Monopoly Pricing with Optimal Information*

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Abstract

We analyze a monopoly pricing model where information about the buyer's valuation is endogenous. Before the seller sets a price, both the buyer and the seller receive private signals that may be informative about the buyer's valuation. The joint distribution of these signals, as a function of the valuation, is optimally chosen by the players. In general, players have conflicting incentives over the provision of information. As a modelling device, we assume that an aggregation function determines the information structure from the choices of the players, and we characterize the pure strategy equilibrium payoffs for a natural class of aggregation functions. Every equilibrium payoff can be achieved by an information structure that is the result of the seller trying to make both players uninformed while the buyer tries to learn about his valuation. Price discrimination is limited to the seller setting different prices for informed vs uninformed buyers.

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1 Introduction

Recent advances in the economic analysis of monopoly, such as Bergemann, Brooks, and Morris (2015) and Roesler and Szentes (2017), show that the impact of discriminatory pricing on consumer and producer surplus critically depends on the information available to both the seller and the buyer(s). Consequently, *both* parties may desire to influence or manipulate this information, and moreover there may be a conflict of interest regarding what information should be available. The idea that the information structure arises endogenously through the potentially conflicting actions of multiple parties raises several important questions. For example, what information does each party want to have, and what information do they want the other to have? In case they disagree, what information structure will result from their conflict? And given that the information structure arises endogenously, to what extent can the seller price discriminate? Our aim in this paper is to address these questions in a simple and tractable framework.

Answering these questions is not straightforward because it is infeasible to model all the possible ways each party can influence every piece of information provided. Moreover, when the buyer and the seller have different incentives over the information they wish to be provided, it is unclear how this conflict of interest will be resolved. The recent information design literature has generated many insights about the information structures that are likely to arise by carefully studying the incentives of some (metaphorical or literal) information designer who can choose from all possible information structures. However, with a few exceptions discussed in Section 2, only the case of a single information designer has been considered. The conceptual challenge of considering multiple designers is that ultimately they must decide on a single information structure, and any model of how a single information structure arises from the decisions of multiple designers requires non-obvious modelling choices. In short, there is a need for a general framework for modelling situations where the information structure is the result of the actions of *multiple* interested parties; specific instances of this framework must determine how conflicts between the parties are resolved.

We consider such framework to study how optimally chosen information affects monopoly pricing, the latter modelled in the standard way: the seller of a good, produced with zero marginal cost, makes a take-it-or-leave-it price offer to a buyer whose valuation is unknown and drawn from a finite set. Our point of departure from the standard model is that before the seller makes a price offer, both the buyer and the seller can take actions that determine the information they receive. In our model, an information structure is a function from the set of unknown valuations to a set of distributions over message profiles, consisting of one message for each player which he receives privately.[1] The information choices of the players will combine to produce some information structure, but as a tractable *reduced form representation* of the various actions players may take to influence this information structure, we assume that each player (covertly) chooses the information structure directly and the true information structure (that determines the information that each player actually receives) is determined by an *aggregation function* that combines their choices.

As a concrete example, consider a provider of a new, untested product (e.g. a healthcare provider offering a new treatment, a university offering a new course, a financial services firm offering complex insurance or mortgage products). Suppose that the provider must submit the product to a ratings agency, who scrutinizes it, solicits feedback from a (representative) buyer, and releases information to both parties.[2] Each party may ask the ratings agency to provide information in a specific way. In case these requests are contradictory, the agency may provide the information according the buyer's wishes or according to the seller's wishes. Suppose that with probability β_b, the ratings agency favors the buyer and with remaining probability β_s, it favors the seller; this situation can be modelled by specifying an aggregation function that maps the information structures chosen by the players into their convex

[1]To allow the players to acquire sufficiently rich information if they desire, we assume that the set of possible messages for each player is the set of natural numbers.

[2]The introduction of an information intermediary is just one possible interpretation of the model. In Section 4.1.4, we discuss an alternative interpretation where the buyer and seller communicate directly with each other.

combination for fixed weights.[3]

The convex combination aggregator allows for rich information incentives. Full information is possible if both players agree on a fully informative information structure (e.g. both the seller and the buyer ask the ratings agency to produce an honest report). But each may prevent the other from full learning by choosing an information structure that obfuscates the message of the other player, for example by sending a message that is only supposed to be received in one state in every state (e.g. the seller may pressure the ratings agency to certify the product regardless of its quality). Nevertheless, the players have the option to choose information structures with disjoint supports, in which case they will know which information structure a message comes from and interpret its meaning correctly. Whether they will do so depends on how incentives play out in equilibrium.

For our main result, presented in Section 3, we will consider a class of aggregation functions that includes the convex combination aggregator and characterize the equilibrium payoffs of a monopoly pricing game where, before the seller makes a price offer, the players choose their information in the way we have described. We provide an interpretation of our model and discuss the assumptions we impose on the aggregation function in Section 4. Under these assumptions, our model generates a number of lessons about the implications of optimally chosen information structures for monopoly pricing: (i) All equilibrium payoffs can be obtained using a specific class of information structures where (ii) price discrimination is severely limited but (iii) multiple prices can be supported in equilibrium. We now discuss each of these lessons in turn.

In general, many kinds of information incentives may potentially shape equilibria. First, there is buyer-learning: the buyer will try to learn whether he should accept the seller's offer. Second, there is buyer-obfuscation: if the seller sometimes gets a

[3]Since many ratings agencies and other information intermediaries exist in order to provide advice to buyers while receiving kickbacks or commissions from sellers, it is unclear what their true preferences are. Our convex combination aggregator is consistent with the simple assumption that with some probability, the ratings agency's preferences are fully aligned with the seller's and with remaining probability, its preferences are those of the buyer.

message that makes the seller believe that the buyer's valuation is low and leads the seller to offer a low price, then the buyer may try to send this message. Third, there is seller-obfuscation: if the buyer's belief following some message is that his valuation is high, then the seller may try to send this message. Finally, the seller may try to learn the buyer's valuation if the buyer is willing to accept higher prices when (he knows that) his value is high.

In equilibrium, these incentives imply that the seller will offer a single price following all messages from his chosen information structure, which the buyer accepts, and a (possibly different) single price following all messages from the buyer's chosen information structure, which the buyer accepts if his valuation is strictly larger than the price and only if it is at least as large as the price. Moreover, as far as payoffs are concerned, we may focus on equilibria where the buyer becomes informed about his valuation only following messages from his chosen information structure; and where the seller is always uninformed about the valuation but knows whether or not the buyer is informed. In particular, it turns out that all pure strategy equilibrium payoffs can be achieved using an information structure that, through the aggregation function, is the result of:

(a) The buyer choosing an information structure which informs him whether his valuation is at least or at most the price and leaves the seller uninformed.

(b) The seller choosing an information structure which leaves both players uninformed.

(c) The players choosing information structures with disjoint supports so that they know which player is responsible for each message they receive.

Under the assumptions we impose on our aggregation function, property (c) will imply that it becomes common knowledge whether the buyer is informed about his valuation. The seller sets price p_s when he learns that the buyer is uninformed and p_b when he learns that the buyer is informed; we must have $p_b \leq p_s$ (otherwise the buyer will pretend to be uninformed) and p_s must be at most the expected valuation (otherwise the uninformed buyer will not accept).

That the buyer chooses an information structure which informs him whether his valuation is at least or at most the price and leaves the seller uninformed is often seen in reality. For example, this happens when someone tries some clothes at a store without assistance from a salesperson.[4] That the seller chooses an information structure where no information is transmitted is also often seen in reality. For example, manufacturers will often take pre-orders before any reviews of the product have been released; many sellers list items on eBay and other online marketplaces with only very limited information;[5] wine producers in Bordeaux sell their product *en primeur* to négociants, before the wine is bottled and when the quality is still uncertain; and restaurant menus are often short and uninformative.[6] In each case, the buyer would prefer to have more information, and sometimes may be successful in acquiring such information (for example, a product reviewer may release an in-depth review that is recognized by the buyer as being reliable).

That the seller sets the same price for all valuations is due to the difficulty of credibly transmitting information when the information structure is designed by the players themselves (without commitment) as in our framework. For example, if the information structure is designed by the buyer, he will pretend to have whichever valuation gets the lowest price, thus rendering the message that the seller receives uninformative. Similarly, if the information structure is designed by the seller, he will try to make the buyer believe that his valuation is greater than the price. In

[4]For a literary example, in Chapter 9 of Part I of Don Quixote by Miguel de Cervantes, the narrator finds, by quietly browsing the notebooks and old papers that a boy was trying to sell, a manuscript that continues the story of Don Quixote and, thus, finds that the value to him of that manuscript is higher than the price the boy was asking for it.

[5]Moreover, sellers on eBay are allowed to make a single time-limited offer to any buyer who views their listing; many sellers set these offers to be sent out automatically before any information is exchanged.

[6]In an article entitled "Why a minimalist menu can ruin my meal", published in the Financial Times on 10 August 2023, the writer laments the trend towards "obfuscating and withholding all but the most minimal information in the menu" which prevents him from making an informed choice. In our model, information is withheld precisely so that the item can be sold to those who would not have accepted if they were fully informed.

reality, we often see, for example, marketing information provided by the seller that is essentially uninformative as in our model.

A striking feature of our main result is that there is only one instance of price discrimination: the price may be lower for informed buyers than for uninformed ones but each is the same across valuations. This is a realistic feature: most service providers offer better deals for (in some cases, all) customers who call them to negotiate, which can be interpreted as the customer revealing that they are well informed about their valuation.

Several pairs of prices, for informed and uninformed buyers, are possible in equilibrium and this creates a multiplicity of equilibrium payoffs. For instance, when the buyer is uninformed, the price is at most the expected valuation but there are equilibria where it is lower; in fact it can be anything between the expected valuation and the lowest valuation of the buyer. What prevents the seller from raising the price in this case is that the buyer may rationally reject price changes by reasonably attributing it to sellers who know that the valuation is low. In other words, an instance of the classic lemons problem prevents the seller from adjusting the price in a way that would be profitable if the buyer's beliefs were fixed.

The multiplicity of equilibrium payoffs in our setting with optimal information is nevertheless smaller than what several recent papers, discussed in Section 2, have found when there is no requirement that the information is optimally chosen. As discussed above, the requirement that the information structure is optimally chosen imposes restrictions on what information can be exchanged, thus limiting the extent of price discrimination.

The multiplicity of payoffs arises because of the possibility that an uncertain buyer may become pessimistic about his valuation when faced with an unexpected price change. It is then conceivable that this multiplicity may be eliminated by ruling out the possibility that the price offer affects the buyer's belief about his valuation. In Section 4, we consider an equilibrium refinement that forces the buyer not to update his belief based on price offers and show that under this refinement there does not exist a pure strategy equilibrium. Roughly, the reason is that is when the buyer learns about his valuation, he will sometimes believe that his valuation is greater than the expected valuation. With pure strategies, the seller can send the buyer a message that makes him hold this belief. The refinement implies the buyer must then accept a price greater than the expected valuation, so this deviation is profitable (since in equilibrium, the seller cannot get more than the expected valuation, which is the total surplus).

2 Related literature

Our paper is inspired by the information design literature but in contrast to, e.g., Kamenica and Gentzkow (2011) or Bergemann and Morris (2016), we decentralize the role of the designer and relax the commitment assumption. In addition to monopoly pricing, our model can be applied to other strategic situations where multiple players design the information structure in a noncooperative way. For example, we have applied our model in Carmona and Laohakunakorn (2023) to study a repeated game where the monitoring structure is optimally chosen by the players themselves, and in Carmona and Laohakunakorn (2024) to study correlated equilibrium where the correlation structure is optimally chosen by the players themselves. Besides these two papers, Gentzkow and Kamenica (2017) also consider the case of multiple information designers. In their setting, multiple senders choose what information to communicate to a single receiver who observes the realization of all information structures. In contrast, in our model, each player is both a sender and a receiver simultaneously and each observes only the realization of one information structure that aggregates their information choice.

Especially relevant is Carroll (2019), who considers a setting that is conceptually similar to ours. In Carroll (2019), an information game consists of the buyer and seller taking actions, possibly sequentially, and eventually receiving informative signals about each other. Such interaction in information games is exactly what our formalization intends to capture in reduced form; in this light, it is reassuring that, for Carroll's (2019) main result, the relevant information game for his analysis can be summarized by the players choosing between different information structures, which in principle can be modelled using some aggregation function. However, his goal is to find the mechanism that maximizes the players' welfare under the worst-case information game out of all possible games, whereas we consider a narrow class of information games (summarized by aggregation functions satisfying our assumptions) and characterize payoffs for the posted-price monopoly pricing mechanism.

Several recent papers have considered information design in a monopoly pricing setting. Bergemann, Brooks, and Morris (2015) consider a model where the buyer is fully informed and show that any feasible payoff such that the seller gets at least the uniform monopoly profit can be supported in equilibrium for *some* information provided to the seller. As an application of their characterization of extensions of Bayes' correlated equilibria to multi-stage games, Makris and Renou (2023) consider all possible information structures in the monopoly setting (i.e. both the buyer and the seller can become informed) and show that any feasible payoff such that the seller gets at least the lowest valuation of the buyer can be supported in equilibrium. Kartik and Zhong (2023) allow the seller's cost also to be uncertain and characterize the payoffs from all information structures, as well payoffs under different restrictions on information structures. In contrast to these papers, we allow the players to choose their information structure optimally.

Many papers have considered information structures that are optimal for either the seller or the buyer(s). For example, Roesler and Szentes (2017) consider a model where the seller is uninformed and find that under the buyer-optimal information structure, the seller's payoff is less than the uniform monopoly profit.[7] Bergemann, Heumann, Morris, Sorokin, and Winter (2022) consider the revenue-maximizing information structure in a second price auction, and Bergemann, Heumann, and Morris (2023) consider the bidder-optimal information structure in an optimal auction. Bobkova

[7]Moreover any feasible payoff such that the seller receives at least this amount can be supported in equilibrium for *some* information provided to the buyer. Relatedly, Ravid, Roesler, and Szentes (2022) consider a model where the buyer can a purchase a signal about his valuation, and they show that as the cost of information goes to zero, equilibria converge to the Pareto-worst free-learning equilibrium.

(2024) compares the efficiency of different auctions when bidders can choose to learn about different components of their value. In contrast, we consider a setting where both the buyer and the seller receive information, the information structure is the result of noncooperative optimal choices by both players, and the players can learn about each other's information as well as about the buyer's valuation.

Finally, a few papers allow agents to learn about the learning of others. Pernoud and Gleyze (2023) allow agents to acquire costly information and find that agents will typically choose to learn about others' preferences even when they are not directly payoff relevant. Denti (2023) and Denti and Ravid (2023) consider a model where players can learn, at a cost, directly about each other's signals as well as the state, and introduces an equilibrium concept which is robust to such information acquisition.[8] Unlike in these papers, we assume that information acquisition actions are costless; what prevents agents from choosing, for example, full information in our model is that they are constrained by the information choices of others.

3 Model and main result

3.1 Model

A monopolist seller of a good makes a take-it-or-leave-it price offer to a buyer whose valuation is unknown and who chooses to buy the good at that price or not. In addition, before the seller makes a price offer, both the buyer and the seller choose an information structure.

The set of players is represented by $N = \{b, s\}$ with player b being the buyer and player s being the seller. The buyer's valuation of the seller's good belongs to the set $V = \{v_1, \ldots, v_K\}$ with $0 < v_1 < \cdots < v_K$; it is unknown to both players, and its prior distribution is $\zeta \in \Delta(V)$ which is fully supported.

Each player chooses an information design which sends messages to both players. The set of messages each player $i \in N$ can potentially receive is $M_i = \mathbb{N}$. This

[8] See also Hellwig and Veldkamp (2009) who study a beauty contest game where agents can learn about what others know.

avoids imposing a bound on the number of different messages that each player can receive; to avoid unnecessary technical complications, we focus on (arbitrary) finite subsets of messages. Letting F be the set of finitely supported probability measures on $M = \prod_{i \in N} M_i = \mathbb{N}^2$, an *information design* consists of a function $\phi : V \to F$. Let Φ be the set of such functions. We use the following notation: For each $m \in M$, $1_m \in F$ denotes the probability measure degenerate on m and, for each $\gamma \in F$, $\operatorname{supp}(\gamma)$ denotes the support of γ and γ_{M_i} denotes the marginal of γ on M_i.

The players' interaction is then described by the following extensive-form game G. At the beginning of the game, each player $i \in N$ chooses an information design $\phi_i \in \Phi$. After all players have chosen their information designs, a profile of buyer's valuations and messages $(v, m) \in V \times M$ is realized according to $\phi \in \Delta(V \times M)$ defined by setting, for each $(v, m) \in V \times M$,

$$\phi[v, m] = \zeta[v]\beta(\phi_b(v), \phi_s(v))[m],$$

where $\beta : F^2 \to F$ is a function that aggregates the information choices of the players. That is, if the buyer chooses information structure ϕ_b, the seller chooses information structure ϕ_s and the buyer's valuation is v, the message profile m is drawn from $\beta(\phi_b(v), \phi_s(v))$.

Let $(\beta_b, \beta_s) \in \mathbb{R}^2$ such that $\beta_b > 0$, $\beta_s > 0$ and $\beta_b + \beta_s = 1$. The leading example of an aggregation function in our framework is the convex combination aggregator: $\beta(\gamma, \gamma') = \beta_b \gamma + \beta_s \gamma'$; this can be interpreted as each player choosing the information structure they want to be implemented and nature choosing who was successful. More generally, β_i will be interpreted as the power player i has to determine the information structure. We will be interested in characterizing the equilibrium payoffs that can be obtained by some aggregation function $\beta : F^2 \to F$ satisfying the following properties:

1. For each $m \in M$ and $(\gamma, \gamma') \in F^2$, if $\gamma[m] = \gamma'[m] = 0$, then $\beta(\gamma, \gamma')[m] = 0$.

2. For each $m \in M$, $\gamma \neq 1_m$ and $\tilde{\gamma} \in F$:

   (a) $\beta(1_m, \tilde{\gamma})[m] > \beta(\gamma, \tilde{\gamma})[m]$ and $\beta(\tilde{\gamma}, 1_m)[m] > \beta(\tilde{\gamma}, \gamma)[m]$,

(b) $\beta(1_m, \tilde{\gamma})[m'] \leq \beta(\gamma, \tilde{\gamma})[m']$ and $\beta(\tilde{\gamma}, 1_m)[m'] \leq \beta(\tilde{\gamma}, \gamma)[m']$ for all $m' \neq m$, with strict inequality if $\gamma[m'] > 0$.

3. For each $(\gamma_b, \gamma_s) \in F^2$, $\beta(\gamma_b, \gamma_s)[\text{supp}(\gamma_b)] \geq \beta_b$ and $\beta(\gamma_b, \gamma_s)[\text{supp}(\gamma_s)] \geq \beta_s$.

Note that the convex combination aggregator satisfies the three properties.

Property 1 requires that if both players agree that some message profile should arise with zero probability, then that message profile indeed arises with zero probability. Property 2 requires that if a player chooses to send a message profile with probability 1, then the probability of that message profile should go up, and the probability of all other message profiles should go down independently of the choice of the other player. Property 3 requires that the realized message profile comes from the seller's information structure with probability at least β_s and from the buyer's information structure with probability at least β_b. Thus, β_i can be thought of as the amount of control player i has over the information structure in the sense that if he chooses to send a particular message profile with probability 1, that message profile will realize with probability at least β_i.

We defer an interpretative discussion of the model and aggregation function, as well as further examples, to Section 4. Here, we comment briefly on the assumption that the information designs map into finitely supported distributions. Under the convex combination aggregator, this implies that each player has the option of learning when his information design is the one that was chosen by nature. Indeed, if player i chooses ϕ_i such that $\cup_v \text{supp}(\phi_{i,M_i}(v)) \subseteq \mathbb{N} \setminus \cup_v \text{supp}(\phi_{j,M_i}(v))$, where $j \neq i$, then whenever he receives a message $m_i \in \cup_v \text{supp}(\phi_{i,M_i}(v))$ he knows that the true information structure is ϕ_i.[9] Thus, he can deviate as if he fully controls the information structure, which facilitates our arguments. With countably infinite supports, each player can still deviate as if he controls the information structure with arbitrarily high probability; hence we conjecture that, by using standard $\varepsilon - \delta$ arguments, our results would continue to hold without the finite support assumption.

[9]Under properties 1 and 3, the same is true if $\phi_i(v)$ is a degenerate distribution for each v, and property 2 implies that we can focus on such distributions when looking for profitable deviations.

Each player $i \in N$ observes $m_i \in M_i$ and his choice $\phi_i \in \Phi$ but neither m_j nor ϕ_j where $j \neq i$. The seller then makes a price offer $p \in [v_1, v_K]$ to the buyer, and the buyer chooses whether to accept ($a = 1$) or reject the offer ($a = 0$). Let $V^* = [v_1, v_K]$ and $A = \{0, 1\}$; payoffs are as follows: For each $(v, p, a) \in V \times V^* \times A$,

$$u_s(p, a) = pa,$$
$$u_b(v, p, a) = (v - p)a.$$

A pure strategy for the seller is $\pi_s = (\pi_s^1, \pi_s^2)$ such that $\pi_s^1 \in \Phi$ and $\pi_s^2 : \mathbb{N} \times \Phi \to V^*$ is measurable.[10] A pure strategy for the buyer is $\pi_b = (\pi_b^1, \pi_b^2)$ such that $\pi_b^1 \in \Phi$ and $\pi_b^2 : \mathbb{N} \times \Phi \times V^* \to A$ is measurable. A pure strategy is $\pi = (\pi_b, \pi_s)$ and let Π be the set of pure strategies. We use sequential equilibrium, defined in Myerson and Reny (2020), as our solution concept: $\pi \in \Pi$ is a *sequential equilibrium* if it is a perfect conditional ε-equilibrium for every $\varepsilon > 0$.[11]

We will focus on pure strategies and thus we often write $\phi_i^* = \pi_i^1$, $p(m_s) = \pi_s^2(m_s, \phi_s^*)$, $a(m_b, p) = \pi_b^2(m_b, \phi_b^*, p)$ and $a(m_b, m_s) = a(m_b, p(m_s))$, where $(m_b, m_s, p) \in \mathbb{N}^2 \times V^*$. Let Π^* be the set of $\pi \in \Pi$ such that $a(m_b, v_1) = 1$ for each $m_b \in M_b$ and we focus on $\pi \in \Pi^*$. This is a mild refinement since, upon receiving any message m_b, the buyer is certain that his valuation is at least v_1 and thus is, at the very least, not worse off by buying at price v_1 than not buying.

3.2 Examples

We present some examples of information structures and ask if they are optimal for the players under specific assumptions about behavior in the resulting monopoly pricing game. The examples feature $V = \{1, 2, 3, 4, 5\}$ with ζ uniform (hence, the expected valuation is 3), and $\beta(\gamma, \gamma') = 0.5\gamma + 0.5\gamma'$ for each $(\gamma, \gamma') \in F^2$.

[10] The set F is endowed with the topology of the weak convergence of probability measures and the corresponding Borel σ-algebra.

[11] Our main result, Theorem 1 below, characterizes sequential equilibrium payoffs and its proof shows that the same characterization is valid for Nash equilibrium payoffs. This holds because the proof of the necessity part of the theorem uses only the properties that sequential equilibrium imposes on the equilibrium path, which are the same as in Nash equilibrium.

Example 1 *The information structure* $\phi : V \to F$ *such that*

$$\phi(v) = 1_{(v,v)} \text{ for each } v \in V$$

corresponds to full information. Suppose that, for each v*, the seller makes the price offer* v *which the buyer accepts. Then* ϕ *cannot be the information structure in equilibrium, i.e. it is not optimal for both players to choose* ϕ *since the seller has a profitable deviation to choose an information structure* $\phi'(v) = 1_{(5,5)}$ *for each* v*. Then the distribution of messages is* $(0.5)1_{(v,v)} + (0.5)1_{(5,5)}$ *for each* v*. Thus, with probability* 0.5*, the seller will receive* $m_s = v$ *and get payoff* $\sum_v \zeta[v]v = 3$ *as in the proposed equilibrium; however, with probability* 0.5*, the seller will get payoff* 5 *instead of* 3*.*

Example 2 *The information structure* $\hat{\phi} : V \to F$ *such that*

$$\hat{\phi}(v) = \begin{cases} 1_{(1,1)} & \text{if } v \in \{1,2\}, \\ 1_{(2,1)} & \text{if } v \in \{3,4,5\} \end{cases}$$

corresponds to a partially informed buyer and an uninformed seller. Here the buyer learns whether his value is less than or at least 3*, and the seller learns nothing. Suppose that the seller makes a price offer that is accepted if* $m_b = 2$ *and rejected if* $m_b = 1$*. Then* $\hat{\phi}$ *cannot be the information structure in equilibrium, since the seller has a profitable deviation to choose an information structure* $\phi'(v) = 1_{(2,1)}$ *for each* v*. In this case, with probability* 0.5*, the seller's price offer will be accepted with probability* 1 *instead of* $\frac{3}{5}$*.*

Example 3 *The information structure* $\bar{\phi} : V \to F$ *such that*

$$\bar{\phi}(v) = 1_{(3,3)} \text{ for each } v \in V$$

corresponds to no information. Suppose that the seller makes a price offer $p \in (1,3]$ *which the buyer accepts. Then* $\bar{\phi}$ *cannot be the information structure in equilibrium since the buyer has a profitable deviation to choose an information structure* $\phi(v) = 1_{(v,3)}$ *for each* v *and to accept only if* $p \leq v$*.*

We now argue that there is an equilibrium where the buyer chooses the information structure $\hat{\phi}$ and the seller chooses the information structure $\bar{\phi}$. First, we specify what happens on the equilibrium path: suppose that $\phi_b^* = \hat{\phi}$, $\phi_s^* = \bar{\phi}$, $p(1) = 3$, $p(3) = 3$ and, if $p = 3$, $a(1, p) = 0$ and $a(2, p) = a(3, p) = 1$. Note that on the equilibrium path, the seller receives messages 1 and 3 and the buyer receives messages 1, 2 and 3. Also, since the seller sets price 3 after messages 1 and 3, the buyer will only see price 3 on the equilibrium path. Thus, the above is a complete description of the strategy for on-path histories.

We now argue that the price offer 3 is optimal for the seller. Crucially, note that any other price offer is off the equilibrium path and thus the belief following such price offer cannot be determined by Bayes' rule. In fact, it is possible to construct perturbations such that the buyer believes that he has valuation 1 after any unexpected price offer. Given such belief, we can specify that the buyer will only accept 1 (by assumption) and the equilibrium price offer, making the equilibrium price offer optimal.

Similarly, to ensure that the information structures are optimally chosen, we can construct perturbations such that following any zero probability message received by the buyer, he believes that his valuation is 1; and following any zero probability message received by the seller, he believes that the buyer would accept 5 (and hence makes price offer 5).[12] Thus, we only have to ensure that the players do not want to deviate by sending different on-path messages to the other player. If the buyer sends message 3 instead of 1 to the seller, the price is the same, and he is making the correct decision ex post conditional on $\hat{\phi}$ being chosen, so $\hat{\phi}$ is optimal. For the seller, conditional on $\bar{\phi} = 1_{(3,3)}$ being chosen, he gets profit 3, which is the same profit he

[12] For example, consider a message profile $\tilde{m} = (\tilde{m}_b, \tilde{m}_s)$, which consists of messages that neither player receives with positive probability in equilibrium. If the buyer believes that the seller's most likely deviation is to send message $\tilde{m}_b$ to the buyer when $v = 1$ and an equilibrium message otherwise, then after receiving $\tilde{m}_b$, the buyer will believe that $v = 1$. Likewise, if the seller believes that the buyer's most likely deviation is to send the message profile $\tilde{m}$ if $v = 5$ and an equilibrium message profile otherwise, then after receiving $\tilde{m}_s$, he will believe that the buyer received $\tilde{m}_b$ and believes that $v = 5$.

can get by sending the buyer message 2 (after which the buyer also accepts price 3) and higher than the profit he can get by sending the buyer message 1 (after which the buyer rejects all prices other than 1). Thus, $\bar{\phi}$ is optimal for the seller.

Note that $\bar{\phi}$ and $\hat{\phi}$ send different messages to each player, so in this equilibrium, the players know which information structure has been chosen. When the realized message profile is sent by the seller, the price is 3 and the buyer accepts. When the realized message profile is sent by the buyer, the price is 3 and the buyer accepts if and only if his valuation is at least 3.

In the next subsection, Theorem 1 will imply that any equilibrium payoff can be achieved using a generalization of the above strategy, with the price following the seller's message being replaced by $p_s \in [1, 3]$, the price following the buyer's message being replaced by $p_b \in [1, p_s]$, and the buyer's information structure being replaced by $\hat{\phi}$ such that $\hat{\phi}(v) = 1_{(1,1)}$ if $v < p_b$, $\hat{\phi}(v) = 1_{(2,1)}$ if $v > p_b$ and $\hat{\phi}(v) = (1-\lambda)1_{(1,1)} + \lambda 1_{(2,1)}$ for any $\lambda \in [0, 1]$ if $v = p_b$.

3.3 Main result

Our main result characterizes the payoffs that can arise in sequential equilibrium for some aggregation function satisfying our assumptions. Let U^* be the set of payoffs of the sequential equilibria $\pi \in \Pi^*$ of any game with an aggregation function β satisfying properties 1–3.[13]

Theorem 1 *A payoff profile* (u_b, u_s) *belongs to* U^* *if and only if there exists* $(p_b, p_s) \in$

[13]Letting $\mathcal{B}$ denote the set of aggregation functions β satisfying Properties 1–3 (given (β_b, β_s)) and $U^*(\beta)$ the set of payoffs of the sequential equilibria $\pi \in \Pi^*$ of the game with aggregation function β, then $U^* = \cup_{\beta \in \mathcal{B}} U^*(\beta)$.

$(V^*)^2$ *and* $\lambda \in [0,1]$ *such that*

$$u_b = \beta_s \left(\sum_v \zeta[v]v - p_s \right) + \beta_b \left(\sum_{v \geq p_b} \zeta[v](v - p_b) \right), \tag{1}$$

$$u_s = \beta_s p_s + \beta_b \left(p_b \sum_{v > p_b} \zeta[v] + p_b \zeta[p_b]\lambda \right), \tag{2}$$

$$p_b \leq p_s \leq \sum_v \zeta[v]v \text{ and} \tag{3}$$

$$v_1 \leq p_b \sum_{v > p_b} \zeta[v] + p_b \zeta[p_b]\lambda. \tag{4}$$

The proof of Theorem 1 establishes that, given any aggregation function satisfying properties 1–3, there are at most two prices p_s and p_b in a pure strategy equilibrium.[14] The price is p_s when the message comes from the seller's information structure, which happens with probability β_s, the price is p_b when the message comes from the buyer's information structure, which happens with probability β_b, and these prices do not contain any information about the buyer's valuation.

Conditions (1) and (2) describe the payoffs from such equilibrium, given that the buyer accepts p_s, accepts p_b whenever his valuation is greater than p_b and rejects p_b whenever his valuation is less than p_b. If the buyer's valuation is exactly p_b, he can accept with any probability λ.[15] Condition (3) requires that $p_b \leq p_s$, otherwise the buyer could deviate by sending the seller the message that results in p_s, and that $p_s \leq \sum_v \zeta[v]v$, otherwise the buyer would not accept p_s. Condition (4) requires that the seller's payoff following each message must be at least v_1, since he can always offer v_1 which will be accepted.

Conversely, each payoff profile satisfying (1) and (2) for some $(p_b, p_s, \lambda) \in (V^*)^2 \times [0,1]$ satisfying (3) and (4) is the payoff of a pure strategy equilibrium given some aggregation function satisfying properties 1–3, namely the convex combination aggregator.

[14] All proofs can be found in the Appendix.

[15] Although the buyer is playing pure strategies, he can choose a nondegenerate information structure that randomizes between sending himself two messages when his valuation is p_b: one where he accepts p_b and another where he rejects p_b.

The focus in Theorem 1 is on payoffs which has the advantage of abstracting from details of equilibrium strategies that are not relevant for the players' welfare. To illustrate this point, note that (u_b, u_s) such that $u_b = \sum_v \zeta[v]v - v_1$ and $u_s = v_1$ is an equilibrium payoff (let $p_b = p_s = v_1$ and $\lambda = 1$), which can be obtained with an equilibrium strategy in which the seller makes price offer v_1 regardless of his information. This then implies that any information structure can be optimally chosen by the players since then any price offer $p > v_1$ is off the equilibrium path and it is possible to construct perturbations such that the buyer optimally rejects any $p > v_1$, making v_1 optimal for the seller and any information structure optimal for each player. This multiplicity of equilibria is however irrelevant for players' welfare as all of them have the same payoff.

As already pointed out, it is possible to construct perturbations such that the buyer optimally rejects price offers that are off the equilibrium path. This allows multiple pairs of prices (p_b, p_s) to be supported in equilibrium and accounts for the multiplicity of payoffs.

To better understand the extent of payoff multiplicity, we provide a further characterization of the set U^* of equilibrium payoffs in the case where the buyer accepts when indifferent. We say that $(u_b, u_s) \in U^*$ is *represented by* (p_b, p_s, λ) if (1) and (2) hold for some $(p_b, p_s, \lambda) \in (V^*)^2 \times [0, 1]$ satisfying (3) and (4). We consider the set U^{**} of $(u_b, u_s) \in U^*$ represented by (p_b, p_s, λ) with $\lambda = 1$.

Following a message from the seller's information structure, the sum of payoffs is the expected valuation (i.e. total surplus), which we denote by $E = \sum_{v \in V} \zeta[v]v$. In contrast, following a message from the buyer's information structure, the buyer accepts the price offer p_b if and only if his valuation is at least p_b. Equivalently, the buyer accepts if and only if his valuation is greater than or equal to some $v_k \in V$, with $p_b \in (v_{k-1}, v_k]$. In addition, p_b must satisfy $p_b Z(v_k) \geq v_1$ and $p_b \leq E$, where $Z(p) = \sum_{v \geq p} \zeta[v]$ is the probability that $v \geq p$, for each $p \in V^*$. Thus, letting

$$C_k = \{p \in (v_{k-1}, v_k] : pZ(v_k) \geq v_1 \text{ and } p \leq E\} \text{ for each } k \in \{2, \ldots, K\},$$

it follows that the buyer will accept if and only if his valuation is at least v_k whenever

$p_b \in C_k$ for some $k \in \{2, \ldots, K\}$. The case $k = 1$ is possible, i.e. the buyer accepts for each valuation, but then $p_b = v_1$. To cover this case, let $C_1 = \{v_1\}$.

When $p_b \in C_k$, the sum of payoffs following a message from the buyer's information structure is the surplus from selling to valuations v_k and above, which we denote by $E(v_k) = \sum_{v \geq v_k} \zeta[v]v$. Thus, equilibrium payoffs (u_b, u_s) in U^{**} satisfy $u_b + u_s = \beta_s E + \beta_b E(v_k)$ for some $k \in \kappa$, where $\kappa = \{k \in \{1, \ldots, K\} : C_k \neq \emptyset\}$. In addition, u_s satisfies the following bounds when $p_b \in C_k$.[16] Let $\underline{v}_k = \inf C_k$ and $\bar{v}_k = \max C_k$ for each $k \in \kappa$;[17] since $p_s \leq E$ and, when $p_b \in C_k$, $u_s = \beta_s p_s + \beta_b p_b Z(v_k)$, it follows that $u_s \leq \beta_s E + \beta_b \bar{v}_k Z(v_k)$. Furthermore, since $p_b \leq p_s$, it follows that $u_s \geq \underline{v}_k(\beta_s + \beta_b Z(v_k))$ and, in fact, $u_s > \underline{v}_k(\beta_s + \beta_b Z(v_k))$ if $\underline{v}_k = v_{k-1}$. Corollary 1 shows that these conditions completely characterize U^{**}.

Corollary 1 *$U^{**} = \cup_{k \in \kappa} U_k$ where, for each $k \in \kappa$ such that $\underline{v}_k > v_{k-1}$,*

$$\begin{aligned} U_k = &\{(u_b, u_s) \in \mathbb{R}^2 : u_b + u_s = \beta_s E + \beta_b E(v_k), \\ &\text{and } \underline{v}_k(\beta_s + \beta_b Z(v_k)) \leq u_s \leq \beta_s E + \beta_b \bar{v}_k Z(v_k)\} \end{aligned}$$

and, for each $k \in \kappa$ such that $\underline{v}_k = v_{k-1}$,

$$\begin{aligned} U_k = &\{(u_b, u_s) \in \mathbb{R}^2 : u_b + u_s = \beta_s E + \beta_b E(v_k), \\ &\text{and } \underline{v}_k(\beta_s + \beta_b Z(v_k)) < u_s \leq \beta_s E + \beta_b \bar{v}_k Z(v_k)\}. \end{aligned}$$

It is clear from Corollary 1 that social surplus $u_s + u_b$ is maximized when $p_b = v_1$ and minimized when $p_b = E$ since $E(v_k)$ is decreasing in v_k. In addition, letting p^* be a solution to $\max_{p \in [v_1, E]} pZ(p)$, the seller's payoff is maximized when $(p_b, p_s) = (p^*, E)$ and minimized when $p_b = p_s = v_1$, i.e. for each $k \in \kappa$:

$$v_1 \leq \underline{v}_k(\beta_s + \beta_b Z(v_k)) \leq \beta_s E + \beta_b \bar{v}_k Z(v_k) \leq \beta_s E + \beta_b p^* Z(p^*).$$

3.4 Two-type example

We illustrate the main logic of our arguments using an example where $V = \{1, 2\}$ and ζ is such that $\zeta[2] = x$. Note that the expected value (and first degree price

[16]Since $u_b = \beta_s E + \beta_b E(v_k) - u_s$, these bounds could also be written in terms of u_b.

[17]Note that $C_k = (v_{k-1}, \min\{E, v_k\}] \cap [\frac{v_1}{Z(v_k)}, \min\{E, v_k\}]$, hence, $\underline{v}_k \in C_k$ if and only if $\underline{v}_k > v_{k-1}$.

discrimination payoff) is $1+x$. When $x \leq 0.5$, the uniform monopoly price is $v^* = 1$, giving uniform monopoly payoff $(u_b^*, u_s^*) = (x, 1)$, and when $x \geq 0.5$, the uniform monopoly price is $v^* = 2$, giving uniform monopoly payoff $(u_b^*, u_s^*) = (0, 2x)$. Recall that Bergemann, Brooks, and Morris (2015) show that any feasible payoff such that the seller gets at least the uniform monopoly profit can be supported under some information structure such that the buyer is fully informed. Dropping the requirement that the buyer must be fully informed, Makris and Renou (2023) show that any feasible payoff where the seller gets at least v_1 can be achieved. In our setting, in contrast, the information structure must be chosen optimally by the players and this implies that only a small subset of the payoffs identified by Makris and Renou (2023) can be sustained in a pure strategy equilibrium.[18]

In any pure strategy equilibrium, there can be at most two prices which are accepted: p_s following a message profile from the seller's information structure and p_b following a message profile from the buyer's information structure. To see this, suppose for simplicity that the buyer fully controls the information structure.[19] If the seller sets price p following m_s and $p' > p$ following m'_s, then $(m'_b, m'_s) \in \text{supp}(\phi_b(2))$ for some m'_b.[20] But then the buyer has a profitable deviation to send message profile (m'_b, m_s) instead of (m'_b, m'_s) when $v = 2$. Thus, there must be a unique price p_b that results from any message profile from the support of the buyer's information structure, and the buyer will accept $p_b = 1$ (by assumption) and $p_b \in (1, 2)$ if and only if $v = 2$. Moreover, this implies that if $p_b > 1$, then $xp_b \geq 1$, otherwise the seller

[18] We note that as $\beta_s \to 1$, the set of equilibrium payoffs in our setting approaches the efficiency frontier. Thus, relative to Makris and Renou (2023), our requirement that the information must be optimal for the seller simply rules out inefficient outcomes (in the limit). As $\beta_b \to 1$, the relevant comparison is with Ravid, Roesler, and Szentes (2022) who characterize equilibria with buyer-optimal learning. However, since sequential equilibrium in our setting allows off-path inference (as a consequence of the possibility that the seller becomes more informed than the buyer), we find a larger set of equilibrium payoffs even as $\beta_b \to 1$.

[19] The assumptions on our aggregation function imply that each player can deviate as if he fully controls the information structure.

[20] Since the buyer controls the information, he will never accept p' when $v = 1$ and if p' is never accepted then it cannot be optimal for the seller.

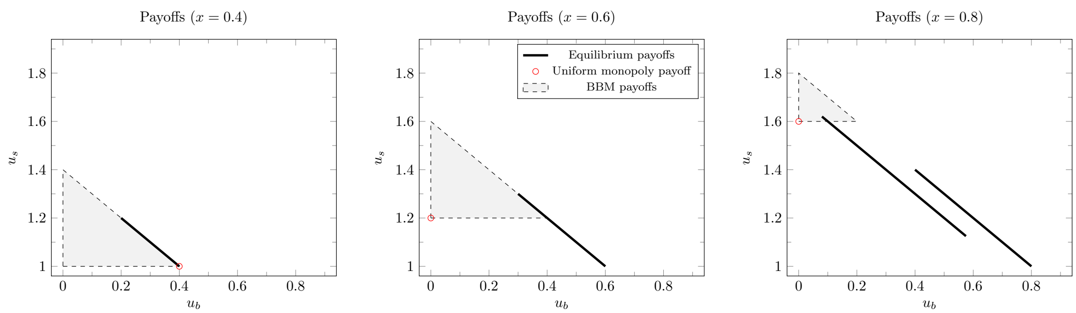

Figure 1: Equilibrium payoffs when $V = \{1, 2\}$, $\zeta[2] = x \in \{0.4, 0.6, 0.8\}$, $\beta_b = \frac{1}{2}$

has a profitable deviation to set price 1 instead of p_b.

Similar considerations imply that the seller would only choose to send message profiles that result in the highest price being accepted; thus, there is a unique price p_s following any message from the seller's information structure which is accepted. This implies that $p_s \leq 1 + x$, otherwise the buyer will not find it optimal to accept, and $p_b \leq p_s$, otherwise the buyer has a profitable deviation to choose a message profile from the seller's information structure instead of his own to induce p_s rather than p_b.

Thus, we have argued that in any equilibrium, with probability β_s, the seller will set price $p_s \leq 1 + x$ which is accepted; with probability β_b, the seller will set price $p_b \leq p_s$ which is accepted if and only if $v \geq p_b$; and either $p_b = 1$ or $xp_b \geq 1$. In the remainder of this section, we illustrate the payoffs that result from varying (p_b, p_s) subject to the above constraints.[21]

First, note that $p_b = 1$ and any $p_s \in [1, 1 + x]$ always satisfies the constraints, resulting in payoffs $\beta_b(x, 1) + \beta_s(1 + x - p_s, p_s)$, for $p_s \in [1, 1 + x]$. In other words, there are equilibria where the payoff is $(x, 1)$ when the buyer's information is chosen and can be any (individually rational) efficient payoff when the seller's information

[21]That any such (p_b, p_s) can be supported in equilibrium follows the arguments of Section 3.2: we can specify that if the buyer receives a message from the seller's information structure and price offer other than p_s or 1, he believes that his value is 1 and rejects; if he receives a message from his own information structure and price offer other than p_b or 1, he believes that his value is 1 and rejects. Such beliefs are credible if the buyer believes that sellers who learn that the valuation is 1 are more likely to deviate.

is chosen. This corresponds to the thick diagonal line that is a subset of the efficient frontier in each of the three panels of Figure 1. In the notation of Corollary 1, these are the payoffs belonging to U_1, where $u_b + u_s = E$.

If $x < \frac{-1+\sqrt{5}}{2} \approx 0.618$, then $[\frac{1}{x}, 1+x] = \emptyset$ and, thus, there are no equilibrium payoffs with $p_b > 1$.[22] In this case, all equilibrium payoffs are as described above, illustrated in the left and middle panels of Figure 1. Note that when $x \geq 0.5$, the uniform monopoly price is $v^* = 2$ and the seller's payoff from setting p_b close to 2 is greater than 1; however, such p_b violates the constraint $p_b \leq p_s$.

On the other hand, when $x \geq \frac{-1+\sqrt{5}}{2}$, there are also equilibria where $p_b \in [\frac{1}{x}, 1+x]$ and $p_s \in [p_b, 1+x]$, giving payoffs $\beta_b(x(2-p_b), xp_b) + \beta_s(1+x-p_s, p_s)$. Note that for such p_b, only buyers with value $v = 2$ accept; hence the surplus is $2x$, xp_b of which goes to the seller. Thus, in this case, there are additional equilibrium payoffs which are convex combinations of $(x(2-p_b), xp_b)$ (when the buyer's information is chosen) and any payoff on the efficient frontier where the seller gets at least p_b (when the seller's information is chosen). This corresponds to the diagonal line away from the efficient frontier in the right panel of Figure 1. In the notation of Corollary 1, these are the payoffs belonging to U_2, where $u_b + u_s = \beta_b E(2) + \beta_s E$.

3.5 More than two types

A special feature of the two-type example is that the buyer always accepts when $p_b \in V$, since $p_b = 2$ is not possible (it violates $p_b \leq p_s \leq 1+x$) and when $p_b = 1$, the buyer accepts by assumption. In general, when $p_b \in V$, it may be possible for the buyer to accept with probability $\lambda < 1$ (from the seller's perspective) by randomizing over the messages he sends himself when $p_b = v$. To illustrate this additional feature, we return to the example from Section 3.2, with $V = \{1, 2, 3, 4, 5\}$ and ζ uniform.

First, focusing on equilibrium payoffs represented by (p_b, p_s, λ) with $\lambda = 1$, note that when $p_b \in (2, 3]$, these payoffs are $\beta_b\left(\frac{(3-p_b)+(4-p_b)+(5-p_b)}{5}, \frac{3}{5}p_b\right) + \beta_s(3-p_s, p_s)$, for $p_s \in [p_b, 3]$. This corresponds to the leftmost line in the left panel of Figure 2.

[22]Since if $p_b > 1$, we must have $\frac{1}{x} \leq p_b \leq p_s \leq 1+x$.

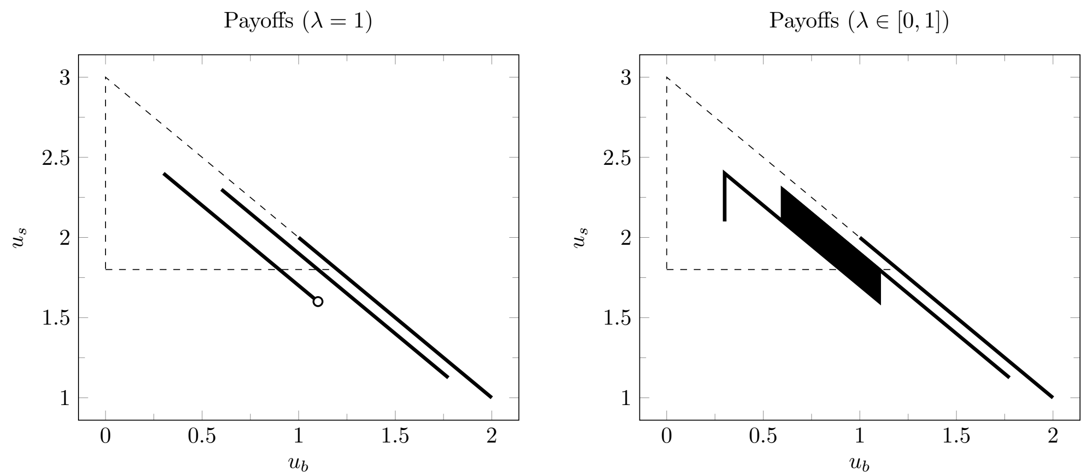

Figure 2: Equilibrium payoffs when $V = \{1, 2, 3, 4, 5\}$, ζ uniform, $\beta_b = \frac{1}{2}$

The bottom endpoint of this line is open because as $p_b \to 2$, the buyer's acceptance probability when $v = 2$ jumps from 0 to 1 in the limit; thus, the payoff represented by $(p_b, p_s, \lambda) = (2, 2, 1)$ is $\beta_b(\frac{6}{5}, \frac{8}{5}) + \beta_s(1, 2)$, which is located on the middle line directly above the bottom endpoint of the leftmost line. Allowing $\lambda = 0$, the payoff represented by $(p_b, p_s, \lambda) = (2, 2, 0)$ is $\beta_b(\frac{6}{5}, \frac{6}{5}) + \beta_s(1, 2)$, the bottom endpoint of the leftmost line, and as λ varies from 0 to 1, the payoffs represented by $(2, 2, \lambda)$ lie on the vertical line joining these two points.

Thus, when we allow all $\lambda \in [0, 1]$, for $p_b = 2$ and each $p_s \in [2, 3]$, we get payoffs $\beta_b(\frac{6}{5}, u_s^b) + \beta_s(3 - p_s, p_s)$, where $u_s^b \in [\frac{6}{5}, \frac{8}{5}]$, corresponding to the shaded region in the right panel of Figure 2. In addition, when $p_b = 3$, the buyer may also accept with any probability $\lambda \in [0, 1]$, giving payoffs $\beta_b(\frac{3}{5}, u_s^b) + \beta_s(0, 3)$, where $u_s^b \in [\frac{6}{5}, \frac{9}{5}]$, which correspond to the leftmost vertical line in the left panel of Figure 2.

As $|V| \to \infty$ with ζ uniform, $v_1 = 1$ and $v_K = 5$, the set of equilibrium payoffs is approximately equal to the set of payoffs illustrated in Figure 3. The shaded region is a union of straight lines, each being the set of convex combinations of payoffs from setting $p_b \in [1, 3]$ and points on the efficient frontier where the seller gets at least p_b. In this figure, the dotted blue line corresponds to payoffs following p_b, for each $p_b \in [1, 3]$. For example, the red circle corresponds to the payoff when $p_b = 2.5$ (the

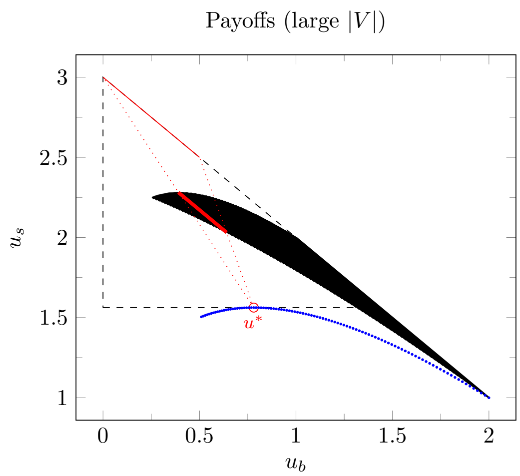

Figure 3: Equilibrium payoffs when $|V|$ is large, ζ uniform, $\beta_b = \frac{1}{2}$

uniform monopoly price in the limit), the red line on the efficient frontier corresponds to payoffs when $p_s \in [2.5, 3]$, and the thick red line corresponds to convex combinations of the payoff when $p_b = 2.5$ with these payoffs on the efficient frontier. The set of equilibrium payoffs is then (approximately) the union of such thick lines, as p_b varies from 1 to 3.[23]

4 Discussion

4.1 Interpretation of the model

Our model of joint information design consists of three key elements: (i) each player chooses a joint information structure, (ii) the information structures are aggregated into a single information structure that generates a message profile, and (iii) each player observes only his private component of the message profile. We will discuss each of these elements in turn.

[23]This is just an approximation because we are ignoring the possibility of varying λ, the effect of which disappears as $|V| \to \infty$.

4.1.1 Choosing information structures

Information is received in the form of private messages. Returning to the example from the introduction, the ratings agency may release a report to each of the parties about its findings. It could issue a public quality rating, which is captured in our model by perfectly correlated messages. A detailed report which is sufficient for the customer to learn his value but does not provide any relevant information to the seller is captured in our model as a perfectly informative message for the buyer and an uninformative message for the seller (interpreting the buyer's message as the combination of the report and his private information about his preferences).

We assume that players choose information structures directly as a reduced form representation of all the actions they can take to influence or manipulate the information they each receive. In particular, our model does not require information to be released simultaneously. Indeed, the private message in our model is just a summary of all the information received and the choice of information structure is just a reduced form representation of the (possibly sequential) information acquisition/revelation actions. In the context of our example, if the seller tells the ratings agency to accurately review the product and inform the buyer of its findings, and to report back all the relevant circumstances of the buyer that is sufficient to determine his valuation, that action is identified with choosing an information structure that provides full information about the valuation to both parties. Of course, if the buyer takes actions to prevent his valuation being revealed, such information structure will not be realized.

The role of the aggregation function is to resolve the conflict between the parties over the information they wish to be provided.

4.1.2 The aggregation function

When both players agree on some information structure they wish to be realized, each can be viewed as an information designer picking his preferred information structure. However, when there is disagreement, the aggregation function is a model of how this conflict is resolved.

The assumptions we make on the aggregation function permit a generalization of the convex combination aggregator and can be interpreted as follows.

1. Property 1 reflects the idea that the players jointly control the information structure. For example, it requires that the ratings agency may release some information that is requested by the buyer or by the seller, but it does not release information that no one asked for.[24]

   Note that property 1 is an assumption on message *profiles*: it requires that message profile m cannot arise if no player chooses a distribution with m in its support, even though both m_b and m_s may belong to the supports of the marginal distributions chosen by the players. For example, if the buyer wants the message profile to be $(1, 1)$ and the seller wants it to be $(2, 2)$, property 1 implies that $(1, 2)$ cannot arise, i.e. the messages will be perfectly correlated and the players will learn each other's message.

   This assumption makes sense in situations where each player can control both his own and the information of the other player equally well. For example, when the ratings agency favours one party, then that party can control what both players learn. This is the opposite extreme to the assumption that each player controls his own information, but has no control over the information of the other player.

2. Property 2 is a weak monotonicity requirement on the aggregation function with respect to the information choices of the players. It implies that a choice of a degenerate distribution can be interpreted as the player taking actions to maximize the probability of a particular message profile being realized. For example, the seller may want the ratings agency to give a good rating regardless of the true state. The actions that he can take to maximize the probability of the agency doing so is captured, in reduced form, by the choice of a degenerate

[24]Any information that is independently provided by the ratings agency can be reflected in the prior.

distribution on the message profile corresponding to the good rating in every state.

3. In the particular case where the players choose information structures with disjoint supports, property 3 requires that β_s (resp. β_b) is the probability that the realized message profile comes from the seller's (resp. buyer's) information structure. The parameter β_i can be interpreted as the amount of control player i has over the true information structure. For example, β_i may represent the resourcefulness or skill of player i in persuading the ratings agency, or alternatively the unmodelled preferences of the agency to favor one player over another.

   The assumption that β_i does not depend on the information structure chosen (and also that the aggregation function is the same in every state) is a simplification that amounts to assuming that whether the information is provided according to a player's wishes does not depend on the content of the information but on, e.g., the bargaining ability of the player.

If we restrict the players to choosing degenerate distributions, then properties 1–3 exactly characterize the convex combination aggregator. When all distributions are allowed, the convex combination aggregator is the unique aggregation function satisfying properties 1–3 that extends (multi) linearly from degenerate distributions, i.e. that satisfies $\beta(\gamma, \gamma') = \sum_{m \in \text{supp}(\gamma)} \sum_{m' \in \text{supp}(\gamma')} \gamma[m]\gamma'[m']\beta(1_m, 1_{m'})$ for each $(\gamma, \gamma') \in F^2$.

We end our discussion of the aggregation function by noting that properties 1 and 3 imply that each player has the ability to learn the buyer's valuation for sure with positive probability: Let $m_i \notin \cup_v \text{supp}(\phi_{-i,M_i}(v))$ be a message that player i never receives from the other player's information structure. If player i chooses ϕ_i such that $\text{supp}(\phi_{i,M_i}(v)) = \{m_i\}$ and $m_i \notin \text{supp}(\phi_{i,M_i}(v'))$ for all $v' \neq v$, then he will receive m_i with positive probability (by property 3) and believe that the buyer's valuation is v for sure (by property 1). A similar argument implies that each player also has the ability (with positive probability) to become certain that the other player has received any given message. This feature of our model will be satisfied in our example if whenever the ratings agency provides information according the seller's (resp. buyer's) wishes, it is willing to inform the seller (resp. buyer) of this fact.

4.1.3 Observability

In our model, each player observes only his own private message (and not the true information structure or the private message of the other player). This captures the idea that players have the ability to covertly manipulate the information structure and such deviations are detected only if the other player receives a message that he was not expecting in equilibrium.

For example, consumers do not directly observe the interaction between the product provider and the ratings agency, and will only detect a deviation if the agency releases an unexpected report; otherwise, each player will interpret the information under his equilibrium belief about the information structure.

Although the true information structure is not observed, under the convex combination aggregator, the players can endogenously choose to learn which information structure realizes by choosing information structures with disjoint supports.

For example, the seller may ask the ratings agency to release a superficial report that is recognised as uninformative by both players, whereas the buyer may request a detailed review that provides key information to the buyer but no new information to the seller. Then on receiving the superficial report, both players will correctly interpret it as uninformative; and on receiving the detailed review, the buyer will know his valuation and the seller will know that the buyer is fully informed.

4.1.4 Alternative interpretation: unstructured communication

An alternative interpretation of our model is as a reduced-form description of unstructured communication. For example, consider a seller negotiating directly with a potential buyer over an object of initially unknown value to the buyer. Suppose that learning takes place through a discussion between the two parties where each reveals relevant facts to the other. In this case, a private message corresponds to the result of the conversation between the buyer and seller and any other information obtained by, for example, inspecting the object.

If the seller tells the buyer all the relevant facts about the condition and history of the object and asks nothing about the buyer, then he is choosing an information structure where the buyer fully learns the valuation and the seller remains uninformed. On the other hand, if the buyer shows no interest in the seller's description and reveals relevant facts about himself, then he is choosing an information structure where the seller fully learns the valuation and the buyer remains uninformed.

This example is consistent with a more metaphorical interpretation of the aggregation function. One can imagine a debate between the buyer and seller about the value of the object, each providing the other with information. The debate concludes with each player settling on some interpretation of the facts. Then β_b can be viewed as the debating skill or persuasiveness of the buyer, so that with probability β_b, the players end up believing the interpretation the buyer advocates, and with remaining probability β_s they agree on the seller's interpretation.

4.2 Refinement

A key feature of the equilibria we construct to support Theorem 1 is that, on observing an unexpected price, the buyer believes that his valuation is low. This feature can arise in any model where the seller may have some information that the buyer does not; it is not a consequence of our requirement that the information structure is optimally chosen.[25] In this section, we introduce a refinement which forces the buyer not to update his belief based on price.

We define $\pi \in \Pi^*$ to be a sequential equilibrium *with price-independent beliefs* if π is a sequential equilibrium and, in addition, following any on-path message $m_b \in$

[25] Indeed, that there is an equilibrium where the seller sets the lowest possible price because an uninformed buyer rejects all other prices even though they are less than that his expected valuation is already established by, for example, Makris and Renou (2023) in a model with exogenous information. However, this feature is absent from Bergemann, Brooks, and Morris (2015) (where the buyer is fully informed) and Roesler and Szentes (2017) (where the seller is uninformed).

$\cup_v \text{supp}(\phi^*_{M_b}(v))$, the buyer's belief about his valuation is the same for all p and is equal to

$$E(v|m_b) = \frac{\sum_v v\zeta[v] \sum_{m_s} \beta(\phi^*_b(v), \phi^*_s(v))[m_b, m_s]}{\sum_v \zeta[v] \sum_{m_s} \beta(\phi^*_b(v), \phi^*_s(v))[m_b, m_s]}$$

(see Appendix B for a formal definition).[26] It turns out that no pure strategy equilibrium survives this refinement.

Theorem 2 *There does not exist* $\pi \in \Pi^*$ *such that* π *is a sequential equilibrium with price-independent beliefs.*

We illustrate Theorem 2 in the two-type example. Suppose that $x = 0.4$ so that $E = 1.4$ and $v^* = 1$. Recall that in any equilibrium, $p_s \leq E = 1.4$. If $p_b = 1$, then the buyer accepts p_b whatever message m_b he receives. But there must be some $m_b \in \text{supp}(\phi^*_{b,M_b}(1)) \cup \text{supp}(\phi^*_{b,M_b}(2))$ such that $E(v|m_b) > E \geq p_s$ or $E(v|m_b) = E$ for all $m_b \in \text{supp}(\phi^*_{b,M_b}(1)) \cup \text{supp}(\phi^*_{b,M_b}(2))$. In the former case, the seller has a profitable deviation to send message m_b, offer price $E(v|m_b) - \varepsilon > p_s$ which is accepted for sure. In the latter case, the seller has a profitable deviation to offer $E - \varepsilon > p_b$ instead of p_b which is accepted for sure. The argument when $p_b > 1$ is similar.[27]

The use of mixed strategies in the first period can restore existence. For example, suppose that the buyer informs himself in the following way: he first picks a number y at random from $\{1, \ldots, Y\}$ and then sends himself message $y+v$ when his valuation v, i.e. let $\phi_{b,y}(v) = 1_{(y+v,1)}$ and the buyer chooses a mixed strategy $\pi^1_b = |Y|^{-1} \sum_{y=1}^{Y} 1_{\phi_{b,y}}$. Following message $m_s = 1$, the seller faces the demand curve of a fully informed buyer; thus, $p_b = v^*$ is optimal. However, when Y is sufficiently large, the buyer's use of mixed strategies prevents the seller from deviating by sending the message corresponding to $v = 2$ because the seller does not know which y the buyer has picked. For example, if the seller sends message $Y + 2$, with probability $\frac{Y-1}{Y}$, the buyer will observe an off-path message; in such case, the buyer is no longer required

[26] We follow Kartik and Zhong (2023) with the terminology of price-independent beliefs.

[27] If $p_b > 1$, then the buyer accepts p_b if and only if $v = 2$, i.e. $\text{supp}(\phi_{b,M_b}(1)) \cap \text{supp}(\phi_{b,M_b}(2)) = \emptyset$, and following (m_b, p) for $m_b \in \text{supp}(\phi_{b,M_b}(2))$, the buyer must believe that his value is 2. But then the seller has a profitable deviation to send m_b and offer price $2 - \varepsilon > p_s$ which is accepted for sure.

to believe that $v = 2$ according to the refinement. The use of randomization by the buyer in this example implies that the meaning of each message is private to the buyer.[28]

A Proofs for Section 3

We start by spelling out the equilibrium implications on the equilibrium path in Section A.1. These will be used to establish general properties that any sequential equilibrium in pure strategies must satisfy. The proof of Theorem 1 is then in Section A.2 (necessity part) and Section A.3 (sufficiency part).

A.1 Preliminary Lemmas

Any sequential equilibrium $\pi \in \Pi$ satisfies the following conditions on the equilibrium path:

$$\begin{aligned}&\sum_v \zeta[v] \sum_m \beta(\phi_b^*(v), \phi_s^*(v))[m] u_s(\pi^2(m, \phi^*)) \geq \\ &\sum_v \zeta[v] \sum_m \beta(\phi_b^*(v), \phi_s(v))[m] u_s(\hat{\pi}^2(m, \phi_b^*, \phi_s)),\end{aligned} \tag{A.1}$$

for each $\phi_s \in \Phi$ and $\hat{\pi}_s^2 : M_s \times \Phi \to V^*$, where

$$\begin{aligned}\pi^2(m, \phi^*) &= (\pi_s^2(m_s, \phi_s^*), \pi_b^2(m_b, \phi_b^*, \pi_s^2(m_s, \phi_s^*))) \text{ and} \\ \hat{\pi}^2(m, \phi_b^*, \phi_s) &= (\hat{\pi}_s^2(m_s, \phi_s), \pi_b^2(m_b, \phi_b^*, \hat{\pi}_s^2(m_s, \phi_s))),\end{aligned}$$

$$\begin{aligned}&\sum_v \zeta[v] \sum_m \beta(\phi_b^*(v), \phi_s^*(v))[m] u_b(v, \pi^2(m, \phi^*)) \geq \\ &\sum_v \zeta[v] \sum_m \beta(\phi_b(v), \phi_s^*(v))[m] u_b(v, \hat{\pi}^2(m, \phi_b, \phi_s^*)),\end{aligned} \tag{A.2}$$

for each $\phi_b \in \Phi$ and $\hat{\pi}_b^2 : M_b \times \Phi \times V^* \to A$, where

$$\hat{\pi}^2(m, \phi_b, \phi_s^*) = (\pi_s^2(m_s, \phi_s^*), \hat{\pi}_b^2(m_b, \phi_b, \pi_s^2(m_s, \phi_s^*))),$$

[28] A similar construction by Deb, Pai, and Said (2024) uses a deliberately vague language to substitute commitment with public communication.

$$\sum_{v,m_b} \frac{\phi^*[v,m_b,m_s]}{\sum_{\hat{v},\hat{m}_b}\phi^*[\hat{v},\hat{m}_b,m_s]} u_s(\pi^2(m,\phi^*)) \geq \sum_{v,m_b} \frac{\phi^*[v,m_b,m_s]}{\sum_{\hat{v},\hat{m}_b}\phi^*[\hat{v},\hat{m}_b,m_s]} u_s(p,\pi_b^2(m_b,\phi_b^*,p)) \tag{A.3}$$

for each $m_s \in \mathbb{N}$ such that $\sum_{v,m_b} \phi^*[v,m_b,m_s] > 0$ and $p \in V^*$, and

$$\sum_{v,m_s} \frac{\phi^*[v,m_b,m_s]\pi_s^2(m_s,\phi_s^*)[p]}{\sum_{\hat{v},\hat{m}_s}\phi^*[\hat{v},m_b,\hat{m}_s]\pi_s^2(\hat{m}_s,\phi_s^*)[p]} u_b(v,p,\pi_b^2(m_b,\phi_b^*,p)) \geq \sum_{v,m_s} \frac{\phi^*[v,m_b,m_s]\pi_s^2(m_s,\phi_s^*)[p]}{\sum_{\hat{v},\hat{m}_s}\phi^*[\hat{v},m_b,\hat{m}_s]\pi_s^2(\hat{m}_s,\phi_s^*)[p]} u_b(v,p,a) \tag{A.4}$$

for each $m_b \in \mathbb{N}$ and $p \in V^*$ such that $\sum_{v,m_s} \phi^*[v,m_b,m_s]\pi_s^2(m_s,\phi_s^*)[p] > 0$ and $a \in A$.

We will use the following notation: For each $(\phi_b,\phi_s) \in \Phi^2$ and $v \in V$, let $\phi(v) = \beta(\phi_b(v),\phi_s(v))$ be the distribution from which messages are drawn when the buyer's valuation is v and players' choice of information designs are ϕ_b and ϕ_s. In the remainder of this section, we establish properties of the messages that each player sends with strictly positive probability. Thus, when the buyer's valuation is v and players' choice of information designs are ϕ_b and ϕ_s, let, for each $i,j \in N$ with $i \neq j$, $m_i \in M_i$ and $m_j \in M_j$, $S_i(v) = \text{supp}(\phi_i(v))$ denote the support of $\phi_i(v)$, i.e. the message profiles that player i sends with strictly positive probability, $S_{i,M_j}(v,m_i) = \{m_j : (m_i,m_j) \in S_i(v)\}$ be the set of messages that player i sends to player j with strictly positive probability when he sends message m_i to himself, $S_{i,M_i}(v,m_j) = \{m_i : (m_i,m_j) \in S_i(v)\}$ be the set of messages that player i sends himself with strictly positive probability when he sends message m_j to player j, $S_{i,M_j}(v) = \cup_{m_i \in M_i} S_{i,M_j}(v,m_i)$ be the set of messages that player i sends to player j with strictly positive probability, $S_{i,M_i}(v) = \cup_{m_j \in M_j} S_{i,M_i}(v,m_j)$ be the set of messages that player i sends himself with strictly positive probability and $S(v) = S_b(v) \cup S_s(v)$ be the set of message profiles that some player sends with strictly positive probability. In particular, $\phi^*(v) = \beta(\phi_b^*(v),\phi_s^*(v))$, $S_i^*(v) = \text{supp}(\phi_i^*(v))$, $S_{i,M_j}^*(v,m_i) = \{m_j : (m_i,m_j) \in S_i^*(v)\}$, $S_{i,M_i}^*(v,m_j) = \{m_i : (m_i,m_j) \in S_i^*(v)\}$, $S_{i,M_j}^*(v) = \cup_{m_i \in M_i} S_{i,M_j}^*(v,m_i)$, $S_{i,M_i}^*(v) = \cup_{m_j \in M_j} S_{i,M_i}^*(v,m_j)$ and $S^*(v) = S_b^*(v) \cup S_s^*(v)$.

Lemma A.1 shows that if a player i sends a message profile m with strictly positive probability when the buyer's valuation is v, then m is optimal for i at v in the sense that the price and acceptance decision that m induces yields a payoff for i at v as high as the price and acceptance decision that any other message m' induces.

Lemma A.1 *If* π *is a sequential equilibrium of* G*, then* $\text{supp}(\phi_i^*(v)) \subseteq \{m \in M : u_i(v, \pi^2(m, \phi^*)) = \sup_{m' \in M} u_i(v, \pi^2(m', \phi^*))\}$ *for each* $i \in N$ *and* $v \in V$*.*

Proof. Suppose not; then there is $i \in N$, $v' \in V$, $m' \in \text{supp}(\phi_i^*(v'))$ and $m^* \in M$ such that $u_i(v', \pi^2(m^*, \phi^*)) > u_i(v', \pi^2(m', \phi^*))$. We may assume in addition that $u_i(v', \pi^2(m^*, \phi^*)) \geq u_i(v', \pi^2(m, \phi^*))$ for all $m \in S^*(v')$ (it is always possible to choose m^* satisfying this condition since $S^*(v')$ is finite).

Consider first the case where $i = s$. Define ϕ_s by setting, for each $v \in V$ and $m \in M$,

$$\phi_s(v)[m] = \begin{cases} 1 & \text{if } v = v' \text{ and } m = m^*, \\ 0 & \text{if } v = v' \text{ and } m \neq m^*, \\ \phi_s^*(v)[m] & \text{otherwise,} \end{cases}$$

and let $\hat{\pi}_s^2 : M_s \times \Phi \to V^*$ be such that $\hat{\pi}_s^2(m_s, \phi_s) = \pi_s^2(m_s, \phi_s^*)$ for each $m_s \in M_s$. Then $\hat{\pi}^2(m, \phi_b^*, \phi_s) = \pi^2(m, \phi^*)$ for each $m \in M$, $\beta(\phi_b^*(v), \phi_s^*(v)) = \beta(\phi_b^*(v), \phi_s(v))$ for each $v \neq v'$, $\beta(\phi_b^*(v'), \phi_s^*(v'))[m] = \beta(\phi_b^*(v'), 1_{m^*})[m] = 0$ for each $m \notin S^*(v') \cup \{m^*\}$

(by Property 1) and

$$
\begin{aligned}
&\sum_v \zeta[v] \sum_m \Big(\beta(\phi_b^*(v), \phi_s(v))[m] u_s(\hat{\pi}^2(m, \phi_b^*, \phi_s)) - \beta(\phi_b^*(v), \phi_s^*(v))[m] u_s(\pi^2(m, \phi^*))\Big) \\
&= \sum_v \zeta[v] \sum_m u_s(\pi^2(m, \phi^*))(\beta(\phi_b^*(v), \phi_s(v))[m] - \beta(\phi_b^*(v), \phi_s^*(v))[m]) \\
&= \zeta[v']\Big((\beta(\phi_b^*(v'), 1_{m^*})[m^*] - \beta(\phi_b^*(v'), \phi_s^*(v'))[m^*]) u_s(\pi^2(m^*, \phi^*)) \\
&- \sum_{m \in S^*(v') \setminus \{m^*, m'\}} (\beta(\phi_b^*(v'), \phi_s^*(v'))[m] - \beta(\phi_b^*(v'), 1_{m^*})[m]) u_s(\pi^2(m, \phi^*)) \\
&- (\beta(\phi_b^*(v'), \phi_s^*(v'))[m'] - \beta(\phi_b^*(v'), 1_{m^*})[m']) u_s(\pi^2(m', \phi^*))\Big) \\
&\geq \zeta[v']\Big(\Big(\beta(\phi_b^*(v'), 1_{m^*})[m^*] - \beta(\phi_b^*(v'), \phi_s^*(v'))[m^*] \\
&- \sum_{m \in S^*(v') \setminus \{m^*, m'\}} (\beta(\phi_b^*(v'), \phi_s^*(v'))[m] - \beta(\phi_b^*(v'), 1_{m^*})[m])\Big) u_s(\pi^2(m^*, \phi^*)) \\
&- (\beta(\phi_b^*(v'), \phi_s^*(v'))[m'] - \beta(\phi_b^*(v'), 1_{m^*})[m']) u_s(\pi^2(m', \phi^*))\Big) \\
&= \zeta[v']\Big(\beta(\phi_b^*(v'), \phi_s^*(v'))[m'] - \beta(\phi_b^*(v'), 1_{m^*})[m']\Big)\Big(u_s(\pi^2(m^*, \phi^*)) - u_s(\pi^2(m', \phi^*))\Big) \\
&> 0
\end{aligned}
$$

where the weak inequality follows because for all $m \in S^*(v') \setminus \{m^*, m'\}$,

$$
\begin{aligned}
&u_s(\pi^2(m^*, \phi^*)) \geq u_s(\pi^2(m, \phi^*)) \text{ and} \\
&\beta(\phi_b^*(v'), \phi_s^*(v'))[m] - \beta(\phi_b^*(v'), 1_{m^*})[m] \geq 0
\end{aligned}
$$

(the latter by Property 2), the last equality follows because

$$
\begin{aligned}
&\sum_{m \in S^*(v') \setminus \{m^*\}} (\beta(\phi_b^*(v'), \phi_s^*(v'))[m] - \beta(\phi_b^*(v'), 1_{m^*})[m]) = \\
&\beta(\phi_b^*(v'), 1_{m^*})[m^*] - \beta(\phi_b^*(v'), \phi_s^*(v'))[m^*]
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\beta(\phi_b^*(v'), 1_{m^*})[m^*] - \beta(\phi_b^*(v'), \phi_s^*(v'))[m^*] \\
&- \sum_{m \in S^*(v') \setminus \{m^*, m'\}} (\beta(\phi_b^*(v'), \phi_s^*(v'))[m] - \beta(\phi_b^*(v'), 1_{m^*})[m]) = \\
&\beta(\phi_b^*(v'), \phi_s^*(v'))[m'] - \beta(\phi_b^*(v'), 1_{m^*})[m'],
\end{aligned}
$$

and the last inequality follows because

$$u_s(\pi^2(m^*, \phi^*)) > u_s(\pi^2(m', \phi^*)) \text{ and}$$
$$\beta(\phi_b^*(v'), \phi_s^*(v'))[m'] - \beta(\phi_b^*(v'), 1_{m^*})[m'] > 0$$

by Property 2. But this is a contradiction since π is a sequential equilibrium of G.

The proof for the case $i = b$ is analogous. Indeed, define ϕ_b by setting, for each $v \in V$ and $m \in M$,

$$\phi_b(v)[m] = \begin{cases} 1 & \text{if } v = v' \text{ and } m = m^*, \\ 0 & \text{if } v = v' \text{ and } m \neq m^*, \\ \phi_b^*(v)[m] & \text{otherwise,} \end{cases}$$

and let $\hat{\pi}_b^2 : M_b \times \Phi \times V^* \to A$ be such that $\hat{\pi}_b^2(m_b, \phi_b, p) = \pi_b^2(m_b, \phi_b^*, p)$ for each $(m_b, p) \in M_b \times V^*$. The remainder of the argument is as in the case $i = s$. ■

Lemma A.2 builds on Lemma A.1 to give a stronger sense in which the message profiles that a player sends with strictly positive probability are optimal. Indeed, when the buyer's valuation is v, then m is optimal for player i in the sense that the message m_{-i} sent to the other player maximizes i's value function at v and i's choice following m_i is a best-reply against $-i$'s choice following m_{-i}. Thus, for each $v \in V$ and $m_s \in M_s$, let $w_b(v, m_s) = \max_{a \in A} u_b(v, \pi_s^2(m_s, \phi_s^*), a)$ and $BR_b(v, m_s) = \{a \in A : u_b(v, \pi_s^2(m_s, \phi_s^*), a) = w_b(v, m_s)\}$ be, respectively, the buyer's value function and best-reply correspondence. Analogously, for each $m_b \in M_b$, let $w_s(m_b) = \sup_{p \in V^*} u_s(p, \pi_b^2(m_b, \phi_b^*, p))$ and $BR_s(m_b) = \{p \in V^* : u_s(p, \pi_b^2(m_b, \phi_b^*, p)) = w_s(m_b)\}$. Furthermore, for each $v \in V$ and $m_b \in M_b$, let $w_s(v, m_b) = w_s(m_b)$ and $BR_s(v, m_b) = BR_s(m_b)$.

Lemma A.2 *If π is a sequential equilibrium of G, then*

$$\text{supp}(\phi_i^*(v)) \subseteq \{m \in M : \; w_i(v, m_{-i}) = \sup_{m'_{-i} \in M_{-i}} w_i(v, m'_{-i}) \; \textit{and } \pi_i^2(m_i, \phi_i^*) \in BR_i(v, m_{-i})\}$$

for each $i \in N$ and $v \in V$, where $\pi_b^2(m_b, \phi_b^*) = \pi_b^2(m_b, \phi_b^*, \pi_s^2(m_s, \phi_s^*))$ for each $m \in M$.

Proof. Suppose not; then there is $i \in N$, $v' \in V$, $m' \in \mathrm{supp}(\phi_i^*(v'))$ and $m^* \in M$ such that (i) $w_i(v', m_{-i}^*) > w_i(v', m_{-i}')$ or (ii) $w_i(v', m_{-i}') = \sup_{\hat{m}_{-i} \in M_{-i}} w_i(v', \hat{m}_{-i})$ and $\pi_i^2(m_i', \phi_{-i}^*) \notin BR_i(v', m_{-i}')$; in case (ii), let $m^* = m'$. In addition, we may assume that $w_i(v', m_{-i}^*) \geq w_i(v', m_{-i})$ for all $m \in S^*(v')$.

Consider the case where $i = b$. Let $a^* \in BR_b(v', m_s^*)$, $\bar{m}_b \in M_b$ be such that $(\bar{m}_b, m_s^*) \notin S^*(v')$,

$$\phi_b(v) = \begin{cases} 1_{(\bar{m}_b, m_s^*)} & \text{if } v = v', \\ \phi_b^*(v) & \text{otherwise,} \end{cases}$$

and $\hat{\pi}_b^2 : M_b \times \Phi \times V^* \to A$ be such that $\hat{\pi}_b^2(\bar{m}_b, \phi_b, \pi_s^2(m_s^*, \phi_s^*)) = a^*$ and $\hat{\pi}_b^2(m_b, \phi_b, p) = \pi_b^2(m_b, \phi_b^*, p)$ for each $(m_b, p) \neq (\bar{m}_b, \pi_s^2(m_s^*, \phi_s^*))$. Then $\hat{\pi}^2(m, \phi_b, \phi_s^*) = \pi^2(m, \phi^*)$ for each $m \in M$ such that $m_b \neq \bar{m}_b$, $\beta(\phi_b^*(v), \phi_s^*(v)) = \beta(\phi_b(v), \phi_s^*(v))$ for each $v \neq v'$, $\beta(\phi_b^*(v'), \phi_s^*(v'))[m] = 0$ for each $m \notin S^*(v')$ and $\beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[m] = 0$ for each $m \notin S^*(v') \cup \{(\bar{m}_b, m_s^*)\}$ (by Property 1), $(\bar{m}_b, m_s^*) \notin S^*(v')$ and

$$\begin{aligned}
&\sum_v \zeta[v] \sum_m \Big(\beta(\phi_b(v), \phi_s^*(v))[m] u_b(v, \hat{\pi}^2(m, \phi_b, \phi_s^*)) \\
&-\beta(\phi_b^*(v), \phi_s^*(v))[m] u_b(v, \pi^2(m, \phi^*))\Big) \\
&= \zeta[v'] \Bigg(\sum_m \Big(\beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[m] u_b(v', \hat{\pi}^2(m, \phi_b, \phi_s^*)) \\
&-\beta(\phi_b^*(v'), \phi_s^*(v'))[m] u_b(v', \pi^2(m, \phi^*))\Big)\Bigg) \\
&= \zeta[v'] \Big(\beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[\bar{m}_b, m_s^*] w_b(v', m_s^*) \\
&- \sum_{m \in S^*(v') \setminus \{m'\}} (\beta(\phi_b^*(v'), \phi_s^*(v'))[m] - \beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[m]) u_b(v', \pi^2(m, \phi^*)) \\
&-(\beta(\phi_b^*(v'), \phi_s^*(v'))[m'] - \beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[m']) u_b(v', \pi^2(m', \phi^*))\Big) \\
&\geq \zeta[v'] \Big(\Big(\beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[\bar{m}_b, m_s^*] \\
&- \sum_{m \in S^*(v') \setminus \{m'\}} (\beta(\phi_b^*(v'), \phi_s^*(v'))[m] - \beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[m])\Big) w_b(v', m_s^*) \\
&-(\beta(\phi_b^*(v'), \phi_s^*(v'))[m'] - \beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[m']) u_b(v', \pi^2(m', \phi^*))\Big) \\
&= \zeta[v'] \Big(\beta(\phi_b^*(v'), \phi_s^*(v'))[m'] - \beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[m']\Big) \\
&\times \Big(w_b(v', m_s^*) - u_b(v', \pi^2(m', \phi^*))\Big)
\end{aligned}$$

where the weak inequality follows because for all $m \in S^*(v') \setminus \{m'\}$,

$$w_b(v', m_s^*) \geq w_b(v', m_s) \geq u_b(v', \pi^2(m, \phi^*)) \text{ and}$$
$$\beta(\phi_b^*(v'), \phi_s^*(v'))[m] - \beta(\phi_b^*(v'), 1_{(\bar{m}_b, m_s^*)})[m] \geq 0$$

(the latter by Property 2), and the last equality follows because

$$\beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[\bar{m}_b, m_s^*] - \sum_{m \in S^*(v') \setminus \{m'\}} (\beta(\phi_b^*(v'), \phi_s^*(v'))[m] - \beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[m])$$
$$= \beta(\phi_b^*(v'), \phi_s^*(v'))[m'] - \beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[m'].$$

By Property 2, $\beta(\phi_b^*(v'), \phi_s^*(v'))[m'] - \beta(1_{(\bar{m}_b, m_s^*)}, \phi_s^*(v'))[m'] > 0$. If $w_b(v', m_s^*) > w_b(v', m_s')$, then

$$w_b(v', m_s^*) - u_b(v', \pi^2(m', \phi^*)) \geq w_b(v', m_s^*) - w_b(v', m_s') > 0;$$

if $w_b(v', m_s^*) = w_b(v', m_s')$, then $\pi_b^2(m_b', \phi_b^*) \notin BR_b(v', m_s')$ and

$$w_b(v', m_s^*) - u_b(v', \pi^2(m', \phi^*)) > w_b(v', m_s^*) - w_b(v', m_s') \geq 0.$$

In either case, it follows that

$$\sum_v \zeta[v] \sum_m (\beta(\phi_b(v), \phi_s^*(v))[m] u_b(v, \hat{\pi}^2(m, \phi_b, \phi_s^*)) >$$
$$\sum_v \zeta[v] \sum_m (\beta(\phi_b^*(v), \phi_s^*(v))[m] u_b(v, \pi^2(m, \phi^*)).$$

But this is a contradiction since π is a sequential equilibrium.

The proof for the case $i = s$ is analogous. Let $\bar{m}_s \in M_s$ be such that $(m_b^*, \bar{m}_s) \notin S^*(v')$ and, for each $k \in \mathbb{N}$, $p_k \in V^*$ be such that $u_s(p_k, \pi_b^2(m_b^*, \phi_b^*, p_k)) > w_s(m_b^*) - 1/k$. Then let

$$\phi_s(v) = \begin{cases} 1_{(m_b^*, \bar{m}_s)} & \text{if } v = v', \\ \phi_s^*(v) & \text{otherwise,} \end{cases}$$

and $\hat{\pi}_s^2 : M_s \times \Phi \to V^*$ be such that $\hat{\pi}_s^2(\bar{m}_s, \phi_s) = p_k$ and $\hat{\pi}_s^2(m_s, \phi_s) = \pi_s^2(m_s, \phi_s^*)$ for each $m_s \neq \bar{m}_s$. An argument analogous to the one for the case $i = b$ then shows that, for each $k \in \mathbb{N}$,

$$\sum_v \zeta[v] \sum_m \Big(\beta(\phi_b^*(v), \phi_s(v))[m] u_s(\hat{\pi}^2(m, \phi_b^*, \phi_s)) - \beta(\phi_b^*(v), \phi_s^*(v))[m] u_s(\pi^2(m, \phi^*))\Big)$$
$$\geq \zeta[v'] \Big(\beta(\phi_b^*(v'), \phi_s^*(v'))[m'] - \beta(\phi_b^*(v'), 1_{(m_b^*, \bar{m}_s)})[m']\Big) \Big(w_s(m_b^*) - u_s(\pi^2(m', \phi^*))\Big)$$
$$-\frac{1}{k} \zeta[v'] \beta(\phi_b^*(v'), 1_{(m_b^*, \bar{m}_s)})[m_b^*, \bar{m}_s].$$

Since

$$\Big(\beta(\phi_b^*(v'), \phi_s^*(v'))[m'] - \beta(\phi_b^*(v'), 1_{(m_b^*, \bar{m}_s)})[m']\Big) \Big(w_s(m_b^*) - u_s(\pi^2(m', \phi^*))\Big) > 0,$$

it follows that, for each k sufficiently large,

$$\sum_v \zeta[v] \sum_m (\beta(\phi_b^*(v), \phi_s(v))[m] u_s(\hat{\pi}^2(m, \phi_b^*, \phi_s)) >$$
$$\sum_v \zeta[v] \sum_m (\beta(\phi_b^*(v), \phi_s^*(v))[m] u_s(\pi^2(m, \phi^*)).$$

But this is a contradiction since π is a sequential equilibrium. ■

A.2 Proof of Theorem 1: Necessity part

Let $\pi \in \Pi^*$ be a sequential equilibrium.

The following lemma shows that there is only one price p_s that is induced from any message that the seller sends himself with strictly positive probability. Furthermore, the buyer accepts p_s after any message that the seller sends him with strictly positive probability. This is a consequence of the optimality of messages described in Lemma A.1.

Lemma A.3 *$p(m_s) = p(m_s')$ for each $m_s, m_s' \in \cup_v S^*_{s,M_s}(v)$ and $a(m_b, p_s) = 1$ for each $m_b \in \cup_v S^*_{s,M_b}(v)$, where p_s is the common value of $p(m_s)$ for $m_s \in \cup_v S^*_{s,M_s}(v)$.*

Proof. Note first that $\max_{m^* \in \cup_v S^*(v)} u_s(\pi^2(m^*, \phi^*)) > 0$. Indeed, $u_s(\pi^2(m^*, \phi^*)) \geq 0$ for each $m^* \in \cup_v S^*(v)$ and if $\max_{m^* \in \cup_v S^*(v)} u_s(\pi^2(m^*, \phi^*)) = 0$, then $u_s(\pi) = 0$ by property 1. But then, letting $\hat{\pi}_s^1 = (\phi_s^*, \hat{\pi}_s^2)$ with $\hat{\pi}_s^2(m_s, \phi) = v_1$ for each $(m_s, \phi) \in M_s \times \Phi$, we have that $u_s(\pi_b, \hat{\pi}_s) = v_1 > 0 = u_s(\pi)$ since the price v_1 is accepted by the buyer with probability 1. But this is a contradiction to the assumption that π is a sequential equilibrium.

Let $m_s, m'_s \in \cup_v S^*_{s,M_s}(v)$ and let $v, v' \in V$ and $m_b, m'_b \in M_b$ be such that $(m_s, m_b) \in S^*_s(v)$ and $(m'_s, m'_b) \in S^*_s(v')$. Then $a(m_b, p(m_s)) = a(m'_b, p(m'_s)) = 1$ since otherwise $\max_{m^* \in \cup_v S^*(v)} u_s(\pi^2(m^*, \phi^*)) = 0$ by Lemma A.1. Hence, by Lemma A.1,

$$p(m_s) = u_s(p(m_s), a(m_b, p(m_s))) = \max_{m^* \in \cup_v S^*(v)} u_s(\pi^2(m^*, \phi^*))$$
$$= u_s(p(m'_s), a(m'_b, p(m'_s))) = p(m'_s)$$

and, since $\max_{m^* \in \cup_v S^*(v)} u_s(\pi^2(m^*, \phi^*)) > 0$, $p_s > 0$. Thus, for each $\hat{m}_b \in \cup_v S^*_{s,M_b}(v)$, $p_s = \max_{m^* \in \cup_v S^*(v)} u_s(\pi^2(m^*, \phi^*)) = p_s a(\hat{m}_b, p_s)$ and, hence, $a(\hat{m}_b, p_s) = 1$. ■

The next result is needed for Lemmas A.5 and A.6 below and simply shows that there is message profile m that the buyer sends with strictly positive probability and which induces him to accept the resulting price.

Lemma A.4 *There exist $v \in V$ and $m \in S^*_b(v)$ such that $a(m_b, p(m_s)) = 1$.*

Proof. We will show that $a(m_b, p(m_s)) = 1$ for each $m \in S^*_b(v_K)$.

We have that $p(m_s) = p_s$ and $a(m_b, p(m_s)) = 1$ for each $m \in \cup_v S^*_s(v)$ by Lemma A.3 and $\phi^*(v)[S^*_s(v)] \geq \beta_s > 0$ for each $v \in V$ by property 3. Thus, for each $v \in V$, there exists $m^v \in S^*_s(v)$ such that $p(m^v_s) = p_s$, $a(m^v_b, p(m^v_s)) = 1$ and $\phi^*[v, m^v_b, m^v_s] > 0$.

We have that $\sum_{(v,m_s):p(m_s)=p_s} \phi^*[v, m^{v_1}_b, m_s] \geq \phi^*[v_1, m^{v_1}_b, m^{v_1}_s] > 0$. Thus, by (A.4),

$$0 \leq \sum_{(v,m_s):p(m_s)=p_s} \phi^*[v, m^{v_1}_b, m_s](v - p_s).$$

If $p_s = v_K$, $(v - p_s) \leq 0$ for all v and $\sum_{(v,m_s):p(m_s)=p_s} \phi^*[v, m^{v_1}_b, m_s](v - p_s) \leq \phi^*[v_1, m^{v_1}_b, m^{v_1}_s](v_1 - p_s) < 0$. Hence, $p_s < v_K$. It follows from $p_s < v_K$ that $0 < v_K - p_s = u_b(v_K, m^{v_K}) \leq u_b(v_K, m)$ for each $m \in S^*_b(v_K)$ by Lemma A.1. Thus, $a(m_b, p(m_s)) = 1$ for each $m \in S^*_b(v_K)$. ■

Lemma A.5 shows that there is only one price p_b that is induced from any message that the buyer sends the seller with strictly positive probability and which induces him to accept at some valuation. Furthermore, the buyer accepts p_b after any message that he sends himself when his valuation is above p_b and rejects p_b after any message that he sends himself when his valuation is below p_b. This is a consequence of the optimality of messages described in Lemmas A.1 and A.2.

Lemma A.5 *$p(m_s) = p(m'_s)$ for each $m, m' \in \cup_v \{\tilde{m} \in S^*_b(v) : a(\tilde{m}_b, p(\tilde{m}_s)) = 1\}$. Furthermore, letting p_b be the common value of $p(m_s)$ for $m \in \cup_v \{\tilde{m} \in S^*_b(v) : a(\tilde{m}_b, p(\tilde{m}_s)) = 1\}$,*

$$a(m_b, p(m_s)) = \begin{cases} 1 & \text{if } v > p_b, \\ 0 & \text{if } v < p_b \end{cases}$$

for each $v \in V$ and $m \in S^*_b(v)$.

Proof. Let $m, m' \in \cup_v \{\tilde{m} \in S^*_b(v) : a(\tilde{m}_b, p(\tilde{m}_s)) = 1\}$ be such that $p(m_s) > p(m'_s)$. Then let $v, v' \in V$ be such that $(m_b, m_s) \in S^*_b(v)$, $a(m_b, p(m_s)) = 1$, $(m'_b, m'_s) \in S^*_b(v')$ and $a(m'_b, p(m'_s)) = 1$.

Consider a deviation by b to a strategy $\hat{\pi}_b = (\phi_b, \hat{\pi}^2_b)$ such that $\phi_b(v) = 1_{(m'_b, m'_s)}$, $\phi_b(\hat{v}) = \phi^*_b(\hat{v})$ for each $\hat{v} \in V \setminus \{v\}$ and $\hat{\pi}^2_b(\hat{m}_b, \phi_b, p) = \pi^2_b(\hat{m}_b, \phi^*_b, p)$ for each $(\hat{m}_b, p) \in \mathbb{N} \times V^*$. This deviation is profitable since $u_b(\hat{\pi}_b, \pi_s) - u_b(\pi)$ equals

$$\zeta[v] \sum_{\tilde{m} \notin \{m', m\}} \big(\beta(\phi_b(v), \phi^*_s(v))[\tilde{m}] - \beta(\phi^*_b(v), \phi^*_s(v))[\tilde{m}]\big) u_b(v, \pi^2(\tilde{m}, \phi^*))$$
$$+ \zeta[v]\big(\beta(\phi_b(v), \phi^*_s(v))[m] - \beta(\phi^*_b(v), \phi^*_s(v))[m]\big) u_b(v, \pi^2(m, \phi^*))$$
$$+ \zeta[v]\big(\beta(\phi_b(v), \phi^*_s(v))[m'] - \beta(\phi^*_b(v), \phi^*_s(v))[m']\big) u_b(v, \pi^2(m', \phi^*)).$$

Lemma A.1 implies that $u_b(v, \pi^2(\tilde{m}, \phi^*)) \leq u_b(v, \pi^2(m, \phi^*))$ for each $\tilde{m} \notin \{m', m\}$ since $m \in S^*_b(v)$. Furthermore, $u_b(v, \pi^2(m, \phi^*)) = v - p(m_s) < v - p(m'_s) = u_b(v, \pi^2(m', \phi^*))$. Thus,

$$u_b(\hat{\pi}_b, \pi_s) - u_b(\pi) \geq$$
$$\zeta[v]\big(\beta(\phi_b(v), \phi^*_s(v))[m'] - \beta(\phi^*_b(v), \phi^*_s(v))[m']\big)\big(u_b(v, \pi^2(m', \phi^*)) - u_b(v, \pi^2(m, \phi^*))\big) =$$
$$\zeta[v]\big(\beta(\phi_b(v), \phi^*_s(v))[m'] - \beta(\phi^*_b(v), \phi^*_s(v))[m']\big)\big(p(m_s) - p(m'_s)\big) > 0$$

since $\beta(\phi_b(v), \phi_s^*(v))[m'] > \beta(\phi_b^*(v), \phi_s^*(v))[m']$ by property 2. But this contradicts the assumption that π is a sequential equilibrium.

Finally, let $v \in V$ and $m \in S_b^*(v)$. First, suppose that $p(m_s) = p_b$. Then $a(m_b, p_b) \in BR_b(v, m_s)$ by Lemma A.2, hence $a(m_b, p_b) = 1$ if $v > p_b$ and $a(m_b, p_b) = 0$ if $v < p_b$. In case $p(m_s) \neq p_b$, $a(m_b, p(m_s)) = 0$. By Lemma A.2, $\sup_{m_s'} w_b(v, m_s') = 0$ and by Lemma A.4, there exists m_s such that $p(m_s) = p_b$; hence $v \leq p_b$. ■

For each $v \in V$, let $\beta[v] = \sum_{m \in S_s^*(v)} \phi^*(v)[m]$ be the $\phi^*(v)$-probability of $S_s^*(v)$. Then

$$\sum_{m \in S_b^*(v) \setminus S_s^*(v)} \phi^*(v)[m] = 1 - \beta[v]$$

by property 1. If $p_b \in V$, let

$$\Lambda = \{m \in S_b^*(p_b) \setminus S_s^*(p_b) : a(m_b, p_b) = 1\}$$

be the set of message profiles that the buyer sends and the seller doesn't at $v = p_b$ and which induce acceptance of p_b, and $\lambda \in [0, 1]$ be the $\phi^*(p_b)$-probability of Λ conditional on $S_b^*(p_b) \setminus S_s^*(p_b)$, i.e.

$$\lambda(1 - \beta[p_b]) = \sum_{m \in \Lambda} \phi^*(p_b)[m];$$

if $p_b \notin V$, then let $\lambda = 0$. We will show in what follows that, adjusting λ if necessary to λ^* (to be defined below), (1)–(4) in the statement of Theorem 1 hold for p_b, p_s and λ^*.

It follows by Lemmas A.3 and A.5 that

$$\begin{aligned} u_b &= \sum_v \zeta[v]\beta[v](v - p_s) + \sum_{v \geq p_b} \zeta[v](1 - \beta[v])(v - p_b) \text{ and} \\ u_s &= p_s \sum_v \zeta[v]\beta[v] + p_b \zeta[p_b](1 - \beta[p_b])\lambda + p_b \sum_{v > p_b} \zeta[v](1 - \beta[v]). \end{aligned}$$

In the case where $p_b \neq p_s$, it follows that, for each $v \in V$, $S_s^*(v) \cap S_b^*(v) = \emptyset$ by Lemmas A.3 and A.5. Indeed, if $m \in S_s^*(v) \cap S_b^*(v)$, then $p(m_s) = p_s$ and $a(m_b, p(m_s)) = 1$ by Lemma A.3. Hence, Lemma A.5 implies that $p(m_s) = p_b \neq p_s = p(m_s)$, a contradiction.

It then follows by property 3 that $\beta[v] = \beta_s$ and $1 - \beta[v] = \beta_b$ and, thus,

$$u_b = \beta_s \left(\sum_v \zeta[v] v - p_s \right) + \beta_b \left(\sum_{v \geq p_b} \zeta[v](v - p_b) \right) \text{ and}$$
$$u_s = \beta_s p_s + \beta_b (p_b \sum_{v > p_b} \zeta[v] + p_b \zeta[p_b] \lambda).$$

Consider next the case $p_b = p_s$ and let $p = p_b = p_s$. Then

$$\begin{aligned} u_b &= \sum_{v>p} \zeta[v](v-p) + \sum_{v \leq p} \zeta[v]\beta[v](v-p) \leq \sum_{v>p} \zeta[v](v-p) + \sum_{v \leq p} \zeta[v]\beta_s(v-p) \\ &= \beta_s \left(\sum_v \zeta[v] v - p_s \right) + \beta_b \left(\sum_{v \geq p_b} \zeta[v](v - p_b) \right) \text{ and} \\ u_s &= p \sum_{v>p} \zeta[v] + p\zeta[p](\beta[p] + (1-\beta[p])\lambda) + p \sum_{v<p} \zeta[v]\beta[v] \\ &\geq p \sum_{v>p} \zeta[v] + p\zeta[p](\beta_s + \beta_b \lambda) + p \sum_{v<p} \zeta[v]\beta_s \\ &= \beta_s p_s + \beta_b (p_b \sum_{v>p_b} \zeta[v] + p_b \zeta[p_b]\lambda). \end{aligned}$$

For each $v \in V$, let $\hat{\beta}[v] = \sum_{m \in S_b^*(v)} \phi^*(v)[m]$. Then

$$\sum_{m \in S_s^*(v) \setminus S_b^*(v)} \phi^*(v)[m] = 1 - \hat{\beta}[v]$$

by property 1. If $p_b \in V$, let

$$\hat{\Lambda} = \{m \in S_b^*(p_b) : a(m_b, p_b) = 1\}, \text{ and} \tag{A.5}$$
$$\hat{\lambda}\hat{\beta}[p_b] = \sum_{m \in \hat{\Lambda}} \phi^*(p_b)[m]; \tag{A.6}$$

if $p_b \notin V$, then let $\hat{\lambda} = 0$. Thus, when $p_b \in V$, $\hat{\lambda} \in [0, 1]$ is the $\phi^*(p_b)$-probability of $\hat{\Lambda}$ conditional on $S_b^*(p_b)$ and $\hat{\Lambda}$ is the set of message profiles that the buyer sends at $v = p_b$ and which induce acceptance of p_b.

It follows by Lemmas A.3 and A.5 that

$$u_b = \sum_v \zeta[v](1 - \hat{\beta}[v])(v - p_s) + \sum_{v \geq p_b} \zeta[v]\hat{\beta}[v](v - p_b) \text{ and}$$
$$u_s = p_s \sum_v \zeta[v](1 - \hat{\beta}[v]) + p_b \zeta[p_b]\hat{\beta}[p_b]\hat{\lambda} + p_b \sum_{v > p_b} \zeta[v]\hat{\beta}[v].$$

Then

$$\begin{aligned}
u_b &= \sum_{v>p} \zeta[v](v-p) + \sum_{v\leq p} \zeta[v](1-\hat{\beta}[v])(v-p) \\
&\geq \sum_{v>p} \zeta[v](v-p) + \sum_{v\leq p} \zeta[v](1-\beta_b)(v-p) \\
&= \beta_s \left(\sum_{v} \zeta[v]v - p_s \right) + \beta_b \left(\sum_{v\geq p_b} \zeta[v](v-p_b) \right) \text{ and} \\
u_s &= p\sum_{v>p} \zeta[v] + p\zeta[p](\hat{\beta}[p]\hat{\lambda} + (1-\hat{\beta}[p])) + p\sum_{v<p} \zeta[v](1-\hat{\beta}[v]) \\
&\leq p\sum_{v>p} \zeta[v] + p\zeta[p](\beta_b\hat{\lambda} + 1 - \beta_b) + p\sum_{v<p} \zeta[v](1-\beta_b) \\
&= \beta_s p_s + \beta_b(p_b \sum_{v>p_b} \zeta[v] + p_b\zeta[p_b]\hat{\lambda}).
\end{aligned}$$

It follows that

$$u_b = \beta_s \left(\sum_{v} \zeta[v]v - p_s \right) + \beta_b \left(\sum_{v\geq p_b} \zeta[v](v-p_b) \right). \tag{A.7}$$

Since $\lambda \geq 0$ and $\hat{\lambda} \leq 1$, we have that

$$\beta_s p_s + \beta_b(p_b \sum_{v>p_b} \zeta[v]) \leq u_s \leq \beta_s p_s + \beta_b(p_b \sum_{v>p_b} \zeta[v] + p_b\zeta[p_b]).$$

Thus, for some $\lambda^* \in [0,1]$,

$$u_s = \beta_s p_s + \beta_b(p_b \sum_{v>p_b} \zeta[v] + \lambda^* p_b\zeta[p_b]). \tag{A.8}$$

It follows by what has been shown above that (1) and (2) in the statement of Theorem 1 hold. Letting $\lambda^* = \lambda$ in the case $p_b \neq p_s$, we now establish conditions (3) and (4).

Lemma A.6 *$p_s \geq p_b$, $p_s \leq \sum_v \zeta[v]v$ and*

$$v_1 \leq p_b \sum_{v>p_b} \zeta[v] + p_b\zeta[p_b]\lambda^*.$$

Proof. Let, by Lemma A.4, $(v,m) \in V \times M$ be such that $m \in S_b^*(v)$ and $a(m_b, p(m_s)) = 1$. Thus, $u_s(\pi^2(m,\phi^*)) = p_b$ by Lemma A.5. Let $m' \in \cup_v S_s^*(v)$; then $u_s(\pi^2(m',\phi^*)) = p_s$ by Lemma A.3. Hence, it follows by Lemma A.1 that $p_s \geq p_b$.

We next show that $p_s \leq \sum_v \zeta[v]v$. Suppose not; then $p_s > \sum_v \zeta[v]v$. Let $\bar{m}_b \in M_b$ be such that $(\bar{m}_b, m_s) \notin \cup_v S^*(v)$ for each $m_s \in M_s$ and $\bar{m}_s \in M_s$ be such that, for some $m_b \in M_b$, $(m_b, \bar{m}_s) \in \cup_v S_b^*(v)$ and $a(m_b, p(\bar{m}_s)) = 1$. We have that $\bar{m}_b$ exists since $\cup_v S^*(v)$ is finite, $\bar{m}_s$ exists by Lemma A.4 and $p(\bar{m}_s) = p_b$ by Lemma A.5. Let ϕ_b be defined by setting, for each $v \in V$,

$$\phi_b(v) = \begin{cases} \phi_b^*(v) & \text{if } v < p_b, \\ 1_{(\bar{m}_b, \bar{m}_s)} & \text{if } v \geq p_b. \end{cases}$$

Let $\phi(v) = \beta(\phi_b(v), \phi_s^*(v))$, $S_s(v) = S_s^*(v)$ and $S_b(v) = \text{supp}(\phi_b(v))$. Then $S_b(v) \cap S_s(v) = \emptyset$ for each $v \in V$. This is clear if $v \geq p_b$ by the choice of $\bar{m}_b$. If $v < p_b$ and $m \in S_b(v) \cap S_s(v) = S_b^*(v) \cap S_s^*(v)$, then $p(m_s) = p_s$ and $a(m_b, p_s) = 1$ by Lemma A.3. Thus, by Lemma A.5, $p(m_s) = p_b$ and, therefore, $p_s = p_b$. Furthermore, $a(m_b, p_b) = 0$ implying that $1 = a(m_b, p_s) = a(m_b, p_b) = 0$, a contradiction.

It then follows that, for each $v \in V$, $\phi(v)[S_b(v)] \geq \beta_b$, $\phi(v)[S_s(v)] \geq \beta_b$ and

$$1 = \phi(v)[S_b(v)] + \phi(v)[S_s(v)] \geq \beta_b + \beta_s = 1$$

by properties 1 and 3. Thus, $\phi(v)[S_b(v)] = \beta_b$ and $\phi(v)[S_s(v)] = \beta_s$ for each $v \in V$.

Consider $\hat{\pi}_b^2$ defined by setting, for each $(m_b, \hat{\phi}, p) \in \mathbb{N} \times \Phi \times V^*$,

$$\hat{\pi}_b^2(m_b, \hat{\phi}, p) = \begin{cases} 1 & \text{if } m_b = \bar{m}_b, \\ 0 & \text{otherwise.} \end{cases}$$

Letting $\hat{\pi}_b = (\phi_b, \hat{\pi}_b^2)$ and $\hat{u}_b = u_b(\hat{\pi}_b, \pi_s)$, it follows that

$$\hat{u}_b = \sum_v \zeta[v] \left(\sum_{m \in S_b(v)} \phi(v)[m] u_b(v, \hat{\pi}^2(m, \phi_b, \phi_s^*)) + \sum_{m \in S_s(v)} \phi(v)[m] u_b(v, \hat{\pi}^2(m, \phi_b, \phi_s^*)) \right).$$

We have that $u_b(v, \hat{\pi}^2(m, \phi_b, \phi_s^*)) = 0$ for each $v \in V$ and $m \in S_s(v)$ since $m \in S_s(v) = S_s^*(v)$ implies that $m_b \neq \bar{m}_b$ and, hence, $\hat{\pi}_b^2(m_b, \phi_b, p_s) = 0$. Similarly, $u_b(v, \hat{\pi}^2(m, \phi_b, \phi_s^*)) = 0$ for each $v < p_b$ and $m \in S_b(v)$ since $m \in S_b(v) = S_b^*(v)$ implies that $m_b \neq \bar{m}_b$ and, hence, $\hat{\pi}_b^2(m_b, \phi_b, p_s) = 0$. Furthermore, $u_b(v, \hat{\pi}^2(m, \phi_b, \phi_s^*)) = v - p_b$ for each $v \geq p_b$ and $m \in S_b(v)$ since $S_b(v) = \{(\bar{m}_b, \bar{m}_s)\}$, $p(\bar{m}_s) = p_b$ and

$\hat{\pi}_b^2(\bar{m}_b, \phi_b, p_b) = 1$. Thus,

$$\hat{u}_b = \sum_{v \geq p_b} \zeta[v]\phi(v)[S_b(v)](v - p_b) = \beta_b \left(\sum_{v \geq p_b} \zeta[v](v - p_b) \right)$$
$$> \beta_s \left(\sum_v \zeta[v]v - p_s \right) + \beta_b \left(\sum_{v \geq p_b} \zeta[v](v - p_b) \right) = u_b$$

since $p_s > \sum_v \zeta[v]v$ and $\beta_s > 0$. But this is a contradiction since π is a sequential equilibrium.

Finally, we show that $v_1 \leq p_b \sum_{v > p_b} \zeta[v] + p_b\zeta[p_b]\lambda^*$. Suppose not; then $v_1 > p_b \sum_{v > p_b} \zeta[v] + p_b\zeta[p_b]\lambda^*$. Let $\bar{m}_s \in M_s$ be such that $(m_b, \bar{m}_s) \notin \cup_v S^*(v)$ for each $m_b \in M_b$ and $\bar{m}_b \in M_b$ be such that, for some $m_s \in M_s$, $(\bar{m}_b, m_s) \in \cup_v S_s^*(v)$. We have that $\bar{m}_s$ exists since $\cup_v S^*(v)$ is finite, $\bar{m}_b$ exists since $\cup_v S_s^*(v) \neq \emptyset$ and $a(\bar{m}_b, p_s) = 1$ by Lemma A.3. Let ϕ_s be defined by setting, for each $v \in V$,

$$\phi_s(v) = 1_{(\bar{m}_b, \bar{m}_s)}.$$

Let $\phi(v) = \beta(\phi_b^*(v), \phi_s(v))$, $S_b(v) = S_b^*(v)$ and $S_s(v) = \text{supp}(\phi_s(v))$. Then $S_b(v) \cap S_s(v) = \emptyset$ for each $v \in V$. It then follows that, for each $v \in V$, $\phi(v)[S_b(v)] \geq \beta_b$, $\phi(v)[S_s(v)] \geq \beta_s$ and

$$1 = \phi(v)[S_b(v)] + \phi(v)[S_s(v)] \geq \beta_b + \beta_s = 1$$

by properties 1 and 3. Thus, $\phi(v)[S_b(v)] = \beta_b$ and $\phi(v)[S_s(v)] = \beta_s$ for each $v \in V$.

Consider $\hat{\pi}_s^2$ defined by setting, for each $(m_s, \hat{\phi}) \in \mathbb{N} \times \Phi$,

$$\hat{\pi}_s^2(m_s, \hat{\phi}) = \begin{cases} p_s & \text{if } m_s = \bar{m}_s, \\ v_1 & \text{otherwise.} \end{cases}$$

Letting $\hat{\pi}_s = (\phi_s, \hat{\pi}_s^2)$ and $\hat{u}_s = u_s(\pi_b, \hat{\pi}_s)$, it follows that

$$\hat{u}_s = \sum_v \zeta[v] \left(\sum_{m \in S_b(v)} \phi(v)[m]u_s(\hat{\pi}^2(m, \phi_b^*, \phi_s)) + \sum_{m \in S_s(v)} \phi(v)[m]u_s(\hat{\pi}^2(m, \phi_b^*, \phi_s)) \right).$$

We have that $u_s(\hat{\pi}^2(m, \phi_b^*, \phi_s)) = p_s$ for each $v \in V$ and $m \in S_s(v)$ since then $m = (\bar{m}_b, \bar{m}_s)$, $\hat{\pi}_s^2(\bar{m}_s, \phi_s) = p_s$ and $a(\bar{m}_b, p_s) = 1$. Furthermore, $u_s(\hat{\pi}^2(m, \phi_b^*, \phi_s)) = v_1$ for each $v \in V$ and $m \in S_b(v)$ since then $m_s \neq \bar{m}_s$, $\hat{\pi}_s^2(m_s, \phi_s) = v_1$ and $a(m_b, v_1) = 1$. Thus,

$$\hat{u}_s = \beta_s p_s + \beta_b v_1 > \beta_s p_s + \beta_b (p_b \sum_{v > p_b} \zeta[v] + p_b \zeta[p_b] \lambda^*) = u_s$$

since $v_1 > p_b \sum_{v > p_b} \zeta[v] + p_b \zeta[p_b] \lambda^*$ and $\beta_b > 0$. But this is a contradiction since π is a sequential equilibrium. ■

The necessity part of the theorem then follows from (A.7), (A.8) and Lemma A.6.

A.3 Proof of Theorem 1: Sufficiency part

Let $(p_b, p_s) \in (V^*)^2$ and $\lambda \in [0, 1]$ be as in the statement of the theorem, u_b defined by (1) and u_s defined by (2). We will show that $(u_b, u_s) \in U^*$ by showing that there is a sequential equilibrium $\pi \in \Pi^*$ with payoff (u_b, u_s) when the aggregation function β is such that $\beta(\gamma, \gamma') = \beta_b \gamma + \beta_s \gamma'$ for each $\gamma, \gamma' \in F$. It is clear that $\beta \in \mathcal{B}$.

We will construct a sequential equilibrium π with the desired payoff. A sequential equilibrium is, by definition, a perfect conditional ε-equilibrium for each $\varepsilon > 0$ and this requires the existence of a net $\{\pi^\alpha, p^\alpha\}_\alpha$ such that the following six properties hold. The first five require that $\{\pi^\alpha\}_\alpha$ is a net of strategies converging to π that assigns strictly positive probability to each choice of information design, price offer and acceptance decision sufficiently far in its tail, and $\{p^\alpha\}_\alpha$ is a net of nature's choices regarding the probability distribution of message profiles for each profile of information designs (ϕ_b, ϕ_s) that converges to $\zeta \otimes (\beta_b \phi_b + \beta_s \phi_s)$ and assigns strictly positive probability to each message profile sufficiently far in its tail:

1. For each α, $p^\alpha : \Phi^2 \to \Delta(V \times M)$ is measurable and π^α is a behavioral strategy,[29]

[29] I.e. $\pi_s^\alpha = (\pi_s^{1,\alpha}, \pi_s^{2,\alpha})$ is such that $\pi_s^{1,\alpha} \in \Delta(\Phi)$ and $\pi_s^{2,\alpha} : \mathbb{N} \times \Phi \to \Delta(V^*)$ is measurable, and $\pi_b^\alpha = (\pi_b^{1,\alpha}, \pi_b^{2,\alpha})$ such that $\pi_b^{1,\alpha} \in \Delta(\Phi)$ and $\pi_b^{2,\alpha} : \mathbb{N} \times \Phi \times V^* \to \Delta(A)$ is measurable.

2. For each $i \in N$, $\sup_{B\in\mathcal{B}(\Phi)} |\pi_i^{1,\alpha}[B] - 1_{\phi_i^*}[B]| \to 0$,[30]

$$\sup_{(m,\phi)\in\mathbb{N}\times\Phi, B\in\mathcal{B}(V^*)} |\pi_s^{2,\alpha}(m,\phi)[B] - \pi_s^2(m,\phi)[B]| \to 0, \text{ and}$$

$$\sup_{(m,\phi,p)\in\mathbb{N}\times\Phi\times V^*, a\in A} |\pi_b^{2,\alpha}(m,\phi,p)[a] - \pi_b^2(m,\phi,p)[a]| \to 0,$$

3. For each $i \in N$, $m \in \mathbb{N}$, $\phi \in \Phi$, $p \in V^*$ and $a \in A$, there is $\bar{\alpha}$ such that $\pi_i^{1,\alpha}[\phi] > 0$, $\pi_s^{2,\alpha}(m,\phi)[p] > 0$ and $\pi_b^{2,\alpha}(m,\phi,p)[a] > 0$ for each $\alpha \geq \bar{\alpha}$,

4. $\sup_{\phi\in\Phi^2, v\in V, B\subseteq M} |p^\alpha(\phi)[\{v\}\times B] - \zeta[v]\sum_{i\in N}\beta_i\phi_i(v)[B]| \to 0$,

5. For each $\phi \in \Phi^2$, $v \in V$ and $m \in M$, there is $\bar{\alpha}$ such that $p^\alpha(\phi)[v,m] > 0$ for each $\alpha \geq \bar{\alpha}$.

A final condition requires that, for each α, π^α is such that the payoff that each player obtains by following it at each information set which is reached with strictly positive probability is within ε of his maximum payoff conditional on that information set:

6.(a) For each $i \in N$ and $\phi_i' \in \Phi$,

$$\sum_{\phi\in\text{supp}(\pi^{1,\alpha})} \pi^{1,\alpha}[\phi] \sum_{(v,m)\in V\times\mathbb{N}^2} p^\alpha(\phi)[v,m]u_i(v,\pi^{2,\alpha}(m,\phi)) \geq$$
$$\sum_{\phi_j\in\text{supp}(\pi_j^{1,\alpha})} \pi_j^{1,\alpha}[\phi_j] \sum_{(v,m)\in V\times\mathbb{N}^2} p^\alpha(\phi_i',\phi_j)[v,m]u_i(v,\pi^{2,\alpha}(m,\phi_i',\phi_j)) - \varepsilon,$$

where $\pi^{1,\alpha} = \prod_{i\in N}\pi_i^{1,\alpha}$, $j \neq i$ and, for each $\phi \in \Phi^2$ and $m \in \mathbb{N}^2$, $\pi^{2,\alpha}(m,\phi) \in \Delta(V^*\times A)$ is defined by setting, for each $(p,a) \in V^*\times A$, $\pi^{2,\alpha}(m,\phi)[p,a] = \pi_s^{2,\alpha}(m_s,\phi_s)[p]\pi_b^{2,\alpha}(m_b,\phi_b,p)[a]$,

6.(b) For each $(m_s,\phi_s) \in \mathbb{N}\times\Phi$ such that $\pi_s^{1,\alpha}[\phi_s]\sum_{\phi_b\in\Phi}\pi_b^{1,\alpha}[\phi_b]p_{M_s}^\alpha(\phi_b,\phi_s)[m_s] > 0$ and $p \in V^*$,

$$\frac{\sum_{\phi_b\in\text{supp}(\pi_b^{1,\alpha})}\pi_b^{1,\alpha}[\phi_b]\left(\sum_{(v,m_b)}p^\alpha(\phi_b,\phi_s)[v,m]u_s(\pi^{2,\alpha}(m,\phi))\right)}{\sum_{\phi_b\in\text{supp}(\pi_b^{1,\alpha})}\pi_b^{1,\alpha}[\phi_b]p_{M_s}^\alpha(\phi_b,\phi_s)[m_s]} \geq$$
$$\frac{\sum_{\phi_b\in\text{supp}(\pi_b^{1,\alpha})}\pi_b^{1,\alpha}[\phi_b]\left(\sum_{(v,m_b)}p^\alpha(\phi_b,\phi_s)[v,m]u_s(p,\pi_b^{2,\alpha}(m_b,\phi_b,p))\right)}{\sum_{\phi_b\in\text{supp}(\pi_b^{1,\alpha})}\pi_b^{1,\alpha}[\phi_b]p_{M_s}^\alpha(\phi_b,\phi_s)[m_s]} - \varepsilon.$$

[30]We let $\mathcal{B}(\Phi)$ denote the class of Borel measurable subsets of Φ and, for each $\phi \in \Phi$, 1_ϕ denote the probability measure on Φ degenerate at ϕ. Analogous definitions apply when Φ is replaced with V^*.

6.(c) For each $(m_b, \phi_b, p) \in \mathbb{N} \times \Phi \times V^*$ such that

$$\pi_b^{1,\alpha}[\phi_b] \sum_{\phi_s \in \text{supp}(\pi_s^{1,\alpha})} \pi_s^{1,\alpha}[\phi_s] \sum_{(v,m_s)} p^\alpha(\phi_s, \phi_b)[v, m_b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s)[p] > 0$$

and $a \in A$,

$$\frac{\sum_{\phi_s \in \text{supp}(\pi_s^{1,\alpha})} \pi_s^{1,\alpha}[\phi_s] \left(\sum_{(v,m_s)} p^\alpha(\phi_b, \phi_s)[v, m]\pi_s^{2,\alpha}(m_s, \phi_s)[p]u_b(v, p, \pi_b^{2,\alpha}(m_b, \phi_b, p))\right)}{\sum_{\phi_s \in \text{supp}(\pi_s^{1,\alpha})} \pi_s^{1,\alpha}[\phi_s] \sum_{(v,m_s)} p^\alpha(\phi_s, \phi_b)[v, m]\pi_s^{2,\alpha}(m_s, \phi_s)[p]} \geq$$
$$\frac{\sum_{\phi_s \in \text{supp}(\pi_s^{1,\alpha})} \pi_s^{1,\alpha}[\phi_s] \left(\sum_{(v,m_s)} p^\alpha(\phi_b, \phi_s)[v, m]\pi_s^{2,\alpha}(m_s, \phi_s)[p]u_b(v, p, a)\right)}{\sum_{\phi_s \in \text{supp}(\pi_s^{1,\alpha})} \pi_s^{1,\alpha}[\phi_s] \sum_{(v,m_s)} p^\alpha(\phi_s, \phi_b)[v, m]\pi_s^{2,\alpha}(m_s, \phi_s)[p]} - \varepsilon.$$

We define π by first specifying that the seller chooses an information design that does not depend on the buyer's valuation and sends a constant message profile $\bar{m}^s$. On the equilibrium path, the seller will set price p_s following message $\bar{m}_s^s$ and the buyer will accept price p_s following message $\bar{m}_b^s$.

We specify the information design of the buyer in such a way that, for each player i, the messages that he can receive are different from $\bar{m}_i^s$. The seller will receive only one message $\bar{m}_s^b \neq \bar{m}_s^s$; the buyer will receive one of two messages, $\bar{m}_b^b \neq \bar{m}_b^s$ and $\tilde{m}_b^b \notin \{\bar{m}_b^b, \bar{m}_b^s\}$, depending on whether his valuation is larger or smaller than p_b. Specifically, the message profile is $\bar{m}^b$ if $v > p_b$ and $(\tilde{m}_b^b, \bar{m}_s^b)$ if $v < p_b$; furthermore, if $v = p_b$, then the message profile is $\bar{m}^b$ with probability λ and $(\tilde{m}_b^b, \bar{m}_s^b)$ with probability $1 - \lambda$. On the equilibrium path, the seller will set price p_b following message $\bar{m}_s^b$; the buyer will accept price p_b if his message is $\bar{m}_b^b$ and reject price p_b if his message is $\tilde{m}_b^b$.

We will then define perturbations such that, following ϕ_b^*, whenever the buyer receives any price offer other than p_b following message $\bar{m}_b^b$ (or other than $\{p_b, p_s\}$ following message $\bar{m}_b^s$), he believes that his value is v_1.[31] In addition, whenever the buyer receives a zero-probability message following ϕ_b^*, he believes that his value is v_1 and whenever the seller receives a zero-probability message, he believes that the buyer knows that his value is v_K and hence sets price v_K.

[31] After the off-path history $(\bar{m}_b^s, \phi_b^*, p_b)$, our perturbations will imply that the buyer's belief is $\sum_v \zeta[v]v$ and so we specify that the buyer accepts following this history; this does not create any incentive for the seller to deviate by offering p_b since $p_b \leq p_s$.

Defining strategies. More formally, let $\bar{m}^b, \bar{m}^s \in M$ with $\bar{m}^b_j \neq \bar{m}^s_j$ for each $j \in \{s, b\}$ and $\tilde{m}^b_b \in M_b \setminus \{\bar{m}^b_b, \bar{m}^s_b\}$. For each $v \in V$, define

$$\phi^*_s(v) = 1_{\bar{m}^s}$$

and

$$\phi^*_b(v) = \begin{cases} 1_{\bar{m}^b} & \text{if } v > p_b, \\ \lambda 1_{\bar{m}^b} + (1-\lambda) 1_{(\tilde{m}^b_b, \bar{m}^b_s)} & \text{if } v = p_b, \\ 1_{(\tilde{m}^b_b, \bar{m}^b_s)} & \text{if } v < p_b. \end{cases}$$

For each $(m_b, m_s, p) \in \mathbb{N}^2 \times V^*$, let

$$\pi^2_s(m_s, \phi^*_s) = \begin{cases} p_s & \text{if } m_s = \bar{m}^s_s, \\ p_b & \text{if } m_s = \bar{m}^b_s, \\ v_K & \text{otherwise} \end{cases}$$

and

$$\pi^2_b(m_b, \phi^*_b, p) = \begin{cases} 1 & \text{if } m_b = \bar{m}^s_b \text{ and } p \in \{p_s, p_b\} \\ 1 & \text{if } m_b = \bar{m}^b_b \text{ and } p = p_b, \\ 1 & \text{if } p = v_1, \\ 0 & \text{otherwise.} \end{cases}$$

The above specifies the strategy of each player i following his own design ϕ^*_i and next we specify each player's strategy following his choice of a design different from ϕ^*_i. For the seller, as above, we specify that whenever he receives a zero-probability message, he believes that the buyer knows that his value is v_K; if he receives a nonzero-probability message, then he best-replies given the conditional probability of the buyer's acceptance.

For each $m_s \in M_s$ and $\phi_s \neq \phi^*_s$ such that $\sum_v \zeta[v](\beta_b \phi^*_b(v) + \beta_s \phi_s(v))_{M_s}[m_s] = 0$, let $\pi^2_s(m_s, \phi_s) = v_K$.

For each $m_s \in M_s$ and $\phi_s \neq \phi^*_s$ such that $\sum_v \zeta[v](\beta_b \phi^*_b(v) + \beta_s \phi_s(v))_{M_s}[m_s] > 0$, let $\pi^2_s(m_s, \phi_s)$ maximize

$$p \sum_{m_b} \frac{\sum_v \zeta[v](\beta_b \phi^*_b(v) + \beta_s \phi_s(v))[m_b, m_s]}{\sum_v \zeta[v](\beta_b \phi^*_b(v) + \beta_s \phi_s(v))_{M_s}[m_s]} \pi^2_b(m_b, \phi^*_b, p).$$

Let $P^* = \{p_b, p_s, v_1\}$; since $\pi_b^2(m_b, \phi_b^*, p) = 0$ for all $p \notin P^*$ and $m_b \in M_b$, the maximum is attained over P^*.

We may assume that $\pi_s^2 : M_s \times \Phi \to V^*$ is measurable. Note first that $M_s \times \Phi = \cup_{r=1}^3 B_r$ with

$$
\begin{aligned}
&B_1 = M_s \times \{\phi_s^*\},\\
&B_2 = \{(m_s, \phi_s) \in (M_s \setminus \{\bar{m}_s^b\}) \times (\Phi \setminus \{\phi_s^*\}) : \sum_v \zeta[v]\phi_s(v)_{M_s}[m_s] = 0\},\\
&B_3 = \{(m_s, \phi_s) \in (M_s \setminus \{\bar{m}_s^b\}) \times (\Phi \setminus \{\phi_s^*\}) : \sum_v \zeta[v]\phi_s(v)_{M_s}[m_s] > 0\}\\
&\quad \cup (\{\bar{m}_s^b\} \times (\Phi \setminus \{\phi_s^*\})).
\end{aligned}
$$

Indeed, B_1 is closed, B_3 is open and B_2 is the intersection of the closed set $\{(m_s, \phi_s) \in M_s \times \Phi : \sum_v \zeta[v]\phi_s(v)_{M_s}[m_s] = 0\}$ with the open set $(M_s \setminus \{\bar{m}_s^b\}) \times (\Phi \setminus \{\phi_s^*\})$. Then, for each measurable $B \subseteq V^*$, note that $(\pi_s^2)^{-1}(B) \cap B_1 = C_1 \cup C_2 \cup C_3$, where:

$$
C_1 = \begin{cases} \{(\bar{m}_s^b, \phi_s^*)\} & \text{if } B \cap \{p_b\} \neq \emptyset, \\ \emptyset & \text{otherwise,} \end{cases}
$$

$$
C_2 = \begin{cases} \{(\bar{m}_s^s, \phi_s^*)\} & \text{if } B \cap \{p_s\} \neq \emptyset, \\ \emptyset & \text{otherwise,} \end{cases}
$$

$$
C_3 = \begin{cases} \{(m_s, \phi_s^*) : m_s \notin \{\bar{m}_s^b, \bar{m}_s^s\}\} & \text{if } B \cap \{v_K\} \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}
$$

Thus, $(\pi_s^2)^{-1}(B) \cap B_1$ is the union of measurable sets, and hence, measurable. For each measurable $B \subseteq V^*$, $(\pi_s^2)^{-1}(B) \cap B_2 = B_2$ if $v_K \in B$ and $(\pi_s^2)^{-1}(B) \cap B_2 = \emptyset$ otherwise; hence $(\pi_s^2)^{-1}(B) \cap B_2$ is measurable. Finally, regarding $(\pi_s^2)^{-1}(B) \cap B_3$, for each $(m_s, \phi_s) \in B_3$, let $f : B_3 \times P^* \to [0, 1]$ be defined by setting, for each $(m_s, \phi_s) \in B_3$ and $p \in P^*$,

$$
f(m_s, \phi_s, p) = \sum_{m_b} \frac{\sum_v \zeta[v](\beta_b\phi_b^*(v) + \beta_s\phi_s(v))[m_b, m_s]}{\sum_v \zeta[v](\beta_b\phi_b^*(v) + \beta_s\phi_s(v))_{M_s}[m_s]} \pi_b^2(m_b, \phi_b^*, p),
$$

and let $\chi : B_3 \rightrightarrows P^*$ be defined by $\chi(m_s, \phi_s) = \arg\max_{p \in P^*} pf(m_s, \phi_s, p)$. Note that for each $p \in P^*$, $\chi^l(\{p\}) = \{(m_s, \phi_s) \in B_3 : pf(m_s, \phi_s, p) \geq p' f(m_s, \phi_s, p')$ for all $p' \in$

$P^*\}$ is closed in B_3, and hence measurable. Thus, χ is weakly measurable and has a measurable selection by the Kuratowski-Ryll-Nardzewski Selection Theorem (e.g. Aliprantis and Border (2006, Theorem 18.13, p. 600)).

For the buyer, if he receives a nonzero-probability message-price pair, then he best-replies given the conditional expected valuation. For each $(m_b, p) \in M_b \times V^*$ and $\phi_b \neq \phi_b^*$ such that $\sum_{\{m_s:\pi_s^2(m_s,\phi_s^*)=p\}} \sum_v \zeta[v](\beta_s\phi_s^*(v) + \beta_b\phi_b(v))[m_b, m_s] > 0$, let $\pi_b^2(m_b, \phi_b, p) = 1$ if and only if:

$$\frac{\sum_{\{m_s:\pi_s^2(m_s,\phi_s^*)=p\}} \sum_v \zeta[v](\beta_b\phi_b(v) + \beta_s\phi_s^*(v))[m_b, m_s]v}{\sum_{\{m_s:\pi_s^2(m_s,\phi_s^*)=p\}} \sum_v \zeta[v](\beta_b\phi_b(v) + \beta_s\phi_s^*(v))[m_b, m_s]} \geq p. \tag{A.9}$$

For each $(m_b, p) \in M_b \times V^*$ and $\phi_b \neq \phi_b^*$ such that $\sum_{\{m_s:\pi_s^2(m_s,\phi_s^*)=p\}} \sum_v \zeta[v](\beta_s\phi_s^*(v)+ \beta_b\phi_b(v))[m_b, m_s] = 0$, we will define $\pi_b^2(m_b, \phi_b, p)$ after the following net $\{\pi^\alpha, p^\alpha\}_\alpha$ has been defined.

Defining perturbations. Consider $\{\pi^\alpha, p^\alpha\}_\alpha$ defined as follows: The index set consists of $(\tau, T, \hat{T}, \tilde{T})$ such that $\tau \in \mathbb{N}$, T is a finite subset of $\mathbb{N}$, $\hat{T}$ is a finite subset of Φ and $\tilde{T}$ is a finite subset of V^*; this set is partially ordered by defining $(\tau', T', \hat{T}', \tilde{T}') \geq (\tau, T, \hat{T}, \tilde{T})$ if $\tau' \geq \tau$, $T \subseteq T'$, $\hat{T} \subseteq \hat{T}'$ and $\tilde{T} \subseteq \tilde{T}'$. If X is a finite set, let $\mathcal{U}_X \in \Delta(X)$ be uniform on X. Let

$$j = \max\{\tau, |T|, |\hat{T}|, |\tilde{T}|\}.$$

The perturbation p^α of nature's choice is such that $p^\alpha(\phi)[v, m]$ is $\zeta[v](\beta_b\phi_b(v)[m]+ \beta_s\phi_s(v)[m])$ with probability $1 - \frac{1}{j^j}$ and uniform on $V \times T^2$ otherwise:

$$p^\alpha(\phi)[v, m] = (1 - j^{-j})\zeta[v](\beta_b\phi_b(v) + \beta_s\phi_s(v))[m] + j^{-j}\mathcal{U}_{V\times T^2}[v, m].$$

Players will be more likely to make mistakes than nature; thus beliefs off-the-equilibrium path will come from perturbations to players' strategies. The perturbation of the seller's strategy is such that the most likely scenario in which (an off-path) message m_b and price p can occur is when $v = v_1$ and a seller's information design that reveals that v is indeed v_1 realizes. This can be achieved by defining, for each $m_b \in M_b$, $\phi_s^{m_b}$ so that it sends the seller a message perfectly revealing the buyer's valuation (e.g. message m_s^v when the valuation is v) and sends the buyer message m_b

only if the valuation is v_1; otherwise it sends the buyer message $\bar{m}_b^b$. We then make $\phi_s^{m_b}$ followed by a second period strategy where the seller sets a random price the most likely mistake of the seller that results in the buyer receiving message m_b and price p.

Formally, for each $(T, \hat{T}, \tilde{T})$, define:

$$\Phi(T,\hat{T}) = \{\phi \in \hat{T} : \operatorname{supp}(\phi) \subseteq T^2\} \text{ and}$$
$$P(T,\hat{T},\tilde{T}) = \tilde{T} \cup \{p_s, p_b, v_K\} \cup \{\pi_s^2(m_s,\phi_s) : m_s \in T \cup \{\bar{m}_s^b\}, \phi_s \in \Phi(T,\hat{T})\}.$$

For each $v \in V$, define $m_s^v \in M_s \setminus \{\bar{m}_s^b\}$ such that $v \mapsto m_s^v$ is one-to-one. For each $m_b \in M_b$, let $\phi_s^{m_b}$ be such that $\phi_s^{m_b}(v_1) = 1_{(m_b, m_s^{v_1})}$ and $\phi_s^{m_b}(v) = 1_{(\bar{m}_b^b, m_s^v)}$ for $v \neq v_1$. Let $\pi_s^{m_b,\alpha}$ be such that $\pi_s^{m_b,1,\alpha} = \phi_s^{m_b}$ and $\pi_s^{m_b,2,\alpha}$ be such that $\pi_s^{m_b,2,\alpha}(m_s^{v_1}, \phi_s^{m_b}) = \mathcal{U}_{P(T,\hat{T},\tilde{T})}$, $\pi_s^{m_b,2,\alpha}(m_s, \phi_s^{m_b}) = p_b$ for $m_s \neq m_s^{v_1}$ and $\pi_s^{m_b,2,\alpha}(m_s,\phi_s) = p_b$ for all $m_s \in M_s$ and $\phi_s \neq \phi_s^{m_b}$. Let $\hat{\pi}_s^\alpha$ be such that $\hat{\pi}_s^{1,\alpha} = \mathcal{U}_{\Phi(T,\hat{T})}$ and $\hat{\pi}_s^{2,\alpha}(m_s,\phi_s) = \mathcal{U}_{P(T,\hat{T},\tilde{T})}$ for all m_s, ϕ_s. Let, for each $t = 1, 2$,

$$\pi_s^{t,\alpha} = (1 - j^{-1})\pi_s^t + j^{-1}(1 - j^{-j})|T|^{-1} \sum_{m_b \in T} \pi_s^{m_b,t,\alpha} + j^{-1} j^{-j} \hat{\pi}_s^{t,\alpha}.$$

The perturbation of the buyer's strategy is analogous. The most likely scenario in which a message m_s can occur is when $v = v_K$ and a buyer's information design that reveals that v is indeed v_K realizes. This can be achieved by defining, for each $m_s \in M_s$, $\phi_b^{m_s}$ so that it sends the buyer a message perfectly revealing the buyer's valuation (e.g. message m_b^v when the valuation is v) and sends the seller message m_s only if the valuation is v_K; otherwise it sends the seller message $\bar{m}_s^s$.

For each $v \in V$, define $m_b^v \in M_b \setminus \{\bar{m}_b^s\}$ such that $v \mapsto m_b^v$ is one-to-one. For each $m_s \in M_s$, let $\phi_b^{m_s}$ be such that $\phi_b^{m_s}(v_K) = 1_{(m_b^{v_K}, m_s)}$ and $\phi_b^{m_s}(v) = 1_{(m_b^v, \bar{m}_s^s)}$ for $v \neq v_K$. Let $\pi_b^{m_s,1,\alpha} = \phi_b^{m_s}$. Let $\hat{\pi}_b^{1,\alpha} = \mathcal{U}_{\Phi(T,\hat{T})}$. Let:

$$\pi_b^{1,\alpha} = (1 - j^{-1})\phi_b^* + j^{-1}(1 - j^{-j})|T|^{-1} \sum_{m_s \in T} \pi_b^{m_s,1,\alpha} + j^{-1} j^{-j} \hat{\pi}_b^{1,\alpha}.$$

It remains to complete the definition of π_b^2 and to define $\pi_b^{2,\alpha}$. Regarding the former, for each $(m_b, p) \in M_b \times V^*$ and $\phi_b \neq \phi_b^*$ such that $\sum_{\{m_s : \pi_s^2(m_s,\phi_s^*) = p\}} \sum_v \zeta[v](\beta_s \phi_s^*(v) +$

$\beta_b\phi_b(v))[m_b, m_s] = 0$, let $\pi_b^2(m_b, \phi_b, p) = 1$ if and only if

$$\lim_{\alpha} \frac{\int_\Phi \left(\sum_{(v,m_s)} p^\alpha(\phi_b, \phi_s)[v, m_b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s)[p]v\right) \mathrm{d}\pi_s^{1,\alpha}[\phi_s]}{\int_\Phi \sum_{(v,m_s)} p^\alpha(\phi_b, \phi_s)[v, m_b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s)[p]\mathrm{d}\pi_s^{1,\alpha}[\phi_s]} \geq p. \tag{A.10}$$

Finally, let $\hat{\pi}_b^{2,\alpha}(m_b, \phi_b, p) = \mathcal{U}_A$ and $\pi_b^{2,\alpha}(m_b, \phi_b, p) = (1 - j^{-1})\pi_b^2(m_b, \phi_b, p) + j^{-1}\hat{\pi}_b^{2,\alpha}(m_b, \phi_b, p)$ for each m_b, ϕ_b, p.

Let $\hat{P} = \{p_s, p_b, v_K\}$ and note that $\pi_s^2(m_s, \phi_s^*) \in \hat{P}$ for each $m_s \in M_s$. Thus,

$$\Big\{(\phi_b, m_b, p) \in (\Phi \setminus \{\phi_b^*\}) \times M_b \times V^* : \\ \sum_{\{m_s : \pi_s^2(m_s, \phi_s^*) = p\}} \sum_v \zeta[v](\beta_b\phi_b(v) + \beta_s\phi_s^*(v))[m_b, m_s] > 0 \text{ and (A.9) holds}\Big\}$$

is measurable since it equals

$$\cup_{m_b \in M_b, p \in \hat{P}} \Big(\Big\{\phi_b \in \Phi \setminus \{\phi_b^*\} : \sum_{\{m_s : \pi_s^2(m_s, \phi_s^*) = p\}} \sum_v \zeta[v](\beta_b\phi_b(v) + \beta_s\phi_s^*(v))[m_b, m_s] > 0\Big\} \\ \cap \Big\{\phi_b \in \Phi : \text{(A.9) holds}\Big\}\Big) \times \{(m_b, p)\},$$

$$\Big\{\phi_b \in \Phi \setminus \{\phi_b^*\} : \sum_{\{m_s : \pi_s^2(m_s, \phi_s^*) = p\}} \sum_v \zeta[v](\beta_b\phi_b(v) + \beta_s\phi_s^*(v))[m_b, m_s] > 0\Big\}$$

is open and

$$\{\phi_b \in \Phi : \text{(A.9) holds}\}$$

is closed. The set

$$\Big\{(\phi_b, m_b, p) \in (\Phi \setminus \{\phi_b^*\}) \times M_b \times V^* : \\ \sum_{\{m_s : \pi_s^2(m_s, \phi_s^*) = p\}} \sum_v \zeta[v](\beta_b\phi_b(v) + \beta_s\phi_s^*(v))[m_b, m_s] = 0 \text{ and (A.10) holds}\Big\}$$

is also measurable since it equals the intersection of the complement of

$$\cup_{m_b \in M_b, p \in \hat{P}} \Big\{\phi_b \in \Phi \setminus \{\phi_b^*\} : \sum_{\{m_s : \pi_s^2(m_s, \phi_s^*) = p\}} \sum_v \zeta[v](\beta_b\phi_b(v) + \beta_s\phi_s^*(v))[m_b, m_s] > 0\Big\} \\ \times \{(m_b, p)\}$$

with the closed set $\Big\{(\phi_b, m_b, p) \in \Phi \times M_b \times V^* : \text{(A.10) holds}\Big\}$ and the open set $\Phi \setminus \{\phi_b^*\} \times M_b \times V^*$.

Proof that the conditions of sequential equilibrium are satisfied. Let $\varepsilon > 0$. We have that conditions 1–5 in the definition of perfect conditional ε-equilibrium hold by construction. Conditions 6.(b) and 6.(c) are imposed on positive probability histories given α; to that end, it will be useful to note that our construction of $\{\pi^\alpha, p^\alpha\}_\alpha$ is such that, for each α and $(m_s, \phi_b, \phi_s) \in \mathbb{N} \times \Phi^2$, $\text{supp}(p^\alpha(\phi_b, \phi_s))$, $\text{supp}(\pi_b^{1,\alpha})$, $\text{supp}(\pi_s^{1,\alpha})$ and $\text{supp}(\pi_s^{2,\alpha}(m_s, \phi_s))$ are all finite. Moreover, if $\pi_b^{1,\alpha}[\phi_b] > 0$ and $\pi_s^{1,\alpha}[\phi_s] > 0$, then

$$\phi_b \in \Phi_b^\alpha := \{\phi_b^*\} \cup \{\phi_b^{m_s} : m_s \in T\} \cup \Phi(T, \hat{T}) \text{ and}$$
$$\phi_s \in \Phi_s^\alpha := \{\phi_s^*\} \cup \{\phi_s^{m_b} : m_b \in T\} \cup \Phi(T, \hat{T}).$$

If $(m_s, \phi_s) \in \mathbb{N} \times \Phi$ is such that $\sum_{\phi_b \in \Phi} \pi_b^{1,\alpha}[\phi_b] p^\alpha_{M_s}(\phi_b, \phi_s)[m_s] > 0$, then $m_s \in \cup_v \text{supp}(\phi_s(v)_{M_s}) \cup \{\bar{m}_s^b\} \cup T$, and if $(m_b, \phi_b, p) \in \mathbb{N} \times \Phi \times V^*$ is such that

$$\sum_{\phi_s \in \text{supp}(\pi_s^{1,\alpha})} \pi_s^{1,\alpha}[\phi_s] \sum_{(v, m_s) \in \text{supp}(p^\alpha(\phi_b, \phi_s))} p^\alpha(\phi_b, \phi_s)[v, m_b, m_s] \pi_s^{2,\alpha}(m_s, \phi_s)[p] > 0,$$

then $m_b \in \cup_v \text{supp}(\phi_b(v)_{M_b}) \cup \{\bar{m}_b^s\} \cup T$ and $p \in P(T, \hat{T}, \tilde{T})$.

We will show that condition 6 holds for some subnet of $\{\pi^\alpha, p^\alpha\}_\alpha$. In particular, for each $(T, \hat{T}, \tilde{T})$, we will show that there exists a $\tau(T, \hat{T}, \tilde{T})$ such that for each $\alpha = (\tau, T, \hat{T}, \tilde{T})$ with $\tau \geq \tau(T, \hat{T}, \tilde{T})$, condition 6 is satisfied.

Consider condition 6.(a) with $i = s$ and $\phi_i' \in \Phi$. The left-hand side converges to $u_s = \beta_b p_b (\sum_{v > p_b} \zeta[v] + \zeta[p_b]\lambda) + \beta_s p_s$ and, when $\varepsilon = 0$, the right-hand side, for any $\phi_s' \in \Phi$, is at most:

$$\begin{aligned}
&(1 - j^{-1})^3 (1 - j^{-j}) \Big(\beta_b p_b (\sum_{v > p_b} \zeta[v] + \zeta[p_b]\lambda) \\
&+ \beta_s \sum_{v,m} \zeta[v] \phi_s'(v)[m_b, m_s] \pi_s^2(m_s, \phi_s') \pi_b^2(m_b, \phi_b^*, \pi_s^2(m_s, \phi_s')) \Big) \\
&+ (1 - (1 - j^{-1})^3 (1 - j^{-j})) v_K \\
&\leq (1 - j^{-1})^3 (1 - j^{-j}) \left(\beta_b p_b (\sum_{v > p_b} \zeta[v] + \zeta[p_b]\lambda) + \beta_s \sum_{v,m} \zeta[v] \phi_s'(v)[m_b, m_s] p_s \right) \\
&+ (1 - (1 - j^{-1})^3 (1 - j^{-j})) v_K
\end{aligned}$$

since v_K is the maximum payoff for the seller and $\pi_b^2(m_b, \phi_b^*, \pi_s^2(m_s, \phi_s')) = 0$ if

$\pi_s^2(m_s, \phi_s') > p_s$. Thus, the inequality holds (uniformly across $\phi_i' \in \Phi$) for each α such that τ (and hence j) is sufficiently large.

Consider next condition 6.(a) with $i = b$. The left-hand side converges to $u_b = \beta_b \sum_{v \geq p_b} \zeta[v](v - p_b) + \beta_s(\sum_v \zeta[v]v - p_s)$ and, when $\varepsilon = 0$, the right-hand side, for any $\phi_b' \in \Phi$, is at most:

$$
\begin{aligned}
&(1-j^{-1})^3(1-j^{-j})\Big(\beta_b \sum_{v,m} \zeta[v]\phi_b'(v)[m_b, m_s](v - \pi_s^2(m_s, \phi_s^*))\pi_b^2(m_b, \phi_b', \pi_s^2(m_s, \phi_s^*)) \\
&+\beta_s(\sum_v \zeta[v]v - p_s)\Big) + (1 - (1-j^{-1})^3(1-j^{-j}))v_K \\
&\leq (1-j^{-1})^3(1-j^{-j})\Big(\beta_b \sum_{v,m} \zeta[v]\phi_b'(v)[m_b, m_s](v - p_b)\pi_b^2(m_b, \phi_b', \pi_s^2(m_s, \phi_s^*)) \\
&+\beta_s(\sum_v \zeta[v]v - p_s)\Big) + (1 - (1-j^{-1})^3(1-j^{-j}))v_K \\
&\leq (1-j^{-1})^3(1-j^{-j})\left(\beta_b \sum_{v \geq p_b} \zeta[v](v - p_b) + \beta_s(\sum_v \zeta[v]v - p_s)\right) \\
&+(1 - (1-j^{-1})^3(1-j^{-j}))v_K
\end{aligned}
$$

since v_K is an upper bound on the buyer's payoff and $\pi_s^2(m_s, \phi_s^*) \geq p_b$ for each $m_s \in M_s$. Thus, the inequality holds (uniformly across $\phi_i' \in \Phi$) for each α such that τ (and hence j) is sufficiently large.

Let τ_a be such that condition 6.(a) holds for each α such that $\tau \geq \tau_a$.

Consider next condition 6.(b). We establish it by considering several cases.

Case 1: $\phi_s = \phi_s^*$ and $m_s = \bar{m}_s^s$. In the limit and when $\varepsilon = 0$, the inequality is $p_s \geq p\pi_b^2(\bar{m}_b^s, \phi_b^*, p)$. It holds since $p_b \leq p_s$ and

$$
p\pi_b^2(\bar{m}_b^s, \phi_b^*, p) = \begin{cases} p & \text{if } p = p_b, \\ 0 & \text{if } p \neq p_b. \end{cases}
$$

By similar arguments as for condition 6.(a), for sufficiently large τ (and hence j), the inequality in fact holds uniformly across all $p \in V^*$. Let τ_{b1} be such that condition 6.(b) holds for $(m_s, \phi_s) = (\bar{m}_s^s, \phi_s^*)$ for α such that $\tau \geq \tau_{b1}$.

Case 2: $\phi_s = \phi_s^*$ and $m_s = \bar{m}_s^b$. In the limit and when $\varepsilon = 0$, the inequality is

$$p_b(\sum_{v>p_b} \zeta[v] + \zeta[p_b]\lambda) \geq p\Big(\sum_{v<p_b} \zeta[v]\pi_b^2(\tilde{m}_b^b, \phi_b^*, p)$$
$$+ \zeta[p_b](\lambda\pi_b^2(\bar{m}_b^b, \phi_b^*, p) + (1-\lambda)\pi_b^2(\tilde{m}_b^b, \phi_b^*, p)) + \sum_{v>p_b} \zeta[v]\pi_b^2(\bar{m}_b^b, \phi_b^*, p)\Big).$$

It holds since $p_b(\sum_{v>p_b} \zeta[v] + \zeta[p_b]\lambda) \geq v_1$ and its right-hand side is equal to v_1 if $p = v_1$ and zero if $p > v_1$ and $p \neq p_b$. Thus, the inequality holds for τ sufficiently large (uniformly across $p \in V^*$). Let τ_{b2} be such that condition 6.(b) holds for $(m_s, \phi_s) = (\bar{m}_s^b, \phi_s^*)$ for α such that $\tau \geq \tau_{b2}$

Case 3: $\phi_s = \phi_s^*$ and $m_s \notin \{\bar{m}_s^s, \bar{m}_s^b\}$. Note that we only need to consider $m_s \in T$ in this case since otherwise $\sum_{\phi_b \in \Phi} \pi_b^{1,\alpha}[\phi_b]p_{M_s}^{\alpha}(\phi_b, \phi_s)[m_s] = 0$. Given that $m_s \in T$, in the limit (as $\tau \to \infty$, i.e. we can keep T fixed) and when $\varepsilon = 0$, the inequality is

$$v_K\pi_b^2(m_b^{v_K}, \phi_b^{m_s}, v_K) \geq p\pi_b^2(m_b^{v_K}, \phi_b^{m_s}, p).$$

We have that $\pi_b^2(m_b^{v_K}, \phi_b^{m_s}, v_K) = 1$ since $\phi_b^{m_s}(v_K)[m_b^{v_K}, m_s] > 0$ and

$$\frac{\sum_{v,\hat{m}_s} v\zeta[v](\beta_b\phi_b^{m_s}(v) + \beta_s\phi_s^*(v))[m_b^{v_K}, \hat{m}_s]}{\sum_{v',m_s'} \zeta[v'](\beta_b\phi_b^{m_s}(v') + \beta_s\phi_s^*(v'))[m_b^{v_K}, m_s']} = v_K.$$

Hence, the inequality holds in the limit and, thus, for τ sufficiently large (uniformly across $p \in V^*$). For each $m_s \in T \setminus \{\bar{m}_s^s, \bar{m}_s^b\}$, let $\tau_{b3}(m_s)$ be such that condition 6.(b) holds for (m_s, ϕ_s^*), for each α such that $\tau \geq \tau_{b3}(m_s)$, and let $\tau_{b3}(T) = \max_{m_s \in T \setminus \{\bar{m}_s^s, \bar{m}_s^b\}} \tau_{b3}(m_s)$. Note that for all $\alpha = (\tau, T, \hat{T}, \tilde{T})$ such that $\tau \geq \tau_{b3}(T)$, condition 6.(b) holds for all $(m_s, \phi_s) \in \{(m_s, \phi_s) : m_s \in T \setminus \{\bar{m}_s^s, \bar{m}_s^b\}, \phi_s = \phi_s^*\}$.

Case 4: $\phi_s \neq \phi_s^*$ and $m_s \in M_s$ such that $\sum_{v,m_b} \zeta[v](\beta_b\phi_b^*(v) + \beta_s\phi_s(v))[m_b, m_s] > 0$. Note that we only have to consider $\phi_s \in \Phi_s^{\alpha} \setminus \{\phi_s^*\}$ and $\Phi_s^{\alpha} \setminus \{\phi_s^*\}$ is finite. In the limit and with $\varepsilon = 0$, the inequality is

$$\pi_s^2(\phi_s, m_s) \sum_{m_b} \frac{\sum_v \zeta[v](\beta_b\phi_b^*(v) + \beta_s\phi_s(v))[m_b, m_s]}{\sum_v \zeta[v](\beta_b\phi_b^*(v) + \beta_s\phi_s(v))_{M_s}[m_s]} \pi_b^2(m_b, \phi_b^*, \pi_s^2(\phi_s, m_s))$$
$$\geq p \sum_{m_b} \frac{\sum_v \zeta[v](\beta_b\phi_b^*(v) + \beta_s\phi_s(v))[m_b, m_s]}{\sum_v \zeta[v](\beta_b\phi_b^*(v) + \beta_s\phi_s(v))_{M_s}[m_s]} \pi_b^2(m_b, \phi_b^*, p),$$

which holds by definition. For each $(T, \hat{T})$, let $\tau_{b4}(T, \hat{T})$ be such that condition 6.(b) holds for all $\phi_s \in \Phi_s^{\alpha} \setminus \{\phi_s^*\}$ and $m_s \in M_s$ such that $\sum_{v,m_b} \zeta[v](\beta_b\phi_b^*(v) + \beta_s\phi_s(v))[m_b, m_s] > 0$, for $\alpha = (\tau, T, \hat{T}, \tilde{T})$ such that $\tau \geq \tau_{b4}(T, \hat{T})$.

Case 5: $\phi_s \neq \phi_s^*$ and $m_s \in M_s$ such that $\sum_{v,m_b} \zeta[v](\beta_b\phi_b^*(v)+\beta_s\phi_s(v))[m_b,m_s] = 0$. This is as in case 3. For each $(T,\hat{T})$, let $\tau_{b5}(T,\hat{T})$ be such that condition 6.(b) holds for all such (m_s,ϕ_s), for $\alpha = (\tau,T,\hat{T},\tilde{T})$ such that $\tau \geq \tau_{b5}(T,\hat{T})$.

For each $(T,\hat{T})$, let $\tau_b(T,\hat{T}) = \max\{\tau_{b1},\tau_{b2},\tau_{b3}(T),\tau_{b4}(T,\hat{T}),\tau_{b5}(T,\hat{T})\}$.

Consider next condition 6.(c). We establish this condition by considering several cases.

Case 1: $\phi_b = \phi_b^*$, $p = p_s$ and $m_b = \bar{m}_b^s$. Since $\pi_b^2(\bar{m}_b^s,\phi_b^*,p_s) = 1$, we may consider $a = 0$. Thus, in the limit and with $\varepsilon = 0$, the inequality is $\sum_v \zeta[v]v - p_s \geq 0$, which holds. Let τ_{c1} be such that condition 6.(c) holds for $(m_b,\phi_b,p) = (\bar{m}_b^s,\phi_b^*,p_s)$, for α such that $\tau \geq \tau_{c1}$.

Case 2: $\phi_b = \phi_b^*$, $p = p_b$ and $m_b = \bar{m}_b^b$. Since $\pi_b^2(\bar{m}_b^b,\phi_b^*,p_b) = 1$, we may consider $a = 0$. Thus, in the limit and with $\varepsilon = 0$, the inequality is

$$\frac{\sum_{v>p_b} \zeta[v](v-p_b) + \zeta[p_b]\lambda(p_b-p_b)}{\sum_{v>p_b} \zeta[v] + \zeta[p_b]\lambda} \geq 0,$$

which holds. Let τ_{c2} be such that condition 6.(c) holds for $(m_b,\phi_b,p) = (\bar{m}_b^b,\phi_b^*,p_b)$, for α such that $\tau \geq \tau_{c2}$.

Case 3: $\phi_b = \phi_b^*$, $p = p_b$ and $m_b = \tilde{m}_b^b$. Since $\pi_b^2(\tilde{m}_b^b,\phi_b^*,p_b) = 0$, we may consider $a = 1$. Thus, in the limit and with $\varepsilon = 0$, the inequality is

$$0 \geq \frac{\sum_{v<p_b} \zeta[v](v-p_b) + \zeta[p_b](1-\lambda)(p_b-p_b)}{\sum_{v<p_b} \zeta[v] + \zeta[p_b](1-\lambda)},$$

which holds. Let τ_{c3} be such that condition 6.(c) holds for $(m_b,\phi_b,p) = (\tilde{m}_b^b,\phi_b^*,p_b)$, for α such that $\tau \geq \tau_{c3}$.

Case 4: $\phi_b = \phi_b^*$, $p \notin \{p_s,p_b\}$ and $m_b = \bar{m}_b^s$. Note that we only have to consider $p \in P(T,\hat{T},\tilde{T})$ in this case. The strategy for the buyer is

$$\pi_b^2(\bar{m}_b^s,\phi_b^*,p) = \begin{cases} 1 & \text{if } p = v_1, \\ 0 & \text{if } p > v_1. \end{cases}$$

We have $p^\alpha(\phi_b^*,\phi_s^*)[v,\bar{m}_b^s,m_s]\pi_s^{2,\alpha}(m_s,\phi_s^*)[p] \leq j^{-j}$ for each $v \in V$ and $m_s \in M_s$ since $(\beta_b\phi_b^*(v)+\beta_s\phi_s^*(v))[\bar{m}_b^s,m_s] = 0$ for $m_s \neq \bar{m}_s^s$ and $\pi_s^2(\bar{m}_s^s,\phi_s^*)[p] = 0$ implies:

$$\zeta[v](\beta_b\phi_b^*(v)+\beta_s\phi_s^*(v))[\bar{m}_b^s,m_s]\pi_s^2(m_s,\phi_s^*)[p] = 0$$

and $\pi_s^{\bar{m}_b^s,2,\alpha}(\bar{m}_s^s,\phi_s^*)[p]=0$ implies:

$$\zeta[v](\beta_b\phi_b^*(v)+\beta_s\phi_s^*(v))[\bar{m}_b^s,m_s]\pi_s^{\bar{m}_b^s,2,\alpha}(m_s,\phi_s^*)[p]=0.$$

If $v\neq v_1$, $m_b\neq\bar{m}_b^s$, or $m_s\neq m_s^{v_1}$, $p^\alpha(\phi_b^*,\phi_s^{m_b})[v,\bar{m}_b^s,m_s]\pi_s^{2,\alpha}(m_s,\phi_s^{m_b})[p]\leq j^{-j}$. This is as follows: (1) if $m_b\neq\bar{m}_b^s$, then $(\beta_b\phi_b^*(v)+\beta_s\phi_s^{m_b}(v))[\bar{m}_b^s,m_s]=0$ for each $v\in V$ and $m_s\in M_s$; (2) if $m_b=\bar{m}_b^s$, $m_s=m_s^{v_1}$ and $v\neq v_1$, then $(\beta_b\phi_b^*(v)+\beta_s\phi_s^{\bar{m}_b^s}(v))[\bar{m}_b^s,m_s^{v_1}]=0$; and (3) if $m_b=\bar{m}_b^s$, $m_s\neq m_s^{v_1}$ and $v\in V$, then (i) $(\beta_b\phi_b^*(v)+\beta_s\phi_s^{\bar{m}_b^s}(v))[\bar{m}_b^s,m_s]=0$ for each $m_s\notin\{m_s^{v'}:v'\in V\}$, (ii) $\pi_s^2(m_s^{v'},\phi_s^{\bar{m}_b^s})[p]=0$ for each $v'\in V$ (since $\pi_b^2(\bar{m}_b^s,\phi_b^*,p)=1$ if and only if $p\in\{p_b,p_s\}$ or $p\leq v_1$, and so $\pi_s^2(m_s^{v'},\phi_s^{\bar{m}_b^s})=p_s$ is optimal), and (iii) $\pi_s^{\bar{m}_b^s,2,\alpha}(m_s,\phi_s^{\bar{m}_b^s})[p]=0$ for each $m_s\neq m_s^{v_1}$ and $\pi_s^{m_b,2,\alpha}(m_s,\phi_s^{\bar{m}_b^s})[p]=0$ for each $m_b\neq\bar{m}_b^s$ and $m_s\in M_s$.

Finally, note that

$$\begin{aligned}\pi_s^{2,\alpha}(m_s^{v_1},\phi_s^{\bar{m}_b^s})[p]&=j^{-1}(1-j^{-j})|T|^{-1}\sum_{m_b\in T}\pi_s^{m_b,2,\alpha}(m_s^{v_1},\phi_s^{\bar{m}_b^s})[p]+O(j^{-j})\\&=j^{-1}(1-j^{-j})|T|^{-1}|P(T,\hat{T},\tilde{T})|^{-1}+O(j^{-j})\end{aligned}$$

since $\pi_s^{m_b,2,\alpha}(m_s^{v_1},\phi_s^{\bar{m}_b^s})=p_b$ for all $m_b\neq\bar{m}_b^s$.

Thus, the denominator of the inequality is (ignoring terms that are $O(j^{-j})$):

$$\begin{aligned}&(1-j^{-1})\sum_{v,m_s}p^\alpha(\phi_b^*,\phi_s^*)[v,\bar{m}_b^s,m_s]\pi_s^{2,\alpha}(m_s,\phi_s^*)[p]\\&+j^{-1}(1-j^{-j})|T|^{-1}\sum_{m_b\in T}\sum_{v,m_s}p^\alpha(\phi_b^*,\phi_s^{m_b})[v,\bar{m}_b^s,m_s]\pi_s^{2,\alpha}(m_s,\phi_s^{m_b})[p]\\&+j^{-1}j^{-j}|\Phi(T,\hat{T})|^{-1}\sum_{\phi\in\Phi(T,\hat{T})}\sum_{v,m_s}p^\alpha(\phi_b^*,\phi)[v,\bar{m}_b^s,m_s]\pi_s^{2,\alpha}(m_s,\phi)[p]\\&=j^{-1}(1-j^{-j})|T|^{-1}(1-j^{-j})\zeta[v_1]\beta_s j^{-1}(1-j^{-j})|T|^{-1}|P(T,\hat{T},\tilde{T})|^{-1}\end{aligned}$$

Likewise, also ignoring terms that are $O(j^{-j})$, the numerator of the right-hand (resp. left-hand) side inequality is

$$j^{-1}(1-j^{-j})|T|^{-1}(1-j^{-j})\zeta[v_1]\beta_s j^{-1}(1-j^{-j})|T|^{-1}|P(T,\hat{T},\tilde{T})|^{-1}(v_1-p)$$

when $p>v_1$ (resp. $p=v_1$).

Thus, when $p > v_1$, the limit inequality (with $a = 1$ and $\varepsilon = 0$) is $0 \geq v_1 - p$. When $p = v_1$, the limit inequality (with $a = 0$ and $\varepsilon = 0$) is $v_1 - v_1 \geq 0$.

For each $p \in P(T, \hat{T}, \tilde{T}) \setminus \{p_s, p_b\}$, let $\tau_{c4}(p)$ be such that condition 6.(c) holds for $(\bar{m}_b^s, \phi_b^*, p)$, for each $\alpha = (\tau, T, \hat{T}, \tilde{T})$ such that $\tau \geq \tau_{c4}(p)$, and let $\tau_{c4}(T, \hat{T}, \tilde{T}) = \max_{p \in P(T, \hat{T}, \tilde{T})} \tau_{c4}(p)$.

Case 5: $\phi_b = \phi_b^*$, $p = p_b < p_s$ and $m_b = \bar{m}_b^s$. Since $\pi_b^2(\bar{m}_b^s, \phi_b^*, p_b) = 1$, we may consider $a = 0$.

We have that $p^\alpha(\phi_b^*, \phi_s^*)[v, \bar{m}_b^s, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^*)[p_b] \leq j^{-j}$ for each $v \in V$ and $m_s \neq \bar{m}_s^s$ since $(\beta_b \phi_b^*(v) + \beta_s \phi_s^*(v))[\bar{m}_b^s, m_s] = 0$ for all $m_s \neq \bar{m}_s^s$. For $m_s = \bar{m}_s^s$, we have $\pi_s^2(\bar{m}_s^s, \phi_s^*)[p_b] = 0$ but $\pi_s^{m_b,2,\alpha}(\bar{m}_s^s, \phi_s^*)[p_b] = 1$ for each $m_b \in M_b$. Thus, $|T^{-1}| \sum_{m_b} \pi_s^{m_b,2,\alpha}(\bar{m}_s^s, \phi_s^*)[p_b] = 1$ and therefore $\pi_s^{2,\alpha}(\bar{m}_s^s, \phi_s^*)[p_b] = j^{-1}(1 - j^{-j})$.

Also, $(\beta_b \phi_b^*(v) + \beta_s \phi_s^{m_b}(v))[\bar{m}_b^s, m_s] > 0$ only if $m_b = \bar{m}_b^s$ and $m_s \in \{m_s^v : v \in V\}$, and $\pi_s^2(m_s^v, \phi_s^{\bar{m}_b^s})[p_b] = 0$ for each $v \in V$ (since $\pi_b^2(\bar{m}_b^s, \phi_b^*, p_s) = 1$). Thus, $\sum_{m_b \in T} \sum_{v, m_s} p^\alpha(\phi_b^*, \phi_s^{m_b})[v, \bar{m}_b^s, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^{m_b})[p_b] = O(j^{-1})$.

Thus, the denominator of the inequality is (ignoring terms that are $O(j^{-j})$):

$$
\begin{aligned}
&(1 - j^{-1}) \sum_{v, m_s} p^\alpha(\phi_b^*, \phi_s^*)[v, \bar{m}_b^s, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^*)[p_b] \\
&+ j^{-1}(1 - j^{-j})|T|^{-1} \sum_{m_b \in T} \sum_{v, m_s} p^\alpha(\phi_b^*, \phi_s^{m_b})[v, \bar{m}_b^s, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^{m_b})[p_b] \\
&= (1 - j^{-1}) \sum_v (1 - j^{-j})\zeta[v]\beta_s j^{-1}(1 - j^{-j}) + j^{-1}(1 - j^{-j})|T|^{-1} O(j^{-1}) \\
&= (1 - j^{-1}) \sum_v (1 - j^{-j})\zeta[v]\beta_s j^{-1}(1 - j^{-j}) + O(j^{-2}) \\
&= (1 - j^{-1})(1 - j^{-j})\beta_s j^{-1}(1 - j^{-j}) \sum_v \zeta[v] + O(j^{-2}) \\
&= (1 - j^{-1})(1 - j^{-j})\beta_s j^{-1}(1 - j^{-j}) + O(j^{-2}).
\end{aligned}
$$

Similarly, ignoring terms that are $O(j^{-j})$ and $O(j^{-2})$, the numerator of the left-hand side of the inequality is

$$
(1 - j^{-1})(1 - j^{-j})\beta_s j^{-1}(1 - j^{-j}) \sum_v \zeta[v](v - p_b).
$$

Thus, the limit inequality is:

$$
\sum_v \zeta[v]v - p_b \geq 0.
$$

Let τ_{c5} be such that condition 6.(c) holds for $(\bar{m}_b^s, \phi_b^*, p_b)$ for each α such that $\tau \geq \tau_{c5}$.

Case 6: $\phi_b = \phi_b^*$, $p \neq p_b$ and $m_b = \bar{m}_b^b$. The strategy for the buyer is

$$\pi_b^2(\bar{m}_b^b, \phi_b^*, p) = \begin{cases} 1 & \text{if } p = v_1, \\ 0 & \text{if } p > v_1. \end{cases}$$

By the same argument as in case 4, we have $p^\alpha(\phi_b^*, \phi_s^*)[v, \bar{m}_b^b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^*)[p] \leq j^{-j}$ for each $v \in V$ and $m_s \in M_s$.

For each $v \neq v_1$, $m_b \neq \bar{m}_b^b$, or $m_s \neq m_s^{v_1}$, $p^\alpha(\phi_b^*, \phi_s^{m_b})[v, \bar{m}_b^b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^{m_b})[p] \leq j^{-j}$. This is because $(\beta_b\phi_b^*(v) + \beta_s\phi_s^{m_b}(v))[\bar{m}_b^b, m_s^{v_1}] = 0$ if $m_b \neq \bar{m}_b^b$ or $v \neq v_1$, $\pi_s^{m_b',2,\alpha}(m_s, \phi_s^{m_b})[p] = 0$ for each $m_s \neq m_s^{v_1}$ and $m_b, m_b' \in M_b$, $\pi_s^{m_b,2,\alpha}(m_s, \phi_s^{\bar{m}_b^b})[p] = 0$ for each $m_b \neq \bar{m}_b^b$ and $m_s \in M_s$, $(\beta_b\phi_b^*(v)+\beta_s\phi_s^{\bar{m}_b^b}(v))[\bar{m}_b^b, m_s] = 0$ for each $m_s \notin M^* = \{m_s^v : v \in V\} \cup \{\bar{m}_s^b\}$, $\pi_s^2(m_s, \phi_s^{\bar{m}_b^b})[p] = 0$ for each $m_s \in M^*$ (since $\pi_s^2(m_s^v, \phi_s^{\bar{m}_b^b}) = p_b$ is optimal for each $v \in V$ and $\pi_s^2(\bar{m}_s^b, \phi_s^{\bar{m}_b^b}) = p_b$), and if $m_b \neq \bar{m}_b^b$, $(\beta_b\phi_b^*(v) + \beta_s\phi_s^{m_b}(v))[\bar{m}_b^b, m_s] = 0$ for each $m_s \notin M' = \{m_s^v : v > v_1\} \cup \{\bar{m}_s^b\}$ and $\pi_s^2(m_s, \phi_s^{m_b})[p] = 0$ for each $m_s \in M'$ (since $\pi_s^2(m_s^v, \phi_s^{m_b}) = p_b$ is optimal for $v > v_1$ and $\pi_s^2(\bar{m}_s^b, \phi_s^{m_b}) = p_b$).

Finally, note that

$$\begin{aligned}\pi_s^{2,\alpha}(m_s^{v_1}, \phi_s^{\bar{m}_b^b})[p] &= j^{-1}(1-j^{-j})|T|^{-1}\sum_{m_b \in T}\pi_s^{m_b,2,\alpha}(m_s^{v_1}, \phi_s^{\bar{m}_b^b})[p] + O(j^{-j}) \\ &= j^{-1}(1-j^{-j})|T|^{-1}|P(T, \hat{T}, \tilde{T})|^{-1} + O(j^{-j}).\end{aligned}$$

Thus, the denominator of the inequality is (ignoring terms that are $O(j^{-j})$):

$$\begin{aligned}&(1-j^{-1})\sum_{v,m_s} p^\alpha(\phi_b^*, \phi_s^*)[v, \bar{m}_b^b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^*)[p] \\ &+ j^{-1}(1-j^{-j})|T|^{-1}\sum_{m_b \in T}\sum_{v,m_s} p^\alpha(\phi_b^*, \phi_s^{m_b})[v, \bar{m}_b^b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^{m_b})[p] \\ &= j^{-1}(1-j^{-j})|T|^{-1}(1-j^{-j})\zeta[v_1]\beta_s j^{-1}(1-j^{-j})|T|^{-1}|P(T, \hat{T}, \tilde{T})|^{-1}.\end{aligned}$$

Likewise, also ignoring terms that are $O(j^{-j})$, the numerator of the right-hand (resp. left-hand) side inequality is

$$j^{-1}(1-j^{-j})|T|^{-1}(1-j^{-j})\zeta[v_1]\beta_s j^{-1}(1-j^{-j})|T|^{-1}|P(T, \hat{T}, \tilde{T})|^{-1}(v_1 - p)$$

when $p > v_1$ (resp. $p = v_1$).

Thus, when $p > v_1$, the limit inequality (with $a = 1$ and $\varepsilon = 0$) is $0 \geq v_1 - p$. When $p = v_1$, the limit inequality (with $a = 0$ and $\varepsilon = 0$) is $v_1 - v_1 \geq 0$.

For each $p \in P(T, \hat{T}, \tilde{T}) \setminus \{p_b\}$, let $\tau_{c6}(p)$ be such that condition 6.(c) holds for each $(\bar{m}_b^b, \phi_b^*, p)$, for each $\alpha = (\tau, T, \hat{T}, \tilde{T})$ such that $\tau \geq \tau_{c6}(p)$, and let $\tau_{c6}(T, \hat{T}, \tilde{T}) = \max_{p \in P(T,\hat{T},\tilde{T})} \tau_{c6}(p)$.

Case 7: $\phi_b = \phi_b^*$ and $m_b \notin \{\bar{m}_b^b, \tilde{m}_b^b, \bar{m}_b^s\}$. The strategy for the buyer is

$$\pi_b^2(m_b, \phi_b^*, p) = \begin{cases} 1 & \text{if } p = v_1, \\ \\ 0 & \text{if } p > v_1. \end{cases}$$

In this case, $p^\alpha(\phi_b^*, \phi_s^*)[v, m_b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^*)[p] \leq j^{-j}$ for all $v \in V$ and $m_s \in M_s$, and $p^\alpha(\phi_b^*, \phi_s^{m_b'})[v, m_b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^{m_b'})[p] \leq j^{-j}$ if $m_b' \neq m_b$, $v \neq v_1$, or $m_s \neq m_s^{v_1}$.

Thus, the denominator of the inequality is (ignoring terms that are $O(j^{-j})$):

$$\begin{aligned} &(1 - j^{-1}) \sum_{v,m_s} p^\alpha(\phi_b^*, \phi_s^*)[v, m_b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^*)[p] \\ &+ j^{-1}(1 - j^{-j})|T|^{-1} \sum_{m_b' \in T} \sum_{v,m_s} p^\alpha(\phi_b^*, \phi_s^{m_b'})[v, m_b, m_s]\pi_s^{2,\alpha}(m_s, \phi_s^{m_b'})[p] \\ &= j^{-1}(1 - j^{-j})|T|^{-1}(1 - j^{-j})\zeta[v_1]\beta_s\pi_s^{2,\alpha}(m_s^{v_1}, \phi_s^{m_b})[p]. \end{aligned}$$

Likewise, also ignoring terms that are $O(j^{-j})$, the numerator of the right-hand (resp. left-hand) side inequality is

$$j^{-1}(1 - j^{-j})|T|^{-1}(1 - j^{-j})\zeta[v_1]\beta_s\pi_s^{2,\alpha}(m_s^{v_1}, \phi_s^{m_b})[p](v_1 - p)$$

when $p > v_1$ (resp. $p = v_1$).

Thus, when $p > v_1$, the limit inequality (with $a = 1$ and $\varepsilon = 0$) is $0 \geq v_1 - p$. When $p = v_1$, the limit inequality (with $a = 0$ and $\varepsilon = 0$) is $v_1 - v_1 \geq 0$.

Let $\tau_{c7}(T, \hat{T}, \tilde{T})$ be such that condition 6.(c) holds for each (m_b, ϕ_b^*, p) such that $m_b \in T \setminus \{\bar{m}_b^b, \tilde{m}_b^b, \bar{m}_b^s\}$ and $p \in P(T, \hat{T}, \tilde{T})$, for each $\alpha = (\tau, T, \hat{T}, \tilde{T})$ such that $\tau \geq \tau_{c7}(T, \hat{T}, \tilde{T})$.

Case 8: For each $m_b \in M_b$ and $\phi_b \neq \phi_b^*$, 6(c) holds in the limit by construction. Let $\tau_{c8}(T, \hat{T}, \tilde{T})$ be such that condition 6.(c) holds for each (m_b, ϕ_b, p) such that $\phi_b \in \Phi_b^\alpha \setminus$

$\{\phi_b^*\}$, $m_b \in \cup_v \text{supp}(\phi_b(v)_{M_b}) \cup \{\bar{m}_b^s\} \cup T$ and $p \in P(T, \hat{T}, \tilde{T})$, for each $\alpha = (\tau, T, \hat{T}, \tilde{T})$ such that $\tau \geq \tau_{c8}(T, \hat{T}, \tilde{T})$.

For each $(T, \hat{T}, \tilde{T})$, let

$$\tau_c(T, \hat{T}, \tilde{T}) = \max\{\tau_{c1}, \tau_{c2}, \tau_{c3}, \tau_{c4}(T, \hat{T}, \tilde{T}), \tau_{c5}, \tau_{c6}(T, \hat{T}, \tilde{T}), \tau_{c7}(T, \hat{T}, \tilde{T}), \tau_{c8}(T, \hat{T}, \tilde{T})\}.$$

The above arguments allow us to define the following subnet $\{\pi^{\varphi(\eta)}, p^{\varphi(\eta)}\}_\eta$ of $\{\pi^\alpha, p^\alpha\}_\alpha$ such that condition 6 holds.

The index set of the subnet $\{\pi^{\varphi(\eta)}, p^{\varphi(\eta)}\}_\eta$ is the same as the one in the net $\{\pi^\alpha, p^\alpha\}_\alpha$. The function $\varphi : \eta \mapsto \alpha$ is defined by setting, for each $\eta = (\tau, T, \hat{T}, \tilde{T})$,

$$\varphi(\eta) = (\max\{\tau_a, \tau_b(T, \hat{T}), \tau_c(T, \hat{T}, \tilde{T})\}, T, \hat{T}, \tilde{T}).$$

It is then clear that condition 6 holds and that, as required by the definition of a subnet, for each α_0, there exists η_0, e.g. $\eta_0 = \alpha_0$, such that $\varphi(\eta) \geq \alpha_0$ for each $\eta \geq \eta_0$.

A.4 Proof of Corollary 1

It is clear that $\beta_s E + \beta_b E(v_K) \leq \beta_s E + \beta_b E(v_k) \leq \beta_s E + \beta_b E(v_1)$ for each $k \in \{1, \ldots, K\}$. Note that, for each $k \in \{2, \ldots, K\}$ and $p \in (v_{k-1}, v_k]$,

$$Z(p) = \sum_{v \geq p} \zeta[v] = \sum_{v \geq v_k} \zeta[v] = Z(v_k), \text{ and} \tag{A.11}$$

$$E(p) = \sum_{v \geq p} \zeta[v]v = \sum_{v \geq v_k} \zeta[v]v = E(v_k). \tag{A.12}$$

When $k = 1$, $Z(p) = Z(v_1)$ and $E(p) = E(v_1)$ for each $p \in C_1 = \{v_1\}$. Thus, for each $k \in \kappa$, $\beta_s E + \beta_b \bar{v}_k Z(v_k) = \beta_s E + \beta_b \bar{v}_k Z(\bar{v}_k) \leq \beta_s E + \beta_b p^* Z(p^*)$ since $\bar{v}_k \leq E$ by the definition of C_k and $\bar{v}_k Z(\bar{v}_k) \leq p^* Z(p^*)$ by the definition of p^*. Thus, we also have that $\underline{v}_k(\beta_s + \beta_b Z(v_k)) \leq \bar{v}_k(\beta_s + \beta_b Z(\bar{v}_k)) \leq \beta_s E + \beta_b p^* Z(p^*)$.

Moreover, for each $k \in \kappa$, $\beta_s E + \beta_b \bar{v}_k Z(v_k) \geq \beta_s E + \beta_b v_1$ by the definition of C_k. Since $\underline{v}_k \geq v_1$ and $\underline{v}_k Z(v_k) \geq v_1$, the latter since $\underline{v}_k = \lim_j p_j$ for some sequence $\{p_j\}_{j=1}^\infty$ such that $p_j \in C_k$ for each $j \in \mathbb{N}$, it follows that $\underline{v}_k(\beta_s + \beta_b Z(v_k)) \geq v_1$.

(Sufficiency) Let $(\hat{u}_b, \hat{u}_s) \in \cup_{k \in \kappa} U_k$ and let $k \in \kappa$ be such that $(\hat{u}_b, \hat{u}_s) \in U_k$. If $\underline{v}_k > v_{k-1}$ (respectively, $\underline{v}_k = v_{k-1}$), then $C_k = [\underline{v}_k, \bar{v}_k]$ (resp. $C_k = (\underline{v}_k, \bar{v}_k]$) by the definition of C_k.

Consider two cases: (a) $\hat{u}_s \leq \bar{v}_k(\beta_s + \beta_b Z(v_k))$ and (b) $\hat{u}_s > \bar{v}_k(\beta_s + \beta_b Z(v_k))$. In case (a), let p_b be such that $\hat{u}_s = p_b(\beta_s + \beta_b Z(v_k))$. Then $p_b \in C_k$ since, by the definition of U_k and of case (a),

$$\underline{v}_k(\beta_s + \beta_b Z(v_k)) \leq \hat{u}_s \leq \bar{v}_k(\beta_s + \beta_b Z(v_k))$$

(resp. $\underline{v}_k(\beta_s + \beta_b Z(v_k)) < \hat{u}_s \leq \bar{v}_k(\beta_s + \beta_b Z(v_k))$). In case (b), let $p_b = \bar{v}_k$. In either case, $p_b \in C_k$ and it follows by (A.11) and the definition of C_k that $p_b Z(p_b) = p_b Z(v_k) \geq v_1$ i.e. (4) holds.

Let $p_s = \frac{\hat{u}_s - \beta_b p_b Z(v_k)}{\beta_s}$. Then it follows by (1), (2), (A.11), (A.12) and the definition of U_k that, in either case,

$$u_s = \beta_s p_s + \beta_b p_b Z(p_b) = \beta_s p_s + \beta_b p_b Z(v_k) = \hat{u}_s$$

and

$$\begin{aligned} u_b &= \beta_s(E - p_s) + \beta_b(E(p_b) - p_b Z(p_b)) \\ &= \beta_s E - \hat{u}_s + \beta_b p_b Z(v_k) + \beta_b E(v_k) - \beta_b p_b Z(v_k) \\ &= \beta_s E + \beta_b E(v_k) - \hat{u}_s = \hat{u}_b. \end{aligned}$$

It remains to show that (3) holds. Since $p_s = \frac{\hat{u}_s - \beta_b p_b Z(v_k)}{\beta_s}$, we have that $p_s \geq p_b$ if and only if $\hat{u}_s \geq p_b(\beta_s + \beta_b Z(v_k))$. This inequality holds in case (a) since then $\hat{u}_s = p_b(\beta_s + \beta_b Z(v_k))$. It also holds in case (b) since then $p_b = \bar{v}_k$ and, by the definition of case (b), $\hat{u}_s > \bar{v}_k(\beta_s + \beta_b Z(v_k))$.

It follows from $p_s = \frac{\hat{u}_s - \beta_b p_b Z(v_k)}{\beta_s}$ that $p_s \leq E$ holds if and only if $\hat{u}_s \leq \beta_s E + \beta_b p_b Z(v_k)$. This inequality holds in case (a) since then $\hat{u}_s = p_b(\beta_s + \beta_b Z(v_k))$ and $p_b \leq E$, the latter because $p_b \in C_k$. It also holds in case (b) since then $p_b = \bar{v}_k$ and $\hat{u}_s \leq \beta_s E + \beta_b \bar{v}_k Z(v_k)$, the latter because $(\hat{u}_b, \hat{u}_s) \in U_k$.

It follows from the above that $(\hat{u}_b, \hat{u}_s)$ is represented by $(p_b, p_s, 1)$ and, hence, $(\hat{u}_b, \hat{u}_s) \in U^{**}$. Since $(\hat{u}_b, \hat{u}_s)$ is arbitrary, it follows that $\cup_{k \in \kappa} U_k \subseteq U^{**}$.

(Necessity) Let $(\hat{u}_b, \hat{u}_s) \in U^{**}$ and let $(p_b, p_s) \in (V^*)^2$ be such that $(\hat{u}_b, \hat{u}_s)$ is represented by $(p_b, p_s, 1)$. Since $\{\{v_1\}, \left((v_{k-1}, v_k]\right)_{1<k\leq K}\}$ is a partition of V^*, let $k = 1$ if $p_b = v_1$ and $k \in \{2, \ldots, K\}$ be such that $p_b \in (v_{k-1}, v_k]$ otherwise. Recall that

$Z(p_b) = Z(v_k)$ and $E(p_b) = E(v_k)$ by (A.11) and (A.12) respectively. Then $p_b \in C_k$ by (3) and (4). Hence, $\underline{v}_k \leq p_b \leq \bar{v}_k$ and, if $\underline{v}_k \notin C_k$ i.e. $\underline{v}_k = v_{k-1}$, $\underline{v}_k < p_b \leq \bar{v}_k$.

By (1) and (2), $\hat{u}_b = \beta_s(E - p_s) + \beta_b(E(v_k) - p_b Z(v_k))$, $\hat{u}_s = \beta_s p_s + \beta_b p_b Z(v_k)$ and, hence,

$$\hat{u}_b + \hat{u}_s = \beta_s E + \beta_b E(v_k). \tag{A.13}$$

Since $p_s \leq E$ by (3),

$$\begin{aligned} \hat{u}_s &= \beta_s p_s + \beta_b p_b Z(v_k) \\ &\leq \beta_s E + \beta_b \bar{v}_k Z(v_k). \end{aligned} \tag{A.14}$$

Moreover,

$$\begin{aligned} \hat{u}_s &= \beta_s p_s + \beta_b p_b Z(v_k) \\ &\geq p_b(\beta_s + \beta_b Z(v_k)) \\ &\geq \underline{v}_k(\beta_s + \beta_b Z(v_k)) \end{aligned} \tag{A.15}$$

and, if $\underline{v}_k \notin C_k$ i.e. $\underline{v}_k = v_{k-1}$,

$$\hat{u}_s > \underline{v}_k(\beta_s + \beta_b Z(v_k)). \tag{A.16}$$

It then follows by (A.13)–(A.16) that $(\hat{u}_b, \hat{u}_s) \in U_k$.

B Refinement

Let $\mu_b^\alpha(\phi_b^*, m_b, p) \in \Delta(V \times \Phi \times M_s)$ be the buyer's belief after observing his choice ϕ_b^*, the message m_b she received and the price p set by the seller, given (π^α, p^α).

Define $\pi \in \Pi^*$ to be a sequential equilibrium *with price-independent beliefs* if π is a sequential equilibrium and, in addition, the defining net $\{\pi^\alpha, p^\alpha\}$ satisfies, for each $(m_b, p) \in \cup_v S^*_{M_b}(v) \times V^*$, $\lim_\alpha \mu_b^\alpha(\phi_b^*, m_b, p) = \mu_b(\phi_b^*, m_b, p) \in \Delta(V \times \Phi \times M_s)$, where $p \mapsto \mu_{b,V}(\phi_b^*, m_b, p)$ is constant, i.e. the buyer's beliefs about v does not depend on p.

In particular, this implies that for each $m_b \in \cup_v S^*_{M_b}(v)$,

$$\sum_v v \mu_{b,V}(\phi_b^*, m_b, p)[v] = E(v|m_b) \tag{B.1}$$

for all $p \in V^*$, where $E(v|m_b)$ is the buyer's expected valuation conditional on m_b. Note also that the expectation of the conditional expected valuations $E(v|m_b)$, with $m_b \in \cup_v S^*_{M_b}(v)$, is the expected valuation, i.e.

$$\sum_{m_b \in \cup_v S^*_{M_b}(v)} \gamma[m_b] E(v|m_b) = E, \tag{B.2}$$

where, for each $m_b \in \cup_v S^*_{M_b}(v)$, $\gamma[m_b] = \sum_v \zeta[v] \sum_{m_s} \beta(\phi^*_b(v), \phi^*_s(v))[m_b, m_s]$ is the probability of m_b.

B.1 Proof of Theorem 2

The following lemma shows that it cannot be that $E(v|m_b) = E$ for each $m_b \in \cup_v S^*_{M_b}(v)$ since if so, then the seller would gain by setting a price slightly below E regardless of his message, which would be accepted.

Lemma B.1 *There exists $m_b \in \cup_v S^*_{M_b}(v)$ such that $E(v|m_b) > E$.*

Proof. If not, then $E(v|m_b) = E$ for each $m_b \in \cup_v S^*_{M_b}(v)$ by (B.2). We have that $u_s = \beta_s p_s + \beta_b p_b(\sum_{v>p_b} \zeta[v] + \lambda\zeta[p_b]) \leq \beta_s E + \beta_b p_b \sum_{v \geq p_b} \zeta[v] < E$. Hence, let $\varepsilon > 0$ be such that $E - \varepsilon > u_s$ and $E - \varepsilon \neq p_b$. Let $p^* = E - \varepsilon$.

By (B.1), for each $m_b \in \cup_v S^*_{M_b}(v)$,

$$\sum_v \mu_{b,V}(\phi^*_b, m_b, p^*)[v](v - p^*) = E(v|m_b) - p^* = E - (E - \varepsilon) = \varepsilon > 0$$

and, hence, $a(m_b, p^*) = 1$.

Consider $\hat{\pi}^2_s$ such that $\hat{\pi}^2_s(m_s, \phi^*_s) = E - \varepsilon$ for each $m_s \in M_s$. Letting $\hat{\pi}_s = (\phi^*_s, \hat{\pi}^2_s)$ and $\hat{u}_s = u_s(\pi_b, \hat{\pi}_s)$, it follows that

$$\hat{u}_s - u_s = p^* - u_s = E - \varepsilon - u_s > 0.$$

But this is a contradiction since π is a sequential equilibrium. ■

We conclude the proof of Theorem 2 by showing that the seller can increase his payoff after his design is chosen by sending message m_b such that $E(v|m_b) > E$ to the buyer and charging a price slightly below $E(v|m_b)$.

Let $m_b^* \in \cup_v S^*_{M_b}(v)$ be such that $E(v|m_b^*) > E$. Let $\varepsilon > 0$ be such that $E(v|m_b^*) - \varepsilon > E$ and $p^* = E(v|m_b^*) - \varepsilon$.

It follows by (B.1) that

$$\sum_v \mu_{b,V}(\phi_b^*, m_b^*, p^*)[v](v - p^*) = E(v|m_b^*) - p^* > 0$$

and, hence, $a(m_b^*, p^*) = 1$. By Lemma A.6, $p_s \leq E < p^*$. Letting $m \in \cup_v S_s^*(v)$, it follows that $w_s(m_b) = p_s < p^* = w_s(m_b^*)$, contradicting Lemma A.2.

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