^*, \phi_s^{m_b})[v, \bar{m}_b^s, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^{m_b})[p_b] = O(j^{-1})$.

Thus, the denominator of the inequality is (ignoring terms that are $O(j^{-j})$):

$$\begin{aligned}
&(1 - j^{-1})\sum_{v,m_s} p^{\alpha}(\phi_b^*, \phi_s^*)[v, \bar{m}_b^s, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^*)[p_b] \\
&+j^{-1}(1 - j^{-j})|F|^{-1}\sum_{m_b\in F}\sum_{v,m_s} p^{\alpha}(\phi_b^*, \phi_s^{m_b})[v, \bar{m}_b^s, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^{m_b})[p_b] \\
&= (1 - j^{-1})\sum_{v}(1 - j^{-j})\nu[v]\beta_s j^{-1}(1 - j^{-j}) + j^{-1}(1 - j^{-j})|F|^{-1}O(j^{-1}) \\
&= (1 - j^{-1})\sum_{v}(1 - j^{-j})\nu[v]\beta_s j^{-1}(1 - j^{-j}) + O(j^{-2}) \\
&= (1 - j^{-1})(1 - j^{-j})\beta_s j^{-1}(1 - j^{-j})\sum_{v}\nu[v] + O(j^{-2}) \\
&= (1 - j^{-1})(1 - j^{-j})\beta_s j^{-1}(1 - j^{-j}) + O(j^{-2}).
\end{aligned}$$

Similarly, ignoring terms that are $O(j^{-j})$ and $O(j^{-2})$, the numerator of the left-hand side of the inequality is

$$(1 - j^{-1})(1 - j^{-j})\beta_s j^{-1}(1 - j^{-j})\sum_{v}\nu[v](v - p_b).$$

Thus, the limit inequality is:

$$\sum_{v}\nu[v]v - p_b \geq 0.$$

Let k_{c5} be such that condition 6.(c) holds for $(\bar{m}_b^s, \phi_b^*, p_b)$ for each α such that $k \geq k_{c5}$.

Case 6: $\phi_b = \phi_b^*$, $p \neq p_b$ and $m_b = \bar{m}_b^b$. The strategy for the buyer is

$$\pi_b^2(\bar{m}_b^b, \phi_b^*, p) = \begin{cases} 1 & \text{if } p = v_1, \\ 0 & \text{if } p > v_1. \end{cases}$$

By the same argument as in case 4, we have $p^\alpha(\phi_b^*, \phi_s^*)[v, \bar{m}_b^b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^*)[p] \leq j^{-j}$ for each $v \in V$ and $m_s \in M_s$.

For each $v \neq v_1$, $m_b \neq \bar{m}_b^b$, or $m_s \neq m_s^{v_1}$, $p^\alpha(\phi_b^*, \phi_s^{m_b})[v, \bar{m}_b^b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^{m_b})[p] \leq j^{-j}$. This is because $(\beta_b\phi_b^*(v) + \beta_s\phi_s^{m_b}(v))[\bar{m}_b^b, m_s^{v_1}] = 0$ if $m_b \neq \bar{m}_b^b$ or $v \neq v_1$, $\pi_s^{m_b',2,\alpha}(m_s, \phi_s^{m_b})[p] = 0$ for each $m_s \neq m_s^{v_1}$ and $m_b, m_b' \in M_b$, $\pi_s^{m_b,2,\alpha}(m_s, \phi_s^{\bar{m}_b^b})[p] = 0$ for each $m_b \neq \bar{m}_b^b$ and $m_s \in M_s$, $(\beta_b\phi_b^*(v) + \beta_s\phi_s^{\bar{m}_b^b}(v))[\bar{m}_b^b, m_s] = 0$ for each $m_s \notin M^* = \{m_s^v : v \in V\} \cup \{\bar{m}_s^b\}$, $\pi_s^2(m_s, \phi_s^{\bar{m}_b^b})[p] = 0$ for each $m_s \in M^*$ (since $\pi_s^2(m_s^v, \phi_s^{\bar{m}_b^b}) = p_b$ is optimal for each $v \in V$ and $\pi_s^2(\bar{m}_s^b, \phi_s^{\bar{m}_b^b}) = p_b$), and if $m_b \neq \bar{m}_b^b$, $(\beta_b\phi_b^*(v) + \beta_s\phi_s^{m_b}(v))[\bar{m}_b^b, m_s] = 0$ for each $m_s \notin M' = \{m_s^v : v > v_1\} \cup \{\bar{m}_s^b\}$ and $\pi_s^2(m_s, \phi_s^{m_b})[p] = 0$ for each $m_s \in M'$ (since $\pi_s^2(m_s^v, \phi_s^{m_b}) = p_b$ is optimal for $v > v_1$ and $\pi_s^2(\bar{m}_s^b, \phi_s^{m_b}) = p_b$).

Finally, note that

$$\begin{aligned}\pi_s^{2,\alpha}(m_s^{v_1}, \phi_s^{\bar{m}_b^b})[p] &= j^{-1}(1 - j^{-j})|F|^{-1}\sum_{m_b \in F}\pi_s^{m_b,2,\alpha}(m_s^{v_1}, \phi_s^{\bar{m}_b^b})[p] + O(j^{-j}) \\ &= j^{-1}(1 - j^{-j})|F|^{-1}|P(F, \hat{F}, \tilde{F})|^{-1} + O(j^{-j}).\end{aligned}$$

Thus, the denominator of the inequality is (ignoring terms that are $O(j^{-j})$):

$$\begin{aligned}&(1 - j^{-1})\sum_{v,m_s} p^\alpha(\phi_b^*, \phi_s^*)[v, \bar{m}_b^b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^*)[p] \\ &+ j^{-1}(1 - j^{-j})|F|^{-1}\sum_{m_b \in F}\sum_{v,m_s} p^\alpha(\phi_b^*, \phi_s^{m_b})[v, \bar{m}_b^b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^{m_b})[p] \\ &= j^{-1}(1 - j^{-j})|F|^{-1}(1 - j^{-j})\nu[v_1]\beta_s j^{-1}(1 - j^{-j})|F|^{-1}|P(F, \hat{F}, \tilde{F})|^{-1}.\end{aligned}$$

Likewise, also ignoring terms that are $O(j^{-j})$, the numerator of the right-hand (resp. left-hand) side inequality is

$$j^{-1}(1 - j^{-j})|F|^{-1}(1 - j^{-j})\nu[v_1]\beta_s j^{-1}(1 - j^{-j})|F|^{-1}|P(F, \hat{F}, \tilde{F})|^{-1}(v_1 - p)$$

when $p > v_1$ (resp. $p = v_1$).

Thus, when $p > v_1$, the limit inequality (with $a = 1$ and $\varepsilon = 0$) is $0 \geq v_1 - p$. When $p = v_1$, the limit inequality (with $a = 0$ and $\varepsilon = 0$) is $v_1 - v_1 \geq 0$.

For each $p \in P(F, \hat{F}, \tilde{F}) \setminus \{p_b\}$, let $k_{c6}(p)$ be such that condition 6.(c) holds for each $(\bar{m}_b^b, \phi_b^*, p)$, for each $\alpha = (k, F, \hat{F}, \tilde{F})$ such that $k \geq k_{c6}(p)$, and let $k_{c6}(F, \hat{F}, \tilde{F}) = \max_{p \in P(F, \hat{F}, \tilde{F})} k_{c6}(p)$.

Case 7: $\phi_b = \phi_b^*$ and $m_b \notin \{\bar{m}_b^b, \tilde{m}_b^b, \bar{m}_b^s\}$. The strategy for the buyer is

$$\pi_b^2(m_b, \phi_b^*, p) = \begin{cases} 1 & \text{if } p = v_1, \\ 0 & \text{if } p > v_1. \end{cases}$$

In this case, $p^\alpha(\phi_b^*, \phi_s^*)[v, m_b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^*)[p] \leq j^{-j}$ for all $v \in V$ and $m_s \in M_s$, and $p^\alpha(\phi_b^*, \phi_s^{m_b'})[v, m_b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^{m_b'})[p] \leq j^{-j}$ if $m_b' \neq m_b$, $v \neq v_1$, or $m_s \neq m_s^{v_1}$.

Thus, the denominator of the inequality is (ignoring terms that are $O(j^{-j})$):

$$\begin{aligned}
&(1 - j^{-1}) \sum_{v,m_s} p^\alpha(\phi_b^*, \phi_s^*)[v, m_b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^*)[p] \\
&+ j^{-1}(1 - j^{-j})|F|^{-1} \sum_{m_b' \in F} \sum_{v,m_s} p^\alpha(\phi_b^*, \phi_s^{m_b'})[v, m_b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^{m_b'})[p] \\
&= j^{-1}(1 - j^{-j})|F|^{-1}(1 - j^{-j})\nu[v_1]\beta_s\pi_s^{2,\alpha}(m_s^{v_1}, \phi_s^{m_b})[p].
\end{aligned}$$

Likewise, also ignoring terms that are $O(j^{-j})$, the numerator of the right-hand (resp. left-hand) side inequality is

$$j^{-1}(1 - j^{-j})|F|^{-1}(1 - j^{-j})\nu[v_1]\beta_s\pi_s^{2,\alpha}(m_s^{v_1}, \phi_s^{m_b})[p](v_1 - p)$$

when $p > v_1$ (resp. $p = v_1$).

Thus, when $p > v_1$, the limit inequality (with $a = 1$ and $\varepsilon = 0$) is $0 \geq v_1 - p$. When $p = v_1$, the limit inequality (with $a = 0$ and $\varepsilon = 0$) is $v_1 - v_1 \geq 0$.

Let $k_{c7}(F, \hat{F}, \tilde{F})$ be such that condition 6.(c) holds for each (m_b, ϕ_b^*, p) such that $m_b \in F \setminus \{\bar{m}_b^b, \tilde{m}_b^b, \bar{m}_b^s\}$ and $p \in P(F, \hat{F}, \tilde{F})$, for each $\alpha = (k, F, \hat{F}, \tilde{F})$ such that $k \geq k_{c7}(F, \hat{F}, \tilde{F})$.

Case 8: For each $m_b \in M_b$ and $\phi_b \neq \phi_b^*$, 6(c) holds in the limit by construction. Let $k_{c8}(F, \hat{F}, \tilde{F})$ be such that condition 6.(c) holds for each (m_b, ϕ_b, p) such that $\phi_b \in \Phi_b^\alpha \setminus \{\phi_b^*\}$, $m_b \in \cup_v \text{supp}(\phi_b(v)_{M_b}) \cup \{\bar{m}_b^s\} \cup F$ and $p \in P(F, \hat{F}, \tilde{F})$, for each $\alpha = (k, F, \hat{F}, \tilde{F})$ such that $k \geq k_{c8}(F, \hat{F}, \tilde{F})$.

For each $(F, \hat{F}, \tilde{F})$, let

$$k_c(F, \hat{F}, \tilde{F}) = \max\{k_{c1}, k_{c2}, k_{c3}, k_{c4}(F, \hat{F}, \tilde{F}), k_{c5}, k_{c6}(F, \hat{F}, \tilde{F}), k_{c7}(F, \hat{F}, \tilde{F}), k_{c8}(F, \hat{F}, \tilde{F})\}.$$

The above arguments allow us to define the following subnet $\{\pi^{\varphi(\eta)}, p^{\varphi(\eta)}\}_\eta$ of $\{\pi^\alpha, p^\alpha\}_\alpha$ such that condition 6 holds.

The index set of the subnet $\{\pi^{\varphi(\eta)}, p^{\varphi(\eta)}\}_\eta$ is the same as the one in the net $\{\pi^\alpha, p^\alpha\}_\alpha$. The function $\varphi : \eta \mapsto \alpha$ is defined by setting, for each $\eta = (k, F, \hat{F}, \tilde{F})$,

$$\varphi(\eta) = (\max\{k_a, k_b(F, \hat{F}), k_c(F, \hat{F}, \tilde{F})\}, F, \hat{F}, \tilde{F}).$$

It is then clear that condition 6 holds and that, as required by the definition of a subnet, for each α_0, there exists η_0, e.g. $\eta_0 = \alpha_0$, such that $\varphi(\eta) \geq \alpha_0$ for each $\eta \geq \eta_0$.

A.3 Proof of Corollary 1

It is clear that $\beta_s E + \beta_b E(v_K) \leq \beta_s E + \beta_b E(v_k) \leq \beta_s E + \beta_b E(v_1)$ for each $k \in \{1, \ldots, K\}$. Note that, for each $k \in \{2, \ldots, K\}$ and $p \in (v_{k-1}, v_k]$,

$$\nu(p) = \sum_{v \geq p} \nu[v] = \sum_{v \geq v_k} \nu[v] = \nu(v_k), \text{ and} \tag{A.9}$$

$$E(p) = \sum_{v \geq p} \nu[v]v = \sum_{v \geq v_k} \nu[v]v = E(v_k). \tag{A.10}$$

When $k = 1$, $\nu(p) = \nu(v_1)$ and $E(p) = E(v_1)$ for each $p \in C_1 = \{v_1\}$. Thus, for each $k \in \kappa$, $\beta_s E + \beta_b \bar{v}_k \nu(v_k) = \beta_s E + \beta_b \bar{v}_k \nu(\bar{v}_k) \leq \beta_s E + \beta_b p^* \nu(p^*)$ since $\bar{v}_k \leq E$ by the definition of C_k and $\bar{v}_k \nu(\bar{v}_k) \leq p^* \nu(p^*)$ by the definition of p^*. Thus, we also have that $\underline{v}_k(\beta_s + \beta_b \nu(v_k)) \leq \bar{v}_k(\beta_s + \beta_b \nu(\bar{v}_k)) \leq \beta_s E + \beta_b p^* \nu(p^*)$.

Moreover, for each $k \in \kappa$, $\beta_s E + \beta_b \bar{v}_k \nu(v_k) \geq \beta_s E + \beta_b v_1$ by the definition of C_k. Since $\underline{v}_k \geq v_1$ and $\underline{v}_k \nu(v_k) \geq v_1$, the latter since $\underline{v}_k = \lim_j p_j$ for some sequence $\{p_j\}_{j=1}^\infty$ such that $p_j \in C_k$ for each $j \in \mathbb{N}$, it follows that $\underline{v}_k(\beta_s + \beta_b \nu(v_k)) \geq v_1$.

(Sufficiency) Let $(\hat{u}_b, \hat{u}_s) \in \cup_{k \in \kappa} U_k$ and let $k \in \kappa$ be such that $(\hat{u}_b, \hat{u}_s) \in U_k$. If $\underline{v}_k > v_{k-1}$ (respectively, $\underline{v}_k = v_{k-1}$), then $C_k = [\underline{v}_k, \bar{v}_k]$ (resp. $C_k = (\underline{v}_k, \bar{v}_k]$) by the definition of C_k.

Consider two cases: (a) $\hat{u}_s \leq \bar{v}_k(\beta_s + \beta_b \nu(v_k))$ and (b) $\hat{u}_s > \bar{v}_k(\beta_s + \beta_b \nu(v_k))$. In case (a), let p_b be such that $\hat{u}_s = p_b(\beta_s + \beta_b \nu(v_k))$. Then $p_b \in C_k$ since, by the definition of U_k and of case (a),

$$\underline{v}_k(\beta_s + \beta_b \nu(v_k)) \leq \hat{u}_s \leq \bar{v}_k(\beta_s + \beta_b \nu(v_k))$$

(resp. $\underline{v}_k(\beta_s+\beta_b\nu(v_k)) < \hat{u}_s \le \bar{v}_k(\beta_s+\beta_b\nu(v_k)))$. In case (b), let $p_b = \bar{v}_k$. In either case, $p_b \in C_k$ and it follows by (A.9) and the definition of C_k that $p_b\nu(p_b) = p_b\nu(v_k) \ge v_1$ i.e. (4) holds.

Let $p_s = \frac{\hat{u}_s - \beta_b p_b \nu(v_k)}{\beta_s}$. Then it follows by (1), (2), (A.9), (A.10) and the definition of U_k that, in either case,

$$u_s = \beta_s p_s + \beta_b p_b \nu(p_b) = \beta_s p_s + \beta_b p_b \nu(v_k) = \hat{u}_s$$

and

$$\begin{aligned} u_b &= \beta_s(E - p_s) + \beta_b(E(p_b) - p_b\nu(p_b)) \\ &= \beta_s E - \hat{u}_s + \beta_b p_b \nu(v_k) + \beta_b E(v_k) - \beta_b p_b \nu(v_k) \\ &= \beta_s E + \beta_b E(v_k) - \hat{u}_s = \hat{u}_b. \end{aligned}$$

It remains to show that (3) holds. Since $p_s = \frac{\hat{u}_s - \beta_b p_b \nu(v_k)}{\beta_s}$, we have that $p_s \ge p_b$ if and only if $\hat{u}_s \ge p_b(\beta_s + \beta_b\nu(v_k))$. This inequality holds in case (a) since then $\hat{u}_s = p_b(\beta_s + \beta_b\nu(v_k))$. It also holds in case (b) since then $p_b = \bar{v}_k$ and, by the definition of case (b), $\hat{u}_s > \bar{v}_k(\beta_s + \beta_b\nu(v_k))$.

It follows from $p_s = \frac{\hat{u}_s - \beta_b p_b \nu(v_k)}{\beta_s}$ that $p_s \le E$ holds if and only if $\hat{u}_s \le \beta_s E + \beta_b p_b \nu(v_k)$. This inequality holds in case (a) since then $\hat{u}_s = p_b(\beta_s + \beta_b\nu(v_k))$ and $p_b \le E$, the latter because $p_b \in C_k$. It also holds in case (b) since then $p_b = \bar{v}_k$ and $\hat{u}_s \le \beta_s E + \beta_b \bar{v}_k \nu(v_k)$, the latter because $(\hat{u}_b, \hat{u}_s) \in U_k$.

It follows from the above that $(\hat{u}_b, \hat{u}_s)$ is represented by $(p_b, p_s, 1)$ and, hence, $(\hat{u}_b, \hat{u}_s) \in U^{**}(\beta_b, \beta_s)$. Since $(\hat{u}_b, \hat{u}_s)$ is arbitrary, it follows that $\cup_{k\in\kappa} U_k \subseteq U^{**}(\beta_b, \beta_s)$.

(Necessity) Let $(\hat{u}_b, \hat{u}_s) \in U^{**}(\beta_b, \beta_s)$ and let $(p_b, p_s) \in (V^*)^2$ be such that $(\hat{u}_b, \hat{u}_s)$ is represented by $(p_b, p_s, 1)$. Since $\{\{v_1\}, \left((v_{k-1}, v_k]\right)_{1<k\le K}\}$ is a partition of V^*, let $k = 1$ if $p_b = v_1$ and $k \in \{2, \ldots, K\}$ be such that $p_b \in (v_{k-1}, v_k]$ otherwise. Recall that $\nu(p_b) = \nu(v_k)$ and $E(p_b) = E(v_k)$ by (A.9) and (A.10) respectively. Then $p_b \in C_k$ by (3) and (4). Hence, $\underline{v}_k \le p_b \le \bar{v}_k$ and, if $\underline{v}_k \notin C_k$ i.e. $\underline{v}_k = v_{k-1}$, $\underline{v}_k < p_b \le \bar{v}_k$.

By (1) and (2), $\hat{u}_b = \beta_s(E - p_s) + \beta_b(E(v_k) - p_b\nu(v_k))$, $\hat{u}_s = \beta_s p_s + \beta_b p_b \nu(v_k)$ and, hence,

$$\hat{u}_b + \hat{u}_s = \beta_s E + \beta_b E(v_k). \tag{A.11}$$

Since $p_s \leq E$ by (3),

$$\begin{aligned} \hat{u}_s &= \beta_s p_s + \beta_b p_b \nu(v_k) \\ &\leq \beta_s E + \beta_b \bar{v}_k \nu(v_k). \end{aligned} \tag{A.12}$$

Moreover,

$$\begin{aligned} \hat{u}_s &= \beta_s p_s + \beta_b p_b \nu(v_k) \\ &\geq p_b(\beta_s + \beta_b \nu(v_k)) \\ &\geq \underline{v}_k(\beta_s + \beta_b \nu(v_k)) \end{aligned} \tag{A.13}$$

and, if $\underline{v}_k \notin C_k$ i.e. $\underline{v}_k = v_{k-1}$,

$$\hat{u}_s > \underline{v}_k(\beta_s + \beta_b \nu(v_k)). \tag{A.14}$$

It then follows by (A.11)–(A.14) that $(\hat{u}_b, \hat{u}_s) \in U_k$.

A.4 Characterization for Corollary 1

An equivalent description of the elements used in Corollary 1, which is useful to actually draw the set of equilibrium payoffs, is as follow. Let $\hat{\kappa} = \{1\} \cup \{k \in \{2, \ldots, K\} : v_{k-1} < E\}$ and define, for each $k \in \hat{\kappa}$, $v_k^* = \min\{v_k, E\}$. Then $\kappa = \{k \in \hat{\kappa} : v_k^* \nu(v_k) \geq v_1\}$ and, for each $k \in \kappa$, $\bar{v}_k = v_k^*$ and

$$\underline{v}_k = \begin{cases} v_1 & \text{if } k = 1, \\ \frac{v_1}{\nu(v_k)} & \text{if } k > 1 \text{ and } \frac{v_1}{\nu(v_k)} \geq v_{k-1} \\ v_{k-1} & \text{if } k > 1 \text{ and } \frac{v_1}{\nu(v_k)} < v_{k-1} \end{cases}$$

as shown in Claims 1–3 below.

Claim 1 $\kappa = \{k \in \hat{\kappa} : v_k^* \nu(v_k) \geq v_1\}$.

Proof. We have that $1 \in \kappa$ and that $1 \in \{k \in \hat{\kappa} : v_k^* \nu(v_k) \geq v_1\}$, the latter since $1 \in \hat{\kappa}$ and $v_1^* = v_1$.

Thus, consider $k > 1$. If $C_k \neq \emptyset$, let $p \in C_k$; hence, $p \in (v_{k-1}, v_k]$, $p\nu(v_k) \geq v_1$ and $p \leq E$. Then $v_{k-1} < p \leq E$ and $p \leq v_k^*$, implying respectively that $k \in \hat{\kappa}$ and $v_k^* \nu(v_k) \geq p\nu(v_k) \geq v_1$.

Conversely, let $k > 1$ be such that $v_{k-1} < E$ and $v_k^* \nu(v_k) \geq v_1$. If $v_k \leq E$, then $v_k^* = v_k$ and $v_k \in C_k$; if $v_k > E$, then $v_k^* = E$ and $E \in C_k$. ■

Claim 2 $\bar{v}_k = v_k^*$ *for each* $k \in \kappa$.

Proof. This is clear when $k = 1$ since $C_1 = \{v_1\}$ and $v_1^* = v_1$. Thus, assume that $k > 1$. If $v_k \leq E$, then $v_k^* = v_k$ and $v_k \in C_k$; hence, $\bar{v}_k = v_k = v_k^*$. If $v_k > E$, then $v_k^* = E$ and $E \in C_k$ (as in the proof of Claim 1); hence, $\bar{v}_k = E = v_k^*$. ■

Claim 3 *For each* $k \in \kappa$,

$$\underline{v}_k = \begin{cases} v_1 & \textit{if } k = 1, \\ \frac{v_1}{\nu(v_k)} & \textit{if } k > 1 \textit{ and } \frac{v_1}{\nu(v_k)} \geq v_{k-1} \\ v_{k-1} & \textit{if } k > 1 \textit{ and } \frac{v_1}{\nu(v_k)} < v_{k-1}. \end{cases}$$

Proof. This is clear when $k = 1$ since $C_1 = \{v_1\}$. Thus, assume that $k > 1$. We then have that $p \geq \frac{v_1}{\nu(v_k)}$ and $p > v_{k-1}$ for each $p \in C_k$.

Consider first the case where $\frac{v_1}{\nu(v_k)} \geq v_{k-1}$ and suppose that there is $\alpha > \frac{v_1}{\nu(v_k)}$ such that $p \geq \alpha$ for each $p \in C_k$. Then $\alpha \leq E$, $\alpha\nu(v_k) > v_1$, $\alpha \leq v_k$ and $\alpha > v_{k-1}$. Letting $\varepsilon > 0$ be such that $\alpha - \varepsilon > \frac{v_1}{\nu(v_k)}$, it follows that $\alpha - \varepsilon \leq E$, $(\alpha - \varepsilon)\nu(v_k) > v_1$, $\alpha - \varepsilon \leq v_k$ and $\alpha - \varepsilon > v_{k-1}$ (since $\alpha - \varepsilon > \frac{v_1}{\nu(v_k)} \geq v_{k-1}$). Hence, $\alpha - \varepsilon \in C_k$, contradicting $p \geq \alpha$ for each $p \in C_k$. Thus, $\underline{v}_k = \frac{v_1}{\nu(v_k)}$.

Consider next the case where $\frac{v_1}{\nu(v_k)} < v_{k-1}$ and suppose that there is $\alpha > v_{k-1}$ such that $p \geq \alpha$ for each $p \in C_k$. Then $\alpha \leq E$, $\alpha\nu(v_k) > v_1$, $\alpha \leq v_k$ and $\alpha > v_{k-1}$. Letting $\varepsilon > 0$ be such that $\alpha - \varepsilon > v_{k-1}$, it follows that $\alpha - \varepsilon \leq E$, $(\alpha - \varepsilon)\nu(v_k) > v_1$ (since $\alpha - \varepsilon > v_{k-1} > \frac{v_1}{\nu(v_k)}$), $\alpha - \varepsilon \leq v_k$ and $\alpha - \varepsilon > v_{k-1}$. Hence, $\alpha - \varepsilon \in C_k$, contradicting $p \geq \alpha$ for each $p \in C_k$. Thus, $\underline{v}_k = v_{k-1}$. ■

The case in Figure 2, where $V = \{1, \ldots, 5\}$, $\nu(1) = 0.1$, $\nu(2) = \nu(3) = \nu(4) = 0.2$, $\nu(5) = 0.3$ and $\beta_s = 1/2$, provides an example where the lower endpoints for u_s are not maximized when $p_b = p^*$.

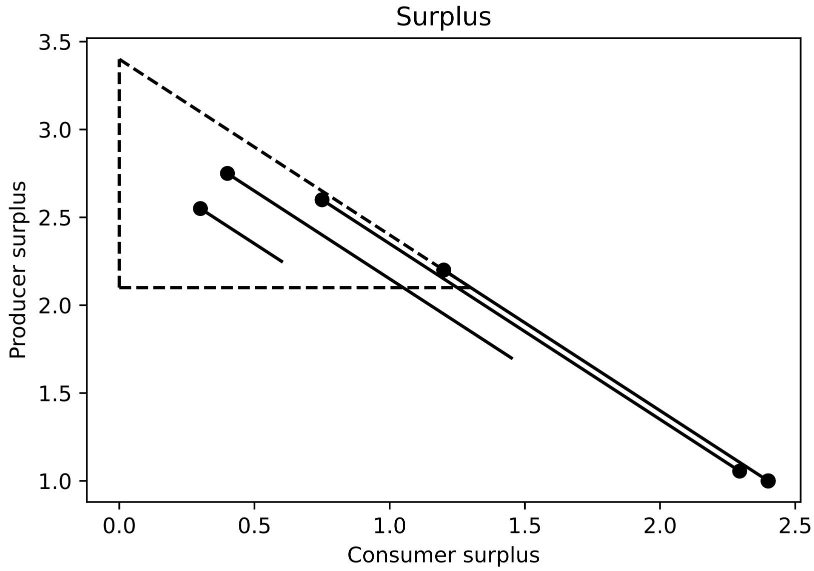

Figure 2: Equilibrium payoffs when $V = \{1, \dots, 5\}$, $\nu(1) = 0.1$, $\nu(2) = \nu(3) = \nu(4) = 0.2$, $\nu(5) = 0.3$ and $\beta_b = 1/2$.

B Proofs for Section 4

B.1 Preliminary Lemmas

Any sequential equilibrium $\pi \in \bar{\Pi}$ satisfies the following condition on the equilibrium path:

$$\begin{aligned}&\sum_{\phi_b} \pi_b^1[\phi_b] \sum_v \nu[v] \sum_m (\beta_b \phi_b(v) + \beta_s \phi_s(v))[m] u_s(\pi^2(m, \phi_b, \phi_s)) \geq \\ &\sum_{\phi_b} \pi_b^1[\phi_b] \sum_v \nu[v] \sum_m (\beta_b \phi_b(v) + \beta_s \phi_s'(v))[m] u_s(\hat{\pi}^2(m, \phi_b, \phi_s')),\end{aligned} \tag{B.1}$$

for each $\phi_s \in \text{supp}(\pi_s^1)$, $\phi_s' \in \Phi$ and $\hat{\pi}_s^2 : M_s \times \Phi \to V^*$, where

$$\begin{aligned}\pi^2(m, \phi_b, \phi_s) &= (\pi_s^2(m_s, \phi_s), \pi_b^2(m_b, \phi_b, \pi_s^2(m_s, \phi_s))) \text{ and} \\ \hat{\pi}^2(m, \phi_b, \phi_s') &= (\hat{\pi}_s^2(m_s, \phi_s'), \pi_b^2(m_b, \phi_b, \hat{\pi}_s^2(m_s, \phi_s'))),\end{aligned}$$

$$\begin{aligned}&\sum_{\phi_s} \pi_s^1[\phi_s] \sum_v \nu[v] \sum_m (\beta_b \phi_b(v) + \beta_s \phi_s(v))[m] u_b(v, \pi^2(m, \phi_b, \phi_s)) \geq \\ &\sum_{\phi_s} \pi_s^1[\phi_s] \sum_v \nu[v] \sum_m (\beta_b \phi_b'(v) + \beta_s \phi_s(v))[m] u_b(v, \hat{\pi}^2(m, \phi_b', \phi_s)),\end{aligned} \tag{B.2}$$

for each $\phi_b \in \text{supp}(\pi_b^1)$, $\phi_b' \in \Phi$ and $\hat{\pi}_b^2 : M_b \times \Phi \times V^* \to A$, where

$$\hat{\pi}^2(m, \phi_b', \phi_s) = (\pi_s^2(m_s, \phi_s), \hat{\pi}_b^2(m_b, \phi_b', \pi_s^2(m_s, \phi_s))),$$

$$\begin{aligned}\sum_{v,m_b,\phi_b} \frac{\pi_b^1[\phi_b]\nu[v](\beta_b\phi_b(v) + \beta_s\phi_s(v))[m_b, m_s]}{\sum_{\hat{v},\hat{m}_b,\hat{\phi}_b} \pi_b^1[\hat{\phi}_b]\nu[\hat{v}](\beta_b\phi_b(\hat{v}) + \beta_s\hat{\phi}_s(\hat{v}))[\hat{m}_b, m_s]} u_s(p, \pi_b^2(m_b, \phi_b, p)) \geq \\ \sum_{v,m_b,\phi_b} \frac{\pi_b^1[\phi_b]\nu[v](\beta_b\phi_b(v) + \beta_s\phi_s(v))[m_b, m_s]}{\sum_{\hat{v},\hat{m}_b,\hat{\phi}_b} \pi_b^1[\hat{\phi}_b]\nu[\hat{v}](\beta_b\phi_b(\hat{v}) + \beta_s\hat{\phi}_s(\hat{v}))[\hat{m}_b, m_s]} u_s(p', \pi_b^2(m_b, \phi_b, p'))\end{aligned} \tag{B.3}$$

for each $\phi_s \in \text{supp}(\pi_s^1)$, $m_s \in \mathbb{N}$ such that

$$\sum_{v,m_b,\phi_b} \pi_b^1[\phi_b]\nu[v](\beta_b\phi_b(v) + \beta_s\phi_s(v))[m_b, m_s] > 0,$$

$p \in \text{supp}(\pi_s^2(m_s, \phi_s))$ and $p' \in V^*$, and

$$\begin{aligned}\sum_{v,m_s,\phi_s} \frac{\pi_s^1[\phi_s]\nu[v](\beta_b\phi_b(v) + \beta_s\phi_s(v))[m_b, m_s]\pi_s^2(m_s, \phi_s)[p]}{\sum_{\hat{v},\hat{m}_s,\hat{\phi}_s} \pi_s^1[\hat{\phi}_s]\nu[\hat{v}](\beta_b\phi_b(\hat{v}) + \beta_s\hat{\phi}_s(\hat{v}))[m_b, \hat{m}_s]\pi_s^2(\hat{m}_s, \hat{\phi}_s)[p]} u_b(v, p, a) \geq \\ \sum_{v,m_s,\phi_s} \frac{\pi_s^1[\phi_s]\nu[v](\beta_b\phi_b(v) + \beta_s\phi_s(v))[m_b, m_s]\pi_s^2(m_s, \phi_s)[p]}{\sum_{\hat{v},\hat{m}_s,\hat{\phi}_s} \pi_s^1[\hat{\phi}_s]\nu[\hat{v}](\beta_b\phi_b(\hat{v}) + \beta_s\hat{\phi}_s(\hat{v}))[m_b, \hat{m}_s]\pi_s^2(\hat{m}_s, \hat{\phi}_s)[p]} u_b(v, p, a')\end{aligned} \tag{B.4}$$

for each $\phi_b \in \text{supp}(\pi_b^1)$, $m_b \in \mathbb{N}$ and $p \in V^*$ such that

$$\sum_{v,m_s,\phi_s} \pi_s^1[\phi_s]\nu[v](\beta_b\phi_b(v) + \beta_s\phi_s(v))[m_b, m_s]\pi_s^2(m_s, \phi_s)[p] > 0,$$

$a \in \text{supp}(\pi_b^2(m_b, \phi_b, p))$ and $a' \in A$.

Lemma B.1 *If π is a sequential equilibrium of G, then*

$$\sum_{\phi_b} \pi_b^1[\phi_b]u_s(\pi^2(m, \phi_b, \phi_s)) \geq \sum_{\phi_b} \pi_b^1[\phi_b]u_s(\pi^2(m', \phi_b, \phi_s))$$

for each $\phi_s \in \text{supp}(\pi_s^1)$, $v \in V$, $m \in \text{supp}(\phi_s(v))$ and $m' \in M$.

Proof. Suppose not; then there is $\tilde{\phi}_s \in \text{supp}(\pi_s^1)$, $\tilde{v} \in V$, $\tilde{m} \in \text{supp}(\tilde{\phi}_s(\tilde{v}))$ and $m' \in M$ such that $\sum_{\phi_b} \pi_b^1[\phi_b]u_s(\pi^2(m', \phi_b, \tilde{\phi}_s)) > \sum_{\phi_b} \pi_b^1[\phi_b]u_s(\pi^2(\tilde{m}, \phi_b, \tilde{\phi}_s))$. Define ϕ_s by setting, for each $v \in V$ and $m \in \text{supp}(\tilde{\phi}_s(v))$,

$$\phi_s(v)[m] = \begin{cases} 0 & \text{if } v = \tilde{v} \text{ and } m = \tilde{m}, \\ \tilde{\phi}_s(\tilde{v})[m'] + \tilde{\phi}_s(\tilde{v})[\tilde{m}] & \text{if } v = \tilde{v} \text{ and } m = m', \\ \tilde{\phi}_s(v)[m] & \text{otherwise,} \end{cases}$$

and let $\hat{\pi}_s^2 : M_s \times \Phi \to V^*$ be such that $\hat{\pi}_s^2(m_s, \phi_s) = \pi_s^2(m_s, \tilde{\phi}_s)$ for each $m_s \in M_s$. Then $\hat{\pi}^2(m, \phi_b, \phi_s) = \pi^2(m, \phi_b, \tilde{\phi}_s)$ for each $\phi_b \in \Phi$ and $m \in M$, and

$$\sum_{\phi_b} \pi_b^1[\phi_b] \Bigg(\sum_v \nu[v] \sum_m \Big((\beta_b \phi_b(v) + \beta_s \phi_s(v))[m] u_s(\hat{\pi}^2(m, \phi_b, \phi_s))$$

$$- (\beta_b \phi_b(v) + \beta_s \tilde{\phi}_s(v))[m] u_s(\pi^2(m, \phi_b, \tilde{\phi}_s)) \Big) \Bigg)$$

$$= \sum_{\phi_b} \pi_b^1[\phi_b] \Bigg(\sum_v \nu[v] \sum_m u_s(\pi^2(m, \phi_b, \tilde{\phi}_s))$$

$$(\beta_b \phi_b(v)[m] + \beta_s \phi_s(v)[m] - \beta_b \phi_b(v)[m] - \beta_s \tilde{\phi}_s(v)[m]) \Bigg)$$

$$= \nu[\tilde{v}] \beta_s \tilde{\phi}_s(\tilde{v})[\tilde{m}] \sum_{\phi_b} \pi_b^1[\phi_b] \Big(u_s(\pi^2(m', \phi_b, \tilde{\phi}_s)) - u_s(\pi^2(\tilde{m}, \phi_b, \tilde{\phi}_s)) \Big) > 0.$$

But this is a contradiction since π is a sequential equilibrium of G. ■

For each $\phi_s \in \text{supp}(\pi_s^1)$, let $u_s^s(\phi_s)$ be the common value of $\sum_{\phi_b} \pi_b^1[\phi_b] u_s(\pi^2(m, \phi_b, \phi_s))$ for each $m \in \text{supp}(\phi_s(v))$ and $v \in V$.

Lemma B.2 *If π is a sequential equilibrium of G, then*

$$u_s^s(\phi_s) = u_s^s(\phi_s')$$

for each $\phi_s, \phi_s' \in \text{supp}(\pi_s^1)$.

Proof. Suppose not; then $u_s^s(\phi_s) > u_s^s(\phi_s')$ for some $\phi_s, \phi_s' \in \text{supp}(\pi_s^1)$. Note that (B.1) holds as an equality for such ϕ_s and ϕ_s'. Letting

$$u_s^b(\phi_s) = \beta_b \sum_{\phi_b} \pi_b^1[\phi_b] \sum_v \nu[v] \sum_m \phi_b(v)[m] u_s(\pi^2(m, \phi_b, \phi_s))$$

and analogously for $u_s^b(\phi_s')$, it follows from (B.1) and Lemma B.1 that

$$u_s(\pi) = \beta_b u_s^b(\phi_s) + \beta_s u_s^s(\phi_s) = \beta_b u_s^b(\phi_s') + \beta_s u_s^s(\phi_s').$$

Hence, $u_s^b(\phi_s) < u_s^b(\phi_s')$.

Let $\hat{m} \in \cup_v \text{supp}(\phi_s(v))$ and $\bar{m}_s \notin \cup_{v \in V, i \in N, \phi_i \in \text{supp}(\pi_i^1)} \text{supp}(\phi_{i,M_s}(v))$. Then define $\bar{\phi}(v) = 1_{(\hat{m}_b, \bar{m}_s)}$ for each $v \in V$ and let $\bar{\pi}_s^2$ be such that $\bar{\pi}_s^2(\bar{m}_s, \bar{\phi}_s) = \pi_s^2(\hat{m}_s, \phi_s)$ and, for each $m_s \neq \bar{m}_s$, $\bar{\pi}_s^2(m_s, \bar{\phi}_s) = \pi_s^2(m_s, \phi_s')$. Letting $\bar{\pi}_s = (\bar{\phi}_s, \bar{\pi}_s^2)$, it follows that

$$u_s(\bar{\pi}_s, \pi_b) = \beta_b u_s^b(\phi_s') + \beta_s u_s^s(\phi_s) > \beta_b u_s^b(\phi_s) + \beta_s u_s^s(\phi_s) = u_s(\pi),$$

a contradiction. ■

Lemma B.3 *If π is a sequential equilibrium of G, then*

$$\sum_{\phi_b} \pi_b^1[\phi_b] u_s(p, \pi_b^2(m_b, \phi_b, p)) \geq \sum_{\phi_b} \pi_b^1[\phi_b] u_s(p', \pi_b^2(m_b, \phi_b, p'))$$

for each $\phi_s \in \operatorname{supp}(\pi_s^1)$, $v \in V$, $m \in \operatorname{supp}(\phi_s(v))$, $p \in \operatorname{supp}(\pi_s^2(m_s, \phi_s))$ and $p' \in V^*$.

Proof. Suppose not; then there is $\hat{\phi}_s \in \operatorname{supp}(\pi_s^1)$, $\hat{v} \in V$, $\hat{m} \in \operatorname{supp}(\phi_i^*(v))$ $\hat{p} \in \operatorname{supp}(\pi_s^2(\hat{m}_s, \hat{\phi}_s))$ and $p' \in V^*$ such that

$$\sum_{\phi_b} \pi_b^1[\phi_b] u_s(\hat{p}, \pi_b^2(\hat{m}_b, \phi_b, \hat{p})) < \sum_{\phi_b} \pi_b^1[\phi_b] u_s(p', \pi_b^2(\hat{m}_b, \phi_b, p')).$$

We may assume that

$$\max_{p \in \operatorname{supp}(\pi_s^2(\hat{m}_s, \hat{\phi}_s))} \sum_{\phi_b} \pi_b^1[\phi_b] u_s(p, \pi_b^2(\hat{m}_b, \phi_b, p)) \leq \sum_{\phi_b} \pi_b^1[\phi_b] u_s(p', \pi_b^2(\hat{m}_b, \phi_b, p'))$$

since if this latter inequality does not hold, we can replace p' with any solution to $\max_{p \in \operatorname{supp}(\pi_s^2(\hat{m}_s, \hat{\phi}_s))} \sum_{\phi_b} \pi_b^1[\phi_b] u_s(p, \pi_b^2(\hat{m}_b, \phi_b, p)$. Hence,

$$\begin{aligned} \sum_{\phi_b} \pi_b^1[\phi_b] u_s(\pi^2(\hat{m}, \phi_b, \phi_s)) &= \sum_{p \in \operatorname{supp}(\pi_s^2(\hat{m}_s, \hat{\phi}_s))} \left(\sum_{\phi_b} \pi_b^1[\phi_b] u_s(p, \pi_b^2(\hat{m}_b, \phi_b, p)) \right) \\ &< \sum_{\phi_b} \pi_b^1[\phi_b] u_s(p', \pi_b^2(\hat{m}_b, \phi_b, p')). \end{aligned} \tag{B.5}$$

Let $\bar{m}_s \notin \cup_v \operatorname{supp}(\phi_{M_s}(v))$,

$$\phi_s(v) = \begin{cases} 1_{(\hat{m}_b, \bar{m}_s)} & \text{if } v = \hat{v}, \\ \hat{\phi}_s(v) & \text{otherwise,} \end{cases}$$

and $\hat{\pi}_s^2 : M_s \times \Phi \to V^*$ be such that $\hat{\pi}_s^2(\bar{m}_s, \phi_s) = p'$ and $\hat{\pi}_s^2(m_s, \phi_s) = \pi_s^2(m_s, \hat{\phi}_s)$ for each $m_s \neq \bar{m}_s$. Then $\hat{\pi}^2(m, \phi_b, \phi_s) = \pi^2(m, \phi_b, \hat{\phi}_s)$ for each $\phi_b \in \Phi$ and $m \in M$ such that $m_s \neq \bar{m}_s$. Thus,

$$\sum_{\phi_b} \pi_b^1[\phi_b] \sum_v \nu[v] \sum_m \Big((\beta_b \phi_b(v) + \beta_s \phi_s(v))[m] u_s(\hat{\pi}^2(m, \phi_b, \phi_s))$$
$$-(\beta_b \phi_b(v) + \beta_s \hat{\phi}_s(v))[m] u_s(\pi^2(m, \phi_b, \hat{\phi}_s)) \Big)$$
$$= \nu[v'] \beta_s \sum_{\phi_b} \pi_b^1[\phi_b] \left(\sum_m \phi_s(v')[m] u_s(\hat{\pi}^2(m, \phi_b, \phi_s)) - \sum_m \hat{\phi}_s(v')[m] u_s(\pi^2(m, \phi_b, \hat{\phi}_s)) \right)$$
$$= \nu[v'] \beta_s \sum_{\phi_b} \pi_b^1[\phi_b] \Big(u_s(p', \pi_b^2(\hat{m}_b, \phi_b)) - \sum_{m \in \text{supp}(\hat{\phi}_s(v'))} \hat{\phi}_s(v')[m] u_s(\pi^2(m, \phi_b, \hat{\phi}_s) \Big)$$
$$= \nu[v'] \beta_s \sum_{\phi_b} \pi_b^1[\phi_b] \Big(u_s(p', \pi_b^2(\hat{m}_b, \phi_b)) - u_s(\pi^2(\hat{m}, \phi_b, \hat{\phi}_s)) \Big) > 0$$

because $\sum_{\phi_b} \pi_b^1[\phi_b] u_s(\pi^2(m, \phi_b, \hat{\phi}_s)) = \sum_{\phi_b} \pi_b^1[\phi_b] u_s(\pi^2(\hat{m}, \phi_b, \hat{\phi}_s))$ for each $m \in \text{supp}(\hat{\phi}_s(v'))$ by Lemma B.1. But this is a contradiction since π is a sequential equilibrium. ■

Lemma B.4 *For each $p \in V^*$, $\phi_s, \phi'_s \in \text{supp}(\pi_s^1)$, $v, v' \in V$ and $m, m' \in M$ such that $m \in \phi_s(v)$, $m' \in \phi_s(v')$ and $p \in \text{supp}(\pi_s^2(m_s, \phi_s)) \cap \text{supp}(\pi_s^2(m'_s, \phi'_s))$,*

$$\sum_{\phi_b} \pi_b^1[\phi_b] \pi_b^2(m_b, \phi_b, p) = \sum_{\phi_b} \pi_b^1[\phi_b] \pi_b^2(m'_b, \phi_b, p).$$

Proof. It follows by Lemmas B.2 and B.3 that

$$p \sum_{\phi_b} \pi_b^1[\phi_b] \pi_b^2(m_b, \phi_b, p) = \sum_{\phi_b} \pi_b^1[\phi_b] u_s(p, \pi_b^2(m_b, \phi_b, p)) = u_s^s$$
$$= \sum_{\phi_b} \pi_b^1[\phi_b] u_s(p, \pi_b^2(m'_b, \phi_b, p) = p \sum_{\phi_b} \pi_b^1[\phi_b] \pi_b^2(m'_b, \phi_b, p).$$

Since $p \geq v_1 > 0$, the conclusion follows. ■

Define $\gamma_s \in \Delta(V^*)$ by setting

$$\gamma_s = \sum_{\phi_s} \pi_s^1[\phi_s] \sum_v \nu[v] \sum_m \phi_s(v)[m] \pi_s^2(m_s, \phi_s)$$

and, for each $p \in \text{supp}(\gamma_s)$, let $a(p)$ be the common value of $\sum_{\phi_b} \pi_b^1[\phi_b] \pi_b^2(m_b, \phi_b, p)$.

Define $\gamma_s(v) \in \Delta(V^*)$ by setting, for each $v \in V$,

$$\gamma_s(v) = \sum_{\phi_s} \pi_s^1[\phi_s] \sum_m \phi_s(v)[m] \pi_s^2(m_s, \phi_s)$$

The above then implies the following lemma.

Lemma B.5 *$u_s^s = pa(p) = p'a(p')$ for each $p, p' \in \text{supp}(\gamma_s)$ and*

$$u_b^s = \sum_v \nu[v] v \sum_p \gamma_s(v)[p] a(p) - u_s^s.$$

Note that Lemma B.5 is consistent with any (u_s^s, u_b^s) such that $v_1 \leq u_s^s \leq E$, $0 \leq u_b^s \leq E - v_1$ and $u_s^s + u_b^s \leq E$. To see this, fix $(\bar{u}_s, \bar{u}_b)$ satisfying $\bar{u}_s \in [v_1, E]$ and $\bar{u}_b = \delta(E - \bar{u}_s)$ for some $\delta \in [0,1]$. Set, for each $v \in V$, $\gamma_s(v) = \delta 1_{\bar{u}_s} + (1-\delta) 1_E$. Let $a(\bar{u}_s) = 1$ and $a(E) = \bar{u}_s / E$. Then $u_s^s = \bar{u}_s a(\bar{u}_s) = E a(E) = \bar{u}_s$ and $u_b^s = E(\delta a(\bar{u}_s) + (1-\delta) a(E)) - u_s^s = E(\delta + (1-\delta)\bar{u}_s / E) - u_s^s = \delta(E - \bar{u}_s) = \bar{u}_b$.

Lemma B.6 *For each $v \in V$, $p \in \text{supp}(\gamma_s(v))$, $m_s \in \text{supp}(\phi_s(v))$ and $\phi_b \in \text{supp}(\pi_b^1)$,*

$$\begin{aligned} &\sum_{v, m_s, \phi_s} \pi_s^1[\phi_s] \nu[v] \phi_s(v)[m_b, m_s] \pi_s^2(m_s, \phi_s)[p] u_b(v, p, a) \geq \\ &\sum_{v, m_s, \phi_s} \pi_s^1[\phi_s] \nu[v] \phi_s(v)[m_b, m_s] \pi_s^2(m_s, \phi_s)[p] u_b(v, p, a') \end{aligned} \tag{B.6}$$

for each $a \in \text{supp}(\pi_b^2(m_b, \phi_b, p))$ and $a' \in A$.

Proof. Suppose not; then let $\bar{v} \in V$, $\bar{p} \in \text{supp}(\gamma_s(v))$, $\bar{m}_s \in \text{supp}(\phi_s(v))$, $\phi_b \in \text{supp}(\pi_b^1)$, $a \in \text{supp}(\pi_b^2(\bar{m}_b, \phi_b, \bar{p}))$ and $a' \in A$ be such that (B.6) fails. Let $f : \cup_v \text{supp}(\phi_{b,M_b}(v)) \to \left(\cup_{v \in V, \phi_s \in \text{supp}(\pi_s^1)} \text{supp}(\phi_{s,M_b}(v))\right)^c$ be 1-1. For each $v \in V$, $m_b \in f(\cup_v \text{supp}(\phi_{b,M_b}(v)))$ and $m_s \in \cup_v \text{supp}(\phi_{b,M_s}(v))$, define

$$\bar{\phi}_b(v)[m_b, m_s] = \phi_b(v)[f^{-1}(m_b), m_s].$$

Furthermore, let $\bar{\pi}_b^2$ be such that, for each $p \in V^*$,

$$\bar{\pi}_b^2(m_s, \bar{\phi}_b, p) = \begin{cases} a' & \text{if } m_s = \bar{m}_s \text{ and } p = \bar{p}, \\ \pi_b^2(f^{-1}(m_b), \phi_s, p) & \text{if } m_b \in f(\cup_v \text{supp}(\phi_{b,M_b}(v))), \\ \pi_b^2(m_b, \phi_s, p) & \text{otherwise.} \end{cases}$$

Let $\bar{\pi}_b^1 = 1_{\bar{\phi}}$. Then $u_b(\bar{\pi}_b, \pi_s) > u_b(\pi)$, which is a contradiction since π is a sequential equilibrium. ■

The following lemmas characterize the payoffs that could arise when the realized message profile belongs to the support of the buyer's information structure.

Lemma B.7 *If π is a sequential equilibrium of G, then*

$$\sum_{\phi_s} \pi_s^1[\phi_s] u_b(v, \pi^2(m, \phi_b, \phi_s)) \geq \sum_{\phi_s} \pi_s^1[\phi_s] u_b(v, \pi^2(m', \phi_b, \phi_s))$$

for each $\phi_b \in \operatorname{supp}(\pi_b^1)$, $v \in V$, $m \in \operatorname{supp}(\phi_b(v))$ and $m' \in M$.

Proof. Suppose not; then there is $\tilde{\phi}_b \in \operatorname{supp}(\pi_b^1)$, $\tilde{v} \in V$, $\tilde{m} \in \operatorname{supp}(\tilde{\phi}_b(\tilde{v}))$ and $m' \in M$ such that $\sum_{\phi_s} \pi_s^1[\phi_s] u_b(\tilde{v}, \pi^2(m', \tilde{\phi}_b, \phi_s)) > \sum_{\phi_s} \pi_s^1[\phi_s] u_b(\tilde{v}, \pi^2(\tilde{m}, \tilde{\phi}_b, \phi_s))$. Define ϕ_b by setting, for each $v \in V$ and $m \in \operatorname{supp}(\tilde{\phi}_b(v))$,

$$\phi_b(v)[m] = \begin{cases} 0 & \text{if } v = \tilde{v} \text{ and } m = \tilde{m}, \\ \tilde{\phi}_b(\tilde{v})[m'] + \tilde{\phi}_b(\tilde{v})[\tilde{m}] & \text{if } v = \tilde{v} \text{ and } m = m', \\ \tilde{\phi}_b(v)[m] & \text{otherwise,} \end{cases}$$

and let $\hat{\pi}_b^2 : M_b \times \Phi \times V^* \to A$ be such that $\hat{\pi}_b^2(m_b, \phi_b, p) = \pi_b^2(m_b, \tilde{\phi}_b, p)$ for each $(m_b, p) \in M_b \times V^*$. Then $\hat{\pi}^2(m, \phi_b, \phi_s) = \pi^2(m, \tilde{\phi}_b, \phi_s)$ for each $m \in M$ and $\phi_s \in \Phi$, and

$$\sum_{\phi_s} \pi_s^1[\phi_s] \Bigg(\sum_v \nu[v] \sum_m \Big((\beta_b \phi_b(v) + \beta_s \phi_s(v))[m] u_b(v, \hat{\pi}^2(m, \phi_b, \phi_s))$$

$$- (\beta_b \tilde{\phi}_b(v) + \beta_s \phi_s(v))[m] u_b(v, \pi^2(m, \phi_b, \tilde{\phi}_s)) \Big) \Bigg)$$

$$= \sum_{\phi_s} \pi_s^1[\phi_s] \Bigg(\sum_v \nu[v] \sum_m u_b(v, \pi^2(m, \tilde{\phi}_b, \phi_s))$$

$$(\beta_b \phi_b(v)[m] + \beta_s \phi_s(v)[m] - \beta_b \tilde{\phi}_b(v)[m] - \beta_s \phi_s(v)[m]) \Bigg)$$

$$= \nu[\tilde{v}] \beta_b \tilde{\phi}_b(\tilde{v})[\tilde{m}] \sum_{\phi_s} \pi_s^1[\phi_s] \Big(u_b(\tilde{v}, \pi^2(m', \tilde{\phi}_b, \phi_s)) - u_b(\pi^2(\tilde{m}, \tilde{\phi}_b, \phi_s)) \Big) > 0.$$

But this is a contradiction since π is a sequential equilibrium of G. ∎

Define, for each $v \in V$ and $\phi_b \in \operatorname{supp}(\pi_b^1)$, $u_b^b(v, \phi_b)$ as the common value of $\sum_{\phi_s} \pi_s^1[\phi_s] u_b(v, \pi^2(m, \phi_b, \phi_s))$ for each $m \in \operatorname{supp}(\phi_b(v))$.

Lemma B.8 *If π is a sequential equilibrium of G, then*

$$u_b^b(v, \phi_b) = u_b^b(v, \phi_b')$$

for each $v \in V$ and $\phi_b, \phi_b' \in \operatorname{supp}(\pi_b^1)$.

Proof. Suppose not; then $u^b_b(\hat{v}, \phi_b) > u^b_b(\hat{v}, \phi'_b)$ for some $\hat{v} \in V$ and $\phi_b, \phi'_b \in \text{supp}(\pi^1_b)$. Note that (B.2) holds as an equality for such ϕ_b and ϕ'_b. Letting

$$u^s_b(v, \phi_b) = \beta_s \sum_{\phi_s} \pi^1_s[\phi_s] \sum_m \phi_s(v)[m] u_b(v, \pi^2(m, \phi_b, \phi_s))$$

and analogously for $u^s_b(\phi'_b)$, it follows from (B.2) and Lemma B.7 that

$$u_b(\pi) = \sum_v \nu[v] \left(\beta_b u^b_b(v, \phi_b) + \beta_s u^s_b(v, \phi_b)\right) = \sum_v \nu[v] \left(\beta_b u^b_b(v, \phi'_b) + \beta_s u^s_b(v, \phi'_b)\right).$$

Let $\hat{m} \in \text{supp}(\phi_b(\hat{v}))$ and $\bar{m}_b \notin \cup_{v \in V, i \in N, \phi_i \in \text{supp}(\pi^1_i)} \text{supp}(\phi_{i,M_b}(v))$. Then define

$$\bar{\phi}_b(v) = \begin{cases} 1_{(\bar{m}_b, \hat{m}_s)} & \text{if } v = \hat{v}, \\ \phi_b(v) & \text{otherwise} \end{cases}$$

for each $v \in V$ and let $\bar{\pi}^2_b$ be such that $\bar{\pi}^2_b(\bar{m}_b, \bar{\phi}_b) = \pi^2_b(\hat{m}_b, \phi_b)$ and, for each $m_b \neq \bar{m}_b$, $\bar{\pi}^2_b(m_b, \bar{\phi}_b) = \pi^2_b(m_b, \phi'_b)$. Letting $\bar{\pi}_b = (\bar{\phi}_b, \bar{\pi}^2_b)$, it follows that

$$u_b(\bar{\pi}_b, \pi_s) = \sum_{v \neq \hat{v}} \nu[v] \left(\beta_b u^b_b(v, \phi'_b) + \beta_s u^s_b(v, \phi'_b)\right) + \nu[\hat{v}] \left(\beta_b u^b_b(\hat{v}, \phi_b) + \beta_s u^s_b(\hat{v}, \phi'_b)\right)$$
$$> \sum_v \nu[v] \left(\beta_b u^b_b(v, \phi'_b) + \beta_s u^s_b(v, \phi'_b)\right) = u_b(\pi),$$

a contradiction. ■

Lemma B.9 *If π is a sequential equilibrium of G, then*

$$\pi^2_b(m_b, \phi_b, p) = \begin{cases} 1 & \text{if } p < v, \\ 0 & \text{if } p > v \end{cases}$$

for each $\phi_b \in \text{supp}(\pi^1_b)$, $\phi_s \in \text{supp}(\pi^1_s)$, $v \in V$, $m \in \text{supp}(\phi_b(v))$ and $p \in \text{supp}(\pi^2_s(m_s, \phi_s))$.

Proof. Note first that, for each $v \in V$ and $\phi_b \in \text{supp}(\pi^1_b)$,

$$u^b_b(\phi_b) = \sum_{\phi_s} \pi^s_1[\phi_s] \sum_{p \in \text{supp}(\pi^2_s(m_s, \phi_s)) : p < v} \pi^2_s(m_s, \phi_s)[p](v - p)\pi^2_b(m_b, \phi_b, p)$$
$$+ \sum_{\phi_s} \pi^s_1[\phi_s] \sum_{p \in \text{supp}(\pi^2_s(m_s, \phi_s)) : p > v} \pi^2_s(m_s, \phi_s)[p](v - p)\pi^2_b(m_b, \phi_b, p).$$

where $m \in \text{supp}(\phi_b(v))$.

Suppose that the conclusion of the lemma fails. Then there is $\hat{\phi}_b \in \text{supp}(\pi_b^1)$, $\hat{\phi}_s \in \text{supp}(\pi_s^1)$, $\hat{v} \in V$, $\hat{m} \in \text{supp}(\hat{\phi}_b(\hat{v}))$ and $\hat{p} \in \text{supp}(\pi_s^2(\hat{m}_s, \hat{\phi}_s))$ such that $\pi_b^2(\hat{m}_b, \hat{\phi}_b, \hat{p}) < 1$ if $\hat{p} < \hat{v}$ or $\pi_b^2(\hat{m}_b, \hat{\phi}_b, \hat{p}) > 0$ if $\hat{p} > \hat{v}$.

Consider the case $\hat{p} < \hat{v}$. Let $\bar{m}_b \notin \cup_{v \in V, i \in N, \phi_i \in \text{supp}(\pi_i^1)} \text{supp}(\phi_{i,M_b}(v))$ and define

$$\bar{\phi}_b(v) = \begin{cases} 1_{(\bar{m}_b, \hat{m}_s)} & \text{if } v = \hat{v}, \\ \phi_b(v) & \text{otherwise} \end{cases}$$

for each $v \in V$ and let $\bar{\pi}_b^2$ be such that $\bar{\pi}_b^2(\bar{m}_b, \bar{\phi}_b) = 1$ and, for each $m_b \neq \bar{m}_b$, $\bar{\pi}_b^2(m_b, \bar{\phi}_b) = \pi_b^2(m_b, \hat{\phi}_b)$. Let

$$\begin{aligned} \bar{u}_b^b(\hat{v}, \hat{\phi}_b) = & \sum_{\phi_s \neq \hat{\phi}_s} \pi_1^s[\phi_s] \sum_{p \in \text{supp}(\pi_s^2(\hat{m}_s, \phi_s)): p < v} \pi_s^2(\hat{m}_s, \phi_s)[p](v - p)\pi_b^2(\hat{m}_b, \hat{\phi}_b, p) \\ & + \pi_1^s[\hat{\phi}_s]\Big(\sum_{p \in \text{supp}(\pi_s^2(\hat{m}_s, \hat{\phi}_s)) \setminus \{\hat{p}\}: p < v} \pi_s^2(\hat{m}_s, \hat{\phi}_s)[p](v - p)\pi_b^2(\hat{m}_b, \hat{\phi}_b, p) \\ & + \pi_s^2(\hat{m}_s, \hat{\phi}_s)[\hat{p}](v - p)\Big) \\ & + \sum_{\phi_s} \pi_1^s[\phi_s] \sum_{p \in \text{supp}(\pi_s^2(m_s, \phi_s)): p > v} \pi_s^2(m_s, \phi_s)[p](v - p)\pi_b^2(m_b, \phi_b, p); \end{aligned}$$

then $\bar{u}_b^b(\hat{v}, \hat{\phi}_b) > u_b^b(\hat{v}, \hat{\phi}_b)$.

Letting $\bar{\pi}_b = (\bar{\phi}_b, \bar{\pi}_b^2)$, it follows that

$$\begin{aligned} u_b(\bar{\pi}_b, \pi_s) = & \sum_{v \neq \hat{v}} \nu[v] \left(\beta_b u_b^b(v, \hat{\phi}_b) + \beta_s u_b^s(v, \hat{\phi}_b) \right) + \nu[\hat{v}] \left(\beta_b \bar{u}_b^b(\hat{v}, \hat{\phi}_b) + \beta_s u_b^s(\hat{v}, \hat{\phi}_b) \right) \\ > & \sum_{v} \nu[v] \left(\beta_b u_b^b(v, \hat{\phi}_b) + \beta_s u_b^s(v, \hat{\phi}_b) \right) = u_b(\pi), \end{aligned}$$

a contradiction.

Finally, the case $\hat{p} > \hat{v}$. In this case, let $\bar{\pi}_b^2$ be such that $\bar{\pi}_b^2(\bar{m}_b, \bar{\phi}_b) = 0$ and, for each $m_b \neq \bar{m}_b$, $\bar{\pi}_b^2(m_b, \bar{\phi}_b) = \pi_b^2(m_b, \hat{\phi}_b)$. ■

Lemma B.10 *If π is a sequential equilibrium of G, $m_s \in \cup_{v \in V, \phi_b \in \text{supp}(\pi_b^1)} \text{supp}(\phi_{b,M_s}(v))$ and $\phi_s \in \text{supp}(\pi_s^1)$, then each $p \in \text{supp}(\pi_s^2(m_s, \phi_s))$ solves*

$$\max_{p' \in V^*} p' \sum_{v, \phi_b, m_b} \frac{\nu[v]\pi_b^1[\phi_b]\phi_b(v)[m_b, m_s]}{\sum_{\hat{v}, \hat{\phi}_b, \hat{m}_b} \nu[\hat{v}]\pi_b^1[\hat{\phi}_b]\hat{\phi}_b(\hat{v})[\hat{m}_b, m_s]} \pi_b^2(m_b, \phi_b, p'). \tag{B.7}$$

Proof. Let $\phi_s \in \text{supp}(\pi_s^1)$ and $m_s \in \cup_{v\in V,\phi_b\in\text{supp}(\pi_b^1)}\text{supp}(\phi_{b,M_s}(v))$. The conclusion of the lemma holds when $m_s \notin \cup_{v\in V,\phi_s\in\text{supp}(\pi_s^1)}\text{supp}(\phi_{s,M_s}(v)$ by (B.3). Thus, assume that $m_s \in \cup_{v\in V,\phi_s\in\text{supp}(\pi_s^1)}\text{supp}(\phi_{s,M_s}(v)$ and suppose that the conclusion of the lemma does not hold. Then there is $v' \in V$, $\phi_s \in \text{supp}(\pi_s^1)$, $m' \in \text{supp}(\phi_s(v'))$ such that $m_s \in \cup_{v\in V,\phi_b\in\text{supp}(\pi_b^1)}\text{supp}(\phi_{b,M_s}(v))$ and $p' \in \text{supp}(\pi_s^2(m'_s,\phi_s))$ that does not solve problem (B.7).

Let p^* be a solution to problem (B.7); then

$$p^* \sum_{v,\phi_b,m_b} \frac{\nu[v]\pi_b^1[\phi_b]\phi_b(v)[m_b,m'_s]}{\sum_{\hat{v},\hat{\phi}_b,\hat{m}_b}\nu[\hat{v}]\pi_b^1[\hat{\phi}_b]\hat{\phi}_b(\hat{v})[\hat{m}_b,m'_s]}\pi_b^2(m_b,\phi_b,p^*) >$$
$$\sum_{p\in\text{supp}(\pi_s^2(m'_s,\phi_s))} \pi_s^2(m'_s,\phi_s)[p]\left(p\sum_{v,\phi_b,m_b}\frac{\nu[v]\pi_b^1[\phi_b]\phi_b(v)[m_b,m'_s]}{\sum_{\hat{v},\hat{\phi}_b,\hat{m}_b}\nu[\hat{v}]\pi_b^1[\hat{\phi}_b]\hat{\phi}_b(\hat{v})[\hat{m}_b,m'_s]}\pi_b^2(m_b,\phi_b,p')\right). \tag{B.8}$$

Let $\bar{m}_s \notin \cup_{v,i\in N,\phi_i\in\text{supp}(\pi_i^1)}\text{supp}(\phi_{i,M_s}(v))$. Define $\bar{\phi}_s \in \Phi$ by setting, for each $v \in V$ and $m \in M$,

$$\bar{\phi}_s(v)[m] = \begin{cases} 1 & \text{if } v = v' \text{ and } m = (m'_b,\bar{m}_s), \\ 0 & \text{if } v = v' \text{ and } m \neq (m'_b,\bar{m}_s), \\ 0 & \text{if } v \neq v' \text{ and } m_s = m'_s, \\ \phi_s(v)[m_b,m'_s] & \text{if } v \neq v' \text{ and } m_s = \bar{m}_s, \\ \phi_s(v)[m] & \text{otherwise} \end{cases}$$

and let $\bar{\pi}_s^2 : M_s \times \Phi \to V^*$ be such that, for each $m_s \in M_s$,

$$\bar{\pi}_s^2(m_s,\bar{\phi}_s) = \begin{cases} p^* & \text{if } m_s = m'_s, \\ \pi_s^2(m'_s,\phi_s) & \text{if } m_s = \bar{m}_s, \\ \pi_s^2(m_s,\phi_s) & \text{otherwise.} \end{cases}$$

Then, letting $\bar{\pi}_s = (\bar{\phi}_s,\bar{\pi}_s^2)$, it follows by Lemma B.1 (recall that $m' \in \text{supp}(\phi_s(v'))$) that

$$u_s(\bar{\pi}_s,\pi_b) - u_s(\pi) = \beta_b \sum_{\phi_b}\pi_b^1[\phi_b]\sum_v \nu[v]\sum_{m_b}\phi_b(v)[m_b,m'_s]$$
$$\Big(p^*\pi_b^2(m_b,\phi_b,p^*) - \sum_{p\in\text{supp}(\pi_s^2(m'_s,\phi_s))}\pi_s^2(m'_s,\phi_s)[p]p\pi_b^2(m_b,\phi_b,p)\Big).$$

It then follows by (B.8) that $u_s(\bar{\pi}_s, \pi_b) > u_s(\pi)$. But this is a contradiction since π is a sequential equilibrium of G. ■

Lemmas B.9 and B.10 imply the following: Fix $\phi_s \in \text{supp}(\pi_s^1)$ and, for each $m_s \in \cup_{(v,\phi_b)\in V\times\text{supp}(\pi_b^1)}\text{supp}(\phi_{b,M_s}(v))$, let $p_{m_s} \in \text{supp}(\pi_s^2(m_s, \phi_s))$. Then

$$p_{m_s} \sum_{v,\phi_b,m_b} \nu[v]\pi_b^1[\phi_b]\phi_b(v)[m_b, m_s]\pi_b^2(m_b, \phi_b, p_{m_s})$$
$$= p \sum_{v,\phi_b,m_b} \nu[v]\pi_b^1[\phi_b]\phi_b(v)[m_b, m_s]\pi_b^2(m_b, \phi_b, p)$$

for each $\phi_s' \in \text{supp}(\pi_s^1)$ and $p \in \text{supp}(\pi_s^2(m_s, \phi_s'))$. Thus,

$$u_s^b = \sum_{m_s} p_{m_s}\Big(\sum_{v>p_{m_s}} \nu[v] \sum_{\phi_b} \pi_b^1[\phi_b] \sum_{m_b} \phi_b(v)[m_b, m_s]$$
$$+ \nu[p_{m_s}] \sum_{\phi_b} \pi_b^1[\phi_b] \sum_{m_b} \phi_b(p_{m_s})[m_b, m_s]\pi_b^2(m_b, \phi_b, p_{m_s})\Big).$$

For each $m_s \in \cup_{(v,\phi_b)\in V\times\text{supp}(\pi_b^1)}\text{supp}(\phi_{b,M_s}(v))$ and $p \in \cup_{\phi_s\in\text{supp}(\pi_s^1)}\text{supp}(\pi_s^2(m_s, \phi_s))$, let

$$\gamma(m_s)[p] = \sum_{\phi_s} \pi_s^1[\phi_s]\pi_s^2(m_s, \phi_s)[p].$$

The following lemma is a consequence of Lemma B.7.

Lemma B.11 *If π is a sequential equilibrium of G, $v \in V$, $m_s, m_s' \in \cup_{\phi_b\in\text{supp}(\pi_b^1)}\text{supp}(\phi_{b,M_s}(v))$ and $\hat{m}_s \in \cup_{(\hat{v},\phi_b)\in V\times\text{supp}(\pi_b^1)}\text{supp}(\phi_{b,M_s}(\hat{v}))$, then*

$$\sum_{p<v} \gamma(m_s)[p](v-p) = \sum_{p<v} \gamma(m_s')[p](v-p) \geq \sum_{p<v} \gamma(\hat{m}_s)[p](v-p).$$

Proof. Let $\phi_b, \phi_b' \in \text{supp}(\pi_b^1)$ and $m_b, m_b' \in M_b$ be such that $(m_b, m_s) \in \text{supp}(\phi_b(v))$ and $(m_b', m_s') \in \text{supp}(\phi_b'(v))$. Then, by Lemma B.9,

$$\sum_{\phi_s} \pi_s^1[\phi_s]u_b(v, \pi^2((m_b, m_s), \phi_b, \phi_s) = \sum_{p}\sum_{\phi_s} \pi_s^1[\phi_s]\pi_s^2(m_s, \phi_s)[p](v-p)\pi_b^2(m_b, \phi_b, p)$$
$$= \sum_{p<v} \gamma(m_s)[p](v-p), \text{ and}$$
$$\sum_{\phi_s} \pi_s^1[\phi_s]u_b(v, \pi^2((m_b', m_s'), \phi_b', \phi_s) = \sum_{p}\sum_{\phi_s} \pi_s^1[\phi_s]\pi_s^2(m_s', \phi_s)[p](v-p)\pi_b^2(m_b', \phi_b', p)$$
$$= \sum_{p<v} \gamma(m_s')[p](v-p).$$

Lemma B.7 implies that $\sum_{p<v} \gamma(m_s)[p](v-p) = \sum_{p<v} \gamma(m'_s)[p](v-p)$.

Suppose that $\sum_{p<v} \gamma(\hat{m}_s)[p](v-p) > \sum_{p<v} \gamma(m_s)[p](v-p)$. Then let $\bar{m}_b \in M_b$ be such that $\bar{m}_b \notin \text{supp}(\phi_{i,M_b}(v'))$ for each $i \in N$, $\phi_i \in \text{supp}(\pi_i^1)$ and $v' \in V$. Let $\bar{\phi}_b$ be such that

$$\bar{\phi}_b(v') = \begin{cases} 1_{(\bar{m}_b, \hat{m}_s)} & \text{if } v' = v, \\ \phi_b(v') & \text{otherwise} \end{cases}$$

and $\bar{\pi}_b^2$ be such that

$$\bar{\pi}_b^2(\bar{m}_b, \bar{\phi}_b, p) = \begin{cases} 1 & \text{if } p < v, \\ 0 & \text{if } p > v, \end{cases}$$

and $\bar{\pi}_b^2(m_s, \bar{\phi}_b, p) = \pi_b^2(m_s, \bar{\phi}_b, p)$ for each $m_s \neq \bar{m}_s$ and $p \in V^*$. Letting $\bar{\pi}_b = (\bar{\phi}_b, \bar{\pi}_b^2)$, we have that

$$u_b(\bar{\pi}_b^2, \pi_s) - u_b(\pi) = \beta_b \nu[v] \left(\sum_{p<v} \gamma(\hat{m}_s)[p](v-p) - \sum_{p<v} \gamma(m_s)[p](v-p) \right) > 0.$$

But this is a contradiction since π is a sequential equilibrium. Thus, $\sum_{p<v} \gamma(\hat{m}_s)[p](v-p) \leq \sum_{p<v} \gamma(m_s)[p](v-p)$. ■

B.2 Proof of Theorem 2

B.2.1 Necessity

For each $v \in V$, let $M_s^v = \cup_{\phi_b \in \text{supp}(\pi_b^1)} \text{supp}(\phi_{b,M_s}(v))$ and for each $m_s \in M_s^v$, let $\mu(m_s) \in \Delta(V)$ and $\tau(m_s) \in (0,1]$ be such that:

$$\mu(m_s)[v] = \frac{\sum_{\phi_b, m_b} \nu[v] \pi_b^1[\phi_b] \phi_b(v)[m_b, m_s]}{\sum_{\hat{v}, \hat{\phi}_b, \hat{m}_b} \nu[\hat{v}] \pi_b^1[\hat{\phi}_b] \hat{\phi}_b(\hat{v})[\hat{m}_b, m_s]}$$
$$\tau(m_s) = \sum_{\hat{v}, \hat{\phi}_b, \hat{m}_b} \nu[\hat{v}] \pi_b^1[\hat{\phi}_b] \hat{\phi}_b(\hat{v})[\hat{m}_b, m_s]$$

In words, conditional on the buyer's information structure being chosen, $\mu(m_s)$ is the seller's belief following m_s and $\tau(m_s)$ is the probability that the seller receives m_s. Note that $v \in \text{supp}(\mu(m_s))$ if and only if $m_s \in M_s^v$.

As is standard, the expected posterior belief is equal to the prior.

Lemma B.12 $\sum_{m_s\in M_s^v}\tau(m_s)\mu(m_s)[v]=\nu[v]$ *for each* $v\in V$*.*

Proof. By definition:

$$\begin{aligned}\sum_{m_s\in M_s^v}\tau(m_s)\mu(m_s)[v] &= \sum_{m_s\in M_s^v}\sum_{\hat{v},\hat{\phi}_b,\hat{m}_b}\nu[\hat{v}]\pi_b^1[\hat{\phi}_b]\hat{\phi}_b(\hat{v})[\hat{m}_b,m_s]\frac{\sum_{\phi_b,m_b}\nu[v]\pi_b^1[\phi_b]\phi_b(v)[m_b,m_s]}{\sum_{\hat{v},\hat{\phi}_b,\hat{m}_b}\nu[\hat{v}]\pi_b^1[\hat{\phi}_b]\hat{\phi}_b(\hat{v})[\hat{m}_b,m_s]}\\ &= \sum_{m_s\in M_s^v,\phi_b,m_b}\nu[v]\pi_b^1[\phi_b]\phi_b(v)[m_b,m_s]\\ &= \sum_{\phi_b,m_b,m_s}\nu[v]\pi_b^1[\phi_b]\phi_b(v)[m_b,m_s]=\nu[v].\end{aligned}$$

■

The next lemma shows that seller must get the same payoff from any price in the support of $\gamma(m_s)$, given his belief $\mu(m_s)$.

Lemma B.13 *For each* $m_s\in\cup_v M_s^v$ *and* $p,p'\in\operatorname{supp}(\gamma(m_s))$*:*

$$p\Big(\sum_{v>p}\mu(m_s)[v]+\mu(m_s)[p]\xi\Big)=p'\Big(\sum_{v>p'}\mu(m_s)[v]+\mu(m_s)[p']\xi'\Big) \tag{B.9}$$

for some $\xi,\xi'\in[0,1]$*.*

Proof. By Lemma B.9 and Lemma B.10, we have, for $m_s\in\cup_v M_s^v$ and $p,p'\in\operatorname{supp}(\gamma(m_s))$:

$$\begin{aligned}&p\sum_{v,\phi_b,m_b}\frac{\nu[v]\pi_b^1[\phi_b]\phi_b(v)[m_b,m_s]}{\sum_{\hat{v},\hat{\phi}_b,\hat{m}_b}\nu[\hat{v}]\pi_b^1[\hat{\phi}_b]\hat{\phi}_b(\hat{v})[\hat{m}_b,m_s]}\pi_b^2(m_b,\phi_b,p)\\ &=p'\sum_{v,\phi_b,m_b}\frac{\nu[v]\pi_b^1[\phi_b]\phi_b(v)[m_b,m_s]}{\sum_{\hat{v},\hat{\phi}_b,\hat{m}_b}\nu[\hat{v}]\pi_b^1[\hat{\phi}_b]\hat{\phi}_b(\hat{v})[\hat{m}_b,m_s]}\pi_b^2(m_b,\phi_b,p')\\ &\iff p\sum_{v>p}\frac{\sum_{\phi_b,m_b}\nu[v]\pi_b^1[\phi_b]\phi_b(v)[m_b,m_s]}{\sum_{\hat{v},\hat{\phi}_b,\hat{m}_b}\nu[\hat{v}]\pi_b^1[\hat{\phi}_b]\hat{\phi}_b(\hat{v})[\hat{m}_b,m_s]}+p\frac{\sum_{\phi_b,m_b}\nu[p]\pi_b^1[\phi_b]\phi_b(p)[m_b,m_s]}{\sum_{\hat{v},\hat{\phi}_b,\hat{m}_b}\nu[\hat{v}]\pi_b^1[\hat{\phi}_b]\hat{\phi}_b(\hat{v})[\hat{m}_b,m_s]}\pi_b^2(m_b,\phi_b,p)\\ &=p'\sum_{v>p'}\frac{\sum_{\phi_b,m_b}\nu[v]\pi_b^1[\phi_b]\phi_b(v)[m_b,m_s]}{\sum_{\hat{v},\hat{\phi}_b,\hat{m}_b}\nu[\hat{v}]\pi_b^1[\hat{\phi}_b]\hat{\phi}_b(\hat{v})[\hat{m}_b,m_s]}+p'\frac{\sum_{\phi_b,m_b}\nu[p']\pi_b^1[\phi_b]\phi_b(p')[m_b,m_s]}{\sum_{\hat{v},\hat{\phi}_b,\hat{m}_b}\nu[\hat{v}]\pi_b^1[\hat{\phi}_b]\hat{\phi}_b(\hat{v})[\hat{m}_b,m_s]}\pi_b^2(m_b,\phi_b,p')\\ &\iff p\Big(\sum_{v>p}\mu(m_s)[v]+\mu(m_s)[p]\xi\Big)=p'\Big(\sum_{v>p'}\mu(m_s)[v]+\mu(m_s)[p']\xi'\Big),\end{aligned}$$

for some $\xi,\xi'\in[0,1]$. ■

Let $\xi:V^*\times\cup_v M_s^v\to[0,1]$ be such that (B.9) holds for each $m_s\in\cup_v M_s^v$ and $p,p'\in\operatorname{supp}(\gamma(m_s))$, with $\xi(p,m_s)$ in place of ξ and $\xi(p',m_s)$ in place of ξ'.[32]

[32]Note that when $p\notin V$ or $m_s\notin M_s^p$, $\xi(p,m_s)$ can be defined in an arbitrary way as ξ does not appear in (B.9) in that case.

Then, letting $p_{m_s} \in \text{supp}(\gamma(m_s))$ for each $m_s \in \cup_v M_s^v$, we have

$$u_s^b = \sum_{m_s \in \cup_v M_s^v} \tau(m_s) p_{m_s} \Big(\sum_{v > p_{m_s}} \mu(m_s)[v] + \mu(m_s)[p_{m_s}] \xi(p_{m_s}, m_s) \Big).$$

Moreover, Lemma B.10 implies that $p_{m_s}\Big(\sum_{v > p_{m_s}} \mu(m_s)[v] + \mu(m_s)[p_{m_s}] \xi(p_{m_s}, m_s) \Big) \geq v_1$ for each $m_s \in \cup_v M_s^v$.

Regarding u_b^b, by Lemma B.11, for each $v \in V$, $m_s, m_s' \in M_s^v$ and $\hat{m}_s \in \cup_v M_s^v$, $\sum_{p<v} \gamma(m_s)[p](v-p) = \sum_{p<v} \gamma(m_s')[p](v-p) \geq \sum_{p<v} \gamma(\hat{m}_s)[p](v-p)$. Thus, letting $m_s^v \in M_s^v$ for each v and $h : V \to \cup_v M_s^v$,

$$u_b^b = \sum_v \nu[v] \sum_{p<v} \gamma(m_s^v)[p](v-p) \geq \sum_v \nu[v] \sum_{p<v} \gamma(h(v))[p](v-p).$$

Regarding $u^s = (u_b^s, u_s^s)$, we must have $u_s^s + u_b^s \leq E$ (by feasibility), $u_b^s \geq 0$ (because the buyer can reject every offer greater than v_1), and $u_s^s \geq v_1$ (because the seller can offer v_1 which is accepted with probability 1).

Finally, the next two lemmas establish the necessity of (11) and (12).

Lemma B.14 *$E \geq v$ for each $v \leq \min \cup_{m_s} \text{supp}(\gamma(m_s))$ if $u_b^s = 0$.*

Proof. It follows from $u_b^s = 0$ that $u_s^s \leq E$. Let $p^* \in \text{supp}(\gamma_s)$ solve $\max_{p \in \text{supp}(\gamma_s)} a(p)$. Then Lemma B.5 and $u_b^s = 0$ implies that $u_s^s = \sum_v \nu[v] v \sum_p \gamma_s(v)[p] a(p) \leq a(p^*) E$. Thus, $a(p^*) \geq u_s^s / E$. Lemma B.5 also implies that $u_s^s = p^* a(p^*) \geq p^* u_s^s / E$, hence $p^* \leq E$.

Let $v \in V$ be such that $v \leq \min \cup_{m_s} \text{supp}(\gamma(m_s))$ and suppose that $v > E$. Since type v of the buyer gets a zero payoff due to $v \leq \min \cup_{m_s} \text{supp}(\gamma(m_s))$, the buyer can deviate by sending $\phi_b \in \text{supp}(\pi_b^1)$ and changing $\phi_b(v)$ such that the message of the sender leads to p^* with strictly positive probability, which he then accepts to get a payoff of $v - p^* \geq v - E > 0$. ■

Lemma B.15 *$u_s^s \geq v$ for each $v \leq \min \cup_{m_s} \text{supp}(\gamma(m_s))$ if $u_b^s > 0$.*

Proof. Let $p^* \in \text{supp}(\gamma_s)$ solve $\max_{p \in \text{supp}(\gamma_s)} a(p)$. Then $a(p^*) = 1$. Indeed,

$$u_b^s = \sum_{v, \phi_s, m_s, p} \nu[v] \pi_s^1[\phi_s] \phi_{s,M_s}(v)[m_s] \pi_s^2(m_s, \phi_s)[p] a(p)(v-p).$$

Thus, if $a(p^*) = 0$, then $a(p) = 0$ for each $p \in \text{supp}(\gamma_s)$ and $u_b^s = 0$, a contradiction. Moreover, if $0 < a(p^*) < 1$, then

$$\sum_{v,m_s,\phi_s} \pi_s^1[\phi_s]\nu[v]\phi_{s,M_s}(v)[m_s]\pi_s^2(m_s,\phi_s)[p](v-p) = 0$$

by Lemma B.6 and, hence, $u_b^s = 0$, a contradiction. Thus, $a(p^*) = 1$.

Then Lemma B.5 implies that $u_s^s = p^* a(p^*) = p^*$.

Let $v \in V$ be such that $v \leq \min \cup_{m_s} \text{supp}(\gamma(m_s))$ and suppose that $v > u_s^s$. Since type v of the buyer gets a zero payoff due to $v \leq \min \cup_{m_s} \text{supp}(\gamma(m_s))$, the buyer can deviate by sending $\phi_b \in \text{supp}(\pi_b^1)$ and changing $\phi_b(v)$ such that the message of the sender leads to p^* with strictly positive probability, which he then accepts to get a payoff of $v - p^* = v - u_s^s > 0$. ■

The above discussion implies the necessity direction of Theorem 2.

B.2.2 Sufficiency

The following lemma will be used to construct an equilibrium with the desired payoff.

Lemma B.16 *For each v, there exists $\eta(v) \in \Delta(M_s^v)$ such that:*

$$\frac{\nu[v]\eta(v)[m_s]}{\sum_{v'} \nu[v']\eta(v')[m_s]} = \mu(m_s)[v]$$
$$\sum_{v'} \nu[v']\eta(v')[m_s] = \tau(m_s).$$

Proof. For each $v \in V$ and $m_s \in M_s^v$, define $\eta(v)[m_s] = \frac{\mu(m_s)[v]\tau(m_s)}{\nu[v]}$. Then we have:

$$\sum_{v'} \nu[v']\eta(v')[m_s] = \sum_{v'} \nu[v'] \frac{\mu(m_s)[v']\tau(m_s)}{\nu[v']} = \sum_{v'} \mu(m_s)[v']\tau(m_s) = \tau(m_s)$$

and hence $\frac{\nu[v]\eta(v)[m_s]}{\sum_{v'} \nu[v']\eta(v')[m_s]} = \frac{\nu[v]\eta(v)[m_s]}{\tau(m_s)} = \mu(m_s)[v]$. Finally, by condition (9),

$$\sum_{m_s \in M_s^v} \eta(v)[m_s] = \sum_{m_s \in M_s^v} \frac{\mu(m_s)[v]\tau(m_s)}{\nu[v]} = 1,$$

and so $\eta(v) \in \Delta(M_s^v)$ as required. ■

Let M_s^v, $(\gamma(m_s))_{(m_s \in \cup_v M_s^v)}$, $(\mu(m_s))_{m_s \in \cup_v M_s^v}$, $\tau \in \Delta(\cup_v M_s^v)$, and ξ be as in the statement of the theorem, and let u be such that conditions (5)–(9) hold. Define $M_s^b =$

$\cup_v M_s^v$. We will show that $u \in \bar{U}^*(\beta_b, \beta_s)$ by constructing a sequential equilibrium $\pi \in \bar{\Pi}^*$ with payoff u.

First period strategy. Let $Y \in \mathbb{N}$ be such that

(i) $Y \geq \sum_v |M_s^v|$,

(ii) $(\sum_v |M_s^v|)Y^{-1} v_K < v_1$ and

(iii) $Y^{-1}(v_K - u_s^s) < \sum_{p<v} \gamma(m_s^v)[p](v-p)$ for each $v > \min \cup_{m_s \in M_s^b} \text{supp}(\gamma(m_s))$, where $m_s^v \in M_s^v$.[33]

For the seller: For each $y \in \{1, \ldots, Y\}$, let $\phi_{s,y}$ be such that $\phi_{s,y}(v) = 1_{(Y+1,y)}$ for all $v \in V$ (i.e. the seller sends message y to himself and $Y+1$ to the buyer). Let $\Phi_s = \{\phi_{s,y} \in \Phi : y \in \{1, \ldots, Y\}\}$. Let $\pi_s^1 = Y^{-1} \sum_{y=1}^{Y} \phi_{s,y}$.

For the buyer: The buyer will send messages in $M_s^b = \cup_v M_s^v$ to the seller, with type v sending messages in M_s^v; assume without loss of generality that $M_s^b \cap \{1, \ldots, Y\} = \emptyset$. For each v, define $M_b^v \subseteq M_b$ such that for $v \neq v'$, $M_b^v \cap M_b^{v'} = \emptyset$, $|M_b^v| = |M_s^v|$, and assume that $\cup_v M_b^v \subseteq \{1, \ldots, Y\}$.[34] Let $J^v = |M_b^v|$ and, for each $v \in V$, enumerate $M_b^v = \{m_{b,1}^v, \ldots, m_{b,J^v}^v\}$ and $M_s^v = \{m_{s,1}^v, \ldots, m_{s,J^v}^v\}$ so that $m_{b,j}^v$ corresponds to $m_{s,j}^v$.

Let Ψ be the set of all bijections $\psi : \{1, \ldots, Y\} \to \{1, \ldots, Y\}$. Let $\phi_{b,\psi} : V \to F$ be such that $\phi_{b,\psi}(v) = \sum_{j=1}^{J^v} \eta(v)[m_{s,j}^v] 1_{(\psi(m_{b,j}^v), m_{s,j}^v)}$ for each $v \in V$ and let $\pi_b^1 = |\Psi|^{-1} \sum_\psi \phi_{b,\psi}$. Let $\Phi_b = \{\phi_{b,\psi} \in \Phi : \psi \in \Psi\}$.

Second period strategy. For each $(m_s, \phi_s) \in M_s \times \Phi_s$, let:

$$\pi_s^2(m_s, \phi_s) = \begin{cases} \frac{u_b^s}{E-u_s^s} 1_{u_s^s} + (1 - \frac{u_b^s}{E-u_s^s}) 1_E & \text{if } \phi_s = \phi_{s,y} \text{ and } m_s = y, \\ \gamma(m_s) & \text{if } m_s \in M_s^b, \\ 1_{v_K} & \text{otherwise.} \end{cases} \tag{B.10}$$

For convenience, define $\delta = \frac{u_b^s}{E-u_s^s}$; when $u_s^s = E$, let $\delta = 1$.

[33] Note that (7) implies that if (iii) holds for some $m_s^v \in M_s^v$, then it holds for all $m_s^v \in M_s^v$. Also, when $v > \min \cup_{m_s \in M_s^b} \text{supp}(\gamma(m_s))$, then for m_s such that $v > \min \text{supp}(\gamma(m_s))$, $\sum_{p<v} \gamma(m_s^v)[p](v - p) \geq \sum_{p<v} \gamma(m_s)[p](v-p) > 0$, where the weak inequality follows from (7); thus (iii) is satisfied for sufficiently large Y.

[34] Note that $|\cup_v M_b^v| = \sum_v |M_s^v| \leq Y$ by condition (i) in the definition of Y.

For each $m_s \in M_s^b$, define $P_{m_s} = \text{supp}(\gamma(m_s))$ and for each $(m_b, \phi_b, p) \in M_b \times \Phi_b \times V^*$, let:

$$\pi_b^2(m_b, \phi_b, p) = \begin{cases} 1 & \text{if } p = v_1 \\ 1 & \text{if } \phi_b = \phi_{b,\psi}, \psi^{-1}(m_b) = m_{b,j}^v, p \in P_{m_{s,j}^v} \text{ and } p < v \\ \xi(v, m_{s,j}^v) & \text{if } \phi_b = \phi_{b,\psi}, \psi^{-1}(m_b) = m_{b,j}^v, p \in P_{m_{s,j}^v} \text{ and } p = v \\ 1 & \text{if } m_b = Y + 1 \text{ and } p = u_s^s \text{ [only for case } \delta > 0] \\ u_s^s/E & \text{if } m_b = Y + 1 \text{ and } p = E \text{ [only for case } \delta < 1] \\ 0 & \text{otherwise.} \end{cases} \tag{B.11}$$

For convenience, define $P_b = \cup_{m_s \in M_s^b} P_{m_s}$.

For each $\phi_s \notin \Phi_s$ and $m_s \notin M_s^b$ such that $\sum_v \nu[v]\phi_s(v)_{M_s}[m_s] = 0$, let $\pi_s^2(m_s, \phi_s) = 1_{v_K}$. For each $\phi_s \notin \Phi_s$ and $m_s \in M_s$ such that $\sum_v \nu[v]\phi_s(v)_{M_s}[m_s] > 0$ or $m_s \in M_s^b$, let $\pi_s^2(m_s, \phi_s) = 1_p$ for some p that solves:

$$\max_{p \in P^*} p|\Psi|^{-1} \sum_{\psi} \sum_{m_b} \frac{\sum_v \nu[v](\beta_b \phi_{b,\psi}(v) + \beta_s \phi_s(v))[m_b, m_s]}{\sum_v \nu[v](\beta_b \phi_{b,\psi}(v) + \beta_s \phi_s(v))_{M_s}[m_s]} \pi_b^2(m_b, \phi_{b,\psi}, p), \tag{B.12}$$

where $P^* = P_b \cup \{v_1, u_s^s, E\}$. Note that since $\pi_b^2(m_b, \phi_{b,\psi}, p) = 0$ for all $p \notin P^*$, $\pi_s^2(m_s, \phi_s)$ also maximizes the above expression over $\Delta(V^*)$.

We may assume that $\pi_s^2 : M_s \times \Phi \to \Delta(V^*)$ is measurable. Note first that $M_s \times \Phi = \cup_{r=1}^3 B_r$ with

$$B_1 = M_s \times \Phi_s,$$

$$B_2 = \{(m_s, \phi_s) \in (M_s \setminus M_s^b) \times \Phi \setminus \Phi_s : \sum_v \nu[v]\phi_s(v)_{M_s}[m_s] = 0\},$$

$$B_3 = \{(m_s, \phi_s) \in (M_s \setminus M_s^b) \times \Phi \setminus \Phi_s : \sum_v \nu[v]\phi_s(v)_{M_s}[m_s] > 0\} \cup M_s^b \times \Phi \setminus \Phi_s).$$

Indeed, B_1 is closed, B_3 is open and B_2 is the intersection of the closed set $\{(m_s, \phi_s) \in M_s \times \Phi : \sum_v \nu[v]\phi_s(v)_{M_s}[m_s] = 0\}$ with the open set $(M_s \setminus M_s^b) \times (\Phi \setminus \Phi_s)$. Then, for each measurable $B \subseteq \Delta(V^*)$, note that $(\pi_s^2)^{-1}(B) \cap B_1 = C_1 \cup C_2 \cup C_3$, where:

$$C_1 = \begin{cases} M_s^b \times \Phi_s & \text{if } B \cap P_b \neq \emptyset, \\ \emptyset & \text{otherwise} \end{cases}$$

$$C_2 = \begin{cases} \{(y, \phi_{s,y}) : y \in \{1, \dots, Y\}\} & \text{if } B \cap \{u_s^s, E\} \neq \emptyset, \\ \emptyset & \text{otherwise} \end{cases}$$

$$C_3 = \begin{cases} \{(m_s, \phi_{s,y}) : y \in \{1, \dots, Y\} \text{ and } m_s \neq y\} & \text{if } B \cap \{v_K\} \neq \emptyset, \\ \emptyset & \text{otherwise} \end{cases}$$

Thus, $(\pi_s^2)^{-1}(B) \cap B_1$ is the union of measurable sets, and hence, measurable. For each measurable $B \subseteq \Delta(V^*)$, $(\pi_s^2)^{-1}(B) \cap B_2 = B_2$ if $v_K \in B$ and $(\pi_s^2)^{-1}(B) \cap B_2 = \emptyset$ otherwise; hence $(\pi_s^2)^{-1}(B) \cap B_2$ is measurable. Finally, regarding $(\pi_s^2)^{-1}(B) \cap B_3$, for each $(m_s, \phi_s) \in B_3$, let $f : B_3 \times P^* \to [0, 1]$ be defined by setting, for each $(m_s, \phi_s) \in B_3$ and $p \in P^*$,

$$f(m_s, \phi_s, p) = |\Psi|^{-1} \sum_{\psi} \sum_{m_b} \frac{\sum_v \nu[v](\beta_b \phi_{b,\psi}(v) + \beta_s \phi_s(v))[m_b, m_s]}{\sum_v \nu[v](\beta_b \phi_{b,\psi}(v) + \beta_s \phi_s(v))_{M_s}[m_s]} \pi_b^2(m_b, \phi_{b,\psi}, p),$$

and let $\chi : B_3 \rightrightarrows P^*$ be defined by $\chi(m_s, \phi_s) = \arg\max_{p \in P^*} p f(m_s, \phi_s, p)$. Note that for each $p \in P^*$, $\chi^l(\{p\}) = \{(m_s, \phi_s) \in B_3 : p f(m_s, \phi_s, p) \geq p' f(m_s, \phi_s, p')$ for all $p' \in P^*\}$ is closed in B_3, and hence measurable. Thus, χ is weakly measurable and has a measurable selection by the Kuratowski-Ryll-Nardzewski Selection Theorem (e.g. Aliprantis and Border (2006, Theorem 18.13, p. 600)).

For each $\phi_b \notin \Phi_b$ and $(m_b, p) \in M_b \times V^*$ such that $Y^{-1} \sum_{y=1}^{Y} \sum_{v, m_s} \nu[v](\beta_b \phi_b(v) + \beta_s \phi_{s,y}(v))[m_b, m_s] \pi_s^2(m_s, \phi_{s,y})[p] > 0$, let $\pi_b^2(m_b, \phi_b, p) = 1$ if

$$\frac{Y^{-1} \sum_{y=1}^{Y} \sum_{v, m_s} \nu[v](\beta_b \phi_b(v) + \beta_s \phi_{s,y}(v))[m_b, m_s] \pi_s^2(m_s, \phi_{s,y})[p] v}{Y^{-1} \sum_{y=1}^{Y} \sum_{v, m_s} \nu[v](\beta_b \phi_b(v) + \beta_s \phi_{s,y}(v))[m_b, m_s] \pi_s^2(m_s, \phi_{s,y})[p]} \geq p \tag{B.13}$$

and $\pi_b^2(m_b, \phi_b, p) = 0$ otherwise.

For each $\phi_b \notin \Phi_b$ and $(m_b, p) \in M_b \times V^*$ such that $Y^{-1} \sum_{y=1}^{Y} \sum_{v, m_s} \nu[v](\beta_b \phi_b(v) + \beta_s \phi_{s,y}(v))[m_b, m_s] \pi_s^2(m_s, \phi_{s,y})[p] = 0$, we will define $\pi_b^2(m_b, \phi_b, p)$ after the following net $\{\pi^\alpha, p^\alpha\}_\alpha$ has been defined, where,

1. for each α, $p^\alpha : \Phi^2 \to \Delta(V \times M)$ is measurable and π^α is a strategy.

Defining perturbations. Consider $\{\pi^\alpha, p^\alpha\}_\alpha$ defined as follows: The index set consists of $(k, F, \hat{F}, \tilde{F})$ such that $k \in \mathbb{N}$, F is a finite subset of $\mathbb{N}$, $\hat{F}$ is a finite subset of Φ and $\tilde{F}$ is a finite subset of V^*; this set is partially ordered by defining $(k', F', \hat{F}', \tilde{F}') \geq (k, F, \hat{F}, \tilde{F})$ if $k' \geq k$, $F \subseteq F'$, $\hat{F} \subseteq \hat{F}'$ and $\tilde{F} \subseteq \tilde{F}'$. If X is a finite set, let $\mathcal{U}_X \in \Delta(X)$ be uniform on X. For each $(F, \hat{F}, \tilde{F})$, define:

$$\Phi(F, \hat{F}) = \{\phi \in \hat{F} : \text{supp}(\phi) \subseteq F^2\} \text{ and}$$
$$P(F, \hat{F}, \tilde{F}) = \tilde{F} \cup \{u_s^s, E, v_K\} \cup P_b \cup (\cup_{m_s \in F \cup M_s^b, \phi_s \in \Phi(F, \hat{F})} \text{supp}(\pi_s^2(m_s, \phi_s))).$$

For each $m_b \in M_b$, let $\phi_s^{m_b}$ be such that $\phi_s^{m_b}(v_1) = 1_{(m_b, 1)}$ and $\phi_s^{m_b}(v_k) = 1_{(Y+1, k)}$ for $v_k \neq v_1$ (i.e. $\phi_s^{m_b}$ sends the seller message k when the value is v_k and sends the buyer message m_b only if the value is v_1; otherwise it sends the buyer message $Y + 1$). Let $\pi_s^{m_b, \alpha}$ be such that $\pi_s^{m_b,1,\alpha} = \phi_s^{m_b}$ and $\pi_s^{m_b,2,\alpha}$ be such that $\pi_s^{m_b,2,\alpha}(1, \phi_s^{m_b}) = \mathcal{U}_{P(F,\hat{F},\tilde{F})}$ and $\pi_s^{m_b,2,\alpha}(m_s, \phi_s) = 1_{v_1}$ for all $(m_s, \phi_s) \neq (1, \phi_s^{m_b})$. Let $\hat{\pi}_s^\alpha$ be such that $\hat{\pi}_s^{1,\alpha} = \mathcal{U}_{\Phi(F,\hat{F})}$ and $\hat{\pi}_s^{2,\alpha}(m_s, \phi_s) = \mathcal{U}_{P(F,\hat{F},\tilde{F})}$ for all m_s, ϕ_s. Let, for each $t = 1, 2$,

$$\pi_s^{t,\alpha} = (1 - j^{-1})\pi_s^t + j^{-1}(1 - j^{-j})|F|^{-1} \sum_{m_b \in F} \pi_s^{m_b,t,\alpha} + j^{-1} j^{-j} \hat{\pi}_s^{t,\alpha},$$

where $j = \max\{k, |F|, |\hat{F}|, |\tilde{F}|\}$.

For each $m_s \in M_s$, let $\phi_b^{m_s}$ be such that $\phi_b^{m_s}(v_K) = 1_{(K, m_s)}$ and $\phi_b^{m_s}(v_k) = 1_{(k, Y+1)}$ for $v_k \neq v_K$ (i.e. $\phi_b^{m_s}$ sends the buyer message k when the value is v_k and sends the seller message m_s only if the value is v_K; otherwise it sends the seller message $Y + 1$). Let $\pi_b^{m_s,1,\alpha} = \phi_b^{m_s}$. Let $\hat{\pi}_b^{1,\alpha} = \mathcal{U}_{\Phi(F,\hat{F})}$. Let:

$$\pi_b^{1,\alpha} = (1 - j^{-1})\pi_b^1 + j^{-1}(1 - j^{-j})|F|^{-1} \sum_{m_s \in F} \pi_b^{m_s,1,\alpha} + j^{-1} j^{-j} \hat{\pi}_b^{1,\alpha}.$$

Let

$$p^\alpha(\phi)[v, m] = (1 - j^{-j})\nu[v](\beta_s \phi_s(v) + \beta_b \phi_b(v))[m] + j^{-j} \mathcal{U}_{V \times F^2}[v, m].$$

Note that, for each α and $(\phi_b, \phi_s, m_s) \in \Phi^2 \times M_s$, $\text{supp}(p^\alpha(\phi_b, \phi_s))$, $\text{supp}(\pi_b^{1,\alpha})$, $\text{supp}(\pi_s^{1,\alpha})$, and $\text{supp}(\pi_s^{2,\alpha}(m_s, \phi_s)))$ are finite.

For each $\phi_b \notin \Phi_b$ and $(m_b, p) \in M_b \times V^*$ such that $Y^{-1} \sum_{y=1}^{Y} \sum_{v,m_s} \nu[v](\beta_b\phi_b(v) + \beta_s\phi_{s,y}(v))[m_b, m_s]\pi_s^2(m_s, \phi_{s,y})[p] = 0$, let $\pi_b^2(m_b, \phi_b, p) = 1$ if

$$\lim_{\alpha} \frac{\int_\Phi \sum_{(v,m_s)} p^\alpha(\phi_b, \phi_s)[v, m_b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s)[p]v\mathrm{d}\pi_s^{1,\alpha}[\phi_s]}{\int_\Phi \sum_{(v,m_s)} p^\alpha(\phi_b, \phi_s)[v, m_b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s)[p]\mathrm{d}\pi_s^{1,\alpha}[\phi_s]} \geq p \tag{B.14}$$

and $\pi_b^2(m_b, \phi_b, p) = 0$ otherwise.

Finally, let $\hat{\pi}_b^{2,\alpha}(m_b, \phi_b, p) = \mathcal{U}_A$ and $\pi_b^{2,\alpha}(m_b, \phi_b, p) = (1 - j^{-1})\pi_b^2(m_b, \phi_b, p) + j^{-1}\hat{\pi}_b^{2,\alpha}(m_b, \phi_b, p)$ for each (m_b, ϕ_b, p). Let $\hat{P} = \{u_s^s, E, v_K\} \cup P_b$ and note that $\cup_{y,m_s}\mathrm{supp}(\pi_s^2(m_s, \phi_{s,y})) \subseteq \hat{P}$. Thus, $\{(\phi_b, m_b, p) \in (\Phi \setminus \Phi_b) \times M_b \times V^* :$ $Y^{-1} \sum_{y=1}^{Y} \sum_{v,m_s} \nu[v](\beta_b\phi_b(v) + \beta_s\phi_{s,y}(v))[m_b, m_s]\pi_s^2(m_s, \phi_{s,y})[p] > 0$ and (B.13) holds$\}$ is measurable since it equals

$$\cup_{m_b \in M_b, p \in \hat{P}} \left(\{\phi_b : \Phi \setminus \Phi_b : Y^{-1} \sum_{y=1}^{Y} \sum_{v,m_s} \nu[v](\beta_b\phi_b(v) + \beta_s\phi_{s,y}(v))[m_b, m_s]\pi_s^2(m_s, \phi_{s,y})[p] > 0\}\right.$$
$$\left.\cap \{\phi_b \in \Phi : \text{(B.13) holds}\}\right) \times \{(m_b, p)\},$$

$$\{\phi_b : \Phi \setminus \Phi_b : Y^{-1} \sum_{y=1}^{Y} \sum_{v,m_s} \nu[v](\beta_b\phi_b(v) + \beta_s\phi_{s,y}(v))[m_b, m_s]\pi_s^2(m_s, \phi_{s,y})[p] > 0\}$$

is open and

$$\{\phi_b \in \Phi : \text{(B.13) holds}\}$$

is closed. The set $\{(\phi_b, m_b, p) \in (\Phi \setminus \Phi_b) \times M_b \times V^* : Y^{-1} \sum_{y=1}^{Y} \sum_{v,m_s} \nu[v](\beta_b\phi_b(v) + \beta_s\phi_{s,y}(v))[m_b, m_s]\pi_s^2(m_s, \phi_{s,y})[p] = 0$ and (B.14) holds$\}$ is also measurable since it equals the intersection of the complement of $\{(\phi_b, m_b, p) \in (\Phi \setminus \Phi_b) \times M_b \times V^* :$ $Y^{-1} \sum_{y=1}^{Y} \sum_{v,m_s} \nu[v](\beta_b\phi_b(v) + \beta_s\phi_{s,y}(v))[m_b, m_s]\pi_s^2(m_s, \phi_{s,y})[p] > 0\}$, the latter being equal to

$$\cup_{m_b \in M_b, p \in \hat{P}} \{\phi_b : \Phi \setminus \Phi_b : Y^{-1} \sum_{y=1}^{Y} \sum_{v,m_s} \nu[v](\beta_b\phi_b(v) + \beta_s\phi_{s,y}(v))[m_b, m_s]\pi_s^2(m_s, \phi_{s,y})[p] > 0\}$$
$$\times \{(m_b, p)\},$$

and the closed set

$$\cup_{m_b \in M_b, p \in \hat{P}}\{\phi_b \in \Phi : \text{(B.14) holds}\} \times \{(m_b, p)\}.$$

The above argument regarding the openness and closedness of certain sets uses (some of) the following conditions, which clearly hold:

2. For each $i \in N$, $\sup_{B \in \mathcal{B}(\Phi)} |\pi_i^{1,\alpha}[B] - \pi_i^1[B]| \to 0$,

$$\sup_{(m,\phi) \in M_s \times \Phi, B \in \mathcal{B}(V^*)} |\pi_s^{2,\alpha}(m,\phi)[B] - \pi_s^2(m,\phi)[B]| \to 0, \text{ and}$$

$$\sup_{(m,\phi,p) \in M_b \times \Phi \times V^*, a \in A} |\pi_b^{2,\alpha}(m,\phi,p)[a] - \pi_b^2(m,\phi,p)[a]| \to 0,$$

3. For each $i \in N$, $m \in \mathbb{N}$, $\phi \in \Phi$, $p \in V^*$ and $a \in A$, there is $\bar{\alpha}$ such that $\pi_i^{1,\alpha}[\{\phi\}] > 0$, $\pi_s^{2,\alpha}(m,\phi)[\{p\}] > 0$ and $\pi_b^{2,\alpha}(m,\phi,p)[a] > 0$ for each $\alpha \geq \bar{\alpha}$,

4. $\sup_{\phi \in \Phi^2, v \in V, B \subseteq M} |p^\alpha(\phi)[\{v\} \times B] - \nu[v] \sum_{i \in N} \beta_i \phi_i(v)[B]| \to 0$,

5. For each $\phi \in \Phi^2$, $v \in V$ and $m \in M$, there is $\bar{\alpha}$ such that $p^\alpha(\phi)[v,m] > 0$ for each $\alpha \geq \bar{\alpha}$.

Note that if $\phi_b \in \operatorname{supp}(\pi_b^{1,\alpha})$ and $\phi_s \in \operatorname{supp}(\pi_s^{1,\alpha})$, then

$$\phi_b \in \Phi_b^\alpha := \Phi_b \cup \{\phi_b^{m_s} : m_s \in F\} \cup \Phi(F, \hat{F}) \text{ and}$$
$$\phi_s \in \Phi_s^\alpha := \Phi_s \cup \{\phi_s^{m_b} : m_b \in F\} \cup \Phi(F, \hat{F}).$$

Thus, to show that π is a sequential equilibrium, it suffices to show that the following conditions hold for each $\varepsilon > 0$ and α:

6.(a) For each $i \in N$ and $\phi'_i \in \Phi$,

$$\int_{\Phi^2} \sum_{(v,m) \in \operatorname{supp}(p^\alpha(\phi))} p^\alpha(\phi)[v,m] u_i(v, \pi^{2,\alpha}(m,\phi)) \mathrm{d}\pi^{1,\alpha}[\phi] \geq$$
$$\int_{\Phi} \sum_{(v,m) \in \operatorname{supp}(p^\alpha(\phi'_i,\phi_j))} p^\alpha(\phi'_i,\phi_j)[v,m] u_i(v, \pi^{2,\alpha}(m,\phi'_i,\phi_j)) \mathrm{d}\pi_j^{1,\alpha}[\phi_j] - \varepsilon,$$

where $\pi^{1,\alpha} = \prod_{i \in N} \pi_i^{1,\alpha}$, $j \neq i$ and, for each $\phi \in \Phi^2$ and $m \in M$, $\pi^{2,\alpha}(m,\phi) \in \Delta(V^* \times A)$ is defined by setting, for each $(p,a) \in V^* \times A$, $\pi^{2,\alpha}(m,\phi)[p,a] = \pi_s^{2,\alpha}(m_s,\phi_s)[p] \pi_b^{2,\alpha}(m_b,\phi_b,p)[a]$,

6.(b) For each $(m_s,\phi_s) \in M_s \times \Phi_s^\alpha$ such that $\sum_{\phi_b \in \operatorname{supp}(\pi_b^{1,\alpha})} p^\alpha(\phi_b,\phi_s)_{M_s}[m_s] \pi_b^{1,\alpha}[\phi_b] > 0$ and $p \in V^*$,

$$\frac{\sum_{\phi_b \in \operatorname{supp}(\pi_b^{1,\alpha})} \pi_b^{1,\alpha}[\phi_b] \sum_{(v,m_b)} p^\alpha(\phi_b,\phi_s)[v,m] u_s(\pi^{2,\alpha}(m,\phi))}{\sum_{\phi_b \in \operatorname{supp}(\pi_b^{1,\alpha})} \pi_b^{1,\alpha}[\phi_b] p^\alpha(\phi_b,\phi_s)_{M_s}[m_s]} \geq$$
$$\frac{\sum_{\phi_b \in \operatorname{supp}(\pi_b^{1,\alpha})} \pi_b^{1,\alpha}[\phi_b] \sum_{(v,m_b)} p^\alpha(\phi_b,\phi_s)[v,m] u_s(p, \pi_b^{2,\alpha}(m_b,\phi_b,p))}{\sum_{\phi_b \in \operatorname{supp}(\pi_b^{1,\alpha})} \pi_b^{1,\alpha}[\phi_b] p^\alpha(\phi_b,\phi_s)_{M_s}[m_s]} - \varepsilon.$$

6.(c) For each $(m_b, \phi_b, p) \in M_b \times \Phi_b^\alpha \times V^*$ such that

$$\int_\Phi \sum_{(v,m_s)\in \text{supp}(p^\alpha(\phi_b,\phi_s))} p^\alpha(\phi_s, \phi_b)[v, m_b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s)[p]\mathrm{d}\pi_s^{1,\alpha}[\phi_s] > 0$$

and $a \in A$,

$$\frac{\int_\Phi \sum_{(v,m_s)} p^\alpha(\phi_b, \phi_s)[v, m_b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s)[p]u_b(v, p, \pi_b^{2,\alpha}(m_b, \phi_b, p))\mathrm{d}\pi_s^{1,\alpha}[\phi_s]}{\int_\Phi \sum_{(v,m_s)} p^\alpha(\phi_s, \phi_b)[v, m_b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s)[p]\mathrm{d}\pi_s^{1,\alpha}[\phi_s]} \geq$$
$$\frac{\int_\Phi \sum_{(v,m_s)} p^\alpha(\phi_b, \phi_s)[v, m_b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s)[p]u_b(v, p, a)\mathrm{d}\pi_s^{1,\alpha}[\phi_s]}{\int_\Phi \sum_{(v,m_s)} p^\alpha(\phi_s, \phi_b)[v, m_b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s)[p]\mathrm{d}\pi_s^{1,\alpha}[\phi_s]} - \varepsilon.$$

Note that if (m_s, ϕ_s) is such that $\sum_{\phi_b \in \text{supp}(\pi_b^{1,\alpha})} p^\alpha(\phi_b, \phi_s)_{M_s}[m_s]\pi_b^{1,\alpha}[\phi_b] > 0$, then $m_s \in \cup_v \text{supp}(\phi_s(v)_{M_s}) \cup M_s^b \cup F$, and if (m_b, ϕ_b, p) is such that

$$\int_\Phi \sum_{(v,m_s)\in \text{supp}(p^\alpha(\phi_b,\phi_s))} p^\alpha(\phi_s, \phi_b)[v, m_b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s)[p]\mathrm{d}\pi_s^{1,\alpha}[\phi_s] > 0,$$

then $m_b \in \cup_v \text{supp}(\phi_b(v)_{M_b}) \cup \{Y + 1\} \cup F$ and $p \in P(F, \hat{F}, \tilde{F})$.

Proof that the conditions of sequential equilibrium are satisfied. Let $\varepsilon > 0$. We will show that these conditions holds for some subnet of $\{\pi^\alpha, p^\alpha\}_\alpha$. In particular, for each $(F, \hat{F}, \tilde{F})$, we will show that there exists a $k(F, \hat{F}, \tilde{F})$ such that for each $\alpha = (k, F, \hat{F}, \tilde{F})$ with $k \geq k(F, \hat{F}, \tilde{F})$, condition 6 is satisfied.

Consider condition 6.(a) with $i = s$. The left-hand side converges to $u_s = \beta_s u_s^s + \beta_b u_s^b$. When $\varepsilon = 0$, the right-hand side of the inequality, for any $\phi_s' \in \Phi$, is at most (where for each m_s, $p'(m_s) \in \text{supp}(\pi_s^2(m_s, \phi_s'))$:

$$\begin{aligned}
&(1 - j^{-1})^3(1 - j^{-j})\Big(\beta_s|\Psi|^{-1} \sum_\psi \sum_{v,m} \nu[v]\phi_s'(v)[m_b, m_s]p'(m_s)\pi_b^2(m_b, \phi_{b,\psi}, p'(m_s)) + \beta_b u_s^b\Big) \\
&+(1 - (1 - j^{-1})^3(1 - j^{-j}))v_K \\
&\leq (1 - j^{-1})^3(1 - j^{-j}) \left(\beta_s \sum_{v,m} \nu[v]\phi_s'(v)[m_b, m_s]u_s^s + \beta_b u_s^b\right) \\
&+(1 - (1 - j^{-1})^3(1 - j^{-j}))v_K,
\end{aligned}$$

since v_K is the maximum payoff for the seller and (i) for $m_b \neq Y+1$: if $p = v_1$, then for each ψ, $v_1\pi_b^2(m_b, \phi_{b,\psi}, v_1) = v_1 \leq u_s^s$ and if $p > v_1$, then $|\Psi|^{-1} \sum_\psi p\pi_b^2(m_b, \phi_{b,\psi}, p) \leq$

$|\cup_v M_b^v|Y^{-1}p < v_1 \le u_s^s$;[35] and (ii) for $m_b = Y+1$, for each ψ, $E\pi_b^2(Y+1, \phi_{b,\psi}, E) = E(u_s^s/E) = u_s^s$, $u_s^s\pi_b^2(Y+1, \phi_{b,\psi}, u_s^s) = u_s^s$, and $p\pi_b^2(Y+1, \phi_{b,\psi}, p) = 0$ for all $p \notin \{u_s^s, E\}$.[36] Thus, the inequality holds (uniformly across $\phi_i' \in \Phi$) for each α such that k (and hence j) is sufficiently large.

Consider next condition 6.(a) with $i = b$. The left-hand side converges to $u_b = \beta_b u_b^b + \beta_s u_b^s$. Let $v \mapsto m_s^v$ be such that $m_s^v \in M_s^v$ for each $v \in V$. When $\varepsilon = 0$, the right-hand side of the inequality, for any $\phi_b' \in \Phi$, is at most:

$$
\begin{aligned}
&(1-j^{-1})^3(1-j^{-j})\Big(\beta_s u_b^s + \\
&\beta_b \sum_{v,m} \nu[v]\phi_b'(v)[m_b, m_s] Y^{-1} \sum_{y=1}^{Y} \int_{V^*} (v-p)\pi_b^2(m_b, \phi_b', p)\mathrm{d}\pi_s^2(m_s, \phi_{s,y})[p]\Big) \\
&+(1-(1-j^{-1})^3(1-j^{-j}))v_K \\
&\le (1-j^{-1})^3(1-j^{-j})\left(\beta_s u_b^s + \beta_b \sum_{v,m} \nu[v]\phi_b'(v)[m_b, m_s] \sum_{p<v} \gamma(m_s^v)[p](v-p)\right) \\
&+(1-(1-j^{-1})^3(1-j^{-j}))v_K \\
&\le (1-j^{-1})^3(1-j^{-j})\left(\beta_s u_b^s + \beta_b \sum_{v} \nu[v] \sum_{p<v} \gamma(m_s^v)[p](v-p)\right) \\
&+(1-(1-j^{-1})^3(1-j^{-j}))v_K
\end{aligned}
$$

since v_K is an upper bound on the buyer's payoff and for each $(v, m_b, m_s) \in V \times M$, $Y^{-1}\sum_{y=1}^{Y}\int_{V^*}(v-p)\pi_b^2(m_b, \phi_b', p)\mathrm{d}\pi_s^2(m_s, \phi_{s,y})[p] \le \sum_{p<v}\gamma(m_s^v)[p](v-p)$. The latter follows because for each $m_s \notin M_s^b$, $\pi_s^2(m_s, \phi_{s,y}) = 1_{v_K}$ for each $y \ne m_s$ and $\pi_s^2(m_s, \phi_{s,y}) = \delta 1_{u_s^s} + (1-\delta)1_E$ for $y = m_s$ by (B.10), and thus

(i) for $v \le \min P_b$, $Y^{-1}\sum_{y=1}^{Y}\int_{V^*}(v-p)\pi_b^2(m_b, \phi_b', p)\mathrm{d}\pi_s^2(m_s, \phi_{s,y})[p] \le (1-Y^{-1})(v-v_K)\pi_b^2(m_b, \phi_b', v_K) + Y^{-1}(\delta(v-u_s^s)\pi_b^2(m_b, \phi_b', u_s^s) + (1-\delta)(v-E)\pi_b^2(m_b, \phi_b', E)) \le 0$ since $E \ge u_s^s \ge v$ by (6) and (11),[37]

[35] Since for each m_b, $|\Psi|^{-1}|\{\psi : m_b \in \psi(\cup_v M_b^v)\}| \le |\cup_v M_b^v|Y^{-1}$, $\pi_b^2(m_b, \phi_{b,\psi}, p) = 0$ for $p > v_1$ and m_b such that $m_b \notin \psi(\cup_v M_b^v)$ by (B.11), and $|\cup_v M_b^v|Y^{-1}p \le v_1$ by condition (ii) in the definition of Y.

[36] If $\delta = 1$, then $E\pi_b^2(Y+1, \phi_{b,\psi}, E) = 0$ and if $\delta = 0$, then $u_s^s\pi_b^2(Y+1, \phi_{b,\psi}, u_s^s) = 0$.

[37] When $\delta = 0$, i.e. when $u_b^s = 0$, note that $v \le E$ is sufficient for the conclusion (i.e. there is no need to require $v \le u_s^s$), and this weaker condition follows from (12).

(ii) for $v > \min P_b$, $Y^{-1}\sum_{y=1}^{Y}\int_{V^*}(v-p)\pi_b^2(m_b,\phi_b',p)\mathrm{d}\pi_s^2(m_s,\phi_{s,y})[p] \leq (1-Y^{-1})(v-v_K)\pi_b^2(m_b,\phi_b',v_K) + Y^{-1}(v_K - u_s^s) < \sum_{p<v}\gamma(m_s^v)[p](v-p)$ by condition (iii) in the definition of Y,

and for $m_s \in M_s^b$, $\pi_s^2(m_s,\phi_{s,y}) = \gamma(m_s)$ for each y, and $\sum_p \gamma(m_s)[p](v-p)\pi_b^2(m_b,\phi_b',p) \leq \sum_{p<v}\gamma(m_s)[p](v-p) \leq \sum_{p<v}\gamma(m_s^v)[p](v-p)$ by (7). Thus, the inequality holds (uniformly across $\phi_i' \in \Phi$) for each α such that k (and hence j) is sufficiently large.

Let k_a be such that condition 6.(a) holds for each α such that $k \geq k_a$.

Consider next condition 6.(b). We establish it by considering several cases.

Case 1: $\phi_s = \phi_{s,y}$ and $m_s = y$. In the limit and when $\varepsilon = 0$, the inequality is $u_s^s \geq p|\Psi|^{-1}\sum_\psi \pi_b^2(0,\phi_{b,\psi},p)$. It holds since $u_s^s \geq v_1$ (because $u_s^s \in V^*$) and by (B.11):

$$p|\Psi|^{-1}\sum_{\psi}\pi_b^2(0,\phi_{b,\psi},p) = \begin{cases} u_s^s & \text{if } p = u_s^s \text{ [and } \delta > 0], \\ u_s^s & \text{if } p = E \text{ [and } \delta < 1], \\ v_1 & \text{if } p = v_1 \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the inequality holds in the limit. By similar arguments as for condition 6.(a), for sufficiently large k (and hence j), the inequality in fact holds uniformly across all $y \in \{1,\ldots,Y\}$ and $p \in V^*$. Let k_{b1} be such that condition 6.(b) holds for all $(m_s,\phi_s) \in \{(y,\phi_{s,y}) : y \in \{1,\ldots,Y\}\}$, for α such that $k \geq k_{b1}$.

Case 2: $\phi_s \in \Phi_s$ and $m_s = m_{s,j}^v$. In the limit and when $\varepsilon = 0$, the inequality is:

$$\begin{aligned}&\sum_{p\in\operatorname{supp}(\gamma(m_s))}\gamma(m_s)[p]p\Big(\sum_{v>p}\mu(m_s)[v] + \mu(m_s)[p]\xi(p,m_s)\Big)\\ &\geq p|\Psi|^{-1}\sum_{\psi}\Big(\sum_{v}\mu(m_s)[v]\pi_b^2(\psi(m_{b,j}^v),\phi_{b,\psi},p)\Big).\end{aligned}$$

By (8), it suffices to show that, for some $p_{m_s} \in \operatorname{supp}(\gamma(m_s))$:

$$\begin{aligned}&p_{m_s}\Big(\sum_{v>p_{m_s}}\mu(m_s)[v] + \mu(m_s)[p_{m_s}]\xi(p_{m_s},m_s)\Big)\\ &\geq p|\Psi|^{-1}\sum_{\psi}\Big(\sum_{v}\mu(m_s)[v]\pi_b^2(\psi(m_{b,j}^v),\phi_{b,\psi},p)\Big).\end{aligned}$$

This holds since for $p \in \text{supp}(\gamma(m_s))$, the right hand side is $p\Big(\sum_{v>p}\mu(m_s)[v] + \mu(m_s)[p]\xi(p,m_s)\Big)$ by (B.11), which is equal to the left hand side by (8); for $p = v_1$, the right hand side is $v_1 \leq p_{m_s}\Big(\sum_{v>p_{m_s}}\mu(m_s)[v] + \mu(m_s)[p_{m_s}]\xi(p_{m_s},m_s)\Big)$ by (10); and for $p \notin \text{supp}(\gamma(m_s)) \cup \{v_1\}$, the right hand side is 0 by (B.11). Thus, the inequality holds for k sufficiently large (uniformly across $m_s \in M_s^b$, $\phi_s \in \Phi_s$ and $p \in V^*$). Let k_{b2} be such that condition 6.(b) holds for all $(m_s,\phi_s) \in M_s^b \times \Phi_s$, for α such that $k \geq k_{b2}$.

Case 3: $\phi_s = \phi_{s,y}$ and $m_s \notin \{y\} \cup M_s^b$. Note that we only need to consider $m_s \in F$ in this case (since otherwise $\sum_{\phi_b \in \text{supp}(\pi_b^{1,\alpha})} p^\alpha(\phi_b,\phi_s)_{M_s}[m_s]\pi_b^{1,\alpha}[\phi_b] = 0$). Given that $m_s \in F$, in the limit (as $k \to \infty$, i.e. we can keep F fixed) and when $\varepsilon = 0$, the inequality is

$$v_K \pi_b^2(K, \phi_b^{m_s}, v_K) \geq p\pi_b^2(K, \phi_b^{m_s}, p).$$

We have that $\pi_b^2(K, \phi_b^{m_s}, v_K) = 1$ by (B.13) since $\phi_b^{m_s}(v_K)[K, m_s] > 0$ and

$$\frac{Y^{-1}\sum_{y=1}^{Y}\sum_{v,\hat{m}_s} v\nu[v](\beta_b\phi_b^{m_s}(v) + \beta_s\phi_{s,y}(v))[K,\hat{m}_s]}{Y^{-1}\sum_{y=1}^{Y}\sum_{v',m'_s}\nu[v'](\beta_b\phi_b^{m_s}(v') + \beta_s\phi_{s,y}(v'))[K,m'_s]} = v_K.$$

Hence, for $m_s \in F$, the inequality holds in the limit and, thus, for each k sufficiently large (uniformly across $y \in \{1,\ldots,Y\} \setminus \{m_s\}$ and $p \in V^*$). For each $m_s \in F \setminus M_s^b$, let $k_{b3}(m_s)$ be such that condition 6.(b) holds for all $\phi_{s,y}$ such that $y \neq m_s$, for each α such that $k \geq k_{b3}(m_s)$, and let $k_{b3}(F) = \max_{m_s \in F \setminus M_s^b} k_{b3}(m_s)$. Note that for all $\alpha = (k, F, \hat{F}, \tilde{F})$ such that $k \geq k_{b3}(F)$, condition 6.(b) holds for all (m_s,ϕ_s) such that $m_s \in F \setminus M_s^b$ and $\phi_s = \phi_{s,y}$ for $y \neq m_s$.

Case 4: $\phi_s \notin \Phi_s$ and $m_s \in M_s$ such that $\sum_v \nu[v]\phi_s(v)_{M_s}[m_s] > 0$ or $m_s \in M_s^b$. Note that we only need to consider $\phi_s \in \Phi_s^\alpha \setminus \Phi_s$ in this case and that $\Phi_s^\alpha \setminus \Phi_s$ is finite. In the limit and with $\varepsilon = 0$, the inequality is (in this case $\pi_s^2(m_s,\phi_s)$ is a pure strategy)

$$\pi_s^2(m_s,\phi_s)|\Psi|^{-1}\sum_\psi\sum_{m_b}\frac{\sum_v \nu[v](\beta_b\phi_{b,\psi}(v) + \beta_s\phi_s(v))[m_b,m_s]}{\sum_v \nu[v](\beta_b\phi_{b,\psi}(v) + \beta_s\phi_s(v))_{M_s}[m_s]}\pi_b^2(m_b,\phi_{b,\psi},\pi_s^2(m_s,\phi_s))$$
$$\geq p|\Psi|^{-1}\sum_\psi\sum_{m_b}\frac{\sum_v \nu[v](\beta_b\phi_{b,\psi}(v) + \beta_s\phi_s(v))[m_b,m_s]}{\sum_v \nu[v](\beta_b\phi_{b,\psi}(v) + \beta_s\phi_s(v))_{M_s}[m_s]}\pi_b^2(m_b,\phi_{b,\psi},p),$$

which holds by (B.12). For each $(F, \hat{F})$, let $k_{b4}(F, \hat{F})$ be such that condition 6.(b) holds for all $\phi_s \in \Phi_s^\alpha \setminus \Phi_s$ and $m_s \in M_s$ such that $\sum_v \nu[v]\phi_s(v)_{M_s}[m_s] > 0$ or $m_s \in M_s^b$, for $\alpha = (k, F, \hat{F}, \tilde{F})$ such that $k \geq k_{b4}(F, \hat{F})$.

Case 5: $\phi_s \notin \Phi_s$ and $m_s \notin M_s^b$ such that $\sum_v \nu[v]\phi_s(v)_{M_s}[m_s] = 0$. This is as in case 3. For each $(F, \hat{F})$, let $k_{b5}(F, \hat{F})$ be such that condition 6.(b) holds for all such (m_s, ϕ_s), for $\alpha = (k, F, \hat{F}, \tilde{F})$ such that $k \geq k_{b5}(F, \hat{F})$.

For each $(F, \hat{F})$, let $k_b(F, \hat{F}) = \max\{k_{b1}, k_{b2}, k_{b3}(F), k_{b4}(F, \hat{F}), k_{b5}(F, \hat{F})\}$.

Consider next condition 6.(c). We establish this condition by considering several cases.

Case 1: $\phi_b \in \Phi_b$, $p \in \{u_s^s, E\}$ and $m_b = Y + 1$.[38] First, consider $p = u_s^s$. Since $\pi_b^2(0, \phi_b, u_s^s) = 1$, we may consider $a = 0$. Thus, in the limit and with $\varepsilon = 0$, the inequality is $\sum_v \nu[v]v - u_s^s \geq 0$, which holds.

For $p = E$, $\pi_b^2(Y+1, \phi_b, E) = u_s^s/E$; thus, we need to show that $a \in \{0, 1\}$ results in the same payoff. In the limit and with $\varepsilon = 0$, both actions give 0 payoff, and thus the inequality holds.

Let k_{c1} be such that condition 6.(c) holds for all $(m_b, \phi_b, p) \in \{Y+1\} \times \Phi_b \times \{u_s^s, E\}$, for α such that $k \geq k_{c1}$.

Case 2: $\phi_b = \phi_{b,\psi}$, $p \in \text{supp}(\gamma(m_{s,j}^v))$ and $m_b = \psi(m_{b,j}^v)$. First, let $p < v$. Since $\pi_b^2(\psi(m_{b,j}^v), \phi_{b,\psi}, p) = 1$, we may consider $a = 0$. Thus, in the limit and with $\varepsilon = 0$, the inequality is $v - p \geq 0$, which holds. Next, let $p > v$. Since $\pi_b^2(\psi(m_{b,j}^v), \phi_{b,\psi}, p) = 0$, we may consider $a = 1$. Thus, in the limit and with $\varepsilon = 0$, the inequality is $0 \geq v - p$, which holds. Finally, let $v = p$. Since $\pi_b^2(\psi(m_{b,j}^v), \phi_{b,\psi}, v) = \xi(p, m_{s,j}^v)$, we need to show that both actions give the same payoff. In the limit and with $\varepsilon = 0$, both actions give 0 payoff and so the inequality holds. Let k_{c2} be such that condition 6.(c) holds for $\{(\psi(m_{b,j}^v), \phi_{b,\psi}, p) : \psi \in \Psi, v \in V, j \in \{1, \ldots, J^v\}, p \in \text{supp}(m_{s,j}^v)\}$, for α such that $k \geq k_{c2}$.

Case 3: $\phi_b \in \Phi_b$, $p \notin \{u_s^s, E\}$ and $m_b = Y + 1$. Note that we only need to consider $p \in P(F, \hat{F}, \tilde{F})$ in this case.

[38] If $\delta = 0$ (resp. $\delta = 1$), then $p = u_s^s$ (resp. $p = E$) belongs to case 3 below.

The strategy for the buyer is

$$\pi_b^2(Y+1,\phi_b,p) = \begin{cases} 1 & \text{if } p = v_1, \\ 0 & \text{if } p > v_1. \end{cases}$$

When $p = v_1$, $\pi_b^2(Y+1,\phi_b,p) = 1$ gives payoff at least 0, which is also the payoff from $a = 0$; thus the inequality holds (in the limit).

We now consider the case $p > v_1$. In this case

$$Y^{-1}\sum_{y=1}^{Y} p^\alpha(\phi_b,\phi_{s,y})[v,Y+1,m_s]\pi_s^{2,\alpha}(m_s,\phi_{s,y})[p] \le j^{-j}$$

for each $v \in V$ and $m_s \in M_s$ since for each $y \in \{1,\dots,Y\}$, $(\beta_b\phi_b(v)+\beta_s\phi_{s,y}(v))[Y+1,m_s] = 0$ for $m_s \neq y$ and $\pi_s^2(y,\phi_{s,y})[p] = 0$ implies:

$$\nu[v](\beta_b\phi_b(v)+\beta_s\phi_{s,y}(v))[Y+1,m_s]\pi_s^2(m_s,\phi_{s,y})[p] = 0$$

and $\pi_s^{Y+1,2,\alpha}(y,\phi_{s,y})[p] = 0$ implies:

$$\nu[v](\beta_b\phi_b(v)+\beta_s\phi_{s,y}(v))[Y+1,m_s]\pi_s^{Y+1,2,\alpha}(m_s,\phi_{s,y})[p] = 0.$$

If $v \neq v_1$, $m_b \neq Y+1$, or $m_s \neq 1$, $p^\alpha(\phi_b,\phi_s^{m_b})[v,Y+1,m_s]\pi_s^{2,\alpha}(m_s,\phi_s^{m_b})[p] \le j^{-j}$. This is as follows: (1) if $m_b \neq Y+1$, then $(\beta_b\phi_b(v)+\beta_s\phi_s^{m_b}(v))[Y+1,m_s] = 0$ for each $v \in V$ and $m_s \in M_s$; (2) if $m_b = Y+1$, $m_s = 1$ and $v \neq v_1$, then $(\beta_b\phi_b(v)+\beta_s\phi_s^{Y+1}(v))[Y+1,1] = 0$; and (3) if $m_b = Y+1$, $m_s \neq 1$ and $v \in V$, then (i) $(\beta_b\phi_b(v)+\beta_s\phi_s^{Y+1}(v))[Y+1,m_s] = 0$ for each $m_s \notin \{1,\dots,K\}$, (ii) $\pi_s^2(k',\phi_s^{Y+1})[p] = 0$ for each $k' \in \{1,\dots,K\}$ (since, for each ψ, $\pi_b^2(Y+1,\phi_{b,\psi},p) = 1$ if and only if $p \in \{u_s^s,E,v_1\}$, and so $\mathrm{supp}(\pi_s^2(k',\phi_s^{Y+1})) \subseteq \{u_s^s,E,v_1\}$), and (iii) $\pi_s^{Y+1,2,\alpha}(m_s,\phi_s^{Y+1})[p] = 0$ for each $m_s \neq 1$ and $\pi_s^{m_b,2,\alpha}(m_s,\phi_s^{Y+1})[p] = 0$ for each $m_b \neq Y+1$ and $m_s \in M_s$.

Finally, note that

$$\begin{aligned}\pi_s^{2,\alpha}(1,\phi_s^{Y+1})[p] &= j^{-1}(1-j^{-j})|F|^{-1}\sum_{m_b\in F}\pi_s^{m_b,2,\alpha}(1,\phi_s^{Y+1})[p] + O(j^{-j}) \\ &= j^{-1}(1-j^{-j})|F|^{-1}|P(F,\hat{F},\tilde{F})|^{-1} + O(j^{-j})\end{aligned}$$

since $\pi_s^{m_b,2,\alpha}(1, \phi_s^{Y+1}) = 1_{v_1}$ for all $m_b \neq Y + 1$.

Thus, the denominator of the inequality is (ignoring terms that are $O(j^{-j})$):

$$\begin{aligned}
&(1 - j^{-1})Y^{-1} \sum_{y=1}^{Y} \sum_{v,m_s} p^\alpha(\phi_b, \phi_{s,y})[v, Y + 1, m_s]\pi_s^{2,\alpha}(m_s, \phi_{s,y})[p] \\
&+ j^{-1}(1 - j^{-j})|F|^{-1} \sum_{m_b \in F} \sum_{v,m_s} p^\alpha(\phi_b, \phi_s^{m_b})[v, Y + 1, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^{m_b})[p] \\
&+ j^{-1}j^{-j}|\Phi(F, \hat{F})|^{-1} \sum_{\phi_s \in \Phi(F,\hat{F})} \sum_{v,m_s} p^\alpha(\phi_b, \phi_s)[v, Y + 1, m_s]\pi_s^{2,\alpha}(m_s, \phi_s)[p] \\
&= j^{-1}(1 - j^{-j})|F|^{-1}(1 - j^{-j})\nu[v_1]\beta_s j^{-1}(1 - j^{-j})|F|^{-1}|P(F, \hat{F}, \tilde{F})|^{-1}
\end{aligned}$$

Likewise, also ignoring terms that are $O(j^{-j})$, the numerator of the right-hand side of the inequality is

$$j^{-1}(1 - j^{-j})|F|^{-1}(1 - j^{-j})\nu[v_1]\beta_s j^{-1}(1 - j^{-j})|F|^{-1}|P(F, \hat{F}, \tilde{F})|^{-1}(v_1 - p).$$

Thus, the limit inequality (with $a = 1$ and $\varepsilon = 0$) is $0 \geq v_1 - p$. For each $p \in P(F, \hat{F}, \tilde{F}) \setminus \{u_s^s, E\}$, let $k_{c3}(p)$ be such that condition 6.(c) holds for each $(m_b, \phi_b, p) \in \{Y + 1\} \times \Phi_b \times \{p\}$, for each $\alpha = (k, F, \hat{F}, \tilde{F})$ such that $k \geq k_{c3}(p)$, and let $k_{c3}(F, \hat{F}, \tilde{F}) = \max_{p \in P(F,\hat{F},\tilde{F})} k_{c3}(p)$.

Case 4: $\phi_b = \phi_{b,\psi}$, $p \notin \text{supp}(\gamma(m_{s,l}^v))$ and $m_b = \psi(m_{b,l}^v)$. The strategy for the buyer is

$$\pi_b^2(\psi(m_{b,l}^v), \phi_{b,\psi}, p) = \begin{cases} 1 & \text{if } p = v_1, \\ 0 & \text{if } p > v_1. \end{cases}$$

Again, we focus on the case $p > v_1$.

By the same argument as in case 3, we have that for each $v \in V$ and $m_s \in M_s$, $Y^{-1} \sum_{y=1}^{Y} \sum_y p^\alpha(\phi_{b,\psi}, \phi_{s,y})[v, \psi(m_{b,l}^v), m_s]\pi_s^{2,\alpha}(m_s, \phi_{s,y})[p] \leq j^{-j}$.

For each $v \neq v_1$, $m_b \neq \psi(m_{b,l}^v)$, or $m_s \neq 1$,

$$p^\alpha(\phi_{b,\psi}, \phi_s^{m_b})[v, \psi(m_{b,l}^v), m_s]\pi_s^{2,\alpha}(m_s, \phi_s^{m_b})[p] \leq j^{-j}.$$

This is because $(\beta_b\phi_{b,\psi}(v) + \beta_s\phi_s^{m_b}(v))[\psi(m_{b,l}^v), 1] = 0$ if $m_b \neq \psi(m_{b,l}^v)$ or $v \neq v_1$, $\pi_s^{m_b',2,\alpha}(m_s, \phi_s^{m_b})[p] = 0$ for each $m_s \neq 1$ and $m_b, m_b' \in M_b$, $\pi_s^{m_b,2,\alpha}(m_s, \phi_s^{\psi(m_{b,l}^v)})[p] = 0$ for each $m_b \neq \psi(m_{b,l}^v)$ and $m_s \in M_s$, $(\beta_b\phi_{b,\psi}(v) + \beta_s\phi_s^{\psi(m_{b,l}^v)}(v))[\psi(m_{b,l}^v), m_s] = 0$ for each $m_s \notin M^* = \{1, \ldots, K\} \cup \{m^v_{s,l}\}$, $\pi^2_s(m_s, \phi^{\psi(m^v_{b,l})}_s)[p] = 0$ for each $m_s \in M^*$ (since $\pi^2_s(m_s, \phi^{\psi(m^v_{b,l})}_s) = v_1$ is optimal for each $m_s \in \{1, \ldots, K\}$[39] and $\text{supp}(\pi^2_s(m^v_{s,l}, \phi^{\psi(m^v_{b,l})}_s)) \subseteq \text{supp}(\gamma(m^v_{s,l}))$, and if $m_b \neq \psi(m^v_{b,l})$, $(\beta_b \phi_{b,\psi}(v) + \beta_s \phi^{m_b}_s(v))[\psi(m^v_{b,l}), m_s] = 0$ for each $m_s \neq m^v_{s,l}$ and $\pi^2_s(m^v_{s,l}, \phi^{m_b}_s)[p] = 0$.

Finally, note that

$$\begin{aligned}\pi^{2,\alpha}_s(1, \phi^{\psi(m^v_{b,l})}_s)[p] &= j^{-1}(1 - j^{-j})|F|^{-1} \sum_{m_b \in F} \pi^{m_b,2,\alpha}_s(1, \phi^{\psi(m^v_{b,l})}_s)[p] + O(j^{-j}) \\ &= j^{-1}(1 - j^{-j})|F|^{-1}|P(F, \hat{F}, \tilde{F})|^{-1} + O(j^{-j}).\end{aligned}$$

Thus, the denominator of the inequality is (ignoring terms that are $O(j^{-j})$):

$$\begin{aligned}&(1 - j^{-1})Y^{-1} \sum_{y=1}^{Y} \sum_{v,m_s} p^\alpha(\phi_{b,\psi}, \phi_{s,y})[v, \psi(m^v_{b,l}), m_s]\pi^{2,\alpha}_s(m_s, \phi_{s,y})[p] \\ &+ j^{-1}(1 - j^{-j})|F|^{-1} \sum_{m_b \in F} \sum_{v,m_s} p^\alpha(\phi_{b,\psi}, \phi^{m_b}_s)[v, \psi(m^v_{b,l}), m_s]\pi^{2,\alpha}_s(m_s, \phi^{m_b}_s)[p] \\ &= j^{-1}(1 - j^{-j})|F|^{-1}(1 - j^{-j})\nu[v_1]\beta_s j^{-1}(1 - j^{-j})|F|^{-1}|P(F, \hat{F}, \tilde{F})|^{-1}.\end{aligned}$$

Likewise, also ignoring terms that are $O(j^{-j})$, the numerator of the right-hand side of the inequality is

$$j^{-1}(1 - j^{-j})|F|^{-1}(1 - j^{-j})\nu[v_1]\beta_s j^{-1}(1 - j^{-j})|F|^{-1}|P(F, \hat{F}, \tilde{F})|^{-1}(v_1 - p).$$

Thus, the limit inequality (with $a = 1$ and $\varepsilon = 0$) is $0 \geq v_1 - p$. For each $p \in P(F, \hat{F}, \tilde{F}) \setminus \text{supp}(\gamma(m^v_{s,l}))$, let $k_{c4}(p)$ be such that condition 6.(c) holds for $\{(\psi(m^v_{b,l}), \phi_{b,\psi}, p) : \psi \in \Psi, v \in V, l \in \{1, \ldots, J^v\}, p \notin \text{supp}(\gamma(m^v_{s,l})\}$, for each $\alpha = (k, F, \hat{F}, \tilde{F})$ such that $k \geq k_{c4}(p)$, and let $k_{c4}(F, \hat{F}, \tilde{F}) = \max_{p \in P(F, \hat{F}, \tilde{F})} k_{c4}(p)$.

Case 5: $\phi_b = \phi_{b,\psi}$ and $m_b \notin \psi(\cup_v M^v_b) \cup \{Y + 1\}$. The strategy for the buyer is

$$\pi^2_b(m_b, \phi_{b,\psi}, p) = \begin{cases} 1 & \text{if } p = v_1, \\ 0 & \text{if } p > v_1. \end{cases}$$

We focus on the case $p > v_1$.

[39] From the perspective of the seller, when the buyer receives message $\psi(m^v_{b,l})$, he will reject any $p > v_1$ with probability at least $|\cup_v M^v_b|Y^{-1}$, giving payoff at most $|\cup_v M^v_b|Y^{-1}v_K < v_1$

In this case, $Y^{-1}\sum_{y=1}^{Y} p^\alpha(\phi_{b,\psi},\phi_{s,y})[v,m_b,m_s]\pi_s^{2,\alpha}(m_s,\phi_{s,y})[p] \leq j^{-j}$ for all $v \in V$ and $m_s \in M_s$, and $p^\alpha(\phi_{b,\psi},\phi_s^{m_b'})[v,m_b,m_s]\pi_s^{2,\alpha}(m_s,\phi_s^{m_b'})[p] \leq j^{-j}$ if $m_b' \neq m_b$, $v \neq v_1$, or $m_s \neq 1$.

Thus, the denominator of the inequality is (ignoring terms that are $O(j^{-j})$):

$$
\begin{aligned}
&(1-j^{-1})Y^{-1}\sum_{y=1}^{Y}\sum_{v,m_s} p^\alpha(\phi_{b,\psi},\phi_{s,y})[v,m_b,m_s]\pi_s^{2,\alpha}(m_s,\phi_{s,y})[p] \\
&+j^{-1}(1-j^{-j})|F|^{-1}\sum_{m_b'\in F}\sum_{v,m_s} p^\alpha(\phi_{b,\psi},\phi_s^{m_b'})[v,m_b,m_s]\pi_s^{2,\alpha}(m_s,\phi_s^{m_b'})[p] \\
&= j^{-1}(1-j^{-j})|F|^{-1}(1-j^{-j})\nu[v_1]\beta_s\pi_s^{2,\alpha}(1,\phi_s^{m_b})[p].
\end{aligned}
$$

Likewise, also ignoring terms that are $O(j^{-j})$, the numerator of the right-hand side of the inequality is

$$
j^{-1}(1-j^{-j})|F|^{-1}(1-j^{-j})\nu[v_1]\beta_s\pi_s^{2,\alpha}(1,\phi_s^{m_b})[p](v_1-p).
$$

Thus, the limit inequality (with $a=1$ and $\varepsilon=0$) is $0 \geq v_1 - p$. Let $k_{c5}(F,\hat{F},\tilde{F})$ be such that condition 6.(c) holds for each $(m_b,\phi_{b,\psi},p)$ such that $\psi \in \Psi$, $m_b \in F \setminus (\psi(\cup_v M_b^v) \cup \{Y+1\})$ and $p \in P(F,\hat{F},\tilde{F})$, for each $\alpha = (k,F,\hat{F},\tilde{F})$ such that $k \geq k_{c5}(F,\hat{F},\tilde{F})$.

Case 6: For each $m_b \in M_b$ and $\phi_b \notin \Phi_b$, 6.(c) holds in the limit by construction. Let $k_{c6}(F,\hat{F},\tilde{F})$ be such that condition 6.(c) holds for each $\phi_b \in \Phi_b^\alpha \setminus \Phi_b$, $m_b \in \cup_v \mathrm{supp}(\phi_b(v)_{M_b}) \cup \{Y+1\} \cup F$ and $p \in P(F,\hat{F},\tilde{F})$, for each $\alpha = (k,F,\hat{F},\tilde{F})$ such that $k \geq k_{c6}(F,\hat{F},\tilde{F})$.

For each $(F,\hat{F},\tilde{F})$, let

$$
k_c(F,\hat{F},\tilde{F}) = \max\{k_{c1},k_{c2},k_{c3}(F,\hat{F},\tilde{F}),k_{c4}(F,\hat{F},\tilde{F}),k_{c5}(F,\hat{F},\tilde{F}),k_{c6}(F,\hat{F},\tilde{F})\}.
$$

The above arguments allow us to define the following subnet $\{\pi^{\varphi(\eta)},p^{\varphi(\eta)}\}_\eta$ of $\{\pi^\alpha,p^\alpha\}_\alpha$ such that condition 6 holds.

The index set of the subnet $\{\pi^{\varphi(\eta)},p^{\varphi(\eta)}\}_\eta$ is the same as the one in the net $\{\pi^\alpha,p^\alpha\}_\alpha$. The function $\varphi : \eta \mapsto \alpha$ is defined by setting, for each $\eta = (k,F,\hat{F},\tilde{F})$,

$$
\varphi(\eta) = (\max\{k_a,k_b(F,\hat{F}),k_c(F,\hat{F},\tilde{F})\},F,\hat{F},\tilde{F}).
$$

It is then clear that condition 6 holds and that, as required by the definition of a subnet, for each α_0, there exists η_0, e.g. $\eta_0 = \alpha_0$, such that $\varphi(\eta) \geq \alpha_0$ for each $\eta \geq \eta_0$.