Stable Matching in Large Markets with Occupational Choice*

Guilherme Carmona† Krittanai Laohakunakorn‡

University of Surrey University of Surrey

June 22, 2024

Abstract

We introduce a framework and stability notion for large many-to-one matching markets in distributional form with occupational choice. Occupational choice means that each individual can choose which side of the market to belong to and implies that the sets of agents to match are determined endogenously. Our model generalizes the setting and stability notion of Greinecker and Kah (2021), which focused on one-to-one matching and did not allow for occupational choice. We show that stable matchings exist under mild assumptions; in particular, both complementarities and externalities can be accommodated. Applications include Gale and Shapley's (1962) roommate problem with non-atomic participants and the frameworks of Lucas (1978), Rosen (1982), Garicano and Rossi-Hansberg (2004) and Garicano and Rossi-Hansberg (2006).

*We wish to thank Michael Greinecker, Ravi Jagadeesan, Fuhito Kojima, Karolina Vocke, Rakesh Vohra, three anonymous referees and seminar participants at the University of Bath, the 2022 SAET conference (Canberra), the 2022 Many Player Games and Applications Workshop (Berlin), the 2023 Lisbon Meetings in Game Theory and Applications, the 2023 EWET (Naples) and the 2023 PEJ conference (Braga) for helpful comments. Any remaining errors are, of course, ours.

†Address: University of Surrey, School of Economics, Guildford, GU2 7XH, UK; email: g.carmona@surrey.ac.uk.

‡Address: University of Surrey, School of Economics, Guildford, GU2 7XH, UK; email: k.laohakunakorn@surrey.ac.uk.

1 Introduction

This paper establishes the existence of many-to-one stable matchings in large markets with complementarities, externalities and occupational choice. Having all these features present simultaneously in the same model is important for at least the following reasons.

Labor markets match a large number of workers and managers in a many-to-one way, since there are typically many workers matched with a firm or manager. Unlike in other scenarios, such as college applications or the marriage market, where the groups to match are determined a priori, being a worker or a manager is up to the individuals to decide and depends on the actual matching that takes place in the labor market. Thus, real-world labor matchings typically feature occupational choice, many-to-one matching and a large number of participants.

Complementarities and externalities also arise naturally in labor markets. For example, managers typically want to hire workers with complementary skills and recent graduates may prefer to enter the same industry as their peers. In addition, knowledge spillovers may imply that the productivity of a manager depends on the aggregate quality of those who take managerial roles according to the matching.

Thus, a model that aims to capture relevant features of real-world labor markets should be able to accomodate complementarities, externalities and occupational choice. As we discuss in Section 2, prior work has established the existence of stable matchings in models that contain a strict subset of these elements.

Our framework for large many-to-one matching markets with occupational choice is sufficiently general to have several important special cases and, thus, to provide a unified view of existing and future work on issues featuring such elements. It generalizes the two-sided one-to-one matching setting in distributional form of Greinecker and Kah (2021) (GK henceforth) by adding many-to-one matching and occupational choice; in particular, our existence result implies existence in GK's one-to-one matching market.[1]

[1]In Section 6.1, we show that GK's setting can be represented as a special case of our gen-

In addition, we show how several classical models that feature occupational choice, many-to-one matching and a large number of participants, such as those of Lucas (1978), Rosen (1982), Garicano and Rossi-Hansberg (2004) and Garicano and Rossi-Hansberg (2006), can be seen as particular cases of our framework. These models also feature a continuum of types, which can be accomodated in our framework. To illustrate the flexibility of our setting and its technical advantages, we provide a detailed analysis of Rosen's (1982) model. We show that stable matchings exist even though some of the assumptions of our general existence result do not hold and we fully characterize stable matchings.

Our setting is not restricted to the analysis of labor markets but has many other possible applications. We illustrate this by formalizing a non-atomic version of Gale and Shapley's (1962) roommate problem as a special case of our setting – in fact, one in which individuals are indifferent between the two occupations – and we show that our existence results imply the existence of stable matchings for the non-atomic roommate problem.

We present our model and stability notion in Section 4 after a brief literature review in Section 2 and a motivating example in Section 3. This example illustrates that stability in the presence of occupational choice differs considerably from the stability notion for two-sided many-to-one matching markets. This happens because individuals no longer have a fixed occupation and, therefore, stability is about someone being unable to find a better match even if this involves a change of occupation.

Our existence results are in Section 5. In particular, we show that stable matchings exist in markets with occupational choice whenever preferences are rational and continuous and the set of feasible measures that managers can match with is bounded

eral framework and that, specialized to this setting, our stability notion coincides with theirs. In Appendix B.1, we generalize these findings by introducing a new two-sided many-to-one matching model that generalizes GK to allow for many-to-one matching (but not occupational choice). We show that this model is also a particular case of our framework and, specialized to this setting, our stability notion coincides with other stability concepts for two-sided markets where both sides are large, such as Azevedo and Hatfield's (2018) (see Appendix B.12 for the comparison with Azevedo and Hatfield (2018)).

and rich.[2] Thus, we can accommodate externalities as long as preferences depend on the matching in a continuous way and we do not impose any substitutability requirement – complementarities cause no problem for existence in our model. In addition, as is standard in models with a continuum of agents, preferences are not required to be convex.

Section 6 contains applications of our framework to marriage markets (Section 6.1), the roommate problem (Section 6.2) and Rosen's (1982) model (Section 6.3), and a brief discussion of the settings of Lucas (1978), Garicano and Rossi-Hansberg (2004) and Garicano and Rossi-Hansberg (2006). Section 7 contains some concluding remarks. The proofs of our results are in Appendix A. Some additional details are in Appendix B.

2 Literature review

There have been many recent advances in the formalization and analysis of large matching markets without occupational choice. These include, among others, Azevedo and Leshno (2016), Fisher and Hafalir (2016), Ashlagi, Kanoria, and Leshno (2017), Eeckhout and Kircher (2018), Fuentes and Tohmé (2018), Nöldeke and Samuelson (2018) and Che and Tercieux (2019). See GK for a survey of this literature.

In this paper, we analyze a general matching model featuring occupational choice, many-to-one matching and a large number of participants. As we describe in what follows, some papers have formalized general matching models with some (but not all) of these features.

The roommate problem can be seen, as we will show in Section 6.2, as a matching problem with occupational choice where participants are indifferent between the occupations. Thus, Chiappori, Galichon, and Salanié (2014), Pęski (2017) and Azevedo and Hatfield (2018) study general matching models with a particular form of occupational choice and a large number of participants; see Section 6.2 for a more detailed

[2] Richness is a weak technical condition that implies that small perturbations of feasible measures are feasible.

discussion of these papers. There is one-to-one matching in the roommate problem and, thus, in relation to these papers, our framework adds many-to-one matching and a general formalization of occupational choice.

More generally, Jagadeesan and Vocke (2021) consider a many-to-many matching model where a continuum of agents of finitely many types can sign multiple contracts with each other. They show that tree-stable outcomes exist without any restriction on who can match with whom; this implies the existence of stable outcomes when only many-to-one matching is allowed since, in this case, tree-stability is equivalent to stability. In particular, they do not require that the market is two-sided and hence their existence result holds in the presence of occupational choice. However, their assumption that the set of contracts available to each agent is finite makes it less convenient to capture settings such as Rosen (1982) which was part of our motivation. In comparison with Jagadeesan and Vocke (2021), we do not allow for many-to-many matching, but we consider more general type and contract spaces and we allow preferences to depend on the matching.[3]

Wu (2021) provides a general existence result for a broad class of finite-type matching models under a convexity condition. Like Jagadeesan and Vocke (2021), he allows for many-to-many matching and arbitrary contracting networks. However, Wu's (2021) result does not apply to our setting where preferences may depend on the entire matching because his convexity requirement that certain combinations of unblocked matchings remain unblocked will fail under the appropriate notion of blocking in a model with general externalities.[4]

Two-sided matching models with many-to-one matching and a large number of participants have been considered in Che, Kim, and Kojima (2019) (CKK henceforth), among others. There is no occupational choice in their setting and, while there is a continuum of workers, the set of managers (or firms) is finite. CKK also considered a simplified version of Azevedo and Hatfield's (2018) model which has a continuum

[3] Another modelling difference is that our managers are matched with measures of workers, whereas theirs are matched with sets of workers.

[4] In Appendix B.13, we provide an example of a roommate market with externalities to which Wu's (2021) result does not apply.

of managers but no occupational choice.[5]

Making the workers negligible allowed CKK to obtain the existence of stable matchings in two-sided many-to-one matching markets where managers' preferences exhibit complementarities. This result solved a longstanding problem in matching theory since, with finitely many workers and managers, Kelso and Crawford (1982), Hatfield and Milgrom (2005) and Hatfield and Kojima (2008) have shown that managers need to have substitutable preferences to guarantee the existence of stable matchings. In Carmona and Laohakunakorn (2023a), we show that CKK's result holds even when workers have weak preferences that may depend on the matching (i.e. we do not require strict preferences and allow for externalities), under an assumption (discussed below) about how the agents in a blocking coalition expect the matching to change.

Externalities cause problems for the existence of stable matchings and raise some conceptual issues in finite markets. Indeed, when individuals' preferences depend on the matching, whether or not an individual gains by being part of a blocking coalition depends on the matching that results from such blocking. Thus, the definition of stability has to specify the (set of possible) matchings that result from each blocking coalition, and many such definitions have been proposed by e.g. Sasaki and Toda (1996), Dutta and Massó (1997), Echenique and Yenmez (2007), Hafalir (2008), Mumcu and Saglam (2010), Bando (2012) and Fisher and Hafalir (2016).

When there are finitely managers but a continuum of workers and only workers' preferences depend on the matching, Cox, Fonseca, and Pakzad-Hurson (2022), Leshno (2022) and Carmona and Laohakunakorn (2023a) define stability by specifying that each worker in a blocking coalition expects the matching to remain unchanged. They then establish the existence of stable matchings for this stability notion. In contrast to these papers, we consider the case where all agents are negligible and, thus, a blocking coalition of one (prospective) manager and a measure of (prospective) workers is negligible and, indeed, has no impact on the matching. Hence, externalities

[5] In Appendices B.11 and B.12, we formally compare our framework with the ones of CKK and Azevedo and Hatfield (2018) respectively.

cause no conceptual issue in our framework. The existence of stable matchings is also not an issue; indeed, we establish it in the presence of both externalities and complementarities.

3 Motivating example

There are two types of individuals, 1 and 2. Individuals have preferences that are fully described by their types and their population is described by a measure ν over the type space $Z = \{1, 2\}$. Let $\nu(1) = \nu(2) = \frac{1}{2}$.

Each individual can be a manager, a worker or self-employed. For each type $z \in \{1, 2\}$, some individuals of type z can be managers and some others can be workers; furthermore, those who are managers (if any) can be matched with workers of type z or of type $z' \neq z$. Those who are managers can hire a workforce, which we represent as a measure over worker types and contracts, from the set X, where each $\delta \in X$ is a measure over $Z \times C$ with C being the set of contracts. For this example, let $C = \mathbb{R}_+$ and $X = \{n1_{(z,c)} : z \in Z, n, c \in \mathbb{R}_+\}$.[6] Specifically, each manager can be matched with a measure $n1_{(z,c)}$, where $z \in Z$ denotes the type of workers he employs, $n \in \mathbb{R}_+$ denotes their number and $c \in \mathbb{R}_+$ denotes the wage paid to them.

The preferences of each individual depend on her type, her occupation and on her match. In this example, we specify that if someone of type $z \in \{1, 2\}$ chooses to be a manager and is matched with $n1_{(z',c)}$, then her payoff is $U_z(m, n1_{(z',c)}) = z^{1+\alpha}n^{1-\alpha} - cn$, where $\alpha \in (0, 1)$. If she chooses to be a worker and is matched with manager z' at wage c, then her payoff is the wage: $U_z(w, 1_{(z',c)}) = c$. An individual can also choose to be unmatched, in which case she receives a payoff of zero.

This example is a particular case of the model in Rosen (1982) which we reformulate and analyse in our framework in Section 6.3. The managers' rents are obtained via a production function of the form $g(z)z^{\alpha}n^{1-\alpha}$, with $g(z) = z$, which has labor and managers' type as inputs, the latter being interpreted as the managers' quality.

In the context of this example, a matching is a measure μ over $Z \times X$ with

[6] If Y is a metric space and $y \in Y$, 1_y denotes the probability measure degenerate on y.

$\mu(z, n1_{(z',c)})$ describing the measure of type z who are managers and hire n workers of type z' at wage c.

Consider first the case where each individual's occupation is fixed, with type 1 individuals being managers and type 2 individuals being workers. There is a unique stable matching in this example without occupational choice: $\mu(1, 1_{(2,1-\alpha)}) = \frac{1}{2}$. In such matching, all workers (i.e. type 2 individuals) are matched with a manager (i.e. a type 1 individual), each manager hires a workforce consisting of a measure $n = 1$ of workers at wage $c = 1 - \alpha$. Since both managers and workers obtain a strictly positive utility in this matching and zero if they were unmatched, such matching is individually rational. Furthermore, no manager and group of workers can block this matching since hiring a measure one of workers is optimal given the wage; hence, the manager cannot gain by changing his workforce since at least the newly hired workers would require a wage higher than $1 - \alpha$.

In the example without occupational choice, type 1 individuals can only be managers and type 2 individuals can only be workers; these restrictions are now removed by the introduction of occupational choice. The specification of our example implies that individuals of type 2 are better managers than those of type 1 since they have higher quality. This then means that the stable matching μ for the setting without occupational choice is intuitively not stable when occupational choice is allowed. For instance, any type 2 individual could choose to be a manager and attract, for example, a measure one of workers of type 2 by paying them $1 - \alpha + \varepsilon$ to obtain a rent of $2^{1+\alpha} - (1 - \alpha) - \varepsilon$; for sufficiently small $\varepsilon > 0$, such workers are willing to work for her and her payoff is higher than $1 - \alpha$ which is her payoff in the matching μ.

Thus, stability in the presence of occupational choice is more demanding than the stability notion for two-sided many-to-one matching markets. The latter roughly requires that no manager can improve his well being by changing the number of workers who work for him or by employing (an optimal number of) workers that he can target, which are those who would prefer to work for him at the proposed wage rather than for the manager with whom they are currently matched.[7] With

[7]Stability also requires individual rationality for the workers.

occupational choice, since anyone can choose to be a manager, this condition must hold not just for those who are managers in the current match but also for those who are workers and unmatched. Similarly, since anyone can be a worker, the targets of a prospective manager are no longer restricted to be the current workers but rather can include current managers and unmatched individuals.

When $\alpha = 1/2$, the unique stable matching in the above example is for all type 2 individuals to be managers, each of them being matched with a measure one of type 1 individuals at wage $w \simeq 1.41$.[8] At this wage, the firm size is optimal for type 2 managers. Their rent is equal to w, so that type 2 individuals are actually indifferent between being a manager or a worker. Type 1 individuals would get a rent approximately equal to 0.18 if they were to hire an optimal number of workers at wage w and, thus, they strictly prefer to be workers rather than managers. It follows from these properties that this matching is indeed stable.[9]

4 Matching with occupational choice

The setting we introduce in this paper is that of a matching market featuring occupational choice, many-to-one matching and a large number of participants. We frame this problem in the context of a labor market for simplicity, so that individuals have a choice of being a manager, a worker or self-employed.

4.1 Environment and matching

Individuals are (potentially) heterogenous in e.g. their talent or knowledge. This is captured by a (nonempty, Polish) *set Z of types.* The population of individuals is described by a nonzero, finite, Borel measure ν on Z; ν is the *type distribution.* A

[8] In Appendix B.7, we fully characterize the stable matchings in this example for each $\alpha \in (0, 1)$; in fact, there is a unique stable matching for each α.

[9] Our general framework allows for externalities and their presence is often natural. In the context of the above example, it might be that the production function depends on the aggregate managerial quality in an analogous way to Romer (1986), so that the rent of a manager with quality z is e.g. $\left(\int_{Z\times X} \hat{z} \mathrm{d}\mu(\hat{z}, \delta)\right) z^{1+\alpha} n^{1-\alpha} - cn$ when the matching is μ.

dummy type $\emptyset \notin Z$ is used to represent unmatched i.e. self-employed individuals, and we let $Z_\emptyset = Z \cup \{\emptyset\}$, with the assumption that $\emptyset$ is an isolated point in $Z_\emptyset$.

A manager of type z may be matched with a worker of type z' under some contract c. In particular, there is a (nonempty, Polish) *set C of contracts* and a *contract correspondence* $\mathbb{C} : Z \times Z_\emptyset \rightrightarrows C$ describing the set $\mathbb{C}(z, z')$ of contracts that are feasible for a manager of type z and a worker of type z' (when $z' = \emptyset$ the manager is, in fact, self-employed and $\mathbb{C}(z, \emptyset)$ describes the feasible contracts for a self-employed individual of type z).

A manager is allowed to hire as many workers as he likes; to capture the many-to-one aspect of matching, a manager is matched with a measure of workers and contracts $\delta \in \mathcal{M}(Z \times C)$.[10] The definition of a matching below will impose feasibility constraints on δ via the contract correspondence $\mathbb{C}$ and, thus, constrain the contracts that the manager can offer to each of his employees. These constraints are of the form $c \in \mathbb{C}(z, z')$ and are, therefore, independent across workers. To capture interdependent and other feasibility constraints, we let X be a subset of $\mathcal{M}(Z \times C)$ and require that managers be matched with $\delta \in X$.

Self-employed (or unmatched) managers are those matched with the dummy type $\emptyset$. To specify his contract (e.g. the number of hours worked as self-employed), we use matches of the form $(z, 1_{(\emptyset,c)})$ to describe a self-employed individual of type z with contract c. To unify the two cases, we let $X_\emptyset = X \cup \{1_{(\emptyset,c)} : c \in C\}$ be the *set of possible matches of managers and self-employed individuals.*

The *set of occupations* is $A = \{w, s, m\}$, where w stands for worker, s for self-employed and m for manager. The choice set of each individual depends on his occupation; namely, a worker chooses among managers' types and contracts, a self-employed individual among contracts, and a manager among measures $\delta \in X$ describing whom to hire and the contracts offered. To capture these differences, let $X_m = X$, $X_s = \{1_{(\emptyset,c)} : c \in C\}$, $X_w = \{1_{(z,c)} : (z, c) \in Z \times C\}$ and $\Delta = \{(a, \delta) : \delta \in X_a\}$.[11]

[10] Whenever Y is a metric space, $\mathcal{M}(Y)$ denotes the set of finite, Borel measures on Y endowed with the weak (narrow) topology (see Varadarajan (1958) for details). We often focus on $\mathcal{M}_R(Y)$ where, for each $R > 0$, $\mathcal{M}_R(Y) = \{\delta \in \mathcal{M}(Y) : \delta(Y) \leq R\}$.

[11] We do not distinguish between (z, c) and $1_{(z,c)}$ for each $(z, c) \in Z_\emptyset \times C$, hence it would be

The set Δ is the *choice set of each individual* as she can choose her occupation and a match feasible for the chosen occupation.

We allow for externalities and, thus, preferences are allowed to depend on the matching. Matchings with occupational choice are elements of $\mathcal{M}(Z \times X_\emptyset)$ satisfying certain properties described below. The *preferences* of an individual of type z are then described by a relation $\succ_z$ defined on $\Delta \times \mathcal{M}(Z \times X_\emptyset)$ for each $z \in Z$.

In summary, a *matching market with occupational choice* (a market, henceforth) is $E = (Z, \nu, C, \mathbb{C}, X, (\succ_z)_{z\in Z})$.

A *matching with occupational choice* (a matching, henceforth) is a Borel measure $\mu \in \mathcal{M}(Z \times X_\emptyset)$ such that

1. $\{z\} \times \operatorname{supp}(\delta) \subseteq \operatorname{graph}(\mathbb{C})$ for each $(z, \delta) \in \operatorname{supp}(\mu)$, and

2. $\nu_M + \nu_S + \nu_W = \nu$

where, for each Borel subset B of Z, $\nu_M(B) = \mu(B \times X)$, $\nu_S(B) = \mu(B \times (X_\emptyset \setminus X))$ and $\nu_W(B) = \int_{Z\times X} \delta(B \times C)\mathrm{d}\mu(z, \delta)$.

The interpretation of μ is as follows. First, μ describes the occupational choices by the place in the match (z, δ), namely, the first coordinate refers to managers and the second to workers (as part of a firm) when $\delta \in X$ and, when $\delta \in X_\emptyset \setminus X$, the first coordinate refers to a self-employed individual and the second, which is equal to $1_{(\emptyset,c)}$ for some $c \in C$, describes the individual's contract. Condition 1 requires that the contract is feasible according to the contract correspondence. Condition 2 requires that everyone in the market is accounted for, as follows: For each Borel subset B of Z, $\mu(B \times X)$ is the measure of managers whose type belongs to B and we call it $\nu_M(B)$. Similarly, $\mu(B \times (X_\emptyset \setminus X))$ is the measure of self-employed individuals whose type belongs to B and we call it $\nu_S(B)$. Finally, $\int_{Z\times X} \delta(B \times C)\mathrm{d}\mu(z, \delta)$ is the measure of workers whose type belongs to B and, thus, we call it $\nu_W(B)$.[12] Since an

simpler to replace the latter with the former in the definition of X_s and X_w. The formalization we use above provides an unified notation which simplifies the exposition elsewhere.

[12]For each Borel subset E of a metric space Y, the function $\delta \mapsto \delta(E) : \mathcal{M}(Y) \to \mathbb{R}$ is Borel measurable. This follows by the argument in Aliprantis and Border (2006, Theorem 15.13, p. 514) together with Varadarajan (1958, Theorem 3.1).

individual must be either a manager, or a worker or self-employed, condition 2 must hold if everyone in the market is accounted for.

4.2 Stability

Heading towards the definition of stable matchings, we start by defining the targets of individuals at a given matching and then define the stability set of a matching.

Targets at a given matching μ depend on the type z and on the occupational choice a, and are denoted by $T_z^a(\mu)$. Because one's occupation is a choice and not a fixed characteristic, these targets are for someone planning to choose occupation a, i.e. if someone chooses occupation a, then his targets are $T_z^a(\mu)$. The targets for the prospective self-employed are simply the contracts that are feasible when someone is unmatched: For each $z \in Z$, let $T_z^s(\mu) = \{\emptyset\} \times \mathbb{C}(z, \emptyset)$.

The targets of prospective managers and workers are more complicated as they consist of contracts and types of people on the other side of the market that managers or workers can attract. But with occupational choice, there is not a fixed "other side of the market" since anyone can change his occupation. In more detail, even if all individuals of type z^* are managers in the matching μ, any type z^* person can choose to became a worker. In particular, if such z^* person gains by becoming a worker and by working for a manager of type z at some contract c, then (z^*, c) is a target for those of type z planning to be a manager, i.e. it belongs to $T_z^m(\mu)$. We then let, for each $z \in Z$, $T_z^m(\mu)$ be the set of $(z^*, c) \in Z \times C$ such that $c \in \mathbb{C}(z, z^*)$ and there exists

(a) $(z', c', \delta') \in Z \times C \times X$ such that $(z', \delta') \in \text{supp}(\mu)$, $(z^*, c') \in \text{supp}(\delta')$ and $(w, 1_{(z,c)}, \mu) \succ_{z^*} (w, 1_{(z',c')}, \mu)$, or

(b) $\delta' \in X_\emptyset \setminus X$ such that $(z^*, \delta') \in \text{supp}(\mu)$ and $(w, 1_{(z,c)}, \mu) \succ_{z^*} (s, \delta', \mu)$, or

(c) $\delta' \in X$ such that $(z^*, \delta') \in \text{supp}(\mu)$ and $(w, 1_{(z,c)}, \mu) \succ_{z^*} (m, \delta', \mu)$.

Anyone of type z can be a manager if he finds workers, here of type z^*, who prefer to work for him than to be in their current occupation. Each of these workers can be someone who was already a worker in μ as described in condition (a), or self-employed as described by condition (b), or even a manager as described by condition (c).

The targets of prospective workers are defined analogously. Thus, for each $z \in Z$, let $T_z^w(\mu)$ be the set of $(z^*, c) \in Z \times C$ such that $c \in \mathbb{C}(z^*, z)$ and there is $\delta \in X$ such that $(z, c) \in \text{supp}(\delta)$ and

(a) $\text{supp}(\delta) \setminus \{(z, c)\} \subseteq T_{z^*}^m(\mu)$ and there is $(z', c', \delta') \in Z \times C \times X$ such that $(z', \delta') \in \text{supp}(\mu)$, $(z^*, c') \in \text{supp}(\delta')$ and $(m, \delta, \mu) \succ_{z^*} (w, 1_{(z',c')}, \mu)$, or

(b) $\text{supp}(\delta) \setminus \{(z, c)\} \subseteq T_{z^*}^m(\mu)$ and there is $\delta' \in X_\emptyset \setminus X$ such that $(z^*, \delta') \in \text{supp}(\mu)$ and $(m, \delta, \mu) \succ_{z^*} (s, \delta', \mu)$, or

(c) there is $\delta' \in X$ such that $\text{supp}(\delta) \setminus \{(z, c)\} \subseteq T_{z^*}^m(\mu) \cup \text{supp}(\delta')$, $(z^*, \delta') \in \text{supp}(\mu)$ and $(m, \delta, \mu) \succ_{z^*} (m, \delta', \mu)$.

As above, anyone of type z can be a worker if she finds a manager, here of type z^*, that hires her, possibly alongside other workers as described by $\delta \in X$, and both agree on a feasible contract $c \in \mathbb{C}(z^*, z)$. This manager can be someone who was already a manager in μ as described in condition (c), or self-employed as described by condition (b), or even a worker as described by condition (a).

The stability set $S(\mu)$ of matching μ is the set of $(z, \delta) \in Z \times X_\emptyset$ such that, if $\delta \in X$, then

(i) there does not exist $(a, \delta') \in \Delta$ such that $\text{supp}(\delta') \subseteq T_z^a(\mu) \cup \text{supp}(\delta)$ if $a = m$, $\text{supp}(\delta') \subseteq T_z^a(\mu)$ if $a \neq m$, and $(a, \delta', \mu) \succ_z (m, \delta, \mu)$,

(ii) for each $(z', c) \in \text{supp}(\delta)$, there does not exist $(a, \delta') \in \Delta$ such that $\text{supp}(\delta') \subseteq T_{z'}^a(\mu)$ and $(a, \delta', \mu) \succ_{z'} (w, 1_{(z,c)}, \mu)$,

and, if $\delta \in X_\emptyset \setminus X$, then

(iii) there does not exist $(a, \delta') \in \Delta$ such that $\text{supp}(\delta') \subseteq T_z^a(\mu)$ and $(a, \delta', \mu) \succ_z (s, \delta, \mu)$.

The set $S(\mu)$ describes matches (z, δ) that do not suffer from instability. Instability could come from those who are managers in μ if a manager of type z can find a match δ' that is better than his current one δ by employing workers of the types currently employed or those of his targets. In addition, he could instead be better off by changing his occupation and matching with some of his targets for the alternative occupation. Condition (i) rules out instability arising from the current managers, whereas condition (ii) does the same for current workers and (iii) for self-employed. A matching μ is *stable* if $\text{supp}(\mu) \subseteq S(\mu)$.

Theorem 1 provides a characterization of stable matchings that is simpler to use. Let $S_M(\mu)$ be defined as $S(\mu)$ but with "$(a, \delta') \in \Delta$" being replaced with "$(a, \delta') \in \Delta$ such that $a = m$" and, analogously, $IR(\mu)$ be defined as $S(\mu)$ but with "$(a, \delta') \in \Delta$" being replaced with "$(a, \delta') \in \Delta$ such that $a = s$".

Theorem 1 *A matching μ is stable if and only if* $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$.

4.3 Discussion

We conclude this section with some comments on our definition of stability. First, note that it focuses on the support of the matching. In some cases, however, not all elements of $\text{supp}(\delta)$ in a match (z, δ) are pairs of worker types and contracts that are matched with a manager of type z. This may happen, for example, if $\delta = \sum_{k=1}^{\infty} 2^{-k} 1_{(z_k, c_k)}$ for some countable subset $D = \{(z_k, c_k)\}_{k=1}^{\infty}$ of $Z \times C$. In this case, it would seem more appropriate to require only that $\{z\} \times D \subseteq \text{graph}(\mathbb{C})$ instead of $\{z\} \times \text{supp}(\delta) \subseteq \text{graph}(\mathbb{C})$ in the definition of a matching. When the correspondence $\mathbb{C}$ is continuous, this issue does not arise since then the two requirements are equivalent. Similar considerations apply to the definition of stability when preferences are also continuous. For instance, when a market E also satisfies a richness condition, we have that a matching μ is stable if and only if $S(\mu)$ has full μ-measure.[13]

A more important issue concerns what we require for a manager of type z, currently matched with δ, and a potential workforce δ' to qualify as a blocking coalition.[14] In the simpler case where preferences do not depend on the matching, we require that

[13] See Section 5 for the notion of continuity and richness we use and Appendix B.6 for a proof of this claim.

[14] I.e. what condition (i) of the definition of $S(\mu)$ for $a = m$ rules out.

$(m, \delta') \succ_z (m, \delta)$ and $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$. This requirement is unusual in that it is between weak and strong domination – but as we now argue, it is the weakest requirement for blocking (and hence associated with the strongest stability notion) such that stable matchings exist under general conditions.

We illustrate the above with the following example, where for simplicity contracts are omitted in addition to preferences not depending on the matching. Let $Z = \{1, 2\}$, $\nu(1) = \nu(2) = \frac{1}{2}$ and $X = \{n1_z : n \leq 1, z \in Z\}$. Let preferences be represented by: $u_z(m, n1_{z'}) = 2nz'$, $u_z(w, 1_{z'}) = z'$, and $u_z(s, 1_\emptyset) = 0$. It is easy to see that μ such that $\mu(2, 1_1) = \frac{1}{2}$ is a stable matching. Here, every individual gets payoff 2 (thus the matching is individually rational and $\text{supp}(\mu) \subseteq IR(\mu)$), and since being a worker yields payoff at most 2, $T_1^m(\mu) = T_2^m(\mu) = \emptyset$. Since $\text{supp}(\mu) = \{(2, 1_1)\}$, $(w, 1_2) \succeq_1 (m, \delta)$ for all $\delta \in X$ such that $\text{supp}(\delta) \subseteq \emptyset$ and $(m, 1_1) \succeq_2 (m, \delta)$ for all $\delta \in X$ such that $\text{supp}(\delta) \subseteq \{1\}$, it follows that $\text{supp}(\mu) \subseteq S_M(\mu)$.

The strongest notion of stability is the one that defines a blocking coalition via *weak domination*, i.e. to require that every individual in the coalition is weakly better off with at least one individual being strictly better off. Let $\nu_W^{\preceq z}(\mu)$ be the measure of types who would *weakly prefer* to work for type z than remain in their current match, given μ. Under weak domination, our requirement that $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ would be replaced with $\text{supp}(\delta') \subseteq \text{supp}(\nu_W^{\preceq z}(\mu))$.[15] Note that $T_z^m(\mu) \cup \text{supp}(\delta) \subseteq \text{supp}(\nu_W^{\preceq z}(\mu))$ since $T_z^m(\mu)$ is the set of types[16] that would *strictly prefer* to work for type z given matching μ and those in $\text{supp}(\delta)$ are currently working for type z and hence indifferent; thus, the resulting notion of stability is stronger.

[15] To see this in the context of the current example, suppose that $(z, \delta) \in \text{supp}(\mu)$ and there exists δ' such that $(m, \delta') \succ_z (m, \delta)$ and $\text{supp}(\delta') \subseteq \text{supp}(\nu_W^{\preceq z}(\mu))$. Then there is a nonnull coalition S of individuals, described by a measure $\nu^S = \nu_M^S + \nu_W^S$, and a matching μ^S for the coalition such that $\text{supp}(\nu_M^S) = \{z\}$, $\mu^S(z, \delta') = \nu_M^S(z)$, $\mu^S(z, \delta')\delta'(z') = \nu_W^S(z')$ for each $z' \in Z$, each manager in ν_M^S is strictly better off and each worker in ν_W^S is weakly better off. Indeed, let $\nu_M^S = \varepsilon 1_z$ and $\nu_W^S(z') = \varepsilon\delta'(z')$ for each $z' \in Z$. For each $z' \in \text{supp}(\nu_W^S) = \text{supp}(\delta')$, we have that $z' \in \text{supp}(\nu_W^{\preceq z}(\mu))$; thus for ε sufficiently small, $\nu_W^S(z') = \varepsilon\delta'(z') \leq \nu_W^{\preceq z}(\mu)(z')$ for each $z' \in \text{supp}(\nu_W^S)$ and so the coalition can be chosen such that each worker is weakly better off. In addition, for ε sufficiently small, $\nu_M^S(z) = \varepsilon \leq \mu(z, \delta)$ and so the coalition can be chosen such that each manager is strictly better off.

[16] Recall that we are omitting contracts for simplicity.

However, this yields a notion of stability for which there are no stable matchings in the current example. In the matching of the previous paragraph, we now have $\text{supp}(\nu_W^{\preceq 2}(\mu)) = \{1, 2\}$, $\text{supp}(1_2) \subseteq \text{supp}(\nu_W^{\preceq 2}(\mu))$ and $(m, 1_2) \succ_2 (m, 1_1)$. It is easy to see that there are no other stable matchings; in any stable matching all type 2 individuals must be managers and employ type 2 individuals but this is impossible.

We could alternatively use *strong domination* to define a blocking coalition, i.e. to require that every individual in the coalition is strictly better off. Then for type z, currently a manager and matched with δ, to form a blocking coalition with potential workforce δ', we would need $(m, \delta') \succ_z (m, \delta)$ and $\text{supp}(\delta') \subseteq T_z^m(\mu)$. Note that $T_z^m(\mu) \subseteq T_z^m(\mu) \cup \text{supp}(\delta) \subseteq \text{supp}(\nu_W^{\preceq z}(\mu))$; hence stability defined via weak domination is the strongest notion, followed by ours, followed by the one defined via strong domination.

Our existence result, Theorem 2, shows generally that, when managers can only hire a bounded number of workers as in the above example, a stable matching exists when blocking coalitions are defined using our requirement $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ (and hence when they are defined via strong domination). Our reason for adopting our stability notion is that it is a refinement of the stability notion defined via strong domination but its existence is nevertheless guaranteed under general conditions. We prefer our notion to the one defined via strong domination because our notion implies existing stability notions in special cases (see Section 6).

5 Existence of stable matchings

In this section we establish the existence of stable matchings and discuss the conditions needed to prove this result.

One requirement in our existence result is that preferences are rational. We say that a market is *rational* if $\succ_z$ is asymmetric and negative transitive for each $z \in Z$.[17] Note that $\succ_z$ is asymmetric and negative transitive if and only if $\succeq_z$ is complete and

[17] A relation $\succ$ on a set Y is asymmetric if, for each $x, y \in Y$, if $x \succ y$ then $\neg(y \succ x)$. It is negative transitive if, for each $x, y, z \in Y$, if $\neg(x \succ y)$ and $\neg(y \succ z)$, then $\neg(x \succ z)$.

transitive (i.e. rational).[18] Rational preferences can be represented by an utility function and this plays an important role in our proof.

Another basic requirement in our existence results is some form of continuity. We say that a market E is *continuous* if $\{(a,\delta,\mu,a',\delta',\mu',z) \in (\Delta \times \mathcal{M}(Z \times X_\emptyset))^2 \times Z : (a,\delta,\mu) \succ_z (a',\delta',\mu')\}$ is open,[19] $\mathbb{C}$ is continuous with nonempty and compact values, and X is closed.

Stable matchings may fail to exist in the absence of a bound on the measure of workers a manager can hire. This existence problem arises because each manager is negligible and, therefore, is effectively unconstrained by the size of the market. In Section A.7, we provide an example showing that, without any boundedness assumptions on X, a stable matching fails to exist.[20] Thus, we focus on bounded markets, defined as follows: We say that a market E is *bounded* if there exists $R > 0$ such that $\delta(Z \times C) \leq R$ for each $\delta \in X$. More succinctly, E is bounded if $X \subseteq \mathcal{M}_R(Z \times C)$ for some $R > 0$.

Note that boundedness is essentially a uniform satiation condition. Indeed, suppose that there exists $R > 0$ such that, for each $z \in Z$ and $\mu \in \mathcal{M}(Z \times X_\emptyset)$, there exists $\delta \in X$ such that $\delta(Z \times C) \leq R$ and $(m,\delta,\mu) \succeq_z (m,\delta',\mu)$ for each $\delta' \in X$. In this case, as far as existence of stable matchings is concerned, we may focus on $\delta \in \mathcal{M}_R(Z \times C)$ and, thus, assume that the market is bounded.

We will also focus on rich markets. The reason is that our approach to the existence problem consists in first addressing discrete markets where Z, C and X are finite. In such markets, managers are matched with measures of workers that are finitely supported and richness will then allow us to extend our existence results from discrete to general markets. We say that a market E is *rich* if the correspondences $\Lambda : Z \times X \times \mathcal{M}(Z \times X_\emptyset) \rightrightarrows X$ and $\Lambda_0 : Z \times \mathcal{M}(Z \times X_\emptyset) \rightrightarrows X$ defined by setting, for each $(z,\delta,\mu) \in Z \times X \times \mathcal{M}(Z \times X_\emptyset)$, $\Lambda(z,\delta,\mu) = \{\delta' \in X : \text{supp}(\delta') \subseteq \text{supp}(\delta) \cup T^m_z(\mu)\}$

[18] The relation $\succeq_z$ is defined as usual by setting, for each $(a,\delta,\mu),(a',\delta',\mu') \in \Delta \times \mathcal{M}(Z \times X_\emptyset)$, $(a,\delta,\mu) \succeq_z (a',\delta',\mu')$ if and only if $(a,\delta,\mu) \succ_z (a',\delta',\mu')$ or $\neg\big((a',\delta',\mu') \succ_z (a,\delta,\mu)\big)$.

[19] The set A of occupations is endowed with the discrete topology.

[20] A stable matching would fail to exist even under the weakest form of stability we discuss in Section 4 which is defined via strong domination.

and $\Lambda_0(z,\mu) = \{\delta' \in X : \text{supp}(\delta') \subseteq T_z^m(\mu)\}$ are lower hemicontinuous.

The richness assumption is a mild requirement which is satisfied in several special cases, including those of CKK and GK where, respectively, $X = \mathcal{M}_1(Z \times C)$ and $X = \{1_{(z,c)} : (z,c) \in Z \times C\}$ (the boundedness assumption is clearly also satisfied in these two cases). This can be seen by noting that, for a market to be rich, it is sufficient that the set of finitely supported measures on $Z \times C$ is dense in X (this is (β) below) and that measures δ obtained via a small perturbation to the support of a finitely supported measure in X remain in X (this is (α) below). More formally, the following conditions are sufficient for richness:[21]

(α) For each $\delta \in X$ such that $\delta = \sum_{j=1}^J a_j 1_{(z_j,c_j)}$ for some $J \in \mathbb{N}$, $a_j \in \mathbb{R}_{++}$ and $(z_j,c_j) \in Z \times C$ for each $j = 1,\ldots,J$ and each open neighborhood V_δ of δ in X, there exist open neighborhoods $V_{(z_j,c_j)}$ of (z_j,c_j) for each $j = 1,\ldots,J$ such that, whenever $(\hat{z}_j,\hat{c}_j) \in V_{(z_j,c_j)}$ for each $j = 1,\ldots,J$, there exists $\hat{a} = (\hat{a}_1,\ldots,\hat{a}_J) \in \mathbb{R}_+^J$ such that $\sum_{j=1}^J \hat{a}_j 1_{(\hat{z}_j,\hat{c}_j)} \in V_\delta$.

(β) For each $\delta \in X$ and open neighborhood V_δ of δ in X, there exists $\hat{\delta} \in V_\delta$ such that $\text{supp}(\hat{\delta})$ is a finite subset of $\text{supp}(\delta)$.

The following is our main existence result. As GK's framework is a special case of ours, it has GK's Theorem 5 as a special case.

Theorem 2 *Every rational, continuous, bounded and rich market has a stable matching.*

When preferences do not depend on externalities, the rationality of E can be replaced with the requirement that preferences are acyclic. This is because when Z, C and X are finite, acyclic preferences defined on Δ (as opposed to $\Delta \times \mathcal{M}(Z \times X_\emptyset)$) can be extended to linear orders, which are rational. We say that E is a *market without externalities* if, for each $z \in Z$ and $(a,\delta),(a',\delta') \in \Delta$, if $(a,\delta,\hat{\mu}) \succ_z (a',\delta',\hat{\mu})$ for some $\hat{\mu} \in \mathcal{M}(Z \times X_\emptyset)$, then $(a,\delta,\mu) \succ_z (a',\delta',\mu)$ for all $\mu \in \mathcal{M}(Z \times X_\emptyset)$. Moreover,

[21] See Appendix B.5 for a proof of this claim.

we say that E is *acyclic* if $\succ_z$ is acyclic for each $z \in Z$.[22] We then obtain the following corollary which has GK's Theorem 1 as a special case.

Corollary 1 *Every acyclic, continuous, bounded and rich market without externalities has a stable matching.*

6 Applications

6.1 Marriage markets

We consider in this section GK's model of two-sided one-to-one matching markets (marriage markets, henceforth) and show that it is a special case of the framework of markets with occupational choice in Section 4. For simplicity, we consider the case where preferences do not depend on the matching.

A *marriage market* is $E = (W, M, \nu_W, \nu_M, C, \mathbb{C}, (\succ_w)_{w\in W}, (\succ_m)_{m\in M})$ satisfying the following conditions and having the following interpretation. The sets M and W are Polish spaces of types of men (or managers) and women (or workers) respectively. To these sets correspond nonzero, finite, Borel measures ν_W and ν_M on W and M, respectively, describing the population of managers and workers. In addition, there is a dummy type $\emptyset \notin W \cup M$, which is an isolated point in $W_\emptyset = W \cup \{\emptyset\}$ and in $M_\emptyset = M \cup \{\emptyset\}$, to represent unmatched individuals. The set C is a Polish space of contracts and $\mathbb{C} : M_\emptyset \times W_\emptyset \rightrightarrows C$ is a contract correspondence. Workers' preferences are described by $(\succ_w)_{w\in W}$ and managers' preferences by $(\succ_m)_{m\in M}$; for each $w \in W$, $\succ_w$ is defined on $M_\emptyset \times C$ and, for each $m \in M$, $\succ_m$ is defined on $W_\emptyset \times C$. In addition, $\succ_\emptyset$ denotes the empty relation under which no elements are comparable and the following conditions hold: **(GK1)** $\succ_w$ (resp. $\succ_m$) is acyclic for each $w \in W$ (resp. $m \in M$), **(GK2)** $\{(m, c, m', c', w) \in (M_\emptyset \times C)^2 \times W : (m, c) \succ_w (m', c')\}$ and $\{(w, c, w', c', m) \in (W_\emptyset \times C)^2 \times M : (w, c) \succ_m (w', c')\}$ are open, and **(GK3)** $\mathbb{C}$ is continuous with nonempty and compact values.

[22] A relation $\succ$ on a set Y is acyclic if there is no finite sequence $y_1, y_2, \ldots, y_n$ in Y such that $y_1 \succ y_2 \succ \cdots \succ y_n \succ y_1$.

A matching for a marriage market E (a *marriage matching*, henceforth) is a Borel measure $\mu \in \mathcal{M}(M_\emptyset \times W_\emptyset \times C)$ such that **(M1)** $\mu(B \times W_\emptyset \times C) = \nu_M(B)$ for each Borel subset B of M, **(M2)** $\mu(M_\emptyset \times B \times C) = \nu_W(B)$ for each Borel subset B of W, **(M3)** $\text{supp}(\mu) \subseteq \text{graph}(\mathbb{C})$, and **(M4)** $\mu(\{(m,w,c) : m = w = \emptyset\}) = 0$.

GK define an instability set I containing the pairs of couples such that there is some instability. The precise definition of I is given in GK; here, we just note that I is a subset of $(M_\emptyset \times W_\emptyset \times C)^2$. A marriage market is *stable* if $\mu \otimes \mu(I) = 0$.

We now show how to represent a marriage market E as a market with occupational choice $\hat{E}$ and characterize the stable matchings of E in terms of those of $\hat{E}$. We may assume that W and M are disjoint (if not, we could consider $\hat{W} = \{w\} \times W$ and $\hat{M} = \{m\} \times M$ where $w \neq m$) and let $Z = W \cup M$ be the set of types in $\hat{E}$; we may assume that W and M are closed subsets of Z.[23] The type distribution ν is defined by setting, for each Borel subset B of Z, $\nu(B) = \nu_M(M \cap B) + \nu_W(W \cap B)$. The set of contracts in $\hat{E}$ is C as in E. The constraint correspondence $\hat{\mathbb{C}}$ is defined from $\mathbb{C}$ just by adjusting the order in which elements are listed and by arbitrarily defining the feasible contracts of two types in M and two types in W, as follows: let $\bar{c} \in C$ be given and set, for each $z \in Z$ and $z' \in Z_\emptyset$, $\hat{\mathbb{C}}(z,z') = \mathbb{C}(z,z')$ if $z \in M$ and $z' \in W \cup \{\emptyset\}$, $\hat{\mathbb{C}}(z,z') = \mathbb{C}(z',z)$ if $z \in W$ and $z' \in M \cup \{\emptyset\}$, and $\hat{\mathbb{C}}(z,z') = \{\bar{c}\}$ if $(z,z') \in M^2$ or $(z,z') \in W^2$.

The set of feasible matches for managers in $\hat{E}$ is $X = \{1_{(z,c)} : (z,c) \in W \times C\}$. Finally, preferences are defined as follows. For each $z \in M$, $a, a' \in \{m, s\}$, $1_{(\bar{z},\bar{c})} \in X_a$, $1_{(z',c')} \in X_{a'}$ and $\tilde{\delta} \in X_w$, (i) $(a, 1_{(\bar{z},\bar{c})}) \hat{\succ}_z (a', 1_{(z',c')})$ if and only if $(\bar{z},\bar{c}) \succ_z (z',c')$, and (ii) $(a,\delta) \hat{\succ}_z (w, \tilde{\delta})$. Similarly, for each $z \in W$, let $\tilde{X} = \{1_{(z,c)} : (z,c) \in M_\emptyset \times C\}$ and define, for each $a, a' \in \{w, s\}$, $1_{(\bar{z},\bar{c})} \in X_a \cap \tilde{X}$, $1_{(z',c')} \in X_{a'} \cap \tilde{X}$, $\hat{\delta} \in X_w \setminus \tilde{X}$ and $\tilde{\delta} \in X_m$, (i) $(a, 1_{(\bar{z},\bar{c})}) \hat{\succ}_z (a', 1_{(z',c')})$ if and only if $(\bar{z},\bar{c}) \succ_z (z',c')$, (ii) $(a,\delta) \hat{\succ}_z (m, \tilde{\delta})$, and (iii) $(a, 1_{(\bar{z},\bar{c})}) \hat{\succ}_z (w, \hat{\delta})$. In both cases, condition (i) says that preferences in $\hat{E}$ are derived from those in E when comparing the choices that individuals can make in E, namely, being a manager or self-employed in the case of someone with type in M and being a worker or self-employed in the case of someone with type in W. Condition

[23] Define a metric d on Z based on those in M and W, and set $d(w,m) = 1$ for each $(w,m) \in W \times M$.

(ii) says that being a worker is always worse than being a manager or self-employed for someone with type in M and being a manager is always worse than being a worker or self-employed for someone with type in W. Finally, condition (iii) says that, for someone with type in W, being a worker matched with someone in W is always worse than being a worker matched with someone in M or self-employed. We say that $\hat{E}$ is the *market associated with* E.

The following result shows that the stable matchings of $\hat{E}$ are the same as those of E up to an homeomorphism. Let

$$Y = (M \times X_\emptyset) \cup (W \times (X_\emptyset \setminus X)) \text{ and } Y' = (M \times W_\emptyset \times C) \cup (\{\emptyset\} \times W \times C).$$

Theorem 3 *Let E be a marriage matching market, $\hat{E}$ be its associated market, $\mathcal{S}$ be the set of stable matchings of E and $\hat{\mathcal{S}}$ be the set of stable matchings of $\hat{E}$. Then there is an homeomorphism $h : Y \to Y'$ with inverse f such that $\hat{\mathcal{S}} = \{\mu \circ f^{-1} : \mu \in \mathcal{S}\}$ and $\mathcal{S} = \{\hat{\mu} \circ h^{-1} : \hat{\mu} \in \hat{\mathcal{S}}\}$.*

6.2 Roommate market

Gale and Shapley (1962) considered a roommate problem in which an "even number of boys wish to divide up into pairs of roommates." This is an example of matching with occupational choice since there aren't two exogenously given sets of individuals to match; it has also the particular feature that individuals are indifferent between the different occupations.

In this section we formulate a general version of the roommate problem with a continuum of individuals in distributional form, a roommate market, as a special case of our framework. We show that, in contrast to the case of finitely many individuals of Gale and Shapley (1962), a stable matching exists in roommate markets that are acyclic and continuous when preferences do not depend on externalities; when preferences depend on externalities, our existence result requires preferences to be rational and continuous. In particular, roommate markets are always bounded and rich.

The importance of large markets for the existence of stable matchings in the roommate problem has been established by Chiappori, Galichon, and Salanié (2014), Pęski (2017), Azevedo and Hatfield (2018), Wu (2021) and Jagadeesan and Vocke (2021). Both Chiappori, Galichon, and Salanié (2014) and Pęski (2017) show the existence of approximately stable matchings in roommate problems with a large finite number of individuals, respectively, with and without transferable utility, and Azevedo and Hatfield (2018) establish the existence of (exact) stable matchings with a continuum of individuals and with transferable utility. Our existence result for the roommate problem, like Wu's (2021) and Jagadeesan and Vocke's (2021), dispenses with the requirement of transferable utility in Azevedo and Hatfield's (2018) result and allows us to cover a setting which is exactly as in Gale and Shapley (1962) except for the cardinality of the set of individuals to be matched; in contrast to Jagadeesan and Vocke (2021) and Wu (2021), we also allow preferences to depend on the entire matching.

A roommate market can be defined in our setting as a market where matching is restricted to be one-to-one and preferences and the contract correspondence satisfy certain restrictions that reflect the fact that the roles of worker and manager have no meaning in the roommate setup. In particular, we define a roommate market as a market $E = (Z, \nu, C, \mathbb{C}, X, (\succ_z)_{z \in Z})$ satisfying the following restrictions:

(R1) $X = \{1_{(z,c)} : (z, c) \in Z \times C\}$,

(R2) $(m, 1_{(z',c)}, \mu) \sim_z (w, 1_{(z',c)}, \mu)$ for each $z, z' \in Z$, $c \in C$ and $\mu \in \mathcal{M}(Z \times X_\emptyset)$,

(R3) $\mathbb{C}(z, z') = \mathbb{C}(z', z)$ for each $z, z' \in Z$, and

(R4) $(m, 1_{(z',c)}, \mu) \sim_z (m, 1_{(z',c)}, \mu \circ f^{-1})$ for each $z, z' \in Z$, $c \in C$ and measurable $f \in F$,

where $F = \{f : f(z, z') = (z, z')$ or $f(z, z') = (z', z)$ for each $z, z' \in Z,$ and $f(z, \emptyset) = (z, \emptyset)$ for each $z \in Z\}$. (R1) requires that matching in a roommate market is one-to-one. (R2) requires that each type cares only about who he is matched with (and not the role he occupies in the match). (R3) requires that switching the roles of two types in a match does not affect the set of feasible contracts, and (R4) requires that matchings that differ only according to who occupies which role in a match are treated the same way.

The particular setting of a roommate market allows for some simplification in its description. In fact, we can identify $1_{(z,c)}$ with (z,c) for each $(z,c) \in Z_\emptyset \times C$ and, thus, we can write (R1) as requiring $X = Z \times C$ and $X_\emptyset = Z_\emptyset \times C$. In particular, a matching is $\mu \in \mathcal{M}(Z \times Z_\emptyset \times C)$.

(R2) implies that individual preferences can be defined on $Z_\emptyset \times C \times \mathcal{M}(Z \times Z_\emptyset \times C)$.[24] In light of this comment and the one in the previous paragraph, we can equivalently define a *roommate market* as $E = (Z, \nu, C, \mathbb{C}, (\succ_z)_{z\in Z})$ such that $(Z, \nu, C, \mathbb{C})$ are as in the general framework of Section 4, $\mathbb{C}$ satisfies $\mathbb{C}(z,z') = \mathbb{C}(z',z)$ for each $z, z' \in Z$, and $\succ_z$ is defined on $Z_\emptyset \times C \times \mathcal{M}(Z \times Z_\emptyset \times C)$ and satisfies $(z', c, \mu) \sim_z (z', c, \mu \circ f^{-1})$ for each $z \in Z$, $(z', c) \in Z_\emptyset \times C$ and measurable $f \in F$.

A matching, which we refer to as a *roommate matching*, is then a Borel measure $\mu \in \mathcal{M}(Z \times Z_\emptyset \times C)$ such that $\operatorname{supp}(\mu) \subseteq \operatorname{graph}(\mathbb{C})$ and $\nu_W + \nu_S + \nu_M = \nu$ where, for each Borel subset B of Z, $\nu_M(B) = \mu(B \times Z \times C)$, $\nu_W(B) = \mu(Z \times B \times C)$ and $\nu_S(B) = \mu(B \times \{\emptyset\} \times C)$.

The targets become

$$T_z^m(\mu) = T_z^w(\mu) = \{(z^*, c) \in Z \times C : c \in \mathbb{C}(z, z^*) \text{ and } \exists (z', c') \in Z \times C \text{ such that } \operatorname{supp}(\mu) \cap \{(z^*, z', c'), (z', z^*, c')\} \neq \emptyset \text{ and } (z, c, \mu) \succ_{z^*} (z', c', \mu)\}$$

and $T_z^s(\mu) = \{\emptyset\} \times \mathbb{C}(z, \emptyset)$. Let $T_z(\mu) = T_z^m(\mu) \cup T_z^s(\mu)$. Then $S(\mu)$ becomes the set of $(z, z', c) \in Z \times Z_\emptyset \times C$ such that

(i) there does not exist $(\hat{z}, \hat{c}) \in T_z(\mu)$ such that $(\hat{z}, \hat{c}, \mu) \succ_z (z', c, \mu)$, and

(ii) if $z' \neq \emptyset$, there does not exist $(\hat{z}, \hat{c}) \in T_{z'}(\mu)$ such that $(\hat{z}, \hat{c}, \mu) \succ_{z'} (z, c, \mu)$.

[24] Indeed, given $\hat{\succ}_z$ defined on $\Delta \times \mathcal{M}(Z \times Z_\emptyset \times C)$, define $\succ_z$ on $Z_\emptyset \times C \times \mathcal{M}(Z \times Z_\emptyset \times C)$ by setting, for each $z', z'' \in Z$, $c', c'' \in C$, and $\mu', \mu'' \in \mathcal{M}(Z \times Z_\emptyset \times C)$, (i) $(z', c', \mu') \succ_z (z'', c'', \mu'')$ if and only if $(m, z', c', \mu') \hat{\succ}_z (m, z'', c'', \mu'')$, (ii) $(z', c', \mu') \succ_z (\emptyset, c'', \mu'')$ if and only if $(m, z', c', \mu') \hat{\succ}_z (s, \emptyset, c'', \mu'')$, (iii) $(\emptyset, c', \mu') \succ_z (z'', c'', \mu'')$ if and only if $(s, \emptyset, c', \mu') \hat{\succ}_z (m, z'', c'', \mu'')$, and (iv) $(\emptyset, c', \mu') \succ_z (\emptyset, c'', \mu'')$ if and only if $(s, \emptyset, c', \mu') \hat{\succ}_z (s, \emptyset, c'', \mu'')$. These four conditions, together with (R2), also define $\hat{\succ}_z$ on $\Delta \times \mathcal{M}(Z \times Z_\emptyset \times C)$ from $\succ_z$ on $Z_\emptyset \times C \times \mathcal{M}(Z \times Z_\emptyset \times C)$.

Since a roommate market is a particular case of the setting of Section 4, the existence of a stable matching for each roommate market follows from Theorem 2.

Corollary 2 *If E is a rational and continuous roommate market or an acyclic and continuous roommate market without externalities, then E has a stable roommate matching.*

To illustrate our stability condition for the roommate market, first consider the example from Gale and Shapley (1962) with four individuals α, β, γ and δ. Preferences are given by: $\beta \succ_\alpha \gamma \succ_\alpha \delta \succ_\alpha \emptyset$, $\gamma \succ_\beta \alpha \succ_\beta \delta \succ_\beta \emptyset$, and $\alpha \succ_\gamma \beta \succ_\gamma \delta \succ_\gamma \emptyset$. As Gale and Shapley (1962) argue, a stable matching does not exist regardless of δ's preferences. In a finite market, someone must be matched with δ or unmatched. But whoever is matched with δ or unmatched would prefer to be matched with either of the other two individuals, one of whom must also prefer to be matched with him.

Suppose instead that there is a continuum of individuals with four types of agents α, β, γ and δ, where each type of agent has the same preference as the single individual of that type given above,[25] and let the measure of each type of agent be $\nu(z) = 1$ for $z \in Z = \{\alpha, \beta, \gamma, \delta\}$. We will now argue that $\mu(\alpha, \beta) = \mu(\beta, \gamma) = \mu(\gamma, \alpha) = \frac{1}{2}$ and $\mu(\delta, \emptyset) = 1$ is a stable matching in our model.[26]

First, note that $\mu(\{z\} \times Z) + \mu(Z \times \{z\}) + \mu(\{z\} \times \{\emptyset\}) = 1$ for each $z \in Z$, so μ is a roommate matching. The targets are: $T_\alpha(\mu) = \{\gamma, \emptyset\}$, $T_\beta(\mu) = \{\alpha, \emptyset\}$, $T_\gamma(\mu) = \{\beta, \emptyset\}$, and $T_\delta(\mu) = \{\emptyset\}$. Note that $\text{supp}(\mu) = \{(\alpha, \beta), (\beta, \gamma), (\gamma, \alpha), (\delta, \emptyset)\}$. To see that $(\alpha, \beta) \in S(\mu)$, note that type α likes β the most so there is no $\hat{z}$ such that $\hat{z} \succ_\alpha \beta$; thus condition (i) is satisfied. Type β prefers γ to α but $\gamma \notin T_\beta(\mu)$; thus condition (ii) is satisfied. Analogous arguments establish that $\text{supp}(\mu) \subseteq S(\mu)$, and hence μ is stable.

[25] With a continuum of individuals, it is possible for a given type to match with itself. To ensure that this does not happen in a stable matching, we specify for this example that $z' \succ_z z$ for each $(z, z') \in Z \times Z_\emptyset$ with $z' \neq z$.

[26] Formally, a model without contracts can be modelled in our framework by letting C be singleton and $\mathbb{C}(z, z') = C$ for each $z, z' \in Z$, but here we omit contracts altogether for simplicity.

A stable matching exists in this example with a continuum of individuals because it is possible for individuals of type α, β and γ all to be matched with each other, leaving individuals of type δ unmatched. More generally, our results imply that the large market version of the roommate problem admits a stable solution with or without transfers and even in the presence of externalities.

6.3 Rosen (1982)

In this section we consider the setting in Rosen (1982, Section 3), which we briefly describe in what follows.

Individuals are characterized by their general ability, with $Z \subseteq \mathbb{R}$ denoting the set of possible abilities and ν denoting its (nonzero, finite) distribution. Individuals can be workers, managers or self-employed (here more correctly interpreted as unemployed as it will be clear from the individuals' payoffs) and their productivity is determined both by this choice and their ability, with $q = q(z)$ denoting the productivity of someone of ability z who chooses to be a worker and $r = r(z)$ his productivity if he chooses to be a manager; both r and q are non-decreasing functions of the ability z.

A firm consists of one manager and several workers of the same type, i.e. there is many-to-one matching. Managers have one unit of time and need to supervise workers: The output produced by a worker with productivity q in a firm with a manager with productivity r is $g(r)f(tr, q)$, where t is the time spent by the manager supervising the worker, $g(r)$ represents the quality of management decisions of a manager of productivity r, $g : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and $f : \mathbb{R}^2_+ \to \mathbb{R}_+$ is continuously differentiable, homogeneous of degree 1, strictly increasing and strictly concave in each coordinate in the interior of its domain[27] and satisfies $f(0, y) = f(x, 0) = 0$ for each $x, y \in \mathbb{R}_+$. For convenience, we define $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ as $\theta(x) = f(x, 1)$ for each $x \in \mathbb{R}_+$; note that θ is strictly increasing and strictly concave. The output of a firm with a

[27]Meaning that for $(x, y) \in \mathbb{R}^2_{++}$, $\frac{\partial f(x,y)}{\partial x} > 0$, $\frac{\partial f(x,y)}{\partial y} > 0$, and $x \mapsto \frac{\partial f(x,y)}{\partial x}$ and $y \mapsto \frac{\partial f(x,y)}{\partial y}$ are strictly decreasing over $\mathbb{R}_{++}$.

manager of ability r and a measure n of workers with productivity q is

$$ng(r)f\left(\frac{r}{n},q\right) = g(r)f(r,nq) = g(r)nq\theta\left(\frac{r}{nq}\right)$$

since the time spent in each worker is $t = 1/n$.[28] The manager's rent is

$$g(r)f(r,nq) - cn = g(r)nq\theta\left(\frac{r}{nq}\right) - cn,$$

where c is the wage paid by the manager to the workers.

To represent the above setting in the general framework of Section 4, let, in addition to Z and ν as above, the set of contracts be $C = \mathbb{R}_+$, interpreted as the set of possible wages, and the contract correspondence be $\mathbb{C} \equiv C$. The set of feasible matches for managers is $X = \{n1_{(z,c)} : (z,c) \in Z \times C \text{ and } n \in \mathbb{R}_+\}$ since managers can hire several workers all of the same type. Occupations are the same as in the general framework: $A = \{w, s, m\}$. Finally, preferences are defined by specifying payoff functions as follows:

$$U_z(w, 1_{(z',c)}) = c \text{ for each } 1_{(z',c)} \in X_w,$$
$$U_z(s, 1_{(\emptyset,c)}) = 0 \text{ for each } 1_{(\emptyset,c)} \in X_s, \text{ and}$$
$$U_z(m, n1_{(z',c)}) = g(r(z))f(r(z), nq(z')) - cn \text{ for each } n1_{(z',c)} \in X_m.$$

We will establish existence and obtain a characterization of stable matchings for the setting of this section under the following simplifying assumptions. We let $Z = [\underline{z}, \bar{z}]$ with $0 \leq \underline{z} < \bar{z} < \infty$ and assume that $q(\underline{z}) > 0$, $r(\underline{z}) > 0$ and $g(r) > 0$ for each $r > 0$; thus, $g(r(\underline{z})) > 0$. A market satisfying these assumptions as well as the additional specifications described above is a *Rosen market* and denoted by E_{rosen}.

Concerning the existence of stable matchings, note that a Rosen market fails to satisfy two assumptions of our existence result, namely the contract correspondence

[28] This claim follows from the Jensen's integral inequality as follows. Let $\mu \in \mathcal{M}([0,1])$ be a probability distribution of time spent on workers so that $\mu(B)$ is the fraction of workers who get supervision time in B, for each Borel subset B of $[0,1]$, and $1_{\frac{1}{n}} \in \mathcal{M}([0,1])$ be the probability distribution degenerate on $1/n$. Then $n\int t\mathrm{d}\mu(t) = 1$ and $\int g(r)nq\theta\left(\frac{rt}{q}\right)\mathrm{d}\mu(t) = nqg(r)\int\theta\left(\frac{rt}{q}\right)\mathrm{d}\mu(t) \leq nqg(r)\theta\left(\frac{r\int t\mathrm{d}\mu(t)}{q}\right) = nqg(r)q\theta\left(\frac{r}{nq}\right) = \int g(r)nq\theta\left(\frac{rt}{q}\right)\mathrm{d}1_{\frac{1}{n}}(t)$.

fails to be compact-valued and the market fails to be bounded. Nevertheless, by considering a sequence of truncated Rosen markets that satisfy our assumptions, we show that stable matchings exist.

Corollary 3 *Every Rosen market has a stable matching.*

We next provide a characterization of stable matchings in Rosen markets that is analogous to the formulation in Rosen (1982). The following concepts are needed. Let $r \in r(Z)$, $q \in q(Z)$ and $w > 0$. If n solves $\max_{n' \in \mathbb{R}_+} [g(r)f(r, n'q) - wn'q]$, then

$$w = g(r)\frac{\partial f(r, nq)}{\partial y} = g(r)\frac{\partial f\left(\frac{r}{nq}, 1\right)}{\partial y}$$

since $\frac{\partial f}{\partial y}$ is homogeneous of degree zero. Thus, there is a continuous function $\phi : r(Z) \times \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $nq = \phi(r, w)$. The manager's rent is then

$$g(r)\frac{\partial f\left(\frac{r}{nq}, 1\right)}{\partial x} r = g(r)\frac{\partial f\left(\frac{r}{\phi(r,w)}, 1\right)}{\partial x} r.$$

The above functions and formulas are used to define, for each manager of type z, the optimal number of workers of type z' he wants to hire at wage $wq(z')$ and the corresponding rent. Define $n : Z^2 \times \mathbb{R}_{++} \to \mathbb{R}_{++}$ by setting, for each $(z, z', w) \in Z^2 \times \mathbb{R}_{++}$,

$$n(z, z', w) = \frac{\phi(r(z), w)}{q(z')}.$$

Moreover, define $R : Z \times \mathbb{R}_{++} \to \mathbb{R}_+$ by setting, for each $(z, w) \in Z \times \mathbb{R}_{++}$,

$$R(z, w) = g(r(z))\frac{\partial f\left(\frac{r(z)}{\phi(r(z),w)}, 1\right)}{\partial x} r(z).$$

Theorem 4 *A matching μ of a Rosen market is stable if and only if there exists $\lambda \in \mathcal{M}(Z^2)$ and $w > 0$ such that*

$$\lambda(B \times Z) + \int_{Z \times B} n(z, z', w) \mathrm{d}\lambda(z, z') = \nu(B) \text{ for each measurable } B \subseteq Z, \quad (1)$$

$$\mathrm{supp}(\lambda) \subseteq \{z \in Z : R(z, w) \geq wq(z)\} \times \{z \in Z : wq(z) \geq R(z, w)\}, \text{ and} \quad (2)$$

$$\mu = \lambda \circ h^{-1}, \quad (3)$$

where $h : Z^2 \to Z \times X$ is defined by setting, for each $(z, z') \in Z^2$,

$$h(z, z') = (z, n(z, z', w)1_{(z', wq(z'))}).$$

As Theorem 4 illustrates, our framework is tractable and our stability notion admits a simple characterization in applied settings; they can therefore be used to clarify important economic questions and highlight what forces might explain them. We give one such example when $q(z) = r(z) = z$ and the technology takes the form $g(z)z^{\alpha}(nz')^{1-\alpha}$ with $\alpha = 1/2$. If $g \equiv 1$, then each individual is indifferent between being a manager or a worker and each individual of type z has an income (wage or rent) equal to $z/2$.[29] In contrast, if $g(z) = z$, then individual income is no longer necessarily linear in the type. For example, when $Z = \{z_1, \ldots, z_4\}$, it is possible to construct a stable matching where individuals of type z_1 and z_2 are workers, individuals of type z_3 and z_4 are managers, each person strictly prefers his occupation to the alternative one and, for some $w > 0$, workers' income is wz while managers' income is $\frac{z^3}{4w}$.[30] In this latter example, any change that leads to a decrease in w causes an increase in the income of those in the top and a decrease in the income of those in the bottom of the income distribution.[31] In addition, as a result of decrease in w, there is less inequality at the bottom (since the function $z \mapsto wz$ describing the income of those in the bottom of the distribution becomes flatter) and more at the top of the income distribution (since the function $z \mapsto \frac{z^3}{4w}$ describing the income of those in the top of the distribution becomes steeper).

6.4 Further applications

In Appendix B we consider additional applications of our framework, which we summarize here, to illustrate its flexibility.

[29]Indeed, if $\alpha = \frac{1}{2}$ and $g \equiv 1$, then $R(z, w) \geq wq(z)$ if and only if $\frac{1}{2} \geq w$. It then must be that $w = \frac{1}{2}$ in any stable matching since otherwise there would be no worker or no managers; thus, $R(z, w) = wq(z) = \frac{z}{2}$ for each $z \in Z$.

[30]If $\alpha = \frac{1}{2}$, $Z = \{z_1, \ldots, z_4\}$ and g is the identity, then pick $w \in (2z_2, 2z_3)$, which implies that $R(z, w) > wq(z)$ for each $z \in \{z_3, z_4\}$ and $R(z, w) < wq(z)$ for each $z \in \{z_1, z_2\}$. Let ν be such that $\nu(z_3) = \nu(z_4) = 1$, $\nu(z_2) = n(z_4, z_2, w)$ and $\nu(z_3) = n(z_3, z_1, w)$. Then λ such that $\lambda(z_3, z_1) = \lambda(z_4, z_2) = 1$ yields a stable matching. Payoffs are wz for each $z \in \{z_1, z_2\}$ and $R(z, w) = \frac{z^3}{4w}$ for each $z \in \{z_3, z_4\}$.

[31]Such a decrease in w would occur, for example, if $\nu(z_1)$ and $\nu(z_2)$ increase by a small amount.

Specifically, we show how our framework can capture the settings of Garicano and Rossi-Hansberg (2004) and Garicano and Rossi-Hansberg (2006) (in Appendices B.8 and B.9 respectively), and how it can be extended to accommodate Lucas's (1978) model (in Appendix B.10). Both Garicano and Rossi-Hansberg (2004) and Garicano and Rossi-Hansberg (2006) require feasible matches for managers that depend on the types of the workers hired. This dependence arises because the measure of workers that a manager can hire is determined by the time constraint of the manager and is increasing in the quality of the workers. In Garicano and Rossi-Hansberg (2004) all workers have the same quality but in Garicano and Rossi-Hansberg (2006) a manager can hire workers of finitely many different qualities.

In Lucas (1978) there is a capital market in addition to a labor market with occupational choice. The easiest approach to represent this setting is to consider, for each rental price of capital, the resulting market with occupational choice with the amount of capital hired by a firm being included in the contract between the manager and workers. An equilibrium is then a rental price of capital and a matching such that the matching is stable given the rental price and the capital market clears.

In addition, in Appendix B.1 we also introduce a model of two-sided many-to-one matching markets and show that it is a special case of our framework of markets with occupational choice. Nevertheless, it is interesting to consider it explicitly since several real-world matching markets do not feature occupational choice but involve many-to-one matching and a large number of participants on both sides of the market.[32]

7 Concluding remarks

In this paper we provided a formalization of large many-to-one matching markets with occupational choice and a notion of a stable matching for them. This was done with the goal of being able to include the settings of Lucas (1978), Rosen (1982),

[32]Examples include school allocation in urban settings and the matching of doctors to hospitals within a country.

Garicano and Rossi-Hansberg (2004) and Garicano and Rossi-Hansberg (2006) in our framework, while at the same time extending the two-sided, one-to-one matching setting of GK.

The large matching markets we consider are, as in GK, formalized using a distributional approach. Thus, the set of individuals is not explicitly included, rather only the distribution of individuals' types is present in the description of the market. This approach is tractable and this has been illustrated in Section 6.3 in the context of Rosen's (1982) setting where stable matchings are fully characterized.

The above tractability makes our setting potentially useful to address the implications of stability in large labor markets, in particular, for income inequality. We aim to do so in future work.

The representation of Lucas's (1978) setting in our framework required the introduction of capital, which proved to be a relatively easy extension. This suggests that other important elements can be added to our framework.

A Appendix: Proofs

A.1 Preliminary lemmas

This section presents some lemmas on the support of a measure and on the existence of convergent subsequences. Lemma 1 shows that the support of the image $\mu \circ h^{-1}$ of a measure μ under a homeomorphism h is the image of the support of μ.

Lemma 1 *Let Y and Y' be separable metric spaces, $\mu \in \mathcal{M}(Y)$, $h : Y \to Y'$ be a homeomorphism and $\nu = \mu \circ h^{-1}$. Then $\mathrm{supp}(\nu) = h(\mathrm{supp}(\mu))$ and $\mathrm{supp}(\mu) = h^{-1}(\mathrm{supp}(\nu))$.*

Proof. Note first that $\nu(\mathrm{supp}(\nu)) = \nu(Y') = \mu(h^{-1}(Y')) = \mu(Y) = \mu(\mathrm{supp}(\mu))$ and, since $\mathrm{supp}(\mu) = h^{-1}(h(\mathrm{supp}(\mu)))$,

$$\begin{aligned}\nu(\mathrm{supp}(\nu)) &\geq \nu(h(\mathrm{supp}(\mu))) = \mu(h^{-1}(h(\mathrm{supp}(\mu))))\\ &= \mu(\mathrm{supp}(\mu)) \geq \mu(h^{-1}(\mathrm{supp}(\nu))) = \nu(\mathrm{supp}(\nu)).\end{aligned}$$

Thus,

$$\mu(\mathrm{supp}(\mu)) = \mu(h^{-1}(\mathrm{supp}(\nu))) = \nu(h(\mathrm{supp}(\mu))) = \nu(\mathrm{supp}(\nu)).$$

Since $h^{-1}(\mathrm{supp}(\nu))$ is closed, $\mathrm{supp}(\mu) \subseteq h^{-1}(\mathrm{supp}(\nu))$ and, hence, $h(\mathrm{supp}(\mu)) \subseteq h(h^{-1}(\mathrm{supp}(\nu))) = \mathrm{supp}(\nu)$. Letting f denote the inverse of h, we have that $h(F) = f^{-1}(F)$ is closed for each closed subset F of Y. Thus, it follows that $\mathrm{supp}(\nu) \subseteq h(\mathrm{supp}(\mu))$.

It follows from $\mathrm{supp}(\nu) = h(\mathrm{supp}(\mu))$ that $h^{-1}(\mathrm{supp}(\nu)) = h^{-1}(h(\mathrm{supp}(\mu))) = \mathrm{supp}(\mu)$. ■

Lemma 2 shows that the support correspondence is lower hemicontinuous.

Lemma 2 *If Y is a separable metric space, then the correspondence $\mu \mapsto \mathrm{supp}(\mu)$, from $\mathcal{M}(Y)$ to Y, is lower hemicontinuous.*

Proof. We have that $\mathcal{M}(Y)$ is a separable metrizable space by Varadarajan (1958, Theorem 3.1). The conclusion then follows from (the proof of) Aliprantis and Border (2006, Theorem 17.14, p. 563). ■

Lemma 3 characterizes the support of a product measure.

Lemma 3 *Let X be a separable metric space and $\mu \in \mathcal{M}(X)$ be finite. Then $\mathrm{supp}(\mu \otimes \mu) = \mathrm{supp}(\mu)^2$.*

Proof. Let $b = \mu(X)$ and write $\mu^2 = \mu \otimes \mu$. We have that $\mu^2(\mathrm{supp}(\mu)^2) = b^2 = \mu^2(X^2)$ and that $\mathrm{supp}(\mu)^2$ is closed in X^2, hence $\mathrm{supp}(\mu^2) \subseteq \mathrm{supp}(\mu)^2$.

Let $F = \mathrm{supp}(\mu^2)$ and, for each $x \in X$, $F(x) = \{y \in X : (x, y) \in F\}$. Then

$$b^2 = \mu^2(F) = \int_X \left(\int_X 1_F(x, y) \mathrm{d}\mu(y) \right) \mathrm{d}\mu(x).$$

Since $\int_X 1_F(x, y)\mathrm{d}\mu(y) = \mu(F(x)) \leq b$ for each $x \in X$, it follows that $\mu(F(x)) = b$ for μ-a.e. x and, as $F(x)$ is closed, that $\mathrm{supp}(\mu) \subseteq F(x)$ for μ-a.e. x. We have that $\{x \in X : \mathrm{supp}(\mu) \subseteq F(x)\}$ is closed (since F is closed) and $\mu(\{x \in X : \mathrm{supp}(\mu) \subseteq F(x)\}) = b$, hence, $\mathrm{supp}(\mu) \subseteq \{x \in X : \mathrm{supp}(\mu) \subseteq F(x)\}$. Thus,

$$\begin{aligned}\mathrm{supp}(\mu)^2 &= \cup_{x \in \mathrm{supp}(\mu)}(\{x\} \times \mathrm{supp}(\mu)) \\ &\subseteq \cup_{x \in \mathrm{supp}(\mu)}(\{x\} \times F(x)) \subseteq \cup_{x \in X}(\{x\} \times F(x)) = F.\end{aligned}$$

It then follows that $\text{supp}(\mu^2) = \text{supp}(\mu)^2$. ■

Lemma 4 provides conditions for the existence of a convergent subsequence.

Lemma 4 *If Y is a separable metrizable space and $\{\mu_k\}_{k=1}^{\infty}$ is a tight sequence in $\mathcal{M}(Y)$ such that, for some $R > 0$, $\mu_k(Y) \leq R$ for all $k \in \mathbb{N}$, then $\{\mu_k\}_{k=1}^{\infty}$ has a convergent subsequence.*

Proof. The proof reduces to the case of probability measures as follows: Suppose first that there is a subsequence $\{\mu_{k_j}\}_{j=1}^{\infty}$ such that $\mu_{k_j}(Y) \to 0$. Then this subsequence converges to the zero measure. Thus, we may assume that there is $\varepsilon > 0$ such that $\mu_k(Y) \geq \varepsilon$ for all but finitely many k. The sequence $\{\mu_k(Y)\}_k$ is bounded, thus we may assume that it converges; let $\theta = \lim_k \mu_k(Y)$. Consider $\{\nu_k\}_{k=1}^{\infty}$ with $\nu_k(B) = \frac{\mu_k(B)}{\mu_k(Y)}$ for each Borel B. This is a tight family of probability measures, so it has a convergent subsequence $\{\nu_{k_j}\}_{j=1}^{\infty}$; let $\nu = \lim_j \nu_{k_j}$, $\mu = \theta\nu$ and B has μ-null boundary, which happens if and only if it has ν-null boundary since $\theta \geq \varepsilon$. Then $\mu_{k_j}(B) = \mu_{k_j}(Y)\frac{\mu_{k_j}(B)}{\mu_{k_j}(Y)} \to \theta\nu(B)$ and, hence, $\mu_{k_j} \to \mu$. ■

A.2 Proof of Theorem 1

Note first that $\text{supp}(\mu) \subseteq S(\mu)$ implies that $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$ since $S(\mu) \subseteq S_M(\mu) \cap IR(\mu)$.

Conversely, suppose that $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$. Let $(z, \delta) \in \text{supp}(\mu)$ and assume, in order to reach a contradiction, that $(z, \delta) \notin S(\mu)$. Since $(z, \delta) \in \text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$, it follows that there is $(z^*, c) \in Z \times C$ and $\bar{z} \in Z$ such that $(z^*, c) \in T^w_{\bar{z}}(\mu)$, $\bar{z} = z$ or $(\bar{z}, \bar{c}) \in \text{supp}(\delta)$ for some $\bar{c} \in C$, (1) $(w, 1_{(z^*,c)}, \mu) \succ_{\bar{z}} (m, \delta, \mu)$ if $\bar{z} = z$ and $\delta \in X$, (2) $(w, 1_{(z^*,c)}, \mu) \succ_{\bar{z}} (w, 1_{(z,\bar{c})}, \mu)$ if $(\bar{z}, \bar{c}) \in \text{supp}(\delta)$ and (3) $(w, 1_{(z^*,c)}, \mu) \succ_{\bar{z}} (s, \delta, \mu)$ if $\bar{z} = z$ and $\delta \in X_\emptyset \setminus X$. Since $(z^*, c) \in T^w_{\bar{z}}(\mu)$, it follows that $c \in \mathbb{C}(z^*, \bar{z})$.

We now show that $(\bar{z}, c) \in T^m_{z^*}(\mu)$. Indeed, we have that $c \in \mathbb{C}(z^*, \bar{z})$ and $(z, \delta) \in \text{supp}(\mu)$. Thus, in case (1), the conclusion follows by condition (c) in the definition of $T^m_{z^*}(\mu)$ since $\bar{z} = z$ and $(w, 1_{(z^*,c)}, \mu) \succ_z (m, \delta, \mu)$; in case (2), the conclusion follows by condition (a) in the definition of $T^m_{z^*}(\mu)$ since $(\bar{z}, \bar{c}) \in \text{supp}(\delta)$ and $(w, 1_{(z^*,c)}, \mu) \succ_{\bar{z}}$

$(w, 1_{(z,\bar{c})}, \mu)$; and, in case (3), the conclusion follows by condition (b) in the definition of $T^m_{z^*}(\mu)$ since $\bar{z} = z$ and $(w, 1_{(z^*,c)}, \mu) \succ_z (s, \delta, \mu)$.

Since $(z^*, c) \in T^w_{\bar{z}}(\mu)$, there is $\tilde{\delta} \in X$ such that $(\bar{z}, c) \in \text{supp}(\tilde{\delta})$ and (a) or (b) or (c) in the definition of $T^w_{\bar{z}}(\mu)$ holds. In either case, we will show that $\text{supp}(\mu) \subseteq S_M(\mu)$ fails, which is a contradiction to $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$.

Suppose that condition (a) in the definition of $T^w_{\bar{z}}(\mu)$ holds. Then, in addition, $\text{supp}(\tilde{\delta}) \setminus \{(\bar{z}, c)\} \subseteq T^m_{z^*}(\mu)$, and there is $(z', c', \delta') \in Z \times C \times X$ such that $(z', \delta') \in \text{supp}(\mu)$, $(z^*, c') \in \text{supp}(\delta')$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (w, 1_{(z',c')}, \mu)$. Since $(\bar{z}, c) \in T^m_{z^*}(\mu)$, it follows that $(z', \delta') \in \text{supp}(\mu) \setminus S_M(\mu)$ since (ii) of the definition of $S_M(\mu)$ fails. Indeed, $(z', \delta') \in \text{supp}(\mu)$, $(z^*, c') \in \text{supp}(\delta')$, $\text{supp}(\tilde{\delta}) \subseteq T^m_{z^*}(\mu)$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (w, 1_{(z',c')}, \mu)$.

Suppose next that condition (b) in the definition of $T^w_{\bar{z}}(\mu)$ holds. Then, in addition, $\text{supp}(\tilde{\delta}) \setminus \{(\bar{z}, c)\} \subseteq T^m_{z^*}(\mu)$, and there is $\delta' \in X_\emptyset \setminus X$ such that $(z^*, \delta') \in \text{supp}(\mu)$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (s, \delta', \mu)$. Since $(\bar{z}, c) \in T^m_{z^*}(\mu)$, it follows that $(z^*, \delta') \in \text{supp}(\mu) \setminus S_M(\mu)$ since (iii) of the definition of $S_M(\mu)$ fails. Indeed, $(z^*, \delta') \in \text{supp}(\mu)$, $\text{supp}(\tilde{\delta}) \subseteq T^m_{z^*}(\mu)$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (s, \delta', \mu)$.

Finally, suppose that condition (c) in the definition of $T^w_{\bar{z}}(\mu)$ holds. Then, in addition, there is $\delta' \in X_\emptyset \setminus X$ such that $(z^*, \delta') \in \text{supp}(\mu)$, $\text{supp}(\tilde{\delta}) \setminus \{(\bar{z}, c)\} \subseteq T^m_{z^*}(\mu) \cup \text{supp}(\delta')$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (m, \delta', \mu)$. Since $(\bar{z}, c) \in T^m_{z^*}(\mu)$, it follows that $(z^*, \delta') \in \text{supp}(\mu) \setminus S_M(\mu)$ since (i) of the definition of $S_M(\mu)$ fails. Indeed, $(z^*, \delta') \in \text{supp}(\mu)$, $\text{supp}(\tilde{\delta}) \subseteq T^m_{z^*}(\mu) \cup \text{supp}(\delta')$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (m, \delta', \mu)$.

A.3 Proof of Theorem 2

The first step in the proof of our existence result consists in the following lemma, which considers the special case where Z, X and C are finite.[33]

Lemma 5 *If E is a rational and continuous market such that Z, X and C are finite, then E has a stable matching.*

[33]See Appendix B.3 for an outline of the proof of Theorem 2.

Proof. Note first that $Z_\emptyset$, $X_\emptyset$ and Δ are also finite. Define $\bar{\tau} \in \mathbb{R}^{Z\times\Delta}$ by setting, for each $(z,a,\delta) \in Z \times \Delta$,

$$\bar{\tau}(z,a,\delta) = \begin{cases} 0 & \text{if } \{z\} \times \text{supp}(\delta) \nsubseteq \text{graph}(\mathbb{C}) \text{ and } a \neq w, \\ 0 & \text{if } \text{supp}(\delta_Z) \times \{z\} \times \text{supp}(\delta_C) \nsubseteq \text{graph}(\mathbb{C}) \text{ and } a = w, \\ \nu(z) & \text{otherwise.} \end{cases}$$

Let $\bar{\kappa} \in \mathbb{R}^{Z\times Z\times C}$ be such that $\bar{\kappa}(z,z',c) = \nu(z')$ if $(z,z',c) \in \text{graph}(\mathbb{C})$, and $\bar{\kappa}(z,z',c) = 0$ otherwise.

Define

$$\mathcal{T} = \{\tau \in \mathbb{R}_+^{Z\times\Delta} : \tau(z,a,\delta) \leq \bar{\tau}(z,a,\delta) \text{ and } \sum_{(a,\delta)\in\Delta} \tau(z,a,\delta) \leq \nu(z)$$
$$\text{for each } (z,a,\delta) \in Z \times \Delta\} \text{ and}$$
$$\mathcal{K} = \{\kappa \in \mathbb{R}_+^{Z\times Z\times C} : \kappa(z,z',c) \leq \bar{\kappa}(z,z',c) \text{ for each } (z,z',c) \in Z \times Z \times C\}.$$

Note that $\mathcal{T}$ and $\mathcal{K}$ are nonempty, convex, and compact subsets of Euclidean spaces.

Let $u : Z \times \Delta \times \mathcal{M}(Z \times X_\emptyset) \to \mathbb{R}$ be a continuous utility function that represent preferences. We normalize so that $u \geq 1$. For each $n \in \mathbb{N}$, let $u_n = u^n$. Since $x \mapsto x^n$ is strictly increasing on $[1,\infty)$, u_n and u represent the same preferences.

Define $d : \mathcal{T} \to \mathbb{R}_+^{Z\times X_\emptyset}$ by setting, for each $\tau \in \mathcal{T}$ and $(z,\delta) \in Z \times X_\emptyset$,

$$d(\tau)(z,\delta) = \begin{cases} \tau(z,m,\delta) & \text{if } \delta \in X, \\ \tau(z,s,\delta) & \text{if } \delta \in X_\emptyset \setminus X. \end{cases}$$

The function d is continuous. We abuse notation and, for each $(z,a,\delta,\tau) \in Z\times\Delta\times\mathcal{T}$, write $u(z,a,\delta,\tau)$ for $u(z,a,\delta,d(\tau))$ and analogously for u_n. We also write $(a,\delta,\tau) \succ_z (a',\delta',\tau)$ for $(a,\delta,d(\tau)) \succ_z (a',\delta',d(\tau))$, where $(a',\delta') \in \Delta$.

For each $n \in \mathbb{N}$, let $D_n : \mathcal{T} \times \mathcal{K} \rightrightarrows \mathcal{T}$ be defined by setting, for each $(\mu, \kappa) \in \mathcal{T} \times \mathcal{K}$,

$$\begin{aligned} D_n(\mu,\kappa) = \{\tau \in \mathcal{T} : \tau \in \arg\max_{\tau' \in \mathcal{T}} & \sum_{z \in Z, (a,\delta) \in \Delta} u_n(z,a,\delta,\mu)\tau'(z,a,\delta) \\ & \text{subject to } \sum_{(a,\delta)\in\Delta} \tau'(z,a,\delta) = \nu(z) \text{ for all } z \in Z, \\ & \sum_{\delta \in X} \tau'(z,m,\delta)\delta(z',c) \leq \kappa(z,z',c) \text{ for all } (z,z',c) \in Z \times Z \times C, \text{ and} \\ & \tau'(z,w,1_{(z',c)}) \leq \sum_{\delta \in X} \mu(z',m,\delta)\delta(z,c) \text{ for all } (z,z',c) \in Z \times Z \times C\}. \end{aligned}$$

Claim 1 *D_n is upper hemicontinuous with nonempty, compact and convex values.*

Proof. It follows by the linearity of the objective function together with the convexity of the constraint set that D_n has convex values. It follows from Berge's maximum theorem that D_n is upper hemicontinuous with nonempty and compact values. To see this, first note that the objective function is continuous and that the constraint set, denoted by $\Phi(\mu, \kappa)$, is contained in the compact set $\mathcal{T}$. It is clear that Φ is upper hemicontinuous with compact values. To see that Φ has nonempty values, define $\tilde{\tau} \in \mathcal{T}$ as follows. For each $z \in Z$, let $c_z \in \mathbb{C}(z, \emptyset)$, $\tilde{\tau}(z, s, 1_{(\emptyset, c_z)}) = \nu(z)$ and $\tilde{\tau}(z, a, \delta) = 0$ for each $(a, \delta) \in \Delta \backslash \{(s, 1_{(\emptyset, c_z)})\}$. We then have that $\tilde{\tau} \in \Phi(\mu, \kappa)$ for each $(\mu, \kappa) \in \mathcal{T} \times \mathcal{K}$. Finally, to see that Φ is lower hemicontinuous, let $(\mu, \kappa) \in \mathcal{T} \times \mathcal{K}$, $O \subseteq \mathcal{T}$ be an open set such that $\Phi(\mu, \kappa) \cap O \neq \emptyset$, and $\tau \in \Phi(\mu, \kappa) \cap O$. Let $\hat{\tau} = \lambda\tau + (1-\lambda)\tilde{\tau} \in O$ for some $\lambda \in (0, 1)$. Note that for each $z \in Z$, $\sum_{(a,\delta)\in\Delta} \hat{\tau}(z, a, \delta) = \nu(z)$, $\sum_{\delta \in X} \hat{\tau}(z, m, \delta)\delta(z', c) < \kappa(z, z', c)$ for each $(z, z', c) \in Z \times Z \times C$ such that $\kappa(z, z', c) > 0$ and $\hat{\tau}(z, w, 1_{(z',c)}) < \sum_{\delta \in X} \mu(z', m, \delta)\delta(z, c)$ for each $(z, z', c) \in Z \times Z \times C$ such that $\sum_{\delta \in X} \mu(z', m, \delta)\delta(z, c) > 0$. Thus, there is an open neighborhood V of (μ, κ) such that $\hat{\tau} \in \Phi(\mu', \kappa') \cap O$ for each $(\mu', \kappa') \in V$. ■

Claim 2 *If $(\mu, \kappa) \in \mathcal{T} \times \mathcal{K}$, $\tau \in D_n(\mu, \kappa)$ and $(z, a, \delta') \in Z \times \Delta$ is such that $\tau(z, a, \delta') > 0$, then $\tau(z, w, 1_{(\hat{z},\hat{c})}) = \sum_{\delta \in X} \mu(\hat{z}, m, \delta)\delta(z, \hat{c})$ for each $(\hat{z}, \hat{c}) \in Z \times C$ such that $(w, 1_{(\hat{z},\hat{c})}, \mu) \succ_z (a, \delta', \mu)$.*

Proof. If not, then $\tau(z, w, 1_{(\hat{z},\hat{c})}) < \sum_{\delta \in X} \mu(\hat{z}, m, \delta)\delta(z, \hat{c})$ for some $(\hat{z}, \hat{c}) \in Z \times C$ such that $(w, 1_{(\hat{z},\hat{c})}, \mu) \succ_z (w, 1_{(z',c')}, \mu)$. Then $\sum_{\delta \in X} \mu(\hat{z}, m, \delta)\delta(z, \hat{c}) > 0$ and, hence,

$(\hat{z}, z, \hat{c}) \in \text{graph}(\mathbb{C})$. Thus, increase $\tau(z, w, 1_{(\hat{z},\hat{c})})$ and decrease $\tau(z, a, \delta')$ by the same amount $\varepsilon \in (0, \tau(z, a, \delta'))$ such that $\tau(z, w, 1_{(\hat{z},\hat{c})}) + \varepsilon < \sum_{\delta \in X} \mu(\hat{z}, m, \delta)\delta(z, \hat{c})$. This increases the objective function in $D_n(\mu, \kappa)$ while satisfying the constraints, thus contradicting $\tau \in D_n(\mu, \kappa)$. ■

For each $\mu \in \mathcal{T}$ and $(z, z', c) \in Z \times Z \times C$, let

$$W(z, z', c, \mu) = \{(a, \delta) \in \Delta : u(z', w, 1_{(z,c)}, \mu) > u(z', a, \delta, \mu)\}.$$

Let $g : \mathcal{T} \to \mathcal{K}$ be defined by setting, for each $\mu \in \mathcal{T}$ and $(z, z', c) \in Z \times Z \times C$,

$$g(\mu)(z, z', c) = \begin{cases} \sum_{(a,\delta) \in W(z,z',c,\mu)} \mu(z', a, \delta) & \text{if } (z, z', c) \in \text{graph}(\mathbb{C}), \\ 0 & \text{otherwise.} \end{cases}$$

To see that $g(\mu) \in \mathcal{K}$, first note that if $(z, z', c) \notin \text{graph}(\mathbb{C})$, $g(\mu)(z, z', c) = 0$. If $(z, z', c) \in \text{graph}(\mathbb{C})$, then, since $\mu \in \mathcal{T}$, $0 \leq g(\mu)(z, z', c) \leq \nu(z') = \bar{\kappa}(z, z', c)$.

The function g may fail to be continuous and, thus, we will consider a continuous approximation to it. For each $n \in \mathbb{N}$ and $(z, z', c) \in Z \times Z \times C$, let $\alpha_{n,(z,z',c)} : \Delta \times \mathcal{T} \to [0, 1]$ be defined by setting, for each $(a, \delta, \mu) \in \Delta \times \mathcal{T}$,

$$\alpha_{n,(z,z',c)}(a, \delta, \mu) = n \max\left\{0, \min\left\{u(z', w, 1_{(z,c)}, \mu) - u(z', a, \delta, \mu), \frac{1}{n}\right\}\right\}.$$

Let $g_n : \mathcal{T} \to \mathcal{K}$ be defined by setting, for each $\mu \in \mathcal{T}$ and $(z, z', c) \in Z \times Z \times C$,

$$g_n(\mu)(z, z', c) = \begin{cases} \sum_{(a,\delta) \in \Delta} \alpha_{n,(z,z',c)}(a, \delta, \mu)\mu(z', a, \delta) & \text{if } (z, z', c) \in \text{graph}(\mathbb{C}), \\ 0 & \text{otherwise.} \end{cases}$$

We have that g_n is continuous since $\alpha_{n,(z,z',c)}$ is continuous for each $(z, z', c) \in Z \times Z \times C$. Note that

$$\alpha_{n,(z,z',c)}(a, \delta, \mu) \in \begin{cases} \{0\} & \text{if } u(z', a, \delta, \mu) \geq u(z', w, 1_{(z,c)}, \mu), \\ (0, 1) & \text{if } u(z', w, 1_{(z,c)}, \mu) - \frac{1}{n} < u(z', a, \delta, \mu) < u(z', w, 1_{(z,c)}, \mu), \\ \{1\} & \text{if } u(z', a, \delta, \mu) \leq u(z', w, 1_{(z,c)}, \mu) - \frac{1}{n}. \end{cases}$$

Hence, it follows that

$$g_n(\mu)(z, z', c) \leq g(\mu)(z, z', c) \tag{4}$$

for each $\mu \in \mathcal{T}$ and $(z, z', c) \in Z \times Z \times C$ since

$$g(\mu)(z, z', c) = \begin{cases} \sum_{(a,\delta)\in\Delta} \alpha_{(z,z',c)}(a, \delta, \mu)\mu(z', a, \delta) & \text{if } (m, w, c) \in \text{graph}(\mathbb{C}), \\ 0 & \text{otherwise} \end{cases}$$

with

$$\alpha_{(z,z',c)}(a, \delta, \mu) = \begin{cases} 1 & \text{if } u(z', a, \delta, \mu) < u(z', w, 1_{(z,c)}, \mu), \\ 0 & \text{otherwise.} \end{cases}$$

To see that $g_n(\mu) \in \mathcal{K}$, note that $0 \leq g_n(\mu) \leq g(\mu) \leq \bar{\kappa}$.

Let $f_n : \mathcal{T} \to \mathcal{K}$ be defined by setting, for each $\mu \in \mathcal{T}$ and $(z, z', c) \in Z \times Z \times C$,

$$f_n(\mu)(z, z', c) = \mu(z', w, 1_{(z,c)}) + \frac{1}{n} g_n(\mu)(z, z', c).$$

To see that $f_n(\mu) \in \mathcal{K}$, note that if $(z, z', c) \notin \text{graph}(\mathbb{C})$,

$$\mu(z', w, 1_{(z,c)}) = g_n(\mu)(z, z', c) = 0,$$

and hence $f_n(\mu)(z, z', c) = 0$. If $(z, z', c) \in \text{graph}(\mathbb{C})$, then

$$\begin{aligned} 0 &\leq \mu(z', w, 1_{(z,c)}) + \frac{1}{n} g_n(\mu)(z, z', c) \\ &\leq \mu(z', w, 1_{(z,c)}) + g(\mu)(z, z', c) \leq \nu(z') = \bar{\kappa}(z, z', c). \end{aligned}$$

We have that f_n is continuous since so is g_n.

Let $\Psi_n : \mathcal{T} \times \mathcal{K} \rightrightarrows \mathcal{T} \times \mathcal{K}$ be defined by setting, for each $(\mu, \kappa) \in \mathcal{T} \times \mathcal{K}$,

$$\Psi_n(\mu, \kappa) = D_n(\mu, \kappa) \times \{f_n(\mu)\}.$$

It follows from the continuity of f_n and from Claim 1 that Ψ_n is upper hemicontinuous with nonempty, compact and convex values. Hence, by Kakutani fixed point theorem, let (μ_n, κ_n) be a fixed point of Ψ_n. Thus, $\mu_n \in D_n(\mu_n, \kappa_n)$ and $\kappa_n = f_n(\mu_n)$.

Since $\mathcal{T} \times \mathcal{K}$ is compact, taking a subsequence if necessary, we may assume that $\{(\mu_n, \kappa_n)\}_{n=1}^{\infty}$ converges; let $(\mu, \kappa) = \lim_{n\to\infty}(\mu_n, \kappa_n)$. For each n, we have $\kappa_n = f_n(\mu_n)$, and so

$$\kappa(z, z', c) = \lim_{n\to\infty} f_n(\mu_n)(z, z', c) = \mu(z', w, 1_{(z,c)}) \tag{5}$$

for each $(z, z', c) \in Z \times Z \times C$. Let

$$\mu^* = d(\mu)$$

and $\mu_n^* = d(\mu_n)$ for each $n \in \mathbb{N}$.

For each $z \in Z$ and $n \in \mathbb{N}$, it follows from $\mu_n \in D_n(\mu_n, \kappa_n)$ that

$$\sum_{(z',c)\in Z\times C} \mu_n(z, w, 1_{(z',c)}) \leq \sum_{(z',c)\in Z\times C} \sum_{\delta\in X} \mu_n(z', m, \delta)\delta(z, c) \leq \sum_{(z',c)\in Z\times C} \kappa_n(z', z, c).$$

By (5), $\lim_n \sum_{(z',c)\in Z\times C} \kappa_n(z', z, c) = \sum_{(z',c)\in Z\times C} \mu(z, w, 1_{(z',c)})$ and, hence,

$$\sum_{(z',c)\in Z\times C} \mu(z, w, 1_{(z',c)}) = \sum_{(z',c)\in Z\times C} \sum_{\delta\in X} \mu(z', m, \delta)\delta(z, c) \text{ for each } z \in Z.$$

Thus, for each $z \in Z$,

$$\sum_{(z',\delta)\in Z\times X} \mu^*(z', \delta)\delta_Z(z) = \sum_{(z',c)\in Z\times C} \sum_{\delta\in X} \mu(z', m, \delta)\delta(z, c) = \sum_{(z',c)\in Z\times C} \mu(z, w, 1_{(z',c)}). \tag{6}$$

Claim 3 *μ^* is a matching.*

Proof. Condition 1 follows because if $(z, \delta) \in Z \times X_\emptyset$ is such that $\mu^*(z, \delta) > 0$, then $\mu(z, a, \delta) > 0$ for some $a \neq w$ and $\{z\} \times \mathrm{supp}(\delta) \subseteq \mathrm{graph}(\mathbb{C})$ since $\mu \in \mathcal{T}$.

Condition 2 holds since, for each $z \in Z$, $\mu = \lim_n \mu_n$ and $\mu_n \in D_n(\mu_n, \kappa_n)$ for each $n \in \mathbb{N}$ imply that

$$\begin{aligned} \nu(z) &= \sum_{\delta\in X} \mu(z, m, \delta) + \sum_{\delta\in X_\emptyset\setminus X} \mu(z, s, \delta) + \sum_{(z',c)\in Z\times C} \mu(z, w, 1_{(z',c)}) \\ &= \sum_{\delta\in X} \mu^*(z, \delta) + \sum_{\delta\in X_\emptyset\setminus X} \mu^*(z, \delta) + \sum_{(z',\delta)\in Z\times X} \mu^*(z', \delta)\delta_Z(z), \end{aligned}$$

the last equality following by (6). ■

Claim 4 *If $(z, z', c, \delta) \in Z \times Z \times C \times X$ is such that $(z, \delta) \in \mathrm{supp}(\mu^*)$ and $(z', c) \in \mathrm{supp}(\delta)$, then $\mu(z', w, 1_{(z,c)}) > 0$.*

Proof. We have that $\sum_{\delta'\in X} \mu(z, m, \delta')\delta'(z', c) \geq \mu(z, m, \delta)\delta(z', c) > 0$ and, thus, $\kappa(z, z', c) > 0$ since $\mu_n \in D_n(\mu_n, \kappa_n)$ for each $n \in \mathbb{N}$. Hence, (5) implies that $\mu(z', w, 1_{(z,c)}) > 0$. ■

Claim 5 $\text{supp}(\mu^*) \subseteq IR(\mu^*)$.

Proof. Suppose not; then there exists $(z^*, \delta^*) \in \text{supp}(\mu^*) \setminus IR(\mu^*)$. We claim that there exists $z \in Z$, $(a, \delta) \in \Delta$ and $c' \in \mathbb{C}(z, \emptyset)$ such that

1. $\mu(z, a, \delta) > 0$ and
2. $(s, 1_{(\emptyset,c')}, \mu^*) \succ_z (a, \delta, \mu^*)$.

Indeed, in cases (i) and (iii) of the definition of $IR(\mu^*)$, let $(z, \delta) = (z^*, \delta^*)$ in both cases and $a = m$ in case (i) and $a = s$ in case (iii). In case (ii) of the definition of $IR(\mu^*)$, we have that $\delta^* \in X$ and there exist $(z', c) \in \text{supp}(\delta^*)$ and $c' \in \mathbb{C}(z', \emptyset)$ such that $(s, 1_{(\emptyset,c')}, \mu^*) \succ_{z'} (w, 1_{(z^*,c)}, \mu^*)$. Claim 4 implies that $\mu(z', w, 1_{(z^*,c)}) > 0$; hence, let $z = z'$, $a = w$ and $\delta = 1_{(z^*,c)}$.

We then have that $\mu_n(z, a, \delta) > 0$ and $(s, 1_{(\emptyset,c')}, \mu_n^*) \succ_z (a, \delta, \mu_n^*)$ for n sufficiently large. Then decrease $\mu_n(z, a, \delta)$ and increase $\mu_n(z, s, 1_{(\emptyset,c')})$ by the same amount $\varepsilon \in (0, \mu_n(z, a, \delta))$ to increase the objective function in $D_n(\mu_n, \kappa_n)$ while satisfying the constraints. But this is a contradiction to $\mu_n \in D_n(\mu_n, \kappa_n)$. ■

Claim 6 *If $(z, z', c) \in Z \times Z \times C$ is such that $(z', c) \in T_z^m(\mu^*)$, then there is $N_{z,z',c} \in \mathbb{N}$ such that $\sum_{\delta \in X} \mu_n(z, m, \delta)\delta(z', c) < \kappa_n(z, z', c)$ for each $n \geq N_{z,z',c}$.*

Proof. Let $(z', c) \in T_z^m(\mu^*)$. Then $c \in \mathbb{C}(z, z')$. In case (a) of the definition of $T_z^m(\mu^*)$, there exists $(\tilde{z}, \tilde{c}, \tilde{\delta}) \in Z \times C \times X$ such that $(\tilde{z}, \tilde{\delta}) \in \text{supp}(\mu^*)$, $(z', \tilde{c}) \in \text{supp}(\tilde{\delta})$ and $(w, 1_{(z,c)}, \mu^*) \succ_{z'} (w, 1_{(\tilde{z},\tilde{c})}, \mu^*)$. Hence, $\mu(z', w, 1_{(\tilde{z},\tilde{c})}) > 0$ by Claim 4.

In cases (b) and (c) of the definition of $T_z^m(\mu^*)$, there exists $(a, \delta') \in \Delta$ such that $a \neq w$, $(z', \delta') \in \text{supp}(\mu^*)$ and $(w, 1_{(z,c)}, \mu^*) \succ_{z'} (a, \delta', \mu^*)$. Thus, letting $a = w$ and $\delta' = 1_{(\tilde{z},\tilde{c})}$ in case (a), it follows that, in all cases, there exists $(a, \delta') \in \Delta$ such that $(z', a, \delta') \in \text{supp}(\mu)$ and $(w, 1_{(z,c)}, \mu^*) \succ_{z'} (a, \delta', \mu^*)$.

Let $N_{z,z',c} \in \mathbb{N}$ be such that $\mu_n(z', a, \delta') > 0$ and $(w, 1_{(z,c)}, \mu_n^*) \succ_{z'} (a, \delta', \mu_n^*)$ for each $n \geq N_{z,z',c}$. Thus, for each $n \geq N_{z,z',c}$,

$$\mu_n(z', w, 1_{(z,c)}) = \sum_{\delta \in X} \mu_n(z, m, \delta)\delta(z', c)$$

by Claim 2 and, since $\alpha_{n,(z,z',c)}(a,\delta',\mu_n) > 0$,

$$\begin{aligned}\kappa_n(z,z',c) &= \mu_n(z',w,1_{(z,c)}) + \frac{1}{n}g_n(\mu_n)(z,z',c)\\ &\geq \mu_n(z',w,1_{(z,c)}) + \frac{1}{n}\alpha_{n,(z,z',c)}(a,\delta',\mu_n)\mu_n(z',a,\delta')\\ &> \mu_n(z',w,1_{(z,c)}) = \sum_{\delta\in X}\mu_n(z,m,\delta)\delta(z',c).\end{aligned}$$

■

Claim 7 $\text{supp}(\mu^*) \subseteq S_M(\mu^*)$.

Proof. Suppose not; then there exists $(z^*,\delta^*) \in \text{supp}(\mu^*) \setminus S_M(\mu^*)$. We claim that there exists $z \in Z$, $(a,\delta) \in \Delta$ and $\delta' \in X$ such that

1. $\mu(z,a,\delta) > 0$,
2. $\text{supp}(\delta') \subseteq T_z^m(\mu^*) \cup \text{supp}(\delta)$ if $a = m$ and $\text{supp}(\delta') \subseteq T_z^m(\mu^*)$ if $a \neq m$, and
3. $(m,\delta',\mu^*) \succ_z (a,\delta,\mu^*)$.

Indeed, in cases (i) and (iii) of the definition of $S_M(\mu^*)$, let $(z,\delta) = (z^*,\delta^*)$ in both cases and $a = m$ in case (i) and $a = s$ in case (iii). In case (ii) of the definition of $S_M(\mu^*)$, we have that $\delta^* \in X$ and there exist $(z',c) \in \text{supp}(\delta^*)$ and $\delta' \in X$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu^*)$ and $(m,\delta',\mu^*) \succ_{z'} (w,1_{(z^*,c)},\mu^*)$. Claim 4 implies that $\mu(z',w,1_{(z^*,c)}) > 0$; hence, let $z = z'$, $a = w$ and $\delta = 1_{(z^*,c)}$.

Note that $\{z\} \times \text{supp}(\delta') \subseteq \text{graph}(\mathbb{C})$ since $\{z\} \times T_z^m(\mu^*) \subseteq \text{graph}(\mathbb{C})$ and, when $a = m$, $(z,\delta) \in \text{supp}(\mu^*)$ and thus $\{z\} \times \text{supp}(\delta) \subseteq \text{graph}(\mathbb{C})$.

Let $\theta = 1$ if $\text{supp}(\delta) \cap \text{supp}(\delta') = \emptyset$; otherwise, let $(\bar{z},\bar{c}) \in \text{supp}(\delta) \cap \text{supp}(\delta')$ be such that $\frac{\delta(\bar{z},\bar{c})}{\delta'(\bar{z},\bar{c})} \leq \frac{\delta(z,c)}{\delta'(z,c)}$ for each $(z,c) \in \text{supp}(\delta) \cap \text{supp}(\delta')$ and define

$$\theta = \min\left\{1, \frac{\delta(\bar{z},\bar{c})}{\delta'(\bar{z},\bar{c})}\right\}.$$

Let $k \in \mathbb{N}$ be such that $k\theta > 1$.

There is $N \in \mathbb{N}$ such that, for each $n \geq N$,

(i) $\text{supp}(\mu) \subseteq \text{supp}(\mu_n)$,

(ii) $\sum_{\hat{\delta}\in X}\mu_n(z,m,\hat{\delta})\hat{\delta}(z',c) < \kappa_n(z,z',c)$ for each $(z',c)\in T_z^m(\mu^*)$, and

(iii) $u_n(z,m,\delta',\mu_n) \geq k u_n(z,a,\delta,\mu_n)$.

Indeed, (i) is clear since Z and Δ are finite. As for (ii), take $N \geq \max_{(z',c)\in Z\times C} N_{z,z',c}$ where, for each $(z',c)\in Z\times C$, $N_{z,z',c}$ is given by Claim 6. Finally, for (iii), we have that $\frac{u(z,m,\delta',\mu)}{u(z,a,\delta,\mu)} > 1$ and, for all n sufficiently large, $\frac{u(z,m,\delta',\mu_n)}{u(z,a,\delta,\mu_n)} \geq \beta$ for some $\beta > 1$. Hence,

$$\frac{u_n(z,m,\delta',\mu_n)}{u_n(z,a,\delta,\mu_n)} = \left(\frac{u(z,m,\delta',\mu_n)}{u(z,a,\delta,\mu_n)}\right)^n \geq \beta^n > k$$

for all n sufficiently large.

Fix $n \geq N$ and let $c^* \in \mathbb{C}(z,\emptyset)$. For each $\varepsilon > 0$, define π_ε by setting, for each $(\hat{z},\hat{a},\hat{\delta}) \in Z\times\Delta$,

$$\pi_\varepsilon(\hat{z},\hat{a},\hat{\delta}) = \begin{cases} \mu_n(z,a,\delta)-\varepsilon & \text{if } \hat{z}=z, \hat{a}=a \text{ and } \hat{\delta}=\delta, \\ \mu_n(z,m,\delta')+\theta\varepsilon & \text{if } \hat{z}=z, \hat{a}=m \text{ and } \hat{\delta}=\delta', \\ \mu_n(z,s,1_{(\emptyset,c^*)})+(1-\theta)\varepsilon & \text{if } \hat{z}=z, \hat{a}=s \text{ and } \hat{\delta}=1_{(\emptyset,c^*)}, \\ \mu_n(\hat{z},\hat{a},\hat{\delta}) & \text{otherwise.} \end{cases}$$

By (i), $\mu_n(z,a,\delta) > 0$. We have that, for each $\varepsilon \in (0,\mu_n(z,a,\delta))$, $\pi_\varepsilon(\hat{z},\hat{a},\hat{\delta}) \geq 0$ for each $(\hat{z},\hat{a},\hat{\delta}) \in Z\times\Delta$, $\pi_\varepsilon(\hat{z},w,1_{(z',c)}) \leq \mu_n(\hat{z},w,1_{(z',c)}) \leq \sum_{\hat{\delta}\in X}\mu_n(z',m,\hat{\delta})\hat{\delta}(\hat{z},c)$ for each $(\hat{z},z',c)\in Z\times Z\times C$ and

$$\sum_{(\hat{a},\hat{\delta})\in\Delta}\pi_\varepsilon(\hat{z},\hat{a},\hat{\delta}) = \sum_{(\hat{a},\hat{\delta})\in\Delta}\mu_n(\hat{z},\hat{a},\hat{\delta}) = \nu(\hat{z})$$

for each $\hat{z}\in Z$. In particular, $\pi_\varepsilon \in \mathcal{T}$.

We also have that, for some $\varepsilon \in (0,\mu_n(z,a,\delta))$,

$$\sum_{\hat{\delta}\in X}\pi_\varepsilon(\hat{z},m,\hat{\delta})\hat{\delta}(z',c) \leq \kappa_n(\hat{z},z',c) \text{ for all } (\hat{z},z',c)\in Z\times Z\times C. \tag{7}$$

First, note that it is enough to consider $\hat{z}=z$ and that, for each $(z',c)\in Z\times C$,

$$\sum_{\hat{\delta}\in X}\pi_\varepsilon(z,m,\hat{\delta})\hat{\delta}(z',c) = \sum_{\hat{\delta}\in X}\mu_n(z,m,\hat{\delta})\hat{\delta}(z',c) + \varepsilon\left(-\delta(z',c)1_m(a)+\theta\delta'(z',c)\right).$$

Thus, (7) holds if $(z',c)\notin \operatorname{supp}(\delta')$.

If $a = m$ and $(z', c) \in \operatorname{supp}(\delta') \cap \operatorname{supp}(\delta)$, the definition of $(\bar{z}, \bar{c})$ implies that:

$$\sum_{\hat{\delta} \in X} \pi_\varepsilon(z, m, \hat{\delta})\hat{\delta}(z', c) \leq \sum_{\hat{\delta} \in X} \mu_n(z, m, \hat{\delta})\hat{\delta}(z', c) \leq \kappa_n(z, z', c).$$

If $a \neq m$ and $(z', c) \in \operatorname{supp}(\delta')$ or if $a = m$ and $(z', c) \in \operatorname{supp}(\delta') \setminus \operatorname{supp}(\delta)$, then $(z', c) \in T_z^m(\mu^*)$ and, thus, $\sum_{\hat{\delta} \in X} \mu_n(z, m, \hat{\delta})\hat{\delta}(z', c) < \kappa_n(z, z', c)$ by (ii). Hence, there is $\varepsilon(z', c) > 0$ such that

$$\sum_{\hat{\delta} \in X} \pi_\varepsilon(z, m, \hat{\delta})\hat{\delta}(z', c) = \sum_{\hat{\delta} \in X} \mu_n(z, m, \hat{\delta})\hat{\delta}(z', c) + \varepsilon\theta\delta'(z', c) < \kappa_n(z, z', c)$$

for each $0 < \varepsilon < \varepsilon(z', c)$. Thus, letting $B = \operatorname{supp}(\delta')$ if $a \neq m$, $B = \operatorname{supp}(\delta') \setminus \operatorname{supp}(\delta)$ if $a = m$ and $0 < \varepsilon < \min_{(z',c) \in B} \varepsilon(z', c)$, we have that $\sum_{\hat{\delta} \in X} \pi_\varepsilon(\hat{z}, m, \hat{\delta})\hat{\delta}(z', c) \leq \kappa_n(\hat{z}, z', c)$ for each $(\hat{z}, z', c) \in Z \times Z \times C$.

Finally, note that

$$\sum_{\hat{z} \in Z, (\hat{a}, \hat{\delta}) \in \Delta} u_n(\hat{z}, \hat{a}, \hat{\delta}, \mu_n)\pi_\varepsilon(\hat{z}, \hat{a}, \hat{\delta}) > \sum_{\hat{z} \in Z, (\hat{a}, \hat{\delta}) \in \Delta} u_n(\hat{z}, \hat{a}, \hat{\delta}, \mu_n)\mu_n(\hat{z}, \hat{a}, \hat{\delta})$$

since $u_n(z, m, \delta', \mu_n) \geq k u_n(z, a, \delta, \mu_n)$ by (iii) since $n \geq N$, $u_n(z, s, 1_{(\emptyset, c^*)}, \mu_n) \geq 1$ and, hence,

$$\sum_{\hat{z} \in Z, (\hat{\delta}, \hat{a}) \in \Delta} u_n(\hat{z}, \hat{a}, \hat{\delta}, \mu_n) \left(\pi_\varepsilon(\hat{z}, \hat{a}, \hat{\delta}) - \mu_n(\hat{z}, \hat{a}, \hat{\delta})\right) \geq u_n(z, a, \delta, \mu_n)\varepsilon\left(-1 + k\theta\right) > 0.$$

In conclusion $\mu_n \notin D_n(\mu_n, \kappa_n)$, a contradiction. ■

It follows from Claims 3, 5 and 7 that μ^* is a stable matching. ■

In the remainder of the proof, we extend Lemma 5 using the following limit result.

Lemma 6 *Let $\{(E_k, \mu_k)\}_{k=1}^\infty$ be such that $E_k = (Z_k, \nu_k, C_k, \mathbb{C}_k, X_k, (\succ_{z,k})_{z \in Z_k})$ is a market and μ_k is a stable matching for E_k for each $k \in \mathbb{N}$. Let $E = (Z, \nu, C, \mathbb{C}, X, (\succ_z)_{z \in Z})$ be a rational, continuous, bounded and rich market such that $\nu_k \to \nu$ and, for each $k \in \mathbb{N}$, $Z_k \subseteq Z$, $\operatorname{supp}(\nu_k) \subseteq \operatorname{supp}(\nu)$, $C_k \subseteq C$, $\mathbb{C}_k(z, z') \subseteq \mathbb{C}(z, z')$ for each $(z, z') \in Z_k \times Z_{\emptyset,k}$ and $X_k \subseteq X$. Then:*

1. *$\{\mu_k\}_{k=1}^\infty$ has a convergent subsequence in $\mathcal{M}(Z \times X_\emptyset)$.*

Suppose further that $\{\mu_k\}_{k=1}^\infty$ converges and let $\mu = \lim_k \mu_k$. Then:

2. μ *is a matching for* E*.*

Suppose further that $\succ_{z,k}$ *is the restriction of* $\succ_z$ *to* $\Delta_k \times \mathcal{M}(Z_k \times X_{\emptyset,k})$ *for each* $z \in Z_k$*. Then:*

3. $\text{supp}(\mu) \subseteq IR(\mu)$*.*

4. $\text{supp}(\mu) \subseteq S_M(\mu)$ *if*

(a) for each $(z, \delta, \mu) \in Z \times X \times \mathcal{M}(Z \times X_\emptyset)$*,* $\delta' \in \Lambda(z, \delta, \mu)$*, open neighborhood* $V_{\delta'}$ *of* δ'*, subsequence* $\{\mu_{k_j}\}_{j=1}^\infty$ *of* $\{\mu_k\}_{k=1}^\infty$ *and sequence* $\{(z_{k_j}, \delta_{k_j})\}_{j=1}^\infty$ *such that* $(z_{k_j}, \delta_{k_j}) \to (z, \delta)$ *and* $(z_{k_j}, \delta_{k_j}) \in Z_{k_j} \times X_{k_j}$ *for each* $j \in \mathbb{N}$*, there exists* $J \in \mathbb{N}$ *such that* $\{\gamma \in X_{k_j} : \{z_{k_j}\} \times \text{supp}(\gamma) \subseteq \text{graph}(\mathbb{C}_{k_j})\} \cap \Lambda(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ *for each* $j \geq J$*, and*

(b) for each $(z, \mu) \in Z \times \mathcal{M}(Z \times X_\emptyset)$*,* $\delta' \in \Lambda_0(z, \mu)$*, open neighborhood* $V_{\delta'}$ *of* δ'*, subsequence* $\{\mu_{k_j}\}_{j=1}^\infty$ *of* $\{\mu_k\}_{k=1}^\infty$ *and sequence* $\{z_{k_j}\}_{j=1}^\infty$ *such that* $z_{k_j} \to z$ *and* $z_{k_j} \in Z_{k_j}$ *for each* $j \in \mathbb{N}$*, there exists* $J \in \mathbb{N}$ *such that* $\{\gamma \in X_{k_j} : \{z_{k_j}\} \times \text{supp}(\gamma) \subseteq \text{graph}(\mathbb{C}_{k_j})\} \cap \Lambda_0(z_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ *for each* $j \geq J$*.*

Proof. We divide the proof into several parts corresponding to the ones in the statement of the lemma.

Part 1: Since $\mathcal{M}(Z \times X_\emptyset)$ is a separable metrizable space, it suffices to show that $\{\mu_k\}_{k=1}^\infty$ is tight; this follows by Lemma 4 together with $\mu_k(Z \times X_\emptyset) \leq \nu_k(Z)$ for each $k \in \mathbb{N}$ and the fact that $\{\nu_k(Z)\}_{k=1}^\infty$ converges (to $\nu(Z)$) and, hence, is bounded.

Let $\varepsilon > 0$. Since $\{\nu_k\}_{k=1}^\infty$ is tight, there exists a compact subset K_Z of Z such that $\nu_k(Z \setminus K_Z) \leq \varepsilon$ for all k.

For each $n \in \mathbb{N}$, let K_n be a compact subset of Z such that $\hat{\nu}(Z \setminus K_n) < \frac{\varepsilon}{n 2^n}$ for each $\hat{\nu} \in \{\nu_k\}_{k=1}^\infty$. Let $K_n^\emptyset = K_n \cup \{\emptyset\}$ and let $D_n = \bigcup_{(z,z') \in K_Z \times K_n^\emptyset} \mathbb{C}(z, z')$. Note that D_n is compact since $\mathbb{C}$ is continuous and compact-valued, and K_Z and $K_n^\emptyset$ are compact.

Define

$$K_X = \left\{ \delta \in X : \delta(Z \times C \setminus K_n \times D_n) \leq \frac{1}{n} \text{ for each } n \in \mathbb{N} \right\}.$$

Then K_X is closed since if $\delta_j \to \delta$ and $\delta_j \in K_X$ for each $j \in \mathbb{N}$, then $\delta \in X$ since X is closed and, for each $n \in \mathbb{N}$, $\delta(Z \times C \setminus K_n \times D_n) \leq \liminf_j \delta_j(Z \times C \setminus K_n \times D_n) \leq \frac{1}{n}$ since $Z \times C \setminus K_n \times D_n$ is open. Hence, $\delta \in K_X$. In addition, K_X is tight since, for each $\eta > 0$, there is $n \in \mathbb{N}$ such that $\frac{1}{n} < \eta$ and, thus, $\delta(Z \times C \setminus K_n \times D_n) \leq \frac{1}{n} < \eta$ for each $\delta \in K_X$. Let $R > 0$ be such that $X \subseteq \mathcal{M}_R(Z \times C)$. Since K_X is a closed and tight subset of $\mathcal{M}_R(Z \times C)$, it follows that K_X is compact.

For each $n \in \mathbb{N}$, let

$$K_{X,n} = \left\{\delta \in X : \delta(Z \times C \setminus K_n \times D_n) > \frac{1}{n} \text{ and } \delta(Z \times C \setminus K_j \times D_j) \leq \frac{1}{j} \right. \\ \left. \text{for each } j = 1, \dots, n-1 \right\}.$$

Then $X \setminus K_X = \cup_{n=1}^{\infty} K_{X,n}$ and the family $\{K_{X,n}\}_{n=1}^{\infty}$ is pairwise disjoint. Fix $k \in \mathbb{N}$. For each $n \in \mathbb{N}$, we have that

$$\begin{aligned} \frac{\varepsilon}{n2^n} > \nu_k(Z \setminus K_n) &\geq \int_{Z \times X} \delta((Z \setminus K_n) \times C) \mathrm{d}\mu_k(z, \delta) \\ &\geq \int_{K_Z \times K_{X,n}} \delta((Z \setminus K_n) \times C) \mathrm{d}\mu_k(z, \delta) \\ &= \int_{K_Z \times K_{X,n}} \delta(Z \times C \setminus K_n \times D_n) \mathrm{d}\mu_k(z, \delta) > \frac{1}{n} \mu_k(K_Z \times K_{X,n}), \end{aligned}$$

where the equality follows because $\delta(Z \times C \setminus K_n \times D_n) = \delta((Z \setminus K_n) \times C) + \delta(K_n \times (C \setminus D_n))$ and condition 1 of a matching implies that, for each $(z, \delta) \in \operatorname{supp}(\mu_k) \cap (K_Z \times X)$, $\operatorname{supp}(\delta) \cap (K_n \times C) \subseteq D_n$ and, thus, $\delta(K_n \times (C \setminus D_n)) = 0$. Hence,

$$\varepsilon = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} > \sum_{n=1}^{\infty} \mu_k(K_Z \times K_{X,n}) = \mu_k(K_Z \times (X \setminus K_X)).$$

Note that $\cup_{z \in K_Z} \mathbb{C}(z, \emptyset) \subseteq D_1$ and let $K_X^{\emptyset} = K_X \cup \{1_{(\emptyset,c)} : c \in D_1\}$. Then $K_X^{\emptyset}$ is compact since both K_X and D_1 are compact. Moreover, $\mu_k(K_Z \times (X_\emptyset \setminus (K_X^{\emptyset} \cup X))) = \mu_k(K_Z \times \{1_{(\emptyset,c)} : c \notin D_1\}) = 0$ where the first equality follows since $(X_\emptyset \setminus K_X^{\emptyset}) \cap (X_\emptyset \setminus X) = (X_\emptyset \setminus K_X^{\emptyset}) \cap \{1_{(\emptyset,c)} : c \in C\} = \{1_{(\emptyset,c)} : c \notin D_1\}$ and the second by condition 1 of a matching since $\cup_{z \in K_Z} \mathbb{C}(z, \emptyset) \subseteq D_1$. Then, for each $k \in \mathbb{N}$,

$$\begin{aligned} &\mu_k(Z \times X_\emptyset \setminus K_Z \times K_X^{\emptyset}) = \mu_k((Z \setminus K_Z) \times X_\emptyset) + \mu_k(K_Z \times (X_\emptyset \setminus (K_X^{\emptyset} \cup X))) \\ &+ \mu_k(K_Z \times ((X_\emptyset \setminus K_X^{\emptyset}) \cap X)) \leq \nu_k(Z \setminus K_Z) + 0 + \mu_k(K_Z \times (X \setminus K_X)) < 2\varepsilon, \end{aligned}$$

where $\mu_k((Z \setminus K_Z) \times X_\emptyset) \leq \nu_k(Z \setminus K_Z)$ because of condition 2 of a matching.

Part 2: We first consider condition 2 of the definition of a matching. Let $\pi : Z \times X_\emptyset \to Z$ be the projection of $Z \times X_\emptyset$ onto Z and note that, for each Borel subset B of Z, $\nu_M(B) + \nu_S(B) = \mu(B \times X_\emptyset) = \mu \circ \pi^{-1}(B)$ and $\nu_{M,k}(B) + \nu_{S,k}(B) = \mu_k(B \times X_\emptyset) = \mu_k \circ \pi^{-1}(B)$ for each $k \in \mathbb{N}$. Since π is continuous, $\mu_k \circ \pi^{-1} \to \mu \circ \pi^{-1}$. Indeed, for each bounded and continuous $f : Z \to \mathbb{R}$, $\int_Z f \mathrm{d}\mu_k \circ \pi^{-1} = \int_{Z \times X_\emptyset} f \circ \pi \mathrm{d}\mu_k \to \int_{Z \times X_\emptyset} f \circ \pi \mathrm{d}\mu = \int_Z f \mathrm{d}\mu_k \circ \pi^{-1}$ since $f \circ \pi : Z \times X_\emptyset \to \mathbb{R}$ is bounded and continuous. Hence, since $\mathcal{M}(Z \times X_\emptyset)$ is metrizable, $\nu_M + \nu_S = \mu \circ \pi^{-1} = \lim_k \mu_k \circ \pi^{-1} = \lim_k(\nu_{M,k} + \nu_{S,k})$.

Also, for each Borel subset B of Z, $\nu_W(B) = \int_{Z \times X} \delta(B \times C) \mathrm{d}\mu(z, \delta)$ and $\nu_{W,k}(B) = \int_{Z \times X} \delta(B \times C) \mathrm{d}\mu_k(z, \delta)$ for each $k \in \mathbb{N}$. We show that $\nu_{W,k} \to \nu_W$. Let $B \subseteq Z$ be closed and $f : X \to \mathbb{R}$ be defined by setting, for each $\delta \in X$, $f(\delta) = \delta(B \times C)$. Then f is bounded and upper semi-continuous. Hence, by (a suitable adaptation of) Aliprantis and Border (2006, Theorem 15.5), $\limsup_k \nu_{W,k}(B) = \limsup_k \int_{Z \times X} f \mathrm{d}\mu_k \leq \int_{Z \times X} f \mathrm{d}\mu = \nu_W(B)$ and it follows that $\nu_W = \lim_k \nu_{W,k}$ as claimed. Thus,

$$\nu_M + \nu_S + \nu_W = \lim_k(\nu_{M,k} + \nu_{S,k}) + \lim_k \nu_{W,k} = \lim_k (\nu_{M,k} + \nu_{S,k} + \nu_{W,k}) = \nu.$$

Condition 1 holds because, by Carmona and Podczeck (2009, Lemma 12), for each $(z, \delta) \in \operatorname{supp}(\mu)$ and $(z', c) \in \operatorname{supp}(\delta)$, there exists a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding $\{(z_{k_j}, \delta_{k_j}, z'_{k_j}, c_{k_j})\}_{j=1}^\infty$ such that $(z_{k_j}, \delta_{k_j}, z'_{k_j}, c_{k_j}) \to (z, \delta, z', c)$ and, for each $j \in \mathbb{N}$, $(z_{k_j}, \delta_{k_j}) \in \operatorname{supp}(\mu_{k_j})$ and $(z'_{k_j}, c_{k_j}) \in \operatorname{supp}(\delta_{k_j})$. Hence, $c_{k_j} \in \mathbb{C}_{k_j}(z_{k_j}, z'_{k_j}) \subseteq \mathbb{C}(z_{k_j}, z'_{k_j})$ and, since $\mathbb{C}$ is continuous, $c \in \mathbb{C}(z, z')$.

Part 3: Let $(z, \delta) \in \operatorname{supp}(\mu)$ and suppose that $(z, \delta) \notin IR(\mu)$. Then either (i) there exists $c \in \mathbb{C}(z, \emptyset)$ such that $(s, 1_{(\emptyset, c)}, \mu) \succ_z (a(\delta), \delta, \mu)$ where $a(\delta) = m$ if $\delta \in X$ and $a(\delta) = s$ if $\delta \in X_\emptyset \setminus X$, or (ii) there exists $(z', c) \in \operatorname{supp}(\delta)$ and $c' \in \mathbb{C}(z', \emptyset)$ such that $(s, 1_{(\emptyset, c')}, \mu) \succ_{z'} (w, 1_{(z,c)}, \mu)$.

Consider case (i) first. The continuity of $(\succ_z)_{z \in Z}$ and $\mathbb{C}$ implies that there are open neighborhoods V_c, V_z, V_δ and V_μ of c, z, δ and μ, respectively, such that $(s, 1_{(\emptyset, \hat{c})}, \hat{\mu}) \succ_{\hat{z}} (a(\delta), \hat{\delta}, \hat{\mu})$ and $\mathbb{C}(\hat{z}, \emptyset) \cap V_c \neq \emptyset$ for each $\hat{c} \in V_c$, $\hat{z} \in V_z$, $\hat{\delta} \in V_\delta$ and $\hat{\mu} \in V_\mu$. Since $(z, \delta) \in \operatorname{supp}(\mu)$, it follows that $0 < \mu(V_z \times V_\delta) \leq \liminf_k \mu_k(V_z \times V_\delta)$; hence, for each k sufficiently large, $\mu_k(V_z \times V_\delta) > 0$ and $\mu_k \in V_\mu$. This means that, for any such k, there exist $(\hat{z}, \hat{\delta}) \in \text{supp}(\mu_k) \cap (V_z \times V_\delta)$ and $\hat{c} \in \mathbb{C}(\hat{z}, \emptyset) \cap V_c$. But then $(s, 1_{(\emptyset,\hat{c})}, \mu_k) \succ_{\hat{z}} (a(\delta), \hat{\delta}, \mu_k)$ and, hence, $(s, 1_{(\emptyset,\hat{c})}, \mu_k) \succ_{\hat{z},k} (a(\delta), \hat{\delta}, \mu_k)$, contradicting the individual rationality of μ_k.

Consider next case (ii). The continuity of $(\succ_z)_{z\in Z}$ and $\mathbb{C}$ implies that there are open neighborhoods $V_{c'}$, V_c, V_z, $V_{z'}$ and V_μ of c', c, z, z' and μ, respectively, such that $(s, 1_{(\emptyset,\tilde{c})}, \hat{\mu}) \succ_{\tilde{z}} (w, 1_{(\hat{z},\hat{c})}, \hat{\mu})$ and $\mathbb{C}(\emptyset, \tilde{z}) \cap V_{c'} \neq \emptyset$ for each $\tilde{c} \in V_{c'}$, $\hat{c} \in V_c$, $\hat{z} \in V_z$, $\tilde{z} \in V_{z'}$ and $\hat{\mu} \in V_\mu$. Since $(z', c) \in \text{supp}(\delta)$, there is an open neighborhood V_δ of δ such that $\text{supp}(\hat{\delta}) \cap (V_{z'} \times V_c) \neq \emptyset$ for each $\hat{\delta} \in V_\delta$ by Lemma 2. Since $\mu_k \to \mu$ and $(z, \delta) \in \text{supp}(\mu)$, it follows that $0 < \mu(V_z \times V_\delta) \leq \liminf_k \mu_k(V_z \times V_\delta)$; hence, for all k sufficiently large, $\mu_k(V_z \times V_\delta) > 0$ and $\mu_k \in V_\mu$. This means that, for any such k, there exists $(\hat{z}, \hat{\delta}) \in \text{supp}(\mu_k) \cap (V_z \times V_\delta)$, $(\tilde{z}, \hat{c}) \in \text{supp}(\hat{\delta}) \cap (V_{z'} \times V_c)$ and $\tilde{c} \in \mathbb{C}(\emptyset, \tilde{z}) \cap V_{c'}$. But then $(s, 1_{(\emptyset,\tilde{c})}, \mu_k) \succ_{\tilde{z}} (w, 1_{(\hat{z},\hat{c})}, \mu_k)$ and, hence, $(s, 1_{(\emptyset,\tilde{c})}, \mu_k) \succ_{\tilde{z},k} (w, 1_{(\hat{z},\hat{c})}, \mu_k)$, contradicting the individual rationality of μ_k.

Part 4: In this proof, to avoid confusion, we write $T^m_z(\mu; E')$ for $T^m_z(\mu)$ in a market E'.

Let $(z, \delta) \in \text{supp}(\mu)$ and suppose that $(z, \delta) \notin S_M(\mu)$. Then there exists $\delta' \in X$ such that either (i) $\text{supp}(\delta') \subseteq T^m_z(\mu) \cup \text{supp}(\delta)$ and $(m, \delta', \mu) \succ_z (a(\delta), \delta, \mu)$, where $a(\delta) = m$ if $\delta \in X$ and $a(\delta) = s$ if $\delta \in X_\emptyset \setminus X$,[34] or (ii) there exists $(z', c) \in \text{supp}(\delta)$ such that $\text{supp}(\delta') \subseteq T^m_{z'}(\mu)$ and $(m, \delta', \mu) \succ_{z'} (w, 1_{(z,c)}, \mu)$.

Consider case (i) first. Let V_z, $V_{\delta'}$, V_δ and V_μ be open neighborhoods of z, δ', δ and μ, respectively, such that $(m, \gamma', \bar{\mu}) \succ_{\bar{z}} (a(\delta), \gamma, \bar{\mu})$ for each $\bar{z} \in V_z$, $\gamma' \in V_{\delta'}$, $\gamma \in V_\delta$ and $\bar{\mu} \in V_\mu$. Let, by the richness of E, $\tilde{V}_z$, $\tilde{V}_\delta$ and $\tilde{V}_\mu$ be open neighborhoods of z, δ and μ, respectively, such that $\Lambda(\bar{z}, \gamma, \bar{\mu}) \cap V_{\delta'} \neq \emptyset$ for each $(\bar{z}, \gamma, \bar{\mu}) \in \tilde{V}_z \times \tilde{V}_\delta \times \tilde{V}_\mu$.

By Carmona and Podczeck (2009, Lemma 12), there is a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding sequence $\{(z_{k_j}, \delta_{k_j})\}_{j=1}^\infty$ such that $(z_{k_j}, \delta_{k_j}) \to (z, \delta)$ and $(z_{k_j}, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$ for each $j \in \mathbb{N}$.

Let $J \in \mathbb{N}$ be such that $\mu_{k_j} \in V_\mu \cap \tilde{V}_\mu$, $z_{k_j} \in V_z \cap \tilde{V}_z$, $\delta_{k_j} \in V_\delta \cap \tilde{V}_\delta$ and $\{\gamma \in$

[34]Note that when $\delta \in X_\emptyset \setminus X$ and $\delta' \in X$, $\text{supp}(\delta') \subseteq T^m_z(\mu) \cup \text{supp}(\delta)$ if and only if $\text{supp}(\delta') \subseteq T^m_z(\mu)$.

$X_{k_j} : \{z_{k_j}\} \times \text{supp}(\gamma) \subseteq \text{graph}(\mathbb{C}_{k_j})\} \cap \Lambda(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ for all $j \geq J$. Let $j \geq J$ and let $\delta'_{k_j} \in \{\gamma \in X_{k_j} : \{z_{k_j}\} \times \text{supp}(\gamma) \subseteq \text{graph}(\mathbb{C}_{k_j})\} \cap \Lambda(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \cap V_{\delta'}$. Then $\text{supp}(\delta'_{k_j}) \subseteq T^m_{z_{k_j}}(\mu_{k_j}; E) \cup \text{supp}(\delta_{k_j})$ and $(m, \delta'_{k_j}, \mu_{k_j}) \succ_{z_{k_j}} (a(\delta), \delta_{k_j}, \mu_{k_j})$. It then follows that $\text{supp}(\delta'_{k_j}) \subseteq T^m_{z_{k_j}}(\mu_{k_j}; E_{k_j}) \cup \text{supp}(\delta_{k_j})$ and $(m, \delta'_{k_j}, \mu_{k_j}) \succ_{z_{k_j}, k_j} (a(\delta), \delta_{k_j}, \mu_{k_j})$. But this contradicts the stability of μ_{k_j}.

Consider next case (ii). Let V_z, $V_{z'}$, $V_{\delta'}$, V_c and V_μ be open neighborhoods of z, z', δ', c and μ respectively, such that $(m, \hat{\delta}', \hat{\mu}) \succ_{\hat{z}'} (w, 1_{(\hat{z}, \hat{c})}, \hat{\mu})$ for each $\hat{z} \in V_z$, $\hat{z}' \in V_{z'}$, $\hat{\delta}' \in V_{\delta'}$, $\hat{c} \in V_c$ and $\hat{\mu} \in V_\mu$. Let, by the richness of E, $\tilde{V}_{z'}$ and $\tilde{V}_\mu$ be open neighborhoods of z' and μ, respectively, such that $\Lambda_0(\hat{z}', \hat{\mu}) \cap V_{\delta'} \neq \emptyset$ for each $(\hat{z}', \hat{\mu}) \in \tilde{V}_{z'} \times \tilde{V}_\mu$.

By Carmona and Podczeck (2009, Lemma 12), there is a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding sequence $\{(z_{k_j}, \delta_{k_j}, z'_{k_j}, c_{k_j})\}_{j=1}^\infty$ such that $(z_{k_j}, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$ and $(z'_{k_j}, c_{k_j}) \in \text{supp}(\delta_{k_j})$ for each $j \in \mathbb{N}$ and $(z_{k_j}, \delta_{k_j}, z'_{k_j}, c_{k_j}) \to (z, \delta, z', c)$.

Let $J \in \mathbb{N}$ be such that $\delta_{k_j} \in V_\delta$, $z_{k_j} \in V_z$, $z'_{k_j} \in V_{z'} \cap \tilde{V}_{z'}$, $c_{k_j} \in V_c$, $\mu_{k_j} \in V_\mu \cap \tilde{V}_\mu$ and $\{\gamma \in X_{k_j} : \{z'_{k_j}\} \times \text{supp}(\gamma) \subseteq \text{graph}(\mathbb{C}_{k_j})\} \cap \Lambda_0(z'_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ for all $j \geq J$. Let $j \geq J$ and let $\delta'_{k_j} \in \{\gamma \in X_{k_j} : \{z'_{k_j}\} \times \text{supp}(\gamma) \subseteq \text{graph}(\mathbb{C}_{k_j})\} \cap \Lambda_0(z'_{k_j}, \mu_{k_j}) \cap V_{\delta'}$. Then $\text{supp}(\delta'_{k_j}) \subseteq T^m_{z'_{k_j}}(\mu_{k_j}; E)$ and $(m, \delta'_{k_j}, \mu_{k_j}) \succ_{z'_{k_j}} (w, 1_{(z_{k_j}, c_{k_j})}, \mu_{k_j})$. It then follows that $\text{supp}(\delta'_{k_j}) \subseteq T^m_{z'_{k_j}}(\mu_{k_j}; E_{k_j})$ and $(m, \delta'_{k_j}, \mu_{k_j}) \succ_{z'_{k_j}, k_j} (w, 1_{(z_{k_j}, c_{k_j})}, \mu_{k_j})$. But this contradicts the stability of μ_{k_j}. ■

The second step in the proof of our existence result consists in the following lemma, which considers the special case where Z is finite and $X = \mathcal{M}_R(Z \times C)$ for some $R > 0$.

Lemma 7 *If E is a rational and continuous market such that Z is finite and $X = \mathcal{M}_R(Z \times C)$ for some $R > 0$, then E has a stable matching.*

Proof. For each $(z, z') \in Z \times Z_\emptyset$, let $\{c^n_{z,z'}\}_{n=1}^\infty$ be a dense subset of $\mathbb{C}(z, z')$. For each $k \in \mathbb{N}$, define $\mathbb{C}_k(z, z') = \{c^n_{z,z'} : n \leq k\}$ and $C_k = \cup_{(z,z') \in Z \times Z_\emptyset} \mathbb{C}_k(z, z')$.

In addition, enumerate $\mathbb{Q} = \{q_1, q_2, \ldots\}$ and, for each $k \in \mathbb{N}$, let X_k be the set of $\delta \in \mathcal{M}_R(Z \times C)$ such that $\text{supp}(\delta)$ is a subset of $Z \times C_k$ and, for each $(z, c) \in Z \times C_k$, $\delta(z, c) \in \{q_n : n \leq k\}$. Let $X_{\emptyset,k} = X_k \cup \{1_{(\emptyset,c)} : c \in C_k\}$, $X_{m,k} = X_k$, $X_{s,k} = \{1_{(\emptyset,c)} :$

$c \in C_k\}$, $X_{w,k} = \{1_{(z,c)} : (z,c) \in Z \times C_k\}$ and $\Delta_k = \{(a,\delta) : \delta \in X_{a,k}\}$.

For each $k \in \mathbb{N}$, let $E_k = (Z, \nu, C_k, \mathbb{C}_k, X_k, (\succ_z)_{z \in Z})$ be a market where $\succ_z$ is restricted to $\Delta_k \times \mathcal{M}(Z \times X_{\emptyset,k})$ for each $z \in Z$. Let $\mu_k \in \mathcal{M}(Z \times X_{\emptyset,k})$ be a stable matching in E_k, which exists by Lemma 5 since Z, C_k and X_k are finite.

It follows by part 1 of Lemma 6 that we may assume that $\{\mu_k\}_{k=1}^{\infty}$ converges; let $\mu = \lim_k \mu_k$. It then follows by parts 2 and 3 of Lemma 6 that μ is a matching and that $\text{supp}(\mu) \subseteq IR(\mu)$.

The following claim will be used to show that condition (a) of part 4 of Lemma 6 holds.

Claim 8 *Let $(\tilde{z}, \tilde{c}) \in T_z^m(\mu)$ and $V_{\tilde{c}}$ be an open neighborhood of $\tilde{c}$. Then, for all k sufficiently large, there exists $c_k \in \mathbb{C}_k(z, \tilde{z})$ such that $(\tilde{z}, c_k) \in T_z^m(\mu_k) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$.*

Proof. Let $(\tilde{z}, \tilde{c}) \in T_z^m(\mu)$ and $V_{\tilde{c}}$ be an open neighborhood of $\tilde{c}$. Then $\tilde{c} \in \mathbb{C}(z, \tilde{z})$ and either (i) there exists $(\hat{z}, \hat{\delta}, \hat{c})$ such that $(\hat{z}, \hat{\delta}) \in \text{supp}(\mu)$, $(\tilde{z}, \hat{c}) \in \text{supp}(\hat{\delta})$ and $(w, 1_{(z,\tilde{c})}, \mu) \succ_{\tilde{z}} (w, 1_{(\hat{z},\hat{c})}, \mu)$, or (ii) there exists $\tilde{\delta} \in X_\emptyset$ such that $(\tilde{z}, \tilde{\delta}) \in \text{supp}(\mu)$ and $(w, 1_{(z,\tilde{c})}, \mu) \succ_{\tilde{z}} (a(\tilde{\delta}), \tilde{\delta}, \mu)$, where $a(\tilde{\delta}) = s$ if $\tilde{\delta} \in X_\emptyset \setminus X$ and $a(\tilde{\delta}) = m$ if $\tilde{\delta} \in X$.

Consider case (i) first. Let $O_{\tilde{c}}$, $O_{\hat{c}}$, $O_{\hat{\delta}}$ and O_μ be open neighborhoods of $\tilde{c}$, $\hat{c}$, $\hat{\delta}$ and μ, respectively, such that $(w, 1_{(z,\tilde{c}')}, \mu') \succ_{\tilde{z}} (w, 1_{(\hat{z},\hat{c}')}, \mu')$ and $\text{supp}(\hat{\delta}') \cap (\{\tilde{z}\} \times O_{\hat{c}}) \neq \emptyset$ for each $\tilde{c}' \in O_{\tilde{c}}$, $\hat{c}' \in O_{\hat{c}}$, $\hat{\delta}' \in O_{\hat{\delta}}$ and $\mu' \in O_\mu$. Since $0 < \mu(\{\hat{z}\} \times O_{\hat{\delta}}) \leq \liminf_k \mu(\{\hat{z}\} \times O_{\hat{\delta}})$, it follows that, for each k sufficiently large, there is $\hat{\delta}_k \in O_{\hat{\delta}}$ such that $(\hat{z}, \hat{\delta}_k) \in \text{supp}(\mu_k)$ and, for some $\hat{c}_k \in O_{\hat{c}}$, $(\tilde{z}, \hat{c}_k) \in \text{supp}(\hat{\delta}_k)$. In addition, $\mu_k \in O_\mu$ and there exists $c_k \in \mathbb{C}_k(z, \tilde{z}) \cap O_{\tilde{c}} \cap V_{\tilde{c}}$ since, respectively, $\mu_k \to \mu$ and $\mathbb{C}_k(z, \tilde{z})$ increases to a dense subset of $\mathbb{C}(z, \tilde{z})$. Then $(w, 1_{(z,c_k)}, \mu_k) \succ_{\tilde{z}} (w, 1_{(\hat{z},\hat{c}_k)}, \mu_k)$ and, hence, $(\tilde{z}, c_k) \in T_z^m(\mu_k) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$ for all k sufficiently large.

Consider next case (ii). Let $O_{\tilde{c}}$, $O_{\tilde{\delta}}$ and O_μ be open neighborhoods of $\tilde{c}$, $\tilde{\delta}$ and μ, respectively, such that $(w, 1_{(z,\tilde{c}')}, \mu') \succ_{\tilde{z}} (a(\tilde{\delta}), \tilde{\delta}', \mu')$ for each $\tilde{c}' \in O_{\tilde{c}}$, $\tilde{\delta}' \in O_{\tilde{\delta}}$ and $\mu' \in O_\mu$. Since $0 < \mu(\{\tilde{z}\} \times O_{\tilde{\delta}}) \leq \liminf_k \mu_k(\{\tilde{z}\} \times O_{\tilde{\delta}})$, it follows that, for each k sufficiently large there is $\tilde{\delta}_k \in O_{\tilde{\delta}}$ such that $(\tilde{z}, \tilde{\delta}_k) \in \text{supp}(\mu_k)$. In addition, $\mu_k \in O_\mu$ and there exists $c_k \in \mathbb{C}_k(z, \tilde{z}) \cap O_{\tilde{c}} \cap V_{\tilde{c}}$ since, respectively, $\mu_k \to \mu$ and $\mathbb{C}_k(z, \tilde{z})$ increases to a dense subset of $\mathbb{C}(z, \tilde{z})$. Then $(w, 1_{(z,c_k)}, \mu_k) \succ_{\tilde{z}} (a(\tilde{\delta}), \tilde{\delta}_k, \mu_k)$

and, hence, $(\tilde{z}, c_k) \in T_z^m(\mu_k) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$ for all k sufficiently large. ■

We now show that condition (a) of part 4 of Lemma 6 holds. Let $(z, \delta, \mu) \in Z \times X \times \mathcal{M}(Z \times X_\emptyset)$, $\delta' \in \Lambda(z, \delta, \mu)$, $V_{\delta'}$ be an open neighborhood of δ' and $\{(z_{k_j}, \delta_{k_j}, \mu_{k_j})\}_{j=1}^\infty$ be a sequence such that $(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \to (z, \delta, \mu)$ and $(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \in Z_{k_j} \times X_{k_j} \times \mathcal{M}(Z_{k_j} \times X_{\emptyset,k_j})$ for each $j \in \mathbb{N}$.

In particular, $\operatorname{supp}(\delta') \subseteq T_z^m(\mu) \cup \operatorname{supp}(\delta)$ and we may assume that $\operatorname{supp}(\delta')$ is finite, i.e. $\delta' = \sum_{(\tilde{z},\tilde{c}) \in \operatorname{supp}(\delta')} a(\tilde{z}, \tilde{c}) 1_{(\tilde{z},\tilde{c})}$ for some $a = (a(\tilde{z}, \tilde{c}))_{(\tilde{z},\tilde{c}) \in \operatorname{supp}(\delta')}$. Let V_a be an open neighborhood of a and, for each $(\tilde{z}, \tilde{c}) \in \operatorname{supp}(\delta')$, $V_{(\tilde{z},\tilde{c})}$ be an open neighborhood of $(\tilde{z}, \tilde{c})$ be such that

$$\sum_{(\tilde{z},\tilde{c}) \in \operatorname{supp}(\delta')} \hat{a}(\tilde{z}, \tilde{c}) 1_{(z(\tilde{z},\tilde{c}), c(\tilde{z},\tilde{c}))} \in V_{\delta'}$$

whenever $(z(\tilde{z}, \tilde{c}), c(\tilde{z}, \tilde{c})) \in V_{(\tilde{z},\tilde{c})}$ for each $(\tilde{z}, \tilde{c}) \in \operatorname{supp}(\delta')$ and $\hat{a} \in V_a$. Let $\hat{a} = (\hat{a}(\tilde{z}, \tilde{c}))_{(\tilde{z},\tilde{c}) \in \operatorname{supp}(\delta')} \in \mathbb{Q}_+^{|\operatorname{supp}(\delta')|} \cap V_a$ and $V_{\tilde{c}}$ be an open neighborhood of $\tilde{c}$ such that $\{\tilde{z}\} \times V_{\tilde{c}} \subseteq V_{(\tilde{z},\tilde{c})}$.

For each $(\tilde{z}, \tilde{c}) \in \operatorname{supp}(\delta') \cap T_z^m(\mu)$, and for each k sufficiently large, let $c_k(\tilde{z}, \tilde{c}) \in \mathbb{C}_k(z, \tilde{z})$ be such that $(\tilde{z}, c_k(\tilde{z}, \tilde{c})) \in T_z^m(\mu_k) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$, which exists by Claim 8.

If $(\tilde{z}, \tilde{c}) \in \operatorname{supp}(\delta') \setminus T_z^m(\mu)$, then $\delta \in X$, $(\tilde{z}, \tilde{c}) \in \operatorname{supp}(\delta)$ and $0 < \delta(\{\tilde{z}\} \times V_{\tilde{c}}) \leq \liminf_j \delta_{k_j}(\{\tilde{z}\} \times V_{\tilde{c}})$. Hence, for each j sufficiently large, let $c_{k_j}(\tilde{z}, \tilde{c}) \in V_{\tilde{c}}$ be such that $(\tilde{z}, c_{k_j}(\tilde{z}, \tilde{c})) \in \operatorname{supp}(\delta_{k_j})$.

Let $J' \in \mathbb{N}$ be such that, for each $j \geq J'$, $(\tilde{z}, c_{k_j}(\tilde{z}, \tilde{c})) \in T_z^m(\mu_{k_j}) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$ if $(\tilde{z}, \tilde{c}) \in \operatorname{supp}(\delta') \cap T_z^m(\mu)$ and $(\tilde{z}, c_{k_j}(\tilde{z}, \tilde{c})) \in \operatorname{supp}(\delta_{k_j}) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$ if $(\tilde{z}, \tilde{c}) \in \operatorname{supp}(\delta') \setminus T_z^m(\mu)$. Thus, letting $\delta'_{k_j} = \sum_{(\tilde{z},\tilde{c}) \in \operatorname{supp}(\delta')} \hat{a}(\tilde{z}, \tilde{c}) 1_{(\tilde{z}, c_{k_j}(\tilde{z},\tilde{c}))}$ for each $j \geq J'$, we have that $\delta'_{k_j} \in V_{\delta'}$ and $\operatorname{supp}(\delta'_{k_j}) \subseteq T_z^m(\mu_{k_j}) \cup \operatorname{supp}(\delta_{k_j})$. Since $\{\hat{a}(\tilde{z}, \tilde{c}) : (\tilde{z}, \tilde{c}) \in \operatorname{supp}(\delta')\}$ is finite, it follows that there is $J > J'$ such that $\delta'_{k_j} \in X_{k_j}$ for each $j \geq J$.

An analogous argument shows that condition (b) of part 4 of Lemma 6 also holds. Hence, it follows that $\operatorname{supp}(\mu) \subseteq S_M(\mu)$. This together with the fact that μ is a matching and $\operatorname{supp}(\mu) \subseteq IR(\mu)$ shows that μ is stable. ■

The next step of the proof of Theorem 2 extends Lemma 7 by requiring only that E be rich.

Lemma 8 *If E is a rational, continuous, bounded and rich market such that Z is finite, then E has a stable matching.*

Proof. It follows by Debreu (1964, Proposition 3) and by the finiteness of Z that there exists a continuous function $u : Z \times \Delta \times \mathcal{M}(Z \times X_\emptyset) \to [1, 2]$ such that $(a, \delta, \mu) \mapsto u(z, a, \delta, \mu)$ represents $\succeq_z$ for each $z \in Z$, using the fact that $[1, 2]$ and the extended reals are homeomorphic.

Let $R > 0$ be such that $X \subseteq \mathcal{M}_R(Z \times C)$, $\Delta^* = (\{m\} \times \mathcal{M}_R(Z \times C)) \cup (\{w\} \times X_w) \cup (\{s\} \times X_s)$ and $X^* = \mathcal{M}_R(Z \times C) \cup \{1_{(\emptyset,c)} : c \in C\}$. By Tietze Extension Theorem, let $U : Z \times \Delta^* \times \mathcal{M}(Z \times X^*) \to [1, 2]$ be a continuous extension of u.

Let ρ be a metric on $\mathcal{M}_R(Z \times C)$. For each $k \in \mathbb{N}$, let

$$\Delta_k = \{m\} \times \{\delta \in \mathcal{M}_R(Z \times C) : \rho(\delta, X) \geq k^{-1}\}.$$

Let, by Urysohn's Lemma, $g_k : \Delta^* \to [0, 1]$ be a continuous function such that $g_k^{-1}(1) = \Delta$ and $g_k^{-1}(0) = \Delta_k$. Then define $U_k : Z \times \Delta^* \times \mathcal{M}(Z \times X^*) \to \mathbb{R}$ by setting, for each $(z, a, \delta, \mu) \in Z \times \Delta^* \times \mathcal{M}(Z \times X^*)$, $U_k(z, a, \delta, \mu) = g_k(a, \delta) U(z, a, \delta, \mu)$.

Consider the market $E_k = (Z, \nu, C, \mathbb{C}, \mathcal{M}_R(Z \times C), U_k)$, i.e. E_k is equal to E except that X is replaced with $\mathcal{M}_R(Z \times C)$ and u with U_k. Since E_k is rational and continuous with Z finite and $X = \mathcal{M}_R(Z \times C)$, then E_k has a stable matching μ_k by Lemma 7.

Let $E^* = (Z, \nu, C, \mathbb{C}, \mathcal{M}_R(Z \times C), U)$. To avoid confusion, we write $IR(\mu; E')$ for $IR(\mu)$ and $S_M(\mu; E')$ for $S_M(\mu)$ whenever μ is a matching of a market E'. It follows by part 1 of Lemma 6 that we may assume that $\{\mu_k\}_{k=1}^\infty$ converges; let $\mu = \lim_k \mu_k$. It then follows by part 2 of Lemma 6 that μ is a matching of E^*.

The proof of part 3 of Lemma 6 implies that $\text{supp}(\mu) \subseteq IR(\mu; E^*)$ since the requirement that $\succ_{z,k}$ is the restriction of $\succ_z$ to $\Delta_k \times \mathcal{M}(Z_k \times X_{\emptyset,k})$ for each $z \in Z_k$ can be replaced with the following condition: $(s, \delta, \hat{\mu}) \succ_{z,k} (a, \delta', \hat{\mu})$ for each $k \in \mathbb{N}$, $z \in Z_k$, $\delta \in X_{s,k}$, $(a, \delta') \in \Delta_k$ and $\hat{\mu} \in \mathcal{M}(Z_k \times X_{\emptyset,k})$ such that $(s, \delta, \hat{\mu}) \succ_z (a, \delta', \hat{\mu})$. This condition holds because $U_k(z, a, \delta', \hat{\mu}) \leq U(z, a, \delta', \hat{\mu})$ and $U_k(z, s, 1_{(\emptyset,\hat{c})}, \hat{\mu}) = U(z, s, 1_{(\emptyset,\hat{c})}, \hat{\mu})$ for each $k \in \mathbb{N}$, $z \in Z$, $(a, \delta') \in \Delta^*$, $\hat{c} \in C$ and $\hat{\mu} \in \mathcal{M}(Z \times X^*)$ since $(s, 1_{(\emptyset,\hat{c})}) \in \Delta$ and, hence, $g_k(s, 1_{(\emptyset,\hat{c})}) = 1$.

We have that μ belongs to $\mathcal{M}(Z \times X_\emptyset)$. Indeed, let $k \in \mathbb{N}$ and $(z, \delta) \in \text{supp}(\mu_k) \cap \mathcal{M}(Z \times C)$. If $\delta \in X$ and $\rho(\delta, X) \geq k^{-1}$, then let $c \in \mathbb{C}(z, \emptyset)$ and $\delta' = 1_{(\emptyset, c)}$ to obtain that $\text{supp}(\delta') \subseteq T_z^s(\mu_k)$ and $U_k(z, s, \delta', \mu) = U(z, s, \delta', \mu) > 0 = U_k(z, m, \delta, \mu)$, the latter since $(s, \delta') \in \Delta$ and, thus, $g_k(s, \delta') = 1$, $U(z, s, \delta', \mu) \in [1, 2]$ and $g_k(m, \delta) = 0$. But this contradicts the stability of μ_k. Hence, it follows that $\rho(\delta, X) < k^{-1}$.

Thus, for each $k \in \mathbb{N}$,

$$\text{supp}(\mu_k) \subseteq (Z \times \{\delta \in \mathcal{M}_R(Z \times C) : \rho(\delta, X) \leq k^{-1}\}) \cup (Z \times \{1_{(\emptyset, c)} : c \in C\}).$$

Hence, $\text{supp}(\mu) \subseteq Z \times X_\emptyset$ as claimed.

It then follows that μ is a matching of E and that $\text{supp}(\mu) \subseteq IR(\mu; E)$ since $IR(\mu; E^*) \cap (Z \times X_\emptyset) \subseteq IR(\mu; E)$. Claim 9, which is analogous to part 4 of Lemma 6, shows that $\text{supp}(\mu) \subseteq S_M(\mu; E)$.

Claim 9 $\text{supp}(\mu) \subseteq S_M(\mu; E)$.

Proof. Let $(z, \delta) \in \text{supp}(\mu)$ and suppose that $(z, \delta) \notin S_M(\mu; E)$. Then there exists $\delta' \in X$ such that either (i) $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ and $U(z, m, \delta', \mu) > U(z, a(\delta), \delta, \mu)$, where $a(\delta) = m$ if $\delta \in X$ and $a(\delta) = s$ if $\delta \in X_\emptyset \setminus X$ (see Footnote 34), or (ii) there exists $(z', c) \in \text{supp}(\delta)$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu)$ and $U(z', m, \delta', \mu) > U(z', w, 1_{(z,c)}, \mu)$.

Consider case (i) first. Let $V_{\delta'}$, V_δ and V_μ be open neighborhoods of δ', δ and μ, respectively, such that $U(z, m, \gamma', \bar{\mu}) > U(z, a(\delta), \gamma, \bar{\mu})$ for each $\gamma' \in V_{\delta'}$, $\gamma \in V_\delta$ and $\bar{\mu} \in V_\mu$. Let, by the richness of E, $\tilde{V}_\delta$ and $\tilde{V}_\mu$ be open neighborhoods of δ and μ, respectively, such that $\Lambda(z, \gamma, \bar{\mu}) \cap V_{\delta'} \neq \emptyset$ for each $(\gamma, \bar{\mu}) \in \tilde{V}_\delta \times \tilde{V}_\mu$.

By Carmona and Podczeck (2009, Lemma 12), there is a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding sequence $\{\delta_{k_j}\}_{j=1}^\infty$ such that $\delta_{k_j} \to \delta$ and $(z, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$ for each $j \in \mathbb{N}$.

Let $J \in \mathbb{N}$ be such that $\mu_{k_j} \in V_\mu \cap \tilde{V}_\mu$ and $\delta_{k_j} \in V_\delta \cap \tilde{V}_\delta$ for all $j \geq J$ and, for each $j \geq J$, let $\delta'_{k_j} \in \Lambda(z, \delta_{k_j}, \mu_{k_j}) \cap V_{\delta'}$. Then, for each $j \geq J$,

$$U_{k_j}(z, m, \delta'_{k_j}, \mu_{k_j}) = U(z, m, \delta'_{k_j}, \mu_{k_j}) > U(z, a(\delta), \delta_{k_j}, \mu_{k_j}) \geq U_{k_j}(z, a(\delta), \delta_{k_j}, \mu_{k_j})$$

since $\delta'_{k_j} \in X$ by the definition of Λ, and $\text{supp}(\delta'_{k_j}) \subseteq T_z^m(\mu_{k_j}) \cup \text{supp}(\delta_{k_j})$. But this contradicts the stability of μ_{k_j}.

Now assume there exists $(z', c) \in \text{supp}(\delta)$ and $\delta' \in X$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu)$ and $U(z', m, \delta', \mu) > U(z', w, 1_{(z,c)}, \mu)$. Let $V_{\delta'}$, V_c and V_μ be open neighborhoods of δ', c and μ respectively, such that $U(z', m, \hat{\delta}', \hat{\mu}) > U(z', w, 1_{(z,\hat{c})}, \hat{\mu})$ for each $\hat{\delta}' \in V_{\delta'}$, $\hat{c} \in V_c$ and $\hat{\mu} \in V_\mu$. Let, by the richness of E, $\tilde{V}_\mu$ be an open neighborhood of μ such that $\Lambda_0(z', \hat{\mu}) \cap V_{\delta'} \neq \emptyset$ for each $\hat{\mu} \in \tilde{V}_\mu$.

By Carmona and Podczeck (2009, Lemma 12), there is a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding sequence $\{(\delta_{k_j}, c_{k_j})\}_{j=1}^\infty$ such that $(\delta_{k_j}, c_{k_j}) \to (\delta, c)$, $(z, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$ and $(z', c_{k_j}) \in \text{supp}(\delta_{k_j})$ for each $j \in \mathbb{N}$.

Let $J \in \mathbb{N}$ be such that $\delta_{k_j} \in V_\delta$, $c_{k_j} \in V_c$ and $\mu_{k_j} \in V_\mu \cap \tilde{V}_\mu$ for all $j \geq J$ and, for each $j \geq J$, let $\delta'_{k_j} \in \Lambda_0(z', \mu_{k_j}) \cap V_{\delta'}$. Then, for each $j \geq J$, $U_{k_j}(z', m, \delta'_{k_j}, \mu_{k_j}) = U(z', m, \delta'_{k_j}, \mu_{k_j}) > U(z', w, 1_{(z,c_{k_j})}, \mu_{k_j}) \geq U_{k_j}(z', w, 1_{(z,c_{k_j})}, \mu_{k_j})$ since $\delta'_{k_j} \in X$ by the definition of Λ_0, and $\text{supp}(\delta'_{k_j}) \subseteq T_{z'}^m(\mu_{k_j})$. But this contradicts the stability of μ_{k_j}. ■

It follows by $\text{supp}(\mu) \subseteq IR(\mu; E)$ and by Claim 9 that $\text{supp}(\mu) \subseteq S_M(\mu; E) \cap IR(\mu; E)$. Thus, μ is stable. ■

We now complete the proof of our existence result.

Proof of Theorem 2. Let $\{\nu_k\}_{k=1}^\infty$ be such that $\nu_k \to \nu$ and $\text{supp}(\nu_k)$ is a finite subset of Z for each $k \in \mathbb{N}$. Define $Z_k = \text{supp}(\nu_k)$, $Z_{\emptyset,k} = Z_k \cup \{\emptyset\}$, $X_k = X \cap \mathcal{M}(Z_k \times C)$, $X_{\emptyset,k} = X_k \cup \{1_{(\emptyset,c)} : c \in C\}$, $X_{m,k} = X_k$, $X_{s,k} = \{1_{(\emptyset,c)} : c \in C\}$, $X_{w,k} = \{1_{(z,c)} : (z,c) \in Z_k \times C\}$ and $\Delta_k = \{(a, \delta) : \delta \in X_{a,k}\}$ for each $k \in \mathbb{N}$. Note that X_k is closed for each $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$, let $\tilde{E}_k = (Z_k, \nu_k, C, \mathbb{C}, X_k, (\succ_z)_{z \in Z_k})$ be a market where $\succ_z$ is restricted to $\Delta_k \times \mathcal{M}(Z_k \times X_{\emptyset,k})$ for each $z \in Z$. Furthermore, let E_k be exactly as $\tilde{E}_k$, except with X in place of X_k and Z in place of Z_k; more precisely, $E_k = (Z, \nu_k, C, \mathbb{C}, X, (\succ_z)_{z \in Z})$.

Claim 10 *For each $k \in \mathbb{N}$, if μ is a stable matching of $\tilde{E}_k$, then μ is a stable matching of E_k.*

Proof. In this proof, to avoid confusion, we write $IR(\mu; E)$ for $IR(\mu)$ and $S_M(\mu; E)$ for $S_M(\mu)$ whenever μ is a matching of a market E.

Let $k \in \mathbb{N}$ and μ be a stable matching of $\tilde{E}_k$. Clearly, μ is a matching of E_k and $\text{supp}(\mu) \subseteq IR(\mu; E_k)$. We show that $\text{supp}(\mu) \subseteq S_M(\mu; E_k)$. Suppose not; then let $(z, \delta) \in \text{supp}(\mu) \setminus S_M(\mu; E_k)$.

First suppose that there exists $\delta' \in X$ such that $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ and $(m, \delta', \mu) \succ_z (a(\delta), \delta, \mu)$, where $a(\delta) = m$ if $\delta \in X$ and $a(\delta) = s$ if $\delta \in X_\emptyset \setminus X$. We claim that $\delta' \in X_k$, i.e. that $\text{supp}(\delta') \subseteq Z_k \times C$, from which we obtain a contradiction to the stability of μ in $\tilde{E}_k$.

Note that $\text{supp}(\bar{\delta}) \subseteq Z_k \times C$ whenever $\bar{\delta} \in X$ and $(\bar{z}, \bar{\delta}) \in \text{supp}(\mu)$ for some $\bar{z} \in Z_k$ since μ is stable in $\tilde{E}_k$. Thus, it follows that $\text{supp}(\delta') \cap \text{supp}(\delta) \subseteq Z_k \times C$ since if $\text{supp}(\delta') \cap \text{supp}(\delta) \neq \emptyset$, then $\delta \in X$. We also have that $\text{supp}(\delta') \cap T_z^m(\mu) \subseteq Z_k \times C$. Indeed, if $(z', c) \in T_z^m(\mu)$, then $(z', \bar{c}) \in \text{supp}(\bar{\delta})$ and $(\bar{z}, \bar{\delta}) \in \text{supp}(\mu)$ for some $\bar{c} \in C$, $\bar{z} \in Z_k$ and $\bar{\delta} \in X$ whenever $\text{supp}(\delta') \cap T_z^m(\mu) \neq \emptyset$; hence, $z' \in Z_k$. Thus, $\text{supp}(\delta') = (\text{supp}(\delta') \cap \text{supp}(\delta)) \cup (\text{supp}(\delta') \cap T_z^m(\mu)) \subseteq Z_k \times C$ as desired.

Now suppose that there exists $\delta' \in X$ and $(z', c) \in \text{supp}(\delta)$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu)$ and $(m, \delta', \mu) \succ_{z'} (w, 1_{(z,c)}, \mu)$. As above, we obtain a contradiction to the stability of μ in $\tilde{E}_k$ by showing that $\delta' \in X_k$. To establish this claim, it suffices to show that $T_{z'}^m(\mu) \subseteq Z_k \times C$. If $(\tilde{z}, \tilde{c}) \in T_{z'}^m(\mu)$, then $(\tilde{z}, \bar{c}) \in \text{supp}(\bar{\delta})$ and $(\bar{z}, \bar{\delta}) \in \text{supp}(\mu)$ for some $\bar{c} \in C$, $\bar{z} \in Z_k$ and $\bar{\delta} \in X$; hence, $(\tilde{z}, \tilde{c}) \in Z_k \times C$ as required. ■

For each $k \in \mathbb{N}$, let $\mu_k \in \mathcal{M}(Z \times X_{\emptyset,k})$ be a stable matching in E_k, which exists by Lemma 8 (since Z_k is finite and $\tilde{E}_k$ satisfies its assumptions) and Claim 10.[35]

[35] It is clear that $\tilde{E}_k$ is rational, continuous and bounded. It can be shown that $\tilde{E}_k$ is rich along the lines of Claim 10: Let $\Lambda_k : Z_k \times X_k \times \mathcal{M}(Z_k \times X_{\emptyset,k}) \rightrightarrows X_k$ be defined by setting, for each $(z, \delta, \mu) \in Z_k \times X_k \times \mathcal{M}(Z_k \times X_{\emptyset,k})$, $\Lambda_k(z, \delta, \mu) = \{\delta' \in X_k : \text{supp}(\delta') \subseteq \text{supp}(\delta) \cup T_z^m(\mu; \tilde{E}_k)\}$. Let $(z, \delta, \mu) \in Z_k \times X_k \times \mathcal{M}(Z_k \times X_{\emptyset,k})$ and O be open and such that $\Lambda_k(z, \delta, \mu) \cap O \neq \emptyset$. Since $T_z(\mu; \tilde{E}_k) \subseteq T_z(\mu; E)$, it follows that $\Lambda(z, \delta, \mu) \cap O \neq \emptyset$. Hence, there is an open neighborhood V of (z, δ, μ) such that, for each $(\tilde{z}, \tilde{\delta}, \tilde{\mu}) \in V$, $\Lambda(\tilde{z}, \tilde{\delta}, \tilde{\mu}) \cap O \neq \emptyset$. Thus, for each $(\tilde{z}, \tilde{\delta}, \tilde{\mu}) \in V \cap (Z_k \times X_k \times \mathcal{M}(Z_k \times X_{\emptyset,k}))$, let $\delta' \in \Lambda(\tilde{z}, \tilde{\delta}, \tilde{\mu}) \cap O$. We have that $\text{supp}(\tilde{\delta}) \subseteq Z_k \times C$ and that $T_{\tilde{z}}^m(\tilde{\mu}; E) \subseteq (Z_k \times C) \cap T_{\tilde{z}}^m(\tilde{\mu}; \tilde{E}_k)$. Hence, $\delta' \in X_k$ and $\text{supp}(\delta') \subseteq \text{supp}(\tilde{\delta}) \cup T_{\tilde{z}}^m(\tilde{\mu}; \tilde{E}_k)$ and it follows that $\delta' \in \Lambda_k(\tilde{z}, \tilde{\delta}, \tilde{\mu}) \cap O$. Thus, Λ_k is lower hemicontinuous and an analogous argument shows that $\Lambda_{0,k}$ is also lower hemicontinuous.

It follows by part 1 of Lemma 6 that we may assume that $\{\mu_k\}_{k=1}^{\infty}$ converges; let $\mu = \lim_k \mu_k$. It then follows by parts 2 – 4 of Lemma 6 that μ is a matching and that $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$. Hence, μ is stable. ■

A.4 Proof of Theorem 3

Theorem 3 characterizes the stable matching of a marriage market E in terms of the stable matchings of its associated market $\hat{E}$ and vice versa. A matching of $\hat{E}$ is a measure on $Z \times X_\emptyset$ but a matching in E is a measure on $M_\emptyset \times W_\emptyset \times C$. The main difference arises due to how we represent unmatched individuals of type $z \in W$. This is done by a match $(z, 1_{(\emptyset,c)})$ in $\hat{E}$ and by a match $(\emptyset, z, c)$ in E. Therefore, we need to transform the former into the latter to obtain a matching in E from a matching in $\hat{E}$ and vice versa.

Let $\hat{\mu}$ be a stable matching of $\hat{E}$ and note that $\hat{\mu}(W \times X) = 0$. Indeed, if $\hat{\mu}(W \times X) > 0$, then there is $(z, \delta) \in (W \times X) \cap \text{supp}(\hat{\mu})$. Letting $c \in \hat{\mathbb{C}}(z, \emptyset) = \mathbb{C}(\emptyset, z)$, it follows that $\text{supp}(1_{(\emptyset,c)}) \subseteq T_z^s(\hat{\mu})$ and $(s, 1_{(\emptyset,c)}) \hat{\succ}_z (m, \delta)$. But this implies that $(z, \delta) \notin S(\hat{\mu})$, contradicting the stability of $\hat{\mu}$.

The above implies that $\text{supp}(\hat{\mu}) \subseteq (M \times X_\emptyset) \cup (W \times (X_\emptyset \setminus X))$. Recall that

$$Y = (M \times X_\emptyset) \cup (W \times (X_\emptyset \setminus X)) \text{ and } Y' = (M \times W_\emptyset \times C) \cup (\{\emptyset\} \times W \times C)$$

and let $g : W_\emptyset \times C \to X_\emptyset$ defined by $g(z, c) = 1_{(z,c)}$ be the standard homeomorphism between $W_\emptyset \times C$ and $X_\emptyset$. To transform a matching $\hat{\mu}$ of $\hat{E}$ into a matching μ of E, we use the function $h : Y \to Y'$ defined, by setting for each $(z, \delta) \in Y$,

$$h(z, \delta) = \begin{cases} (z, g^{-1}(\delta)) & \text{if } z \in M \text{ and } \delta \in X_\emptyset, \\ (\emptyset, z, c) & \text{if } z \in W, \delta \in X_\emptyset \setminus X, \delta = 1_{(\emptyset,c)} \text{ and } c \in C. \end{cases}$$

Then h is continuous because M and W are both open and closed and $c_k \to c$ whenever $1_{(\emptyset,c_k)} \to 1_{(\emptyset,c)}$. Thus, $\mu = \hat{\mu} \circ h^{-1}$ is a measure on $Y' \subseteq M_\emptyset \times W_\emptyset \times C$.

The function h is actually an homeomorphism between Y and Y', its inverse being

$f : Y' \to Y$ defined, by setting, for each $(z, z', c) \in Y'$,

$$f(z, z', c) = \begin{cases} (z, g(z', c)) & \text{if } z \in M, \\ (z', 1_{(\emptyset, c)}) & \text{if } z = \emptyset \text{ and } (z', c) \in W \times C. \end{cases}$$

Since $\text{supp}(\mu) \subseteq Y'$ for each matching μ of E, $\hat{\mu} = \mu \circ f^{-1}$ is a measure on $Y \subseteq Z \times X_\emptyset$.

Note first that to establish Theorem 3 it suffices to show that: (1) if $\hat{\mu}$ is a stable matching of $\hat{E}$, then $\hat{\mu} \circ h^{-1}$ is a stable matching of E; and (2) if μ is a stable matching of E, then $\mu \circ f^{-1}$ is a stable matching of $\hat{E}$. Indeed, (2) implies that $\{\mu \circ f^{-1} : \mu \in \mathcal{S}\} \subseteq \hat{\mathcal{S}}$. For the converse, let $\hat{\mu} \in \hat{\mathcal{S}}$ and note that (1) implies that $\hat{\mu} \circ h^{-1} \in \mathcal{S}$. Then $\hat{\mu} \in \{\mu \circ f^{-1} : \mu \in \mathcal{S}\}$ since $\hat{\mu} = \hat{\mu} \circ (f \circ h)^{-1} = (\hat{\mu} \circ h^{-1}) \circ f^{-1}$. An analogous argument shows that $\mathcal{S} = \{\hat{\mu} \circ h^{-1} : \hat{\mu} \in \hat{\mathcal{S}}\}$.

We turn to the proof of (1). Let $\hat{\mu} \in \hat{\mathcal{S}}$ and $\mu = \hat{\mu} \circ h^{-1}$. Recall that we have already shown that $\hat{\mu}(W \times X) = 0$.

We now show that μ is a matching of E. For condition (M1), let B be a Borel subset of M and note that $h^{-1}(B \times W_\emptyset \times C) = B \times X_\emptyset$ and that $\int_{Z \times X} \delta(B \times C) \mathrm{d}\hat{\mu}(z, \delta) = 0$, the latter since $\delta \in X = \{1_{(z,c)} : (z, c) \in W \times C\}$ and $B \cap W \subseteq M \cap W = \emptyset$ implies $\delta(B \times C) = 0$. Hence,

$$\begin{aligned} &\mu(B \times W_\emptyset \times C) = \hat{\mu}(h^{-1}(B \times W_\emptyset \times C)) = \hat{\mu}(B \times X_\emptyset) \\ &= \hat{\mu}(B \times X) + \hat{\mu}(B \times (X_\emptyset \setminus X)) + \int_{Z \times X} \delta(B \times C) \mathrm{d}\hat{\mu}(z, \delta) = \nu(B) = \nu_M(B). \end{aligned}$$

For condition (M2), let B be a Borel subset of W and note that $h^{-1}(M_\emptyset \times B \times C) = (M \times \{1_{(z,c)} : (z, c) \in B \times C\}) \cup (B \times (X_\emptyset \setminus X))$. Thus, using $\hat{\mu}(W \times X) = 0$,

$$\begin{aligned} &\mu(M_\emptyset \times B \times C) = \hat{\mu}(M \times \{1_{(z,c)} : (z, c) \in B \times C\}) + \hat{\mu}(B \times (X_\emptyset \setminus X)) \\ &= \hat{\mu}(Z \times \{1_{(z,c)} : (z, c) \in B \times C\}) + \hat{\mu}(B \times (X_\emptyset \setminus X)) \\ &= \int_{Z \times X} \delta(B \times C) \mathrm{d}\hat{\mu}(z, \delta) + \hat{\mu}(B \times (X_\emptyset \setminus X)) + \hat{\mu}(B \times X) = \nu(B) = \nu_W(B). \end{aligned}$$

For condition (M3), let $(m, w, c) \in \text{supp}(\mu)$ and, since $\text{supp}(\mu) = h(\text{supp}(\hat{\mu}))$ by Lemma 1, let $(z, \delta) \in \text{supp}(\hat{\mu})$ be such that $(m, w, c) = h(z, \delta)$. If $m \in M$, then $z = m$, $\delta = 1_{(w,c)}$ and thus $c \in \hat{\mathbb{C}}(m, w) = \mathbb{C}(m, w)$. If $m = \emptyset$, then $z = w$, $\delta = 1_{(\emptyset,c)}$

and $c \in \hat{\mathbb{C}}(w, \emptyset) = \mathbb{C}(\emptyset, w)$. Finally, condition (M4) holds since $h^{-1}(\{(m, w, c) : m = w = \emptyset\}) = \emptyset$.

Before establishing the stability of μ, we note that the set I in GK is defined as a union of open subsets of $(M_\emptyset \times W_\emptyset \times C)^2$. For convenience, we write this definition as follows. Let $I = I_B \cup I_{IN} \cup I_{IR}$, where

$$\begin{aligned}
I_B = \; & \Big\{(m, w, c, m', w', c') \in (M_\emptyset \times W_\emptyset \times C)^2 : (m', c'') \succ_w (m, c) \\
& \text{ and } (w, c'') \succ_{m'} (w', c') \text{ for some } c'' \in \mathbb{C}(m', w)\Big\} \cup \\
& \Big\{(m, w, c, m', w', c') \in (M_\emptyset \times W_\emptyset \times C)^2 : (m, c'') \succ_{w'} (m', c') \\
& \text{ and } (w', c'') \succ_m (w, c) \text{ for some } c'' \in \mathbb{C}(m, w')\Big\}, \\
I_{IN} = \; & \Big\{(m, w, c, m', w', c') \in (M_\emptyset \times W_\emptyset \times C)^2 : (m, c'') \succ_w (m, c) \\
& \text{ and } (w, c'') \succ_m (w, c) \text{ for some } c'' \in \mathbb{C}(m, w)\Big\} \cup \\
& \Big\{(m, w, c, m', w', c') \in (M_\emptyset \times W_\emptyset \times C)^2 : (m', c'') \succ_{w'} (m', c') \\
& \text{ and } (w', c'') \succ_{m'} (w', c') \text{ for some } c'' \in \mathbb{C}(m', w')\Big\}, \text{ and} \\
I_{IR} = \; & \Big\{(m, w, c, m', w', c') \in (M_\emptyset \times W_\emptyset \times C)^2 : (\emptyset, c'') \succ_w (m, c) \\
& \text{ for some } c'' \in \mathbb{C}(\emptyset, w)\} \cup \Big\{(m, w, c, m', w', c') \in (M_\emptyset \times W_\emptyset \times C)^2 : \\
& (\emptyset, c'') \succ_{w'} (m', c') \text{ for some } c'' \in \mathbb{C}(\emptyset, w')\Big\} \cup \\
& \Big\{(m, w, c, m', w', c') \in (M_\emptyset \times W_\emptyset \times C)^2 : (\emptyset, c'') \succ_m (w, c) \\
& \text{ for some } c'' \in \mathbb{C}(m, \emptyset)\Big\} \cup \Big\{(m, w, c, m', w', c') \in (M_\emptyset \times W_\emptyset \times C)^2 : \\
& (\emptyset, c'') \succ_{m'} (w', c') \text{ for some } c'' \in \mathbb{C}(m', \emptyset)\Big\}.
\end{aligned}$$

Let S_{IN} be the complement of $\{(x, y, c) \in M_\emptyset \times W_\emptyset \times C : (x, c') \succ_y (x, c) \text{ and } (y, c') \succ_x (y, c) \text{ for some } c' \in \mathbb{C}(x, y)\}$. It is straightforward to see that

$$I_{IN} = (S_{IN}^c \times (M_\emptyset \times W_\emptyset \times C)) \cup ((M_\emptyset \times W_\emptyset \times C) \times S_{IN}^c).$$

Hence, writing μ^2 for $\mu \otimes \mu$, $\mu^2(I_{IN}) = 0$ if and only if $\mu(S_{IN}^c) = 0$, which holds if and only if $\text{supp}(\mu) \subseteq S_{IN}$.

Analogously, let S_{IR} be the complement of $\{(x, y, c) \in M_\emptyset \times W_\emptyset \times C : (\emptyset, c') \succ_y (x, c) \text{ for some } c' \in \mathbb{C}(\emptyset, y)\} \cup \{(x, y, c) \in M_\emptyset \times W_\emptyset \times C : (\emptyset, c') \succ_x (y, c) \text{ for some } c' \in$

$\mathbb{C}(x, \emptyset)\}$. We then have that

$$I_{IR} = (S_{IR}^c \times (M_\emptyset \times W_\emptyset \times C)) \cup ((M_\emptyset \times W_\emptyset \times C) \times S_{IR}^c) \,.$$

Hence, $\mu^2(I_{IR}) = 0$ if and only if $\mu(S_{IR}^c) = 0$, which holds if and only if $\text{supp}(\mu) \subseteq S_{IR}$.

We now show that μ is stable. Let $(x, y, c, x', y', c') \in \text{supp}(\mu^2)$. It follows by Lemma 3 that both (x, y, c) and (x', y', c') belong to $\text{supp}(\mu)$. Thus, $(x, 1_{(y,c)}) \in \text{supp}(\hat{\mu})$ if $x \in M$ and $(y, 1_{(\emptyset,c)}) \in \text{supp}(\hat{\mu})$ if $x = \emptyset$.

Suppose, in order to reach a contradiction, that $(x, y, c, x', y', c') \in I_B$. If $(x', c'') \succ_y (x, c)$ and $(y, c'') \succ_{x'} (y', c')$ for some $c'' \in \mathbb{C}(x', y)$, it follows that $y \in W$ and $x' \in M$ because $\succ_\emptyset$ is empty and that $(x', c'') \in T_y^w(\hat{\mu})$ by the latter condition together with $(x', y', c') \in \text{supp}(\mu) \Leftrightarrow (x', 1_{(y',c')}) \in \text{supp}(\hat{\mu})$; then the former condition implies that $(x, 1_{(y,c)}) \notin S(\hat{\mu})$ if $x \in M$ and $(y, 1_{(\emptyset,c)}) \notin S(\hat{\mu})$ if $x = \emptyset$.

If $(x, c'') \succ_{y'} (x', c')$ and $(y', c'') \succ_x (y, c)$ for some $c'' \in \mathbb{C}(x, y')$, it follows that $x \in M$, $y' \in W$ and $(y', c'') \in T_x^m(\hat{\mu})$ by the former condition together with $(x', y', c') \in \text{supp}(\mu)$, i.e. $(x', 1_{(y',c')}) \in \text{supp}(\hat{\mu})$ if $x' \in M$ and $(y', 1_{(\emptyset,c')}) \in \text{supp}(\hat{\mu})$ if $x' \in M$; then the latter condition implies that $(x, 1_{(y,c)}) \notin S(\hat{\mu})$. In either case we reach a contradiction, and thus $(x, y, c, x', y', c') \in I_B^c$. It then follows that $\text{supp}(\mu^2) \subseteq I_B^c$ and, thus, $\mu^2(I_B) = 0$.

Suppose next that $(x, y, c) \in S_{IN}^c$. Then there exists $c'' \in \mathbb{C}(x, y)$ such that $(x, c'') \succ_y (x, c)$ and $(y, c'') \succ_x (y, c)$; in particular $x \in M$ and $y \in W$ since $\succ_\emptyset$ is empty. Since $(x, 1_{(y,c)}) \in \text{supp}(\hat{\mu})$, it follows by the former condition that $(y, c'') \in T_x^m(\hat{\mu})$. Then the latter condition shows that $(x, 1_{(y,c)}) \notin S(\hat{\mu})$, a contradiction. It then follows that $\text{supp}(\mu) \subseteq S_{IN}$ and, thus, $\mu^2(I_{IN}) = 0$.

Finally suppose that $(x, y, c) \in S_{IR}^c$ and that $(\emptyset, c'') \succ_y (x, c)$ for some $c'' \in \mathbb{C}(\emptyset, y)$ (the case where $(\emptyset, c'') \succ_x (y, c)$ for some $c'' \in \mathbb{C}(x, \emptyset)$ being analogous). Then $(\emptyset, c'') \in T_y^s(\hat{\mu})$ and $(\emptyset, c'') \succ_y (x, c)$ implies that $(x, 1_{(y,c)}) \notin S(\hat{\mu})$ if $x \in M$ and $(y, 1_{(\emptyset,c)}) \notin S(\hat{\mu})$ if $x = \emptyset$, a contradiction. It then follows that $\text{supp}(\mu) \subseteq S_{IR}$ and, thus, $\mu^2(I_{IR}) = 0$.

It follows from what has been shown above that $0 \leq \mu^2(I) \leq \mu^2(I_B) + \mu^2(I_{IN}) + \mu^2(I_{IR}) = 0$ and, therefore, μ is stable.

We turn to the proof of (2). Let $\mu \in \mathcal{S}$ and $\hat{\mu} = \mu \circ f^{-1}$. We start by showing that $\hat{\mu}$ is a matching. For condition 1, let $(z, \delta) \in \text{supp}(\hat{\mu}) \subseteq Z \times X_\emptyset$ and $(z', c) \in \text{supp}(\delta)$. Then $h(z, \delta) \in \text{supp}(\mu)$ by Lemma 1. Thus, if $z \in M$, then $(z, z', c) \in \text{supp}(\mu)$ and, therefore, $c \in \mathbb{C}(z, z') = \hat{\mathbb{C}}(z, z')$. If $z \in W$, then $(\emptyset, z, c) = h(z, \delta) \in \text{supp}(\mu)$. Thus, $c \in \mathbb{C}(\emptyset, z) = \hat{\mathbb{C}}(z, \emptyset)$. In either case, it follows that $\{z\} \times \text{supp}(\delta) \subseteq \text{graph}(\hat{\mathbb{C}})$.

For condition 2, let B be a Borel subset of Z. We have that $\hat{\mu}(B \times X) = \mu(f^{-1}(B \times X)) = \mu((B \cap M) \times W \times C)$ and that $\hat{\mu}(B \times (X_\emptyset \setminus X)) = \mu(f^{-1}(B \times (X_\emptyset \setminus X))) = \mu((B \cap M) \times \{\emptyset\} \times C)) + \mu(\{\emptyset\} \times (B \cap W) \times C)$. Moreover, $\int_{Z \times X} \delta(B \times C) \mathrm{d}\hat{\mu}(z, \delta) = \mu(M \times (B \cap W) \times C)$ since, for each $(z, \delta) \in \text{supp}(\hat{\mu}) \cap (Z \times X)$, $\delta = 1_{(z',c)}$ for some $(z', c) \in W \times C$ and $1_{(z',c)}(B \times C) = 1$ if and only if $z' \in B$. Thus,

$$\begin{aligned}
&\hat{\mu}(B \times X) + \int_{Z \times X} \delta(B \times C) \mathrm{d}\hat{\mu}(z, \delta) + \hat{\mu}(B \times (X_\emptyset \setminus X)) \\
&= \mu((B \cap M) \times W_\emptyset \times C) + \mu(M_\emptyset \times (B \cap W) \times C) \\
&= \nu_M(B \cap M) + \nu_W(B \cap W) = \nu(B).
\end{aligned}$$

We now show that $\hat{\mu}$ is stable. Since μ is stable, then $\mu^2(I) = 0$ and, hence, that $\text{supp}(\mu^2) \subseteq I^c$ since I is open. Note that $I^c = I_B^c \cap I_{IN}^c \cap I_{IR}^c$.

Let $(x, 1_{(y,c)}) \in \text{supp}(\hat{\mu})$ and first suppose that $x \in M$. Then $(x, y, c) \in \text{supp}(\mu)$. Suppose, in order to reach a contradiction, that $(x, 1_{(y,c)}) \notin S_M(\hat{\mu}) \cap IR(\hat{\mu})$. Then $(y', c'') \succ_x (y, c)$ for some $(y', c'') \in W_\emptyset \times C$ such that $(y', c'') \in T_x^m(\hat{\mu})$ if $y' \in W$ and $(y', c'') \in T_x^s(\hat{\mu})$ if $y' = \emptyset$.

If $y' \in W$, then $(y', c'') \in T_x^m(\hat{\mu})$ implies that $c'' \in \mathbb{C}(x, y')$ and that there is $(x', c') \in M_\emptyset \times C$ such that $(x', y', c') \in \text{supp}(\mu)$ and $(x, c'') \succ_{y'} (x', c')$. Therefore, $(x, y, c, x', y', c') \in \text{supp}(\mu)^2 = \text{supp}(\mu^2)$ by Lemma 3 and $(x, y, c, x', y', c') \in I_B$, a contradiction. If $y' = \emptyset$, then $c'' \in \mathbb{C}(x, \emptyset)$ and $(\emptyset, c'') \succ_x (y, c)$, implying that $(x, y, c) \in S_{IR}^c$ and, thus, $(x, y, c, x, y, c) \in \text{supp}(\mu^2) \cap I_{IR}$, a contradiction.

Thus, we reach a contradiction in either case and it follows that $(x, 1_{(y,c)}) \in S_M(\hat{\mu}) \cap IR(\hat{\mu})$.

Now let $(x, 1_{(y,c)}) \in \text{supp}(\hat{\mu})$ and suppose that $x \in W$, which implies $y = \emptyset$. Then $(\emptyset, x, c) \in \text{supp}(\mu)$. Suppose for a contradiction that $(x, 1_{(\emptyset,c)}) \notin S_M(\hat{\mu}) \cap IR(\hat{\mu})$. Note that $(x, 1_{(\emptyset,c)}) \in S_M(\hat{\mu})$ since $(s, 1_{(\emptyset,c)}) \hat{\succ}_x (m, \tilde{\delta})$ for all $\tilde{\delta} \in X_m$. If $(x, 1_{(\emptyset,c)}) \notin IR(\hat{\mu})$, then $(\emptyset, c') \succ_x (\emptyset, c)$ for some $c' \in \mathbb{C}(\emptyset, x)$, implying that $(\emptyset, x, c) \in S_{IR}^c$ and, thus, $(\emptyset, x, c, \emptyset, x, c) \in \text{supp}(\mu^2) \cap I_{IR}$, a contradiction. Thus, $(x, 1_{(y,c)}) \in S_M(\hat{\mu}) \cap IR(\hat{\mu})$.

Hence, $\text{supp}(\hat{\mu}) \subseteq S_M(\hat{\mu}) \cap IR(\hat{\mu})$ and $\hat{\mu}$ is stable by Theorem 1.

A.5 Proof of Corollary 3

Let E be a Rosen market. For each $k \in \mathbb{N}$, let $\mathbb{C}_k \equiv [0, k]$, $X_k = \{n1_{(z,c)} : (z, c) \in Z \times C \text{ and } n \in [0, k]\}$ and E_k be equal to E except for these changes to $\mathbb{C}_k$ and X_k. It follows by Theorem 2 that there exists a stable matching μ_k of E_k.

Claim 11 $\text{supp}(\mu_k) \subseteq Z \times X$ *for each* $k \in \mathbb{N}$.

Proof. Suppose not; then let $(z, \delta) \in \text{supp}(\mu_k) \cap (Z \times (X_\emptyset \backslash X))$. Let $\varepsilon > 0$ be such that $g(r(z))q(z)\theta\left(\frac{r(z)}{q(z)}\right) - \varepsilon > 0$. Then $(z, \varepsilon) \in T_z^m(\mu_k)$ since $(z, \delta) \in \text{supp}(\mu)$ and $U_z(w, 1_{(z,\varepsilon)}) = \varepsilon > 0 = U_z(s, \delta)$. Thus, letting $\delta' = 1_{(z,\varepsilon)}$, it follows that $\text{supp}(\delta') \subseteq T_z^m(\mu_k)$ and $U_z(m, \delta') = g(r(z))q(z)\theta\left(\frac{r(z)}{q(z)}\right) - \varepsilon > 0 = U_z(s, \delta)$. Hence, $(z, \delta) \notin S(\mu_k)$, a contradiction to the stability of μ_k. ■

Claim 12 *There exist* $K, M \in \mathbb{N}$ *such that, for each* $k \geq K$ *and* $(z, \delta) \in \text{supp}(\mu_k)$, $\delta(Z \times C) \leq M$ *and* $\delta(Z \times ([0, \frac{1}{M}) \cup (M, \infty))) = 0$.

Proof. Suppose not; then, for each $j \in \mathbb{N}$, there exists $k_j \geq j$ and $(z_{k_j}, \delta_{k_j}) \in \text{supp}(\mu_{k_j}) \subseteq Z \times X$ such that $\delta_{k_j}(Z \times C) > j$ or $\delta_{k_j}(Z \times ([0, \frac{1}{j}) \cup (j, \infty))) > 0$.

Suppose first that $\delta_{k_j}(Z \times C) > j$ holds for infinitely many js. Taking a subsequence if needed, we may assume that $\delta_{k_j}(Z \times C) > j$ holds for each j. Thus, for some $(z'_{k_j}, c_{k_j}, n_{k_j}) \in Z \times C \times [0, k_j]$, $\delta_{k_j} = n_{k_j}1_{(z'_{k_j}, c_{k_j})}$ with $n_{k_j} > j$. We have that

$$\begin{aligned} U_{z_{k_j}}(m, n_{k_j}1_{(z'_{k_j}, c_{k_j})}) &\leq g(r(\bar{z}))f(r(\bar{z}), n_{k_j}q(\bar{z})) - c_{k_j}n_{k_j} \\ &= \left[g(r(\bar{z}))q(\bar{z})\theta\left(\frac{r(\bar{z})}{n_{k_j}q(\bar{z})}\right) - c_{k_j}\right] n_{k_j}. \end{aligned}$$

Since μ_{k_j} is stable, it follows that $U_{z_{k_j}}(m, n_{k_j}1_{(z'_{k_j}, c_{k_j})}) \geq 0$ for each j; hence,

$$0 \leq c_{k_j} \leq g(r(\bar{z}))q(\bar{z})\theta\left(\frac{r(\bar{z})}{n_{k_j}q(\bar{z})}\right).$$

Since $n_{k_j} \to \infty$, it follows that $g(r(\bar{z}))q(\bar{z})\theta\left(\frac{r(\bar{z})}{n_{k_j}q(\bar{z})}\right) \to 0$ and, hence, $c_{k_j} \to 0$. Since $g(r(\underline{z}))q(\underline{z})\theta\left(\frac{r(\underline{z})}{q(\underline{z})}\right) > 0$, let $\varepsilon > 0$ be such that

$$g(r(\underline{z}))q(\underline{z})\theta\left(\frac{r(\underline{z})}{q(\underline{z})}\right) - \varepsilon > 0.$$

We have that $(z'_{k_j}, c_{k_j} + \varepsilon) \in T^m_{z'_{k_j}}(\mu_{k_j})$ for each j and that

$$U_{z'_{k_j}}(m, 1_{(z'_{k_j}, c_{k_j}+\varepsilon)}) \geq g(r(\underline{z}))q(\underline{z})\theta\left(\frac{r(\underline{z})}{q(\underline{z})}\right) - c_{k_j} - \varepsilon > c_{k_j}$$

for all j sufficiently large. But this contradicts the stability of μ_{k_j}.

It follows from what has been shown above that $\delta_{k_j}(Z \times ([0, \frac{1}{j}) \cup (j, \infty))) > 0$ holds for each j sufficiently large. Thus, for some $(z'_{k_j}, c_{k_j}, n_{k_j}) \in Z \times C \times [0, k_j]$, $\delta_{k_j} = n_{k_j} 1_{(z'_{k_j}, c_{k_j})}$ with $c_{k_j} > j$ or $c_{k_j} < \frac{1}{j}$. First, suppose that $c_{k_j} < \frac{1}{j}$ holds for infinitely many js. Note that $(z'_{k_j}, \frac{1}{j}) \in T^m_{z'_{k_j}}(\mu_{k_j})$ and

$$U_{z'_{k_j}}(m, 1_{(z'_{k_j}, \frac{1}{j})}) \geq g(r(\underline{z}))q(\underline{z})\theta\left(\frac{r(\underline{z})}{q(\underline{z})}\right) - \frac{1}{j} > \frac{1}{j} > c_{k_j}$$

for j sufficiently large, contradicting the stability of μ_{k_j}.

Now suppose that $c_{k_j} > j$ for all j sufficiently large. Since μ_{k_j} is stable, we then have that

$$0 \leq U_{z_{k_j}}(m, n_{k_j} 1_{(z'_{k_j}, c_{k_j})}) \leq \left[g(r(\bar{z}))q(\bar{z})\theta\left(\frac{r(\bar{z})}{n_{k_j}q(\bar{z})}\right) - c_{k_j}\right] n_{k_j}.$$

Thus, $n_{k_j} \to 0$ as $c_{k_j} \to \infty$ and, hence,

$$U_{z_{k_j}}(m, n_{k_j} 1_{(z'_{k_j}, c_{k_j})}) \leq g(r(z_{k_j}))f(r(z_{k_j}), n_{k_j}q(z_{k_j})) \to 0.$$

Let $\varepsilon > 0$ be such that

$$g(r(\underline{z}))q(\underline{z})\theta\left(\frac{r(\underline{z})}{q(\underline{z})}\right) - \varepsilon > 0.$$

We have that $(z_{k_j}, \varepsilon) \in T^m_{z_{k_j}}(\mu_{k_j})$ and that

$$U_{z_{k_j}}(m, 1_{(z_{k_j}, \varepsilon)}) \geq g(r(\underline{z}))q(\underline{z})\theta\left(\frac{r(\underline{z})}{q(\underline{z})}\right) - \varepsilon > 0$$

for all j sufficiently large. But this contradicts the stability of μ_{k_j}. ■

Claim 12 implies that, for each $k \geq K$, the payoff of a manager in μ_k is bounded above by $\max_{n\in[0,M]} g(r(\bar{z}))f(r(\bar{z}), nq(\bar{z})) = \max_{n\in[0,M]} g(r(\bar{z}))nq(\bar{z})\theta\left(\frac{r(\bar{z})}{nq(\bar{z})}\right)$. In addition, the payoff of a manager is bounded below by $\frac{1}{2}g(r(\underline{z}))q(\underline{z})\theta\left(\frac{r(\underline{z})}{q(\underline{z})}\right)$, since if $(z,\delta) \in \text{supp}(\mu_k)$ and $U_z(m,\delta) < \frac{1}{2}g(r(\underline{z}))q(\underline{z})\theta\left(\frac{r(\underline{z})}{q(\underline{z})}\right)$, then, letting $\varepsilon > 0$ be such that $g(r(\underline{z}))q(\underline{z})\theta\left(\frac{r(\underline{z})}{q(\underline{z})}\right) - \varepsilon > 2U_z(m,\delta)$, it follows that $(z, U_z(m,\delta)+\varepsilon) \in T_z^m(\mu_k)$ and

$$\begin{aligned} U_z(m, 1_{(z,U_z(m,\delta)+\varepsilon)}) &= g(r(z))f(r(z), q(z)) - U_z(m,\delta) - \varepsilon \\ &\geq g(r(\underline{z}))q(\underline{z})\theta\left(\frac{r(\underline{z})}{q(\underline{z})}\right) - U_z(m,\delta) - \varepsilon > U_z(m,\delta), \end{aligned}$$

which contradicts the stability of μ_k.

The payoff of a worker in μ_k is bounded below by $\frac{1}{M}$; since by Claim 11 there is no unemployment, it follows that

$$\min\{U_z(m, n1_{(z',c)}), U_{z'}(w, 1_{(z,c)})\} \geq \min\left\{\frac{1}{M}, \frac{1}{2}g(r(\underline{z}))q(\underline{z})\theta\left(\frac{r(\underline{z})}{q(\underline{z})}\right)\right\} \tag{8}$$

for each $(z, n1_{(z',c)}) \in \text{supp}(\mu_k)$ and $k \geq K$.

Let

$$\bar{M} = \max\left\{M, \max_{n\in[0,M]} g(r(\bar{z}))nq(\bar{z})\theta\left(\frac{r(\bar{z})}{nq(\bar{z})}\right), \frac{2}{g(r(\underline{z}))q(\underline{z})\theta\left(\frac{r(\underline{z})}{q(\underline{z})}\right)}\right\},$$

$n(z,z',c)$ be the solution of $\max_{n\in\mathbb{R}_+}\left[g(r(z))nq(z')\theta\left(\frac{r(z)}{nq(z')}\right) - cn\right]$ for each $z, z' \in Z$ and $c \in [1/\bar{M}, \bar{M}+1]$ and $\bar{n} = \max_{(z,z',c)\in Z^2\times[1/\bar{M},\bar{M}+1]} n(z,z',c)$; the existence of $\bar{n}$ follows by the compactness of $Z^2 \times [1/\bar{M}, \bar{M}+1]$ and the continuity of $(z,z',c) \mapsto n(z,z',c)$.

Let $k > \max\{K, \bar{M}+1, \bar{n}\}$ and $\mu = \mu_k$.

Claim 13 *μ is a stable matching of E.*

Proof. We will explicitly indicate the market we are considering in the stability set of μ, and thus write $S_M(\mu; E)$ and $S_M(\mu; E_k)$. We use analogous notation for $IR(\mu)$ and $T_z^m(\mu)$ for each $z \in Z$.

We first claim that, for each $(z,z',c) \in Z^2 \times C$, if $(z',c) \in T_z^m(\mu;E)$, then $(z', \bar{M}+1) \in T_z^m(\mu;E_k)$. Indeed, $(z',c) \in T_z^m(\mu;E)$ implies that $c = U_{z'}(w, 1_{(z,c)}) > U_{z'}(a,\delta)$ for some $(a,\delta) \in \Delta$ such that:

(a) If $a = w$, then $\delta = 1_{(\hat{z},\hat{c})}$ with $(\hat{z}, \hat{n}1_{(z',\hat{c})}) \in \text{supp}(\mu)$ and, thus, $U_{z'}(w, \delta) = \hat{c} \leq M$ by Claim 12.

(b) If $a = s$, then $U_{z'}(s, \delta) = 0$.

(c) If $a = m$, then $(z', \delta) \in \text{supp}(\mu)$ and, thus, $U_{z'}(m, \delta) \leq \bar{M}$ by Claim 12.

Hence, $U_{z'}(a, \delta) \leq \bar{M}$ and it follows that $(z', \bar{M} + 1) \in T^m_z(\mu; E_k)$ since $k > \bar{M} + 1$.

We now establish that μ is a stable matching of E. Let $(z, \delta) \in \text{supp}(\mu)$. Since μ is a stable matching of E_k, $(z, \delta) \in S_M(\mu; E_k) \cap IR(\mu; E_k)$ and $\delta \in X$ by Claim 11. We have that $U_{z'}(s, \delta') = 0$ for each $(z', \delta') \in Z \times X_s$ and, thus, $IR(\mu; E_k) \subseteq IR(\mu; E)$. Hence, $(z, \delta) \in IR(\mu; E)$.

It thus remains to show that $(z, \delta) \in S_M(\mu; E)$. Let $\delta = n1_{(\tilde{z},c)}$ and let (i) $(\hat{z}, \hat{\delta}) = (z, \delta)$ and $a = m$ or (ii) $(\hat{z}, \hat{\delta}) = (\tilde{z}, 1_{(z,c)})$ and $a = w$. Let $\delta' \in X$ be such that $\text{supp}(\delta') \subseteq T^m_{\hat{z}}(\mu; E)$ and let $\delta' = n^* 1_{(z^*,c^*)}$. Note that $(z^*, c^*) \in T^m_{\hat{z}}(\mu; E)$ implies that $c^* \geq 1/\bar{M}$ by (8). If $c^* \leq \bar{M} + 1$, then $(z^*, c^*) \in T^m_{\hat{z}}(\mu; E_k)$ and

$$U_{\hat{z}}(m, \delta') = U_{\hat{z}}(m, n^* 1_{(z^*,c^*)}) \leq U_{\hat{z}}(m, n(\hat{z}, z^*, c^*)1_{(z^*,c^*)}) \leq U_{\hat{z}}(a, \hat{\delta}),$$

where the last inequality follows from $(z, \delta) \in S_M(\mu; E_k)$ and $k > \bar{n}$. If $c^* > \bar{M} + 1$, then $(z^*, \bar{M} + 1) \in T^m_{\hat{z}}(\mu; E_k)$ and

$$\begin{aligned} &U_{\hat{z}}(m, \delta') = U_{\hat{z}}(m, n^* 1_{(z^*,c^*)}) \leq U_{\hat{z}}(m, n^* 1_{(z^*,\bar{M}+1)}) \leq \\ &U_{\hat{z}}(m, n(\hat{z}, z^*, \bar{M} + 1)1_{(z^*,\bar{M}+1)}) \leq U_{\hat{z}}(a, \hat{\delta}), \end{aligned}$$

where the last inequality follows from $(z, \delta) \in S_M(\mu; E_k)$ and $k > \bar{n}$.

Finally, let $\delta' \in X$ be such that $\text{supp}(\delta') \subseteq \text{supp}(\delta)$ in case (i). Then $\delta' = n' 1_{(\tilde{z},c)}$ for some $n' \in \mathbb{R}_+$. Since $1/M \leq c \leq M$ by Claim 12, it follows that

$$U_z(m, \delta') = U_z(m, n' 1_{(\tilde{z},c)}) \leq U_z(m, n(z, \tilde{z}, c)1_{(\tilde{z},c)}) \leq U_z(m, \delta),$$

where the last inequality follows from $(z, \delta) \in S_M(\mu; E_k)$ and $k > \bar{n}$. This concludes the proof that $(z, \delta) \in S_M(\mu; E)$ and establishes the claim. ■

A.6 Proof of Theorem 4

In this section, we show that the conditions in the statement of Theorem 4 are necessary and sufficient for μ to be a stable matching of the Rosen market.[36] Note that the function h is an homeomorphism between Z^2 and $h(Z^2)$.

Sufficiency. Let $\mu = \lambda \circ h^{-1}$ for some w and λ as in the statement of the Theorem. To see that μ is a matching, note that for each measurable B,

$$\begin{aligned}&\mu(B \times X) + \int_{Z \times X} \delta(B \times C) \mathrm{d}\mu(z, \delta) \\ &= \lambda \circ h^{-1}(B \times X) + \int_{Z \times X} \delta(B \times C) \mathrm{d}\lambda \circ h^{-1}(z, \delta) \\ &= \lambda(B \times Z) + \int_{Z \times B} n(z, z', w) \mathrm{d}\lambda(z, z') = \nu(B).\end{aligned}$$

We now show that μ is stable by establishing that $\mathrm{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$. Let $(z, \delta) \in \mathrm{supp}(\mu)$; then $\delta = n(z, z', w) 1_{(z', wq(z'))}$ for some $z' \in Z$ and $(z, z') \in \mathrm{supp}(\lambda)$ by Lemma 1. To see that $(z, \delta) \in IR(\mu)$, note that $U_z(m, n(z, z', w) 1_{(z', wq(z'))}) = R(z, w) > 0$ and $U_{z'}(w, 1_{(z, wq(z'))}) = wq(z') > 0$.

Suppose that $(z, n(z, z', w) 1_{(z', wq(z'))}) \notin S_M(\mu)$. Then either there exists $(z^*, c^*) \in T_z^m(\mu) \cup \{(z', wq(z'))\}$ such that $U_z(m, n(z, z^*, \frac{c^*}{q(z^*)}) 1_{(z^*, c^*)}) > R(z, w)$ or there exists $(z^*, c^*) \in T_{z'}^m(\mu)$ such that $U_{z'}(m, n(z', z^*, \frac{c^*}{q(z^*)}) 1_{(z^*, c^*)}) > wq(z')$. If $(z^*, c^*) = (z', wq(z'))$, then $U_z(m, n(z, z^*, \frac{c^*}{q(z^*)}) 1_{(z^*, c^*)}) = R(z, w)$. Thus, $(z^*, c^*) \in T_z^m(\mu) \cup T_{z'}^m(\mu)$ and, hence, $c^* > wq(z^*)$; indeed, condition (b) of $T_z^m(\mu) \cup T_{z'}^m(\mu)$ cannot happen since $\mathrm{supp}(\mu) \subseteq Z \times X$, condition (a) implies $c^* > wq(z^*)$ and condition (c) implies that $z^* \in \mathrm{proj}_1(\mathrm{supp}(\lambda))$ and $c^* > R(z^*, w)$ and, thus, that $c^* > wq(z^*)$ since then $R(z^*, w) \geq wq(z^*)$. If $(z^*, c^*) \in T_z^m(\mu)$, then $U_z(m, n(z, z^*, \frac{c^*}{q(z^*)}) 1_{(z^*, c^*)}) < R(z, w) = U_z(m, n(z, z', w) 1_{(z', wq(z'))})$. If $(z^*, c^*) \in T_{z'}^m(\mu)$, then

$$U_{z'}(m, n(z', z^*, \tfrac{c^*}{q(z^*)}) 1_{(z^*, c^*)}) < R(z', w) \leq wq(z'),$$

the last inequality holding since $z' \in \mathrm{proj}_2(\mathrm{supp}(\lambda))$. Thus, $(z, n(z, z', w) 1_{(z', wq(z'))}) \in S_M(\mu)$, and hence μ is stable.

[36] See Appendix B.4 for an illustration of Theorem 4 and its proof in the Cobb-Douglas case.

Necessity. Let μ be a stable matching of a Rosen market. We first show that $\text{supp}(\mu) \subseteq h(Z^2)$. Let $z, z', \hat{z}, \tilde{z} \in Z$ and $\hat{n}, \tilde{n}, c(z), c(z') \in \mathbb{R}_+$ be such that $(\hat{z}, \hat{n}1_{(z,c(z))})$ and $(\tilde{z}, \tilde{n}1_{(z',c(z'))})$ belong to $\text{supp}(\mu)$. Suppose, in order to reach a contradiction, that $\frac{c(z)}{c(z')} \neq \frac{q(z)}{q(z')}$; for concreteness, $c(z) > \frac{c(z')}{q(z')}q(z)$ and let $w = \frac{c(z')}{q(z')}$. It follows that

$$U_{\hat{z}}(m, \hat{n}1_{(z,c(z))}) < \max_n U_{\hat{z}}(m, n1_{(z,wq(z))}) = R(\hat{z}, w) = \max_n U_{\hat{z}}(m, n1_{(z',wq(z'))}).$$

Thus, there is $\varepsilon > 0$ such that $U_{\hat{z}}(m, \hat{n}1_{(z,c(z))}) < R(\hat{z}, w + \varepsilon)$. Since $(w + \varepsilon)q(z') = c(z') + \varepsilon q(z') > c(z')$, it follows that $(z', (w+\varepsilon)q(z')) \in T^m_{\hat{z}}(\mu)$. Thus, $\delta' = n(\hat{z}, z', w+\varepsilon)1_{(z',(w+\varepsilon)q(z'))}$ is such that $\text{supp}(\delta') \subseteq T^m_{\hat{z}}(\mu)$ and

$$U_{\hat{z}}(m, \delta') = R(\hat{z}, w + \varepsilon) > U_{\hat{z}}(m, \hat{n}1_{(z,c(z))}).$$

But this contradicts the stability of μ. It then follows that $\frac{c(z)}{c(z')} = \frac{q(z)}{q(z')}$ and, again letting $w = \frac{c(z')}{q(z')}$, that $c(z) = wq(z)$.

It also follows that $\hat{n} = n(\hat{z}, z, w)$ since otherwise $\delta' = n(\hat{z}, z, w)1_{(z,wq(z))}$ is such that $\text{supp}(\delta') \subseteq \text{supp}(\hat{n}1_{(z,wq(z))})$ and $U_{\hat{z}}(m, \delta') > U_{\hat{z}}(m, \hat{n}1_{(z,wq(z))})$ and, thus, contradicts the stability of μ.

Let $h^{-1} : h(Z^2) \to Z^2$ be the inverse of h and define $\lambda = \mu \circ (h^{-1})^{-1}$. Then (1) and (3) follow.

To see (2), let $(z, z') \in \text{supp}(\lambda)$, which implies that $(z, n(z, z', w)1_{(z',wq(z'))}) \in \text{supp}(\mu)$ by Lemma 1. If $R(z, w) < wq(z)$, then let $\varepsilon > 0$ be such that $wq(z) - \varepsilon > R(z, w)$ and note that $(z, wq(z) - \varepsilon) \in T^m_z(\mu)$ since $(z, n(z, z', w)1_{(z',wq(z'))}) \in \text{supp}(\mu)$ and $U_z(w, 1_{(z,wq(z)-\varepsilon)}) > U_z(m, n(z, z', w)1_{(z',wq(z'))})$. Thus,

$$\begin{aligned} &U_z(m, n(z, z, w)1_{(z,wq(z)-\varepsilon)}) > U_z(m, n(z, z, w)1_{(z,wq(z))}) \\ &= R(z, w) = U_z(m, n(z, z', w)1_{(z',wq(z'))}), \end{aligned}$$

contradicting the stability of μ. Hence, $R(z, w) \geq wq(z)$.

Similarly, if $wq(z') < R(z', w)$, then let $\varepsilon > 0$ be such that

$$U_{z'}(m, n(z', z', w)1_{(z',wq(z')+\varepsilon)}) > wq(z').$$

Note that $(z', wq(z') + \varepsilon) \in T^m_{z'}(\mu)$ since $(z, n(z, z', w)1_{(z', wq(z'))}) \in \text{supp}(\mu)$ and $U_{z'}(w, 1_{(z', wq(z')+\varepsilon)}) > U_{z'}(w, 1_{(z, wq(z'))})$. Thus,

$$U_{z'}(m, n(z', z', w)1_{(z', wq(z')+\varepsilon)}) > wq(z') = U_{z'}(w, 1_{(z, wq(z'))}),$$

contradicting the stability of μ. Hence, $wq(z') \geq R(z', w)$.

A.7 Non-existence example

We show that without the boundedness assumptions on X, a stable matching cannot generally exist, even when stability is defined via strong domination.

Consider the following market E, where for simplicity we omit contracts and preferences do not depend on the matching. Let $Z = [0, 1]$, let ν be the uniform distribution and let $X = \mathcal{M}(Z)$. Preferences are given by $u_z(m, \delta) = \delta(Z)$, $u_z(w, 1_{z'}) = z'$ and $u_z(s, 1_\emptyset) = 0$ for each $z, z' \in Z$ and $\delta \in X$. Then E is rational, continuous and rich but not bounded and it has no stable matching as we next show.

Suppose that E has a stable matching μ. First note that $\mu(Z \times (X_\emptyset \setminus X)) = 0$. If not, then let $\hat{Z} = \{z \in Z : (z, 1_\emptyset) \in \text{supp}(\mu)\}$ and $z \in \hat{Z}$ be such that $z > 0$. Then $\hat{Z} \subseteq T^m_z(\mu)$ and $\hat{Z}$ is closed. Thus, letting $\nu|_{\hat{Z}}$ be the restriction of ν to $\hat{Z}$ (i.e. $\nu|_{\hat{Z}}(B) = \nu(B \cap \hat{Z})$ for each Borel subset B of Z), it follows that $\text{supp}(\nu|_{\hat{Z}}) \subseteq T^m_z(\mu)$ which, together with $(m, \nu|_{\hat{Z}}) \succ_z (s, 1_\emptyset)$, contradicts the stability of μ.

Next note that if $(z, \delta) \in \text{supp}(\mu) \cap \big((Z \setminus \{1\}) \times X\big)$, then $\delta(Z) = 0$. To see this, suppose that $(z, \delta) \in \text{supp}(\mu)$ with $z < 1$, $\delta \in X$ and $\delta(Z) > 0$. Note that for all $z' \in \text{supp}(\delta)$, $z' \in T^m_{z^*}(\mu)$ for each $z^* > z$ since $(w, z^*) \succ_{z'} (w, z)$; thus, $\text{supp}(\delta) \subseteq T_{z^*}(\mu)$. Since $\mu((z, 1] \times X_\emptyset) + \int_{Z \times X} \delta((z, 1]) \mathrm{d}\mu(z, \delta) = \nu((z, 1]) > 0$, it follows that either $\text{supp}(\mu) \cap \big((z, 1] \times X_\emptyset\big) \neq \emptyset$ or $\text{supp}(\hat{\delta}) \cap \big((z, 1]\big) \neq \emptyset$ for some $(\hat{z}, \hat{\delta}) \in \text{supp}(\mu)$. Let $z^* > z$ be such that either $(z^*, \delta^*) \in \text{supp}(\mu)$ for some $\delta^* \in X_\emptyset$ or $z^* \in \text{supp}(\hat{\delta})$ for some $(\hat{z}, \hat{\delta}) \in \text{supp}(\mu)$. Then consider $\delta' = n\delta$ where n is such that $\delta'(Z) = n\delta(Z) > \max\{\delta^*(Z), 1\}$. We have that $\text{supp}(\delta') = \text{supp}(\delta) \subseteq T_{z^*}(\mu)$ and $(m, \delta') \succ_{z^*} (m, \delta^*)$ if $(z^*, \delta^*) \in \text{supp}(\mu)$ and $\delta^* \in X$, $(m, \delta') \succ_{z^*} (s, \delta^*)$ if $(z^*, \delta^*) \in \text{supp}(\mu)$ and $\delta^* \in X_\emptyset \setminus X$, and $(m, \delta') \succ_{z^*} (w, 1_{\hat{z}})$ if $z^* \in \text{supp}(\hat{\delta})$ and $(\hat{z}, \hat{\delta}) \in \text{supp}(\mu)$. This contradicts the stability of μ.

It follows by the above claims that $\mu(Z \times (X_\emptyset \setminus X)) = 0$ and that

$$\int_{Z\times X} \delta(Z)\mathrm{d}\mu(z,\delta) = \int_{\mathrm{supp}(\mu)\cap((Z\setminus\{1\})\times X)} \delta(Z)\mathrm{d}\mu(z,\delta) = 0.$$

Thus, $\mu(Z\times X) = \nu(Z) = 1$ and, since $\delta = 0$ for each $(z,\delta) \in \mathrm{supp}(\mu)\cap\big((Z\setminus\{1\})\times X\big)$, it follows that $\mathrm{supp}(\mu) = Z \times \{0\}$, where $0 \in \mathcal{M}(Z)$ denotes the zero measure on Z. But then $Z \subseteq T_1^m(\mu)$ and $(m,\nu) \succ_1 (m,0)$, contradicting the stability of μ.

B Appendix: Supplementary Material

B.1 Two-sided many-to-one matching markets

We introduce in this section a model of two-sided many-to-one matching markets and show that it is a special case of our framework of markets with occupational choice. The framework of this section can be seen as a hybrid between those of GK and CKK. Indeed, our two-sided many-to-one matching market is as in GK except that they have one-to-one matching and we allow for (but do not impose) many-to-one matching. Our setting is also similar to that of CKK since they consider many-to-one matching with a continuum of workers and finitely many managers and we consider the case of a continuum of managers as well as workers which may be a better description of some labor markets.

A *two-sided many-to-one matching market* is $E = (W, M, \nu_W, \nu_M, C, \mathbb{C}, X, (\succ_w)_{w\in W}, (\succ_m)_{m\in M})$ satisfying the following conditions and having the following interpretation. The sets M and W are Polish spaces of types of managers and workers respectively. To these sets correspond nonzero, finite, Borel measures ν_W and ν_M on W and M, respectively, describing the population of managers and workers. In addition, there is a dummy type $\emptyset \notin W \cup M$, which is an isolated point in $W_\emptyset = W \cup \{\emptyset\}$ and in $M_\emptyset = M \cup \{\emptyset\}$, to represent unmatched individuals. The set C is a Polish space of contracts and $\mathbb{C} : M_\emptyset \times W_\emptyset \rightrightarrows C$ is a contract correspondence. Each manager is matched with a finite measure of workers and contracts $\delta \in \mathcal{M}(W \times C)$ or unmatched, i.e. matched with $1_{(\emptyset,c)}$ for some $c \in C$. The set X is subset of $\mathcal{M}(W \times C)$ and $X_\emptyset = X \cup \{1_{(\emptyset,c)} : c \in C\}$ is the set of possible matches of a manager. Workers can also be unmatched; we use matches of the form $(\emptyset, 1_{(w,c)})$ to represent unmatched workers of type w with contract c. For this reason, we assume that $\{1_{(w,c)} : (w,c) \in W \times C\} \subseteq X$. Workers' preferences are described by $(\succ_w)_{w\in W}$ and managers' preferences by $(\succ_m)_{m\in M}$; for each $w \in W$, $\succ_w$ is defined on $M_\emptyset \times C \times \mathcal{M}(M_\emptyset \times X_\emptyset)$ and, for each $m \in M$, $\succ_m$ is defined on $X_\emptyset \times \mathcal{M}(M_\emptyset \times X_\emptyset)$. In addition, $\succ_\emptyset$ denotes the empty relation under which no elements are comparable.

A *matching* for a two-sided many-to-one matching market E is a Borel measure $\mu \in \mathcal{M}(M_\emptyset \times X_\emptyset)$ such that

(TS1) $\mu(B \times X_\emptyset) = \nu_M(B)$ for each Borel subset B of M,

(TS2) $\int_{M_\emptyset \times X} \delta(B \times C)\mathrm{d}\mu(m,\delta) = \nu_W(B)$ for each Borel subset B of W,

(TS3) $\{m\} \times \mathrm{supp}(\delta) \subseteq \mathrm{graph}(\mathbb{C})$ for each $(m,\delta) \in \mathrm{supp}(\mu)$, and

(TS4) $\mu(\{(m,\delta) : m = \emptyset \text{ and } \delta \in X_\emptyset \setminus \{1_{(w,c)} : (w,c) \in W \times C\}) = 0$.

For each Borel subset B of M and B' of $X_\emptyset$, $\mu(B \times B')$ is the measure of managers whose type belongs to B and whose match belongs to B'. In particular, $\mu(B \times X_\emptyset)$ is the measure of managers whose type belongs to B and, thus, it must equal $\nu_M(B)$ — this is condition (TS1). Regarding workers, $\int_{M_\emptyset \times X} \delta(W' \times C)\mathrm{d}\mu(m,\delta) = \int_{M \times X} \delta(W' \times C)\mathrm{d}\mu(m,\delta) + \int_{\{\emptyset\} \times X} \delta(W' \times C)\mathrm{d}\mu(m,\delta)$ is the measure of workers whose type belongs to W' that are matched with a manager or unmatched and, thus, it must equal $\nu_W(W')$ — this is condition (TS2). Condition (TS3) requires that matches are feasible according to the contract correspondence $\mathbb{C}$ and condition (TS4) says that matches $(m,\delta) \in \mathrm{supp}(\mu)$ with $m = \emptyset$ are such that $\delta \in \{1_{(w,c)} : (w,c) \in W \times C\}$ and this accounts for unmatched workers.

The stability of a matching for two-sided many-to-one matching markets is defined using the following notions of targets and stability sets. For each $m \in M_\emptyset$, the set of *targets of managers* of type m at μ is

$$\begin{aligned} T_m(\mu) = \quad & \{(w,c) \in W \times C : c \in \mathbb{C}(m,w) \text{ and there exists } (m',c',\delta') \in M_\emptyset \times C \times X_\emptyset \\ & \text{such that } (m',\delta') \in \mathrm{supp}(\mu), (w,c') \in \mathrm{supp}(\delta') \text{ and } (m,c,\mu) \succ_w (m',c',\mu)\} \\ & \cup(\{\emptyset\} \times \mathbb{C}(m,\emptyset)). \end{aligned}$$

For each $w \in W_\emptyset$, the set $T_w(\mu)$ of *targets of workers* of type w at μ is union of $\{\emptyset\} \times \mathbb{C}(\emptyset, w)$ with the set of $(m, c) \in M \times C$ such that $c \in \mathbb{C}(m, w)$ and there exists $(\delta, \delta') \in X \times X_\emptyset$ such that $(m, \delta') \in \text{supp}(\mu)$, $(w, c) \in \text{supp}(\delta)$, $\text{supp}(\delta) \setminus \{(w, c)\} \subseteq T_m(\mu) \cup \text{supp}(\delta')$ and $(\delta, \mu) \succ_m (\delta', \mu)$.

The *stability set* $S(\mu)$ of a matching μ is the set of $(m, \delta) \in M_\emptyset \times X_\emptyset$ such that

(TS-i) there does not exist $\delta' \in X_\emptyset$ such that $\text{supp}(\delta') \subseteq T_m(\mu) \cup \text{supp}(\delta)$ and $(\delta', \mu) \succ_m (\delta, \mu)$, and

(TS-ii) for each $(w, c) \in \text{supp}(\delta)$, there does not exist $(m', c') \in T_w(\mu)$ such that $(m', c', \mu) \succ_w (m, c, \mu)$.

A matching μ is *stable* in a two-sided many-to-one matching market E if $\text{supp}(\mu) \subseteq S(\mu)$.

We now show how to represent a two-sided many-to-one matching market E as a market with occupational choice $\hat{E}$ and characterize the stable matchings of E in terms of those of $\hat{E}$. To simplify the exposition, we assume that preferences do not depend on the matching.

We may assume that W and M are disjoint (if not, we could consider $\hat{W} = \{w\} \times W$ and $\hat{M} = \{m\} \times M$ where $w \neq m$) and let $Z = W \cup M$ be the set of types in $\hat{E}$; we may assume that W and M are closed subsets of Z.[37] The type distribution ν is defined by setting, for each Borel subset B of Z, $\nu(B) = \nu_M(M \cap B) + \nu_W(W \cap B)$. The set of contracts in $\hat{E}$ is C, i.e. the same as in E. The constraint correspondence $\hat{\mathbb{C}}$ is defined from $\mathbb{C}$ just by adjusting the order in which elements are listed and by arbitrarily defining the feasible contracts of two types in M and two types in W, as follows: let $\bar{c} \in C$ be given and set, for each $z \in Z$ and $z' \in Z_\emptyset$,

$$\hat{\mathbb{C}}(z, z') = \begin{cases} \mathbb{C}(z, z') & \text{if } z \in M \text{ and } z' \in W \cup \{\emptyset\}, \\ \mathbb{C}(z', z) & \text{if } z \in W \text{ and } z' \in M \cup \{\emptyset\}, \\ \{\bar{c}\} & \text{if } (z, z') \in M^2 \text{ or } (z, z') \in W^2. \end{cases}$$

[37]Define a metric d on Z based on those in M and W, and set $d(w, m) = 1$ for each $w \in W$ and $m \in M$.

The set X of feasible matches for managers in $\hat{E}$ is the same as in E. Finally, preferences are defined as follows. For each $z \in M$, for each $a, a' \in \{m, s\}$, $\delta \in X_a$, $\delta' \in X_{a'}$ and $\tilde{\delta} \in X_w$, (i) $(a, \delta) \hat{\succ}_z (a', \delta')$ if and only if $\delta \succ_z \delta'$, and (ii) $(a, \delta) \hat{\succ}_z (w, \tilde{\delta})$. Similarly, for each $z \in W$, for each $a, a' \in \{w, s\}$, $\delta \in X_a$, $\delta' \in X_{a'}$ and $\tilde{\delta} \in X_m$, (i) $(a, \delta) \hat{\succ}_z (a', \delta')$ if and only if $\text{supp}(\delta) \succ_z \text{supp}(\delta')$ and $\text{supp}(\delta) \subseteq M_\emptyset \times C$, and (ii) $(a, \delta) \hat{\succ}_z (m, \tilde{\delta})$.[38] In both cases, condition (i) says that preferences in $\hat{E}$ are derived from those in E when comparing the choices that individuals can make in E, namely, being a manager or self-employed in the case of someone with type in M and being a worker or self-employed in the case of someone with type in W. Condition (ii) says that being a worker is always worse than being a manager or self-employed for someone with type in M and being a manager is always worse than being a worker or self-employed for someone with type in W. We say that $\hat{E}$ is the *market associated with* E.

The following result shows that the stable matchings of $\hat{E}$ are the same as those of E up to an homeomorphism. Let

$$Y = (M \times X_\emptyset) \cup (W \times (X_\emptyset \setminus X)) \text{ and } Y' = (M \times X_\emptyset) \cup (\{\emptyset\} \times \{1_{(w,c)} : (w, c) \in W \times C\}).$$

Theorem 5 *Let E be an acyclic two-sided many-to-one matching market, $\hat{E}$ be its associated market, $\mathcal{S}$ be the set of stable matchings of E and $\hat{\mathcal{S}}$ be the set of stable matchings of $\hat{E}$.*[39] *Then there is an homeomorphism $h : Y \to Y'$ with inverse f such that $\hat{\mathcal{S}} = \{\mu \circ f^{-1} : \mu \in \mathcal{S}\}$ and $\mathcal{S} = \{\hat{\mu} \circ h^{-1} : \hat{\mu} \in \hat{\mathcal{S}}\}$.*

Proof. Theorem 5 characterizes the stable matching of E in terms of the stable matchings of $\hat{E}$ and vice versa. A matching of $\hat{E}$ is a measure on $Z \times X_\emptyset$ but a matching in E is a measure on $M_\emptyset \times X_\emptyset$. The difference arises due to how we represent unmatched individuals of type $z \in W$. This is done by a match $(z, 1_{(\emptyset,c)})$ in $\hat{E}$ and by

[38]For each $\delta \in X_s \cup X_w$, $\text{supp}(\delta) = \{(w, c)\}$ for some $(w, c) \in W_\emptyset \times C$. The meaning of $\text{supp}(\delta) \succ_z \text{supp}(\delta')$ is then that $(w, c) \succ_z (w', c')$ where (w, c) is the unique element of $\text{supp}(\delta)$ and likewise for (w', c').

[39]A two-sided many-to-one matching market E is *acyclic* if $\succ_w$ is acyclic for each $w \in W$ and $\succ_m$ is acyclic for each $m \in M$.

a match $(\emptyset, 1_{(z,c)})$ in E. Therefore, we need to transform the former into the latter to obtain a matching in E from a matching in $\hat{E}$ and vice versa.

Let $\hat{\mu}$ be a stable matching of $\hat{E}$ and note that $\hat{\mu}(W \times X) = 0$. Indeed, if $\hat{\mu}(W \times X) > 0$, then there is $(z, \delta) \in (W \times X) \cap \text{supp}(\hat{\mu})$. Letting $c \in \hat{\mathbb{C}}(z, \emptyset) = \mathbb{C}(\emptyset, z)$, it follows that $\text{supp}(1_{(\emptyset,c)}) \subseteq T^s_z(\hat{\mu})$ and $(s, 1_{(\emptyset,c)}) \hat{\succ}_z (m, \delta)$. But this implies that $(z, \delta) \notin S(\hat{\mu})$, contradicting the stability of $\hat{\mu}$.

The above implies that $\text{supp}(\hat{\mu}) \subseteq (M \times X_\emptyset) \cup (W \times (X_\emptyset \setminus X))$. Recall that

$$Y = (M \times X_\emptyset) \cup (W \times (X_\emptyset \setminus X)) \text{ and } Y' = (M \times X_\emptyset) \cup (\{\emptyset\} \times \{1_{(w,c)} : (w, c) \in W \times C\});$$

to transform a matching $\hat{\mu}$ of $\hat{E}$ into a matching μ of E, we use the function $h : Y \to Y'$ defined, by setting for each $(z, \delta) \in Y$,

$$h(z, \delta) = \begin{cases} (z, \delta) & \text{if } z \in M \text{ and } \delta \in X_\emptyset, \\ (\emptyset, 1_{(z,c)}) & \text{if } z \in W, \delta \in X_\emptyset \setminus X, \delta = 1_{(\emptyset,c)} \text{ and } c \in C. \end{cases}$$

Then h is continuous because M and W are both open and closed and $c_k \to c$ whenever $1_{(\emptyset,c_k)} \to 1_{(\emptyset,c)}$. Thus, $\mu = \hat{\mu} \circ h^{-1}$ is a measure on $Y' \subseteq M_\emptyset \times X_\emptyset$.

The function h is actually an homeomorphism between Y and Y', its inverse being $f : Y' \to Y$ defined, by setting, for each $(z, \delta) \in Y'$,

$$f(z, \delta) = \begin{cases} (z, \delta) & \text{if } z \in M, \\ (w, 1_{(\emptyset,c)}) & \text{if } z = \emptyset, \delta = 1_{(w,c)} \text{ and } (w, c) \in W \times C. \end{cases}$$

Since $\text{supp}(\mu) \subseteq Y'$ for each matching μ of E, $\hat{\mu} = \mu \circ f^{-1}$ is a measure on $Y \subseteq Z \times X_\emptyset$.

Note first that to establish Theorem 5 it suffices to show that: (1) if $\hat{\mu}$ is a stable matching of $\hat{E}$, then $\hat{\mu} \circ h^{-1}$ is a stable matching of E; and (2) if μ is a stable matching of E, then $\mu \circ f^{-1}$ is a stable matching of $\hat{E}$. Indeed, (2) implies that $\{\mu \circ f^{-1} : \mu \in \mathcal{S}\} \subseteq \hat{\mathcal{S}}$. For the converse, let $\hat{\mu} \in \hat{\mathcal{S}}$ and note that (1) implies that $\hat{\mu} \circ h^{-1} \in \mathcal{S}$. Then $\hat{\mu} \in \{\mu \circ f^{-1} : \mu \in \mathcal{S}\}$ since $\hat{\mu} = \hat{\mu} \circ (f \circ h)^{-1} = (\hat{\mu} \circ h^{-1}) \circ f^{-1}$. An analogous argument shows that $\mathcal{S} = \{\hat{\mu} \circ h^{-1} : \hat{\mu} \in \hat{\mathcal{S}}\}$.

We turn to the proof of (1). Let $\hat{\mu} \in \hat{\mathcal{S}}$ and $\mu = \hat{\mu} \circ h^{-1}$. Recall that we have already shown that $\hat{\mu}(W \times X) = 0$. We next claim that $\delta(B \times C) = 0$ for each $(z, \delta) \in \text{supp}(\hat{\mu})$

and Borel subset B of M. Note first that this is clear when $\delta \in X_\emptyset \setminus X$ since then $\delta = 1_{(\emptyset,c)}$ for some $c \in C$. Thus, assume that $\delta \in X$ and that $\delta(B \times C) > 0$. Then there exists $(z', c) \in \operatorname{supp}(\delta)$ such that $z' \in M$ and pick $c' \in \mathbb{C}(z', \emptyset) = \hat{\mathbb{C}}(z', \emptyset)$. Then $\delta' = 1_{(\emptyset,c')}$ is such that $\operatorname{supp}(\delta') \subseteq T^s_{z'}(\hat{\mu})$ and $(s, \delta') \hat{\succ}_{z'} (w, 1_{(z,c)})$. But this implies that $(z, \delta) \notin S(\hat{\mu})$, contradicting the stability of $\hat{\mu}$. This contradiction shows that $\delta(B \times C) = 0$.

We now show that μ is a matching of E. For condition (TS1), let B be a Borel subset of M and note that $h^{-1}(B \times X_\emptyset) = B \times X_\emptyset$ and that $\int_{Z \times X} \delta(B \times C) \mathrm{d}\hat{\mu}(z, \delta) = 0$, the latter since $\delta(B \times C) = 0$ for each $(z, \delta) \in \operatorname{supp}(\hat{\mu})$. Hence,

$$\begin{aligned} \mu(B \times X_\emptyset) &= \hat{\mu}(h^{-1}(B \times X_\emptyset)) = \hat{\mu}(B \times X_\emptyset) = \hat{\mu}(B \times X) + \hat{\mu}(B \times (X_\emptyset \setminus X)) \\ &= \hat{\mu}(B \times X) + \hat{\mu}(B \times (X_\emptyset \setminus X)) + \int_{Z \times X} \delta(B \times C) \mathrm{d}\hat{\mu}(z, \delta) = \nu(B) = \nu_M(B). \end{aligned}$$

For condition (TS2), let B be a Borel subset of W and note that $h^{-1}(M_\emptyset \times X) = (M \times X) \cup (W \times (X_\emptyset \setminus X))$. For each $(m, \delta) \in M_\emptyset \times X$, write $\gamma(m, \delta) = \delta(B \times C)$ and note that, for each $(z, \delta) \in W \times (X_\emptyset \setminus X)$, $h(z, \delta) = (\emptyset, 1_{(z,c)})$ for some $c \in C$, $\gamma(h(z, \delta)) = 1_{(z,c)}(B \times C)$ and, hence, $\gamma(h(z, \delta)) = 1$ if $z \in B$ and $\gamma(h(z, \delta)) = 0$ otherwise. Thus, using $\hat{\mu}(W \times X) = 0$,

$$\begin{aligned} &\int_{M_\emptyset \times X} \delta(B \times C) \mathrm{d}\mu(m, \delta) = \int_{M_\emptyset \times X} \gamma \mathrm{d}\mu = \int_{M \times X} \gamma \circ h \mathrm{d}\hat{\mu} + \int_{W \times (X_\emptyset \setminus X)} \gamma \circ h \mathrm{d}\hat{\mu} \\ &= \int_{M \times X} \delta(B \times C) \mathrm{d}\hat{\mu}(z, \delta) + \hat{\mu}(B \times (X_\emptyset \setminus X)) \\ &= \int_{Z \times X} \delta(B \times C) \mathrm{d}\hat{\mu}(z, \delta) + \hat{\mu}(B \times (X_\emptyset \setminus X)) + \hat{\mu}(B \times X) = \nu(B) = \nu_W(B). \end{aligned}$$

For condition (TS3), let $(m, \delta) \in \operatorname{supp}(\mu)$ and, since $\operatorname{supp}(\mu) = h(\operatorname{supp}(\hat{\mu}))$ by Lemma 1, let $(z, \delta') \in \operatorname{supp}(\hat{\mu})$ be such that $(m, \delta) = h(z, \delta')$. If $m \in M$, then $(m, \delta) = (z, \delta') \in \operatorname{supp}(\hat{\mu})$, and thus $c \in \hat{\mathbb{C}}(m, z') = \mathbb{C}(m, z')$ for each $(z', c) \in \operatorname{supp}(\delta)$. If $m = \emptyset$, then $\delta = 1_{(z,c)}$ for some $c \in C$, $z \in W$, $\delta' = 1_{(\emptyset,c)}$ and $c \in \hat{\mathbb{C}}(z, \emptyset) = \mathbb{C}(\emptyset, z)$.

Finally, condition (TS4) holds since $h^{-1}(\{\emptyset\} \times (X_\emptyset \setminus \{1_{(w,c)} : (w, c) \in W \times C\})) = \emptyset$.

We establish next some results on the relationship between the target sets in E and $\hat{E}$.

(a) If $(\emptyset, c) \in T_z(\mu)$, then $(\emptyset, c) \in T^s_z(\hat{\mu})$.

Indeed, $(\emptyset, c) \in T_z(\mu)$ implies that $c \in \mathbb{C}(z, \emptyset) = \hat{\mathbb{C}}(z, \emptyset)$ if $z \in M$ and $c \in \mathbb{C}(\emptyset, z) = \hat{\mathbb{C}}(z, \emptyset)$ if $z \in W$. In either case, this implies that $(\emptyset, c) \in T^s_z(\hat{\mu})$.

(b) $T_z(\mu) \cap (W \times C) \subseteq T^m_z(\hat{\mu})$ for each $z \in M$.

Indeed, if $(z^*, c) \in T_z(\mu) \cap (W \times C)$, then $c \in \mathbb{C}(z, z^*) = \hat{\mathbb{C}}(z, z^*)$ and there exists $(z', c', \delta') \in M_\emptyset \times C \times X_\emptyset$ such that $(z', \delta') \in \text{supp}(\mu)$, $(z^*, c') \in \text{supp}(\delta')$ and $(z, c) \succ_{z^*} (z', c')$. It then follows that $\delta' \in X$ since $(z^*, c') \in \text{supp}(\delta')$. If $z' \in M$, then $(z', \delta') \in \text{supp}(\hat{\mu})$ which, together with $(z^*, c') \in \text{supp}(\delta')$ and $(w, 1_{(z,c)}) \hat{\succ}_{z^*} (w, 1_{(z',c')})$, implies that $(z^*, c) \in T^m_z(\hat{\mu})$. If $z' = \emptyset$, then $\delta' = 1_{(z^*,c')}$, $(z^*, 1_{(\emptyset,c')}) \in \text{supp}(\hat{\mu})$ and $(w, 1_{(z,c)}) \hat{\succ}_{z^*} (s, 1_{(\emptyset,c')})$ implies that $(z^*, c) \in T^m_z(\hat{\mu})$.

We now show that μ is stable. Analogously to Theorem 1, the stability of μ is equivalent to $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR_W(\mu)$, where $S_M(\mu)$ is the set of $(m, \delta) \in M_\emptyset \times X_\emptyset$ such that there does not exist $\delta' \in X_\emptyset$ such that $\text{supp}(\delta') \subseteq T_m(\mu) \cup \text{supp}(\delta)$ and $(\delta', \mu) \succ_m (\delta, \mu)$, and $IR_W(\mu)$ is the set of $(m, \delta) \in M_\emptyset \times X_\emptyset$ such that, for each $(w, c) \in \text{supp}(\delta)$, there does not exist $c' \in \mathbb{C}(\emptyset, w)$ such that $(\emptyset, c', \mu) \succ_w (m, c, \mu)$.[40]

Let $(z, \delta) \in \text{supp}(\mu)$ and, since $\text{supp}(\mu) = h(\text{supp}(\hat{\mu}))$, let $(\hat{z}, \hat{\delta}) \in \text{supp}(\hat{\mu})$ be such that $(z, \delta) = h(\hat{z}, \hat{\delta})$. If $z \in M$, then $(z, \delta) = (\hat{z}, \hat{\delta}) \in \text{supp}(\hat{\mu})$; if $z = \emptyset$, then $\hat{z} \in W$, $\delta = 1_{(\hat{z},c)}$ for some $c \in C$ and $\hat{\delta} = 1_{(\emptyset,c)}$.

Suppose, in order to reach a contradiction, that there exists $\delta' \in X_\emptyset$ such that $\text{supp}(\delta') \subseteq T_z(\mu) \cup \text{supp}(\delta)$ and $\delta' \succ_z \delta$. Then $z \in M$ since $\succ_\emptyset$ is empty.

If $\delta' \in X_\emptyset \setminus X$, then $\delta' = 1_{(\emptyset,c')}$ for some $c' \in C$ and $\text{supp}(\delta') \subseteq T_z(\mu)$. The latter follows since $\text{supp}(\delta') \subseteq \{\emptyset\} \times C$ and $\text{supp}(\hat{\delta}) \subseteq W \times C$ when $\delta \in X$ and, when $\delta \in X_\emptyset \setminus X$, because if $\text{supp}(\delta') \cap \text{supp}(\delta) \neq \emptyset$, then $\delta' = \delta$, implying that $\delta \succ_z \delta$, a contradiction to the assumption that $\succ_z$ is acyclic. Thus, $(\emptyset, c) \in T_z(\mu)$ and, by (a) above, $\text{supp}(\delta') \subseteq T^s_z(\hat{\mu})$. Moreover, $\delta' \succ_z \delta$ implies that $(s, \delta') \hat{\succ}_z (m, \delta)$ when $\delta \in X$ and $(s, \delta') \hat{\succ}_z (s, \delta)$ when $\delta \in X_\emptyset \setminus X$. Thus, $(z, \delta) \in \text{supp}(\hat{\mu}) \setminus S(\hat{\mu})$, a contradiction to the stability of $\hat{\mu}$.

If $\delta' \in X$, then $\text{supp}(\delta') \subseteq W \times C$ and, by (b) above, $\text{supp}(\delta') \cap T_z(\mu) \subseteq T^m_z(\hat{\mu})$. Therefore, $\text{supp}(\delta') = (\text{supp}(\delta') \cap T_z(\mu)) \cup (\text{supp}(\delta') \cap \text{supp}(\delta)) \subseteq T^m_z(\hat{\mu}) \cup \text{supp}(\delta)$

[40] I.e. $IR_W(\mu)$ is the set of matches that are individually rational for the workers.

and $(m,\delta')\hat{\succ}_z(m,\delta)$ when $\delta \in X$ and $(m,\delta')\hat{\succ}_z(s,\delta)$ when $\delta \in X_\emptyset \setminus X$. But then $(z,\delta) \in \text{supp}(\hat{\mu}) \setminus S(\hat{\mu})$, contradicting the stability of $\hat{\mu}$. This contradiction, together with the one above, implies that $(z,\delta) \in S_M(\mu)$.

Suppose now that there exists $(z',c) \in \text{supp}(\delta)$ and $c' \in \mathbb{C}(\emptyset,z')$ such that $(\emptyset,c') \succ_{z'} (z,c)$. Then $(\emptyset,c') \in T_{z'}(\mu) \cap T^s_{z'}(\hat{\mu})$ by (a) above and $(s,1_{(\emptyset,c')})\hat{\succ}_{z'}(w,1_{(z,c)})$ when $z \in M$ and $(s,1_{(\emptyset,c')})\hat{\succ}_{z'}(s,1_{(\emptyset,c)})$ when $z = \emptyset$. It follows in either case that $(z,\delta) \in \text{supp}(\hat{\mu}) \setminus S(\hat{\mu})$, contradicting the stability of $\hat{\mu}$. Hence, $(z,\delta) \in IR_W(\mu)$.

We then have that $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR_W(\mu)$ and it follows that μ is stable.

We turn to the proof of (2). Let $\mu \in \mathcal{S}$ and $\hat{\mu} = \mu \circ f^{-1}$. We start by showing that $\hat{\mu}$ is a matching. For condition 1, let $(z,\delta) \in \text{supp}(\hat{\mu}) \subseteq Z \times X_\emptyset$ and $(z',c) \in \text{supp}(\delta)$. Then $h(z,\delta) \in \text{supp}(\mu)$ by Lemma 1. Thus, if $z \in M$, then $(z,\delta) = h(z,\delta) \in \text{supp}(\mu)$ and, therefore, $c \in \mathbb{C}(z,z') = \hat{\mathbb{C}}(z,z')$. If $z \in W$, then $\delta = 1_{(\emptyset,c)}$ and $(\emptyset, 1_{(z,c)}) = h(z,\delta) \in \text{supp}(\mu)$. Thus, $c \in \mathbb{C}(\emptyset,z) = \hat{\mathbb{C}}(z,\emptyset)$. In either case, it follows that $\{z\} \times \text{supp}(\delta) \subseteq \text{graph}(\hat{\mathbb{C}})$.

For condition 2, let B be a Borel subset of Z. We have that $\hat{\mu}(B\times X) = \mu(f^{-1}(B\times X)) = \mu((B\cap M)\times X)$ and that $\hat{\mu}(B \times (X_\emptyset \setminus X)) = \mu(f^{-1}(B \times (X_\emptyset \setminus X))) = \mu((B\cap M) \times (X_\emptyset \setminus X)) + \mu(\{\emptyset\} \times \{1_{(w,c)} : (w,c) \in (B\cap W) \times C\})$. Moreover,

$$\int_{Z\times X} \delta(B\times C)\text{d}\hat{\mu}(z,\delta) = \int_{M\times X} \delta(B\times C)\text{d}\mu(z,\delta) = \int_{M\times X} \delta((B\cap W)\times C)\text{d}\mu(z,\delta),$$

the latter equality holding since $\text{supp}(\delta) \subseteq W \times C$. Note also that

$$\int_{\{\emptyset\}\times X} \delta((B\cap W)\times C)\text{d}\mu(z,\delta) = \mu(\{\emptyset\} \times \{1_{(w,c)} : (w,c) \in (B\cap W)\times C\})$$

since $\mu(\{\emptyset\} \times (X \setminus \{1_{(w,c)} : (w,c) \in W \times C\})) = 0$ and $1_{(w,c)}((B\cap W)\times C) = 1$ if $w \in B\cap W$ and $1_{(w,c)}((B\cap W)\times C) = 0$ otherwise. Thus,

$$\begin{aligned}
&\hat{\mu}(B\times X) + \int_{Z\times X} \delta(B\times C)\text{d}\hat{\mu}(z,\delta) + \hat{\mu}(B\times(X_\emptyset\setminus X)) \\
&= \mu((B\cap M)\times X_\emptyset) + \int_{M_\emptyset\times X} \delta((B\cap W)\times C)\text{d}\mu(z,\delta) \\
&= \nu_M(B\cap M) + \nu_W(B\cap W) = \nu(B).
\end{aligned}$$

We establish next some results on the relationship between the target sets in E and $\hat{E}$.

(c) If $(\emptyset, c) \in T_z^s(\hat{\mu})$, then $(\emptyset, c) \in T_z(\mu)$.

Indeed, $(\emptyset, c) \in T_z^s(\hat{\mu})$ implies that $c \in \hat{\mathbb{C}}(z, \emptyset) = \mathbb{C}(z, \emptyset)$ if $z \in M$ and $c \in \hat{\mathbb{C}}(z, \emptyset) = \mathbb{C}(\emptyset, z)$ if $z \in W$. In either case, this implies that $(\emptyset, c) \in T_z(\mu)$.

(d) $T_z^m(\hat{\mu}) \cap (W \times C) \subseteq T_z(\mu)$ for each $z \in M$.

Indeed, let $(z^*, c) \in T_z^m(\hat{\mu}) \cap (W \times C)$. Then $c \in \hat{\mathbb{C}}(z, z^*) = \mathbb{C}(z, z^*)$. Since $\text{supp}(\hat{\mu}) \subseteq f(\text{supp}(\mu)) \subseteq Y$ by Lemma 1 and $z^* \in W$, there is no $\delta' \in X$ such that $(z^*, \delta') \in \text{supp}(\hat{\mu})$. Suppose first that there exists $\delta' \in X_\emptyset \setminus X$ such that $(z^*, \delta') \in \text{supp}(\hat{\mu})$ and $(w, 1_{(z,c)}) \hat{\succ}_{z^*} (s, \delta')$. Since $\delta' \in X_\emptyset \setminus X$, it follows that $\delta' = 1_{(\emptyset, c')}$ for some $c' \in C$; since $z^* \in W$, we have that $(\emptyset, 1_{(z^*, c')}) = h(z^*, \delta') \in \text{supp}(\mu)$. This, together with $(z, c) \succ_{z^*} (\emptyset, c')$ (which follows from $(w, 1_{(z,c)}) \hat{\succ}_{z^*} (s, \delta')$), implies that $(z^*, c) \in T_z(\mu)$. Suppose next that there is $(z', c', \delta') \in Z \times C \times X$ such that $(z', \delta') \in \text{supp}(\hat{\mu})$, $(z^*, c') \in \text{supp}(\delta')$ and $(w, 1_{(z,c)}) \hat{\succ}_{z^*} (w, 1_{(z', c')})$. Since $\delta' \in X$ and $(z', \delta') \in \text{supp}(\hat{\mu})$, it follows that $z' \in M$ and that $(z', \delta') = h(z', \delta') \in \text{supp}(\mu)$. This, together with $(z^*, c') \in \text{supp}(\delta')$ and $(z, c) \succ_{z^*} (z', c')$ (the latter follows from $(w, 1_{(z,c)}) \hat{\succ}_{z^*} (w, 1_{(z', c')})$), implies that $(z^*, c) \in T_z(\mu)$.

We now show that $\hat{\mu}$ is stable. Let $(z, \delta) \in \text{supp}(\hat{\mu})$ and assume first that $z \in M$. Then $(z, \delta) \in \text{supp}(\mu)$.

Suppose, in order to reach a contradiction, that there exists $(a, \delta') \in \Delta$ such that $a \in \{m, s\}$, $\text{supp}(\delta') \subseteq T_z^a(\hat{\mu}) \cup \text{supp}(\delta)$ if $\delta \in X$ and $a = m$, $\text{supp}(\delta') \subseteq T_z^a(\hat{\mu})$ otherwise, $(a, \delta') \hat{\succ}_z (m, \delta)$ if $\delta \in X$ and $(a, \delta') \hat{\succ}_z (s, \delta)$ if $\delta \in X_\emptyset \setminus X$. If $a = m$, then $\delta' \in X$ (as $\delta' \in X_m = X$). Thus, $\text{supp}(\delta') \subseteq W \times C$, $\text{supp}(\delta') \subseteq T_z(\mu) \cup \text{supp}(\delta)$ by (d) above and $\delta' \succ_z \delta$ by $(m, \delta') \hat{\succ}_z (a', \delta)$ for some $a' \in \{m, s\}$. If $a = s$, then $\delta' = 1_{(\emptyset, c')}$ for some $c' \in C$ (since $\delta' \in X_s$), $(\emptyset, c') \in T_z(\mu)$ by $\text{supp}(\delta') \subseteq T_z^s(\hat{\mu})$ and (c) above, and $1_{(\emptyset, c')} \succ_z \delta$ due to $(s, \delta') \hat{\succ}_z (a', \delta)$ for some $a' \in \{m, s\}$. Thus, in either case, $(z, \delta) \in \text{supp}(\mu) \setminus S(\mu)$, a contradiction to the stability of μ.

Suppose next that there exists $(z', c) \in \text{supp}(\delta)$ and $(a, \delta') \in \Delta$ such that $a \in \{m, s\}$, $\text{supp}(\delta') \subseteq T_{z'}^a(\hat{\mu})$ and $(a, \delta') \hat{\succ}_{z'} (w, 1_{(z,c)})$. Since $\delta \in X$, it follows that $w' \in W$. Moreover, since $(m, \delta') \hat{\succ}_{z'} (w, 1_{(z,c)})$ cannot hold, it follows that $a = s$. Then $\delta' = 1_{(\emptyset, c')}$ for some $c' \in C$, $(\emptyset, c') \in T_{z'}(\mu)$ by $\text{supp}(\delta') \subseteq T_{z'}^s(\hat{\mu})$ and (c) above, and $(\emptyset, c') \succ_{z'} (z, c)$ due to $(s, \delta') \hat{\succ}_{z'} (w, 1_{(z,c)})$. Thus, $(z, \delta) \in \text{supp}(\mu) \setminus S(\mu)$, a contradiction to the stability of μ.

Finally, consider $(z, \delta) \in \text{supp}(\hat{\mu})$ such that $z \in W$. Then $\delta \in X_\emptyset \setminus X$, $\delta = 1_{(\emptyset,c)}$ for some $c \in C$ and $(\emptyset, 1_{(z,c)}) \in \text{supp}(\mu)$.

Suppose, in order to reach a contradiction, that there exists $(a, \delta') \in \Delta$ such that $a \in \{m, s\}$, $\text{supp}(\delta') \subseteq T^a_z(\hat{\mu})$ and $(a, \delta') \hat{\succ}_z (s, 1_{(\emptyset,c)})$. Since $(m, \delta') \hat{\succ}_z (w, 1_{(\emptyset,c)})$ cannot hold, it follows that $a = s$. Then $\delta' = 1_{(\emptyset,c')}$ for some $c' \in C$, $(\emptyset, c') \in T_z(\mu)$ by $\text{supp}(\delta') \subseteq T^s_z(\hat{\mu})$ and (c) above, and $(\emptyset, c') \succ_z (\emptyset, c)$ due to $(s, \delta') \hat{\succ}_z (w, \delta)$. Thus, $(z, \delta) \in \text{supp}(\mu) \setminus S(\mu)$, a contradiction to the stability of μ.

We then conclude that $\text{supp}(\hat{\mu}) \subseteq S_M(\hat{\mu}) \cap IR(\hat{\mu})$ and Theorem 1 implies that $\hat{\mu}$ is stable. ■

B.2 Special cases

Our definition of a two-sided many-to-one matching market in Section B.1 is general enough to allow for widespread externalities so that preferences can depend on the matching itself. In addition, we allow for the possibility that unmatched managers may be able to choose from a set of contracts. In certain applications (e.g. the ones in Sections B.11 and B.12) this generality is not needed and it is convenient to restrict attention to two-sided markets without externalities, or two-sided markets where there are no contracts for the unmatched manager (this latter case makes sense, for example, when the contract is the wage paid to the worker).

A *two-sided market without externalities* is a two-sided many-to-one matching market E where preferences satisfy the following restrictions: for any $m \in M$, $\delta \in X_\emptyset$ and $\delta' \in X_\emptyset$, if $(\delta, \hat{\mu}) \succ_m (\delta', \hat{\mu})$ for some $\hat{\mu} \in \mathcal{M}(M_\emptyset \times X_\emptyset)$, then $(\delta, \mu) \succ_m (\delta', \mu)$ for all $\mu \in \mathcal{M}(M_\emptyset \times X_\emptyset)$; and, for any $w \in W$, $(m, c) \in M_\emptyset \times C$ and $(m', c') \in M_\emptyset \times C$, if $(m, c, \hat{\mu}) \succ_w (m', c', \hat{\mu})$ for some $\hat{\mu} \in \mathcal{M}(M_\emptyset \times X_\emptyset)$, then $(m, c, \mu) \succ_w (m', c', \mu)$ for all $\mu \in \mathcal{M}(M_\emptyset \times X_\emptyset)$.

A *two-sided market without empty workers* is $E = (W, M, \nu_W, \nu_M, C, \mathbb{C}, X, (\succ_w)_{w \in W}, (\succ_m)_{m \in M})$ with the following differences from a two-sided many-to-one matching market in Section B.1. The contract correspondence is $\mathbb{C} : M_\emptyset \times W \rightrightarrows C$, $\succ_m$ is defined on $X \times \mathcal{M}(M_\emptyset \times X)$, $\succ_w$ is defined on $M_\emptyset \times C \times \mathcal{M}(M_\emptyset \times X)$, and we assume in addition that $0 \in X$. In this case, unmatched managers are the ones matched with the zero measure in X. The definitions of a matching and a stable matching for a two-sided market without empty workers are then adapted from the ones in Section B.1 by replacing $X_\emptyset$ with X.

Of course, we can combine the two definitions in the obvious way: A *two-sided market without empty workers or externalities* is a two-sided market without empty workers where preferences satisfy the following restrictions: for any $m \in M$, $\delta \in X$ and $\delta' \in X$, if $(\delta, \hat{\mu}) \succ_m (\delta', \hat{\mu})$ for some $\hat{\mu} \in \mathcal{M}(M_\emptyset \times X)$, then $(\delta, \mu) \succ_m (\delta', \mu)$ for all $\mu \in \mathcal{M}(M_\emptyset \times X)$; and, for any $w \in W$, $(m, c) \in M_\emptyset \times C$ and $(m', c') \in M_\emptyset \times C$, if $(m, c, \hat{\mu}) \succ_w (m', c', \hat{\mu})$ for some $\hat{\mu} \in \mathcal{M}(M_\emptyset \times X)$, then $(m, c, \mu) \succ_w (m', c', \mu)$ for all $\mu \in \mathcal{M}(M_\emptyset \times X)$.

It can be shown that these alternative two-sided markets can be formulated as special cases of our general framework along the lines of Theorem 5 so that our existence results hold for these markets as well. In addition, Theorem 1 can be specialized to this setting and stability in these two-sided markets is equivalent to $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR_W(\mu)$, where $S_M(\mu)$ is defined as $S(\mu)$ but requiring only condition (TS-i) and $IR_W(\mu)$ is the set of $(m, \delta) \in M_\emptyset \times X_\emptyset$ (or $(m, \delta) \in M_\emptyset \times X$ in the case of the two-sided market with empty workers) such that, for each $(w, c) \in \text{supp}(\delta)$, there does not exist $c' \in \mathbb{C}(\emptyset, w)$ such that $(\emptyset, c', \mu) \succ_w (m, c, \mu)$.[41]

B.3 Outline of the proof of Theorem 2

The proof of Theorem 2 deals with several difficulties described in what follows. The first step of the proof is to establish its conclusion in the special case where Z, C and X are finite. Our approach builds on that in Section S.10 in CKK but requires many changes since preferences depend on externalities, workers' preferences are not strict and there is occupational choice. We describe these changes in the following outline of our proof for the case where there are no contracts.

[41] I.e. $IR_W(\mu)$ is the set of matches that are individually rational for the workers.

Without occupational choice, our existence proof would focus on pairs (μ, κ) consisting of a matching μ and a measure of available workers κ, where $\mu(m, \delta)$ is the measure of matches featuring a manager of type m and workforce δ and $\kappa(m, w)$ is the measure of workers of type w that are available to managers of type m. For μ to be stable, (μ, κ) must reflect both managers' and workers' preferences: μ maximizes managers' preferences subject to the constraint requiring that no manager's type hires more than the workers available to him and κ is obtained from an allocation of workers to managers that maximizes workers' preferences subject to the demand of workers by managers. Roughly, this is achieved by obtaining a correspondence Ψ and a fixed point of it, i.e. μ is stable if $(\mu, \kappa) \in \Psi(\mu, \kappa)$. In fact, due to some difficulties that also arise with occupational choice and are described below, we consider a sequence $\{\Psi_n\}_{n=1}^{\infty}$ of correspondences and a converging (sub)sequence $\{(\mu_n, \kappa_n)\}_{n=1}^{\infty}$ of fixed points, $(\mu_n, \kappa_n) \in \Psi_n(\mu_n, \kappa_n)$; the stable matching is then $\lim_n \mu_n$.

The main difficulty with occupational choice is that we no longer have fixed sets of managers and workers. Nevertheless, we use an approach similar to the above. We consider an allocation τ of types to occupations and matches that maximizes preferences subject to some constraints. Specifically, τ specifies the measure $\tau(z, a, \delta)$ of people of type z that are assigned to occupation a and are matched with $\delta \in X_a$. This allocation is done to maximize preferences, represented by an utility function $u : Z \times \Delta \times \mathcal{M}(Z \times X_\emptyset) \to \mathbb{R}$, subject to two constraints analogous to the ones above.

For the above constrained maximization problem to be defined, it needs a matching μ' to determine, in particular, the utility $u(z, a, \delta, \mu')$ that someone of type z obtains if assigned to occupation a and matched with δ when the matching is μ'. Since we are aiming for a fixed point argument, we take as given an allocation $\mu = (\mu(z, a, \delta))_{z \in Z, (a,\delta) \in \Delta}$ and then obtain a matching μ' by setting $\mu'(z, \delta)$ to be the measure $\mu(z, m, \delta)$ of type z people assigned to be a manager and matched with δ if $\delta \in X$; if $\delta \in X_\emptyset \setminus X$ instead, then $\mu'(z, \delta)$ is set to be the measure $\mu(z, s, \delta)$ of type z people assigned to be self-employed and matched with δ.

One of the constraints of the constrained maximization problem then requires that the measure $\tau(z, w, 1_{z'})$ of those individuals of type z that are assigned to be workers and are matched with someone of type z' be no greater than the demand of managers of type z', which is $\sum_{\delta \in X} \mu(z', m, \delta)\delta(z)$. The other constraint of the constrained maximization problem requires that the demand of workers by managers of type z' be no greater than the measure of workers of type z available to managers of type z'. Thus, the constrained maximization problem also takes as given a measure κ of available workers, with $\kappa(z', z)$ specifying the measure of individuals of type z that are available to work for an individual of type z'.

In summary, the set of solutions to the constrained optimization problem is

$$\begin{aligned} D(\mu, \kappa) = \{ & \tau \in \mathbb{R}_+^{Z \times \Delta} : \tau \in \arg\max_{\tau'} \sum_{z \in Z, (a,\delta) \in \Delta} u(z, a, \delta, \mu)\tau'(z, a, \delta) \\ & \text{subject to } \sum_{(a,\delta) \in \Delta} \tau'(z, a, \delta) = \nu(z) \text{ for all } z \in Z, \\ & \sum_{\delta \in X} \tau'(z, m, \delta)\delta(z') \leq \kappa(z, z', c) \text{ for all } (z, z') \in Z \times Z, \text{ and} \\ & \tau'(z, w, 1_{z'}) \leq \sum_{\delta \in X} \mu(z', m, \delta)\delta(z) \text{ for all } (z, z') \in Z \times Z \}. \end{aligned}$$

The measure of available workers is determined from the allocation μ of people to occupations and matches. Specifically, the measure $\kappa(z, z')$ of workers of type z' available to those of type z consists of the measure of those people of type z' who are already workers and matched with someone of type z plus the measure of those of type z' that are in worse occupations or matches, i.e. it equals

$$\mu(z', w, 1_z) + \sum_{(a,\delta) \in W(z,z',\mu)} \mu(z', a, \delta),$$

where $W(z, z', \mu) = \{(a, \delta) \in \Delta : u(z', w, 1_z, \mu) > u(z', a, \delta, \mu)\}$ are those pairs of occupations and matches (a, δ) which are worse for someone of type z' than being a worker working for a manager of type z.

The problem with this approach is that the function

$$\mu \mapsto \sum_{(a,\delta) \in W(z,z',\mu)} \mu(z', a, \delta)$$

may fail to be continuous. Discontinuities may arise because pairs of occupations and matches $(a, \delta) \in W(z, z', \mu)$ get weight 1 and those $(a, \delta) \notin W(z, z', \mu)$ get weight 0 and this depends on μ in a possibly discontinuous way. We solve this problem by using a sequence of continuous weights: For each $n \in \mathbb{N}$ and $(a,\delta) \in \Delta$, the weight of occupation-match pair (a,δ) when the allocation is μ is

$$\alpha_{n,(z,z')}(a,\delta,\mu) = n \max\left\{0, \min\left\{u(z',w,1_z,\mu) - u(z',a,\delta,\mu), \frac{1}{n}\right\}\right\}.$$

We then use $\sum_{(a,\delta)\in\Delta} \alpha_{n,(z,z')}(a,\delta,\mu)\mu(z',a,\delta)$ instead of $\sum_{(a,\delta)\in W(z,z',\mu)} \mu(z',a,\delta)$. In fact, as a measure of available workers of type z' to type z given an allocation μ, we use

$$f_n(\mu)(z,z') = \mu(z',w,1_z) + \frac{1}{n} \sum_{(a,\delta)\in\Delta} \alpha_{n,(z,z')}(a,\delta,\mu)\mu(z',a,\delta)$$

since adding the term $1/n$ facilitates the argument while maintaining the property that, for each n, $f_n(\mu)(z,z')$ is strictly higher than $\mu(z',w,1_z)$ if and only if there are individuals of type z' allocated to pairs of occupations and matches that are worse for z than $(w,1_z)$.

A final difficulty with the above approach is that some instability may exist in an optimal allocation $\mu \in D(\mu,\kappa)$. This would arise, for instance, if there is a strictly positive measure of managers of type z with workforce δ, $\mu(z,m,\delta) > 0$, who can improve their well-being by changing their workforce to δ', and the constraint for the single (for simplicity) type z' of workers employed both in δ and δ', i.e. $\text{supp}(\delta') = \text{supp}(\delta) = \{z'\}$, is binding:

$$\sum_{\hat{\delta}\in X} \mu(z,m,\hat{\delta})\hat{\delta}(z',c) = \kappa(z,z').$$

If $\delta(z) < \delta'(z)$, μ can be optimal since it may not be possible to increase $\mu(z,m,\delta')$ and decrease $\mu(z,m,\delta)$ to increase the objective function in $D(\mu,\kappa)$ while satisfying its constraints. Indeed, due to $\delta(z) < \delta'(z)$ and $\sum_{\hat{\delta}\in X} \mu(z,m,\hat{\delta})\hat{\delta}(z',c) = \kappa(z,z')$, $\mu(z,m,\delta')$ can only increase by a fraction $0 < \theta < 1$ of the decrease of $\mu(z,m,\delta)$ but this increase of $\mu(z,m,\delta')$ may not be big enough to increase the objective function of $D(\mu,\kappa)$.

We solve the above problem by increasing the difference $u(z,m,\delta',\mu) - u(z,m,\delta,\mu)$. Specifically, for each $n \in \mathbb{N}$, we use $u_n(z,\hat{m},\hat{\delta},\hat{\mu}) = u(z,\hat{m},\hat{\delta},\hat{\mu})^n$ to represent preferences with u normalized so that $u \geq 1$. Then, even when $\theta u(z, m, \delta', \mu) < u(z, m, \delta, \mu)$, we have that $\theta u_n(z, m, \delta', \mu) > u_n(z, m, \delta, \mu)$ for all n sufficiently large.

In summary, we establish the existence of stable matchings via the existence, for each $n \in \mathbb{N}$, of a fixed point of the correspondence

$$(\mu, \kappa) \mapsto D_n(\mu, \kappa) \times \{f_n(\mu)\},$$

where D_n is obtained by replacing u with u_n in D. We obtain in this way, and by taking a subsequence if necessary, a convergent sequence $\{(\mu_n, \kappa_n)\}_{n=1}^{\infty}$ and we show that the allocation $\mu = \lim_n \mu_n$ yields a stable matching.

We then use three limit arguments to extend the existence result from discrete to general markets. The first limit argument considers the case where X is $\mathcal{M}_R(Z \times C)$ for some $R > 0$ to dispense with the finiteness of C. It shows that for an appropriately chosen sequence $\{C_k\}_{k=1}^{\infty}$, where C_k is finite for each k, the associated sequence $\{\mu_k\}_{k=1}^{\infty}$ of stable matchings for the discrete markets converges to a stable matching of the market where the set of contracts is C. The argument that the limit matching μ is stable consists in showing that if there is some z who would prefer and is able to become a manager and hire some workforce δ' when the matching is μ, then the richness of $\mathcal{M}_R(Z \times C)$ implies that there is some $\delta'_k \in X_k$, where $\text{supp}(X_k) \subseteq Z \times C_k$, such that z would also prefer and is able to become a manager and hire δ'_k when the matching is μ_k. This contradicts the stability of μ_k.

The second limit argument replaces $X = \mathcal{M}_R(Z \times C)$ with a general X satisfying our assumptions in the case where Z is finite. The finiteness of Z is important to represent each preference relation $\succ_z$ with a continuous and bounded (e.g. by 1 below and 2 above) utility function $u : Z \times \Delta \times \mathcal{M}(Z \times X_\emptyset) \to [1, 2]$. Such function can then be extended by replacing X with $\mathcal{M}_R(Z \times C)$ in the definition of its domain to obtain a market to which the conclusion of the previous limit argument applies. This is done in such a way that there is a utility penalty for managers who choose a workforce δ at a distance greater than $1/k$ from X; specifically, they get a utility of zero if they do so, whereas the utility of choosing a workforce $\delta \in X$ is at least one. Thus, we show that the limit μ of the sequence $\{\mu_k\}_{k=1}^{\infty}$ of stable matchings for the sequence $\{E_k\}_{k=1}^{\infty}$ of such markets is a stable matching of the market with the original X.

The final limit argument then dispenses with the finiteness of Z. This argument considers a sequence E_k of markets with finite sets of types Z_k. As before, we show that the limit μ of the sequence $\{\mu_k\}_{k=1}^{\infty}$ of stable matchings for $\{E_k\}_{k=1}^{\infty}$ is a stable matching for the market where the set of types is Z. For μ to be stable, it must not be the case, for example, that there is $(z, \delta) \in \text{supp}(\mu)$ and $\delta' \in X$ such that $(m, \delta', \mu) \succ_z (m, \delta, \mu)$ and $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$, which can be shown by proving that if such z, δ and δ' exist, then μ_k fails to be stable for each sufficiently large k. However, in general, (z, δ) is not an element of $\text{supp}(\mu_k)$ and, thus, to show that μ_k is not stable, we need to consider z_k close to z and δ_k close to δ; in addition, μ_k differs from μ too. This then implies that $T_{z_k}^m(\mu_k) \cup \text{supp}(\delta_k)$ may differ from $T_z^m(\mu) \cup \text{supp}(\delta)$, which in turn requires us to find $\delta'_k \in X$ close to δ' so that we have $\text{supp}(\delta'_k) \subseteq T_{z_k}^m(\mu_k) \cup \text{supp}(\delta_k)$. As in the previous limit arguments, richness guarantees the existence of such δ'_k.

B.4 Illustration of Theorem 4 in the Cobb-Douglas case

We will illustrate Theorem 4 in the Cobb-Douglas case where $f(r, nq) = r^{\alpha}(nq)^{1-\alpha}$ for some $\alpha \in (0, 1)$. In a stable matching, managers' maximize rent. Therefore, if a manager has productivity r and employs workers with productivity q at wage c, then the number n of workers solves $\max_n [g(r)r^{\alpha}(nq)^{1-\alpha} - cn]$; thus,

$$n = \left[\frac{(1-\alpha)g(r)r^{\alpha}q^{1-\alpha}}{c}\right]^{\frac{1}{\alpha}}.$$

We also have that $cn = (1-\alpha)y$ with $y = g(r)r^{\alpha}(nq)^{1-\alpha}$ and, hence, the managers' rent is $\alpha y = \alpha g(r)r^{\alpha}(nq)^{1-\alpha}$.

If the workers' wages are proportional to their productivity, so that $c(z) = wq(z)$ for each z and some $w > 0$, then the optimal number $n(z, z', w)$ of workers in a firm with a manager of ability z and workers with ability z', and the manager's rent

$R(z,w)$ equal

$$n(z,z',w) = \left(\frac{(1-\alpha)g(r(z))}{w}\right)^{\frac{1}{\alpha}} \frac{r(z)}{q(z')},$$
$$R(z,w) = \alpha g(r(z))^{\frac{1}{\alpha}} r(z) \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}}.$$

In particular, the manager's rent is independent of the ability of the workers.

In fact, workers' wages must be proportional to their productivity in a stable matching. Indeed, if there are workers of type z and z' but $c(z) > \frac{c(z')}{q(z')}q(z)$, then the manager employing workers of type z can be better off by employing workers of type z' and attracting them by paying a wage slightly above $c(z')$. Indeed, any manager of ability $\hat{z}$ employing workers of type z has a rent strictly smaller than $R(\hat{z}, \frac{c(z')}{q(z')})$ and could virtually obtain the latter by employing $n(\hat{z}, z', \frac{c(z')}{q(z')})$ workers of type z' at a wage slightly above $\frac{c(z')}{q(z')}q(z') = c(z')$.

It then follows that there exists $w > 0$ such that $c(z) = wq(z)$ for each $z \in Z$ who are workers. Therefore, in any stable matching, matches are of the form $(z, n(z,z',w)1_{(z',wq(z'))})$. Any worker is indifferent to any manager and vice versa since workers' wages are independent of the manager's ability and managers' rents are independent of the workers' ability. It thus remains to determine who is who in this market; since there is no unemployment in stable matchings of Rosen markets due to $R(\underline{z}, w) > 0$, this can be described by a measure $\lambda \in \mathcal{M}(Z^2)$ with $\lambda(B \times B')$ denoting the measure of individuals of type in B that are managers and are matched with individuals (who are then workers) of type in B', where B and B' are any measurable subsets of Z. Thus, if (z, z') belongs to the support of λ, it must be that $R(z,w) \geq wq(z)$ and $wq(z') \geq R(z',w)$, i.e. type z individuals prefer to be managers and type z' individuals prefer to be workers. Finally, the accounting constraint must hold, i.e. $\lambda(B \times Z) + \int_{Z \times B} n(z,z')\mathrm{d}\lambda(z,z') = \nu(B)$ for each measurable $B \subseteq Z$.

B.5 Sufficient condition for richness

In this section we establish the sufficiency of (α) and (β) for richness.

Claim 14 *Let E be a continuous market. If (α) and (β) hold, then E is rich.*

We use the following lemma in the proof of Claim 14.

Lemma 9 *Let* $\{z_k\}_{k=1}^{\infty}$ *be such that* $z_k \to z$, $(\tilde{z}, \tilde{c}) \in T_z^m(\mu)$, *and* $V_{\tilde{z}}$ *and* $V_{\tilde{c}}$ *be open neighborhoods of* $\tilde{z}$ *and* $\tilde{c}$ *respectively. Then, for all* k *sufficiently large, there exists* $(\tilde{z}_k, \tilde{c}_k) \in T_{z_k}^m(\mu_k) \cap (V_{\tilde{z}} \times V_{\tilde{c}})$.

Proof. Let $(\tilde{z}, \tilde{c}) \in T_z^m(\mu)$. Then $\tilde{c} \in \mathbb{C}(z, \tilde{z})$ and either there exists $(\hat{z}, \hat{\delta}, \hat{c})$ such that $(\hat{z}, \hat{\delta}) \in \text{supp}(\mu)$, $(\tilde{z}, \hat{c}) \in \text{supp}(\hat{\delta})$ and $(w, 1_{(z,\tilde{c})}, \mu) \succ_{\tilde{z}} (w, 1_{(\hat{z},\hat{c})}, \mu)$, or there exists $\tilde{\delta} \in X_\emptyset$ such that $(\tilde{z}, \tilde{\delta}) \in \text{supp}(\mu)$ and $(w, 1_{(z,\tilde{c})}, \mu) \succ_{\tilde{z}} (a(\tilde{\delta}), \tilde{\delta}, \mu)$, where $a(\tilde{\delta}) = s$ if $\tilde{\delta} \in X_\emptyset \setminus X$ and $a(\tilde{\delta}) = m$ if $\tilde{\delta} \in X$.

First assume that there exists $(\hat{z}, \hat{\delta}, \hat{c})$ such that $(\hat{z}, \hat{\delta}) \in \text{supp}(\mu)$, $(\tilde{z}, \hat{c}) \in \text{supp}(\hat{\delta})$ and $(w, 1_{(z,\tilde{c})}, \mu) \succ_{\tilde{z}} (w, 1_{(\hat{z},\hat{c})}, \mu)$. Let O_z, $O_{\tilde{c}}$, $O_{\tilde{z}}$, $O_{\hat{z}}$, $O_{\hat{c}}$, $O_{\hat{\delta}}$ and O_μ be open neighborhoods of z, $\tilde{c}$, $\tilde{z}$, $\hat{z}$, $\hat{c}$, $\hat{\delta}$ and μ, respectively, such that $(w, 1_{(z',\tilde{c}')}, \mu') \succ_{\tilde{z}'} (w, 1_{(\hat{z}',\hat{c}')}, \mu')$, $\text{supp}(\hat{\delta}') \cap ((O_{\tilde{z}} \cap V_{\tilde{z}}) \times O_{\hat{c}}) \neq \emptyset$ and $\mathbb{C}(z', \tilde{z}') \cap (O_{\tilde{c}} \cap V_{\tilde{c}}) \neq \emptyset$ for each $z' \in O_z$, $\tilde{c}' \in O_{\tilde{c}}$, $\tilde{z}' \in O_{\tilde{z}}$, $\hat{z}' \in O_{\hat{z}}$, $\hat{c}' \in O_{\hat{c}}$, $\hat{\delta}' \in O_{\hat{\delta}}$ and $\mu' \in O_\mu$. Since $\mu_k \to \mu$, $0 < \mu(O_{\hat{z}} \times O_{\hat{\delta}}) \leq \liminf_k \mu_k(O_{\hat{z}} \times O_{\hat{\delta}})$ and $z_k \to z$, it follows that, for each k sufficiently large, $\mu_k \in O_\mu$, $z_k \in O_z$ and there is $(\hat{z}_k, \hat{\delta}_k) \in \text{supp}(\mu_k) \cap (O_{\hat{z}} \times O_{\hat{\delta}})$, $(\tilde{z}_k, \hat{c}_k) \in \text{supp}(\hat{\delta}_k) \cap ((O_{\tilde{z}} \cap V_{\tilde{z}}) \times O_{\hat{c}})$ and $\tilde{c}_k \in \mathbb{C}(z_k, \tilde{z}_k) \cap O_{\tilde{c}} \cap V_{\tilde{c}}$. Then $(w, 1_{(z_k,\tilde{c}_k)}, \mu_k) \succ_{\tilde{z}_k} (w, 1_{(\hat{z}_k,\hat{c}_k)}, \mu_k)$ and, hence, $(\tilde{z}_k, \tilde{c}_k) \in T_{z_k}^m(\mu_k) \cap (V_{\tilde{z}} \times V_{\tilde{c}})$.

Next assume there exists $\tilde{\delta} \in X_\emptyset$ such that $(\tilde{z}, \tilde{\delta}) \in \text{supp}(\mu)$ and $(w, 1_{(z,\tilde{c})}, \mu) \succ_{\tilde{z}} (a(\tilde{\delta}), \tilde{\delta}, \mu)$. Let O_z, $O_{\tilde{z}}$, $O_{\tilde{c}}$, $O_{\tilde{\delta}}$ and O_μ be open neighborhoods of z, $\tilde{z}$, $\tilde{c}$, $\tilde{\delta}$ and μ, respectively, such that $(w, 1_{(z',\tilde{c}')}, \mu') \succ_{\tilde{z}} (a(\tilde{\delta}), \tilde{\delta}', \mu')$ and $\mathbb{C}(z', \tilde{z}') \cap (O_{\tilde{c}} \cap V_{\tilde{c}}) \neq \emptyset$ for each $z' \in O_z$, $\tilde{z}' \in O_{\tilde{z}}$, $\tilde{c}' \in O_{\tilde{c}}$, $\tilde{\delta}' \in O_{\tilde{\delta}}$ and $\mu' \in O_\mu$. Since $0 < \mu((O_{\tilde{z}} \cap V_{\tilde{z}}) \times O_{\tilde{\delta}}) \leq \liminf_k \mu_k((O_{\tilde{z}} \cap V_{\tilde{z}}) \times O_{\tilde{\delta}})$, it follows that, for each k sufficiently large there is $(\tilde{z}_k, \tilde{\delta}_k) \in \text{supp}(\mu_k) \cap ((O_{\tilde{z}} \cap V_{\tilde{z}}) \times O_{\tilde{\delta}})$. Since $\mu_k \to \mu$ and $z_k \to z$, it follows that, for each k sufficiently large, $\mu_k \in O_\mu$, $z_k \in O_z$ and there exists $\tilde{c}_k \in \mathbb{C}(z_k, \tilde{z}_k) \cap O_{\tilde{c}} \cap V_{\tilde{c}}$. Then $(w, 1_{(z_k,\tilde{c}_k)}, \mu_k) \succ_{\tilde{z}_k} (a(\tilde{\delta}), \tilde{\delta}_k, \mu_k)$ and, hence, $(\tilde{z}_k, \tilde{c}_k) \in T_{z_k}^m(\mu_k) \cap (V_{\tilde{z}} \times V_{\tilde{c}})$. ■

We can now turn to the proof of the claim.

Proof of Claim 14. Let $(z, \delta, \mu) \in Z \times X \times \mathcal{M}(Z \times X_\emptyset)$ and $V \subseteq X$ be open and such that $\Lambda(z, \delta, \mu) \cap V \neq \emptyset$. Let $\delta' \in \Lambda(z, \delta, \mu) \cap V$, i.e. V is an open neighborhood of δ' and $\text{supp}(\delta') \subseteq \text{supp}(\delta) \cup T_z(\mu)$.

We may assume that $\text{supp}(\delta')$ is finite by condition (β) since $\delta' \in X$. For each $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')$, let $V_{(\tilde{z},\tilde{c})}$ be given by condition (α). Let $V_{\tilde{z}}$ and $V_{\tilde{c}}$ be open neighborhoods of $\tilde{z}$ and $\tilde{c}$ such that $V_{\tilde{z}} \times V_{\tilde{c}} \subseteq V_{(\tilde{z},\tilde{c})}$. By Lemma 2, let O_δ be an open neighborhood of δ such that $\text{supp}(\tilde{\delta}) \cap V_{(\tilde{z},\tilde{c})} \neq \emptyset$ for each $\tilde{\delta} \in O_\delta$.

If Λ fails to be lower hemicontinuous, then there is a sequence $\{(z_k, \delta_k, \mu_k)\}_{k=1}^{\infty}$ such that $(z_k, \delta_k, \mu_k) \to (z, \delta, \mu)$ and

$$\Lambda(z_k, \delta_k, \mu_k) \cap V = \emptyset \text{ for each } k \in \mathbb{N}. \tag{9}$$

For each $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \cap T_z^m(\mu)$ and k sufficiently large, let $z_k(\tilde{z}, \tilde{c}) \in V_{\tilde{z}}$ and $c_k(\tilde{z}, \tilde{c}) \in V_{\tilde{c}}$ be such that $(z_k(\tilde{z}, \tilde{c}), c_k(\tilde{z}, \tilde{c})) \in T_{z_k}^m(\mu_k) \cap V_{(\tilde{z},\tilde{c})}$, which exists by Lemma 9.

For each $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \cap \text{supp}(\delta)$ and k sufficiently large, let $z_k(\tilde{z}, \tilde{c}) \in V_{\tilde{z}}$ and $c_k(\tilde{z}, \tilde{c}) \in V_{\tilde{c}}$ be such that $(z_k(\tilde{z}, \tilde{c}), c_k(\tilde{z}, \tilde{c})) \in \text{supp}(\delta_k) \cap V_{(\tilde{z},\tilde{c})}$.

Let $K \in \mathbb{N}$ be such that, for each $k \geq K$, $(z_k(\tilde{z}, \tilde{c}), c_k(\tilde{z}, \tilde{c})) \in T_{z_k}^m(\mu_k) \cap V_{(\tilde{z},\tilde{c})}$ if $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \cap T_z^m(\mu)$ and $(z_k(\tilde{z}, \tilde{c}), c_k(\tilde{z}, \tilde{c})) \in \text{supp}(\delta_k) \cap V_{(\tilde{z},\tilde{c})}$ if $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \setminus T_z^m(\mu)$. Condition (α) gives $(a_k(\tilde{z}, \tilde{c}))_{(\tilde{z},\tilde{c}) \in \text{supp}(\delta')} \in \mathbb{R}_+^{|\text{supp}(\delta')|}$ such that, setting $\delta'_k = \sum_{(\tilde{z},\tilde{c}) \in \text{supp}(\delta')} a_k(\tilde{z}, \tilde{c}) 1_{(z_k(\tilde{z},\tilde{c}), c_k(\tilde{z},\tilde{c}))}$ for each $k \geq K$, we have that $\delta'_k \in V_{\delta'}$ and $\text{supp}(\delta'_k) \subseteq T_{z_k}^m(\mu_k) \cup \text{supp}(\delta_k)$. But this contradicts (9). Thus, it follows that Λ is lower hemicontinuous.

The proof that Λ_0 is lower hemicontinuous is analogous. ■

B.6 Characterization of stability in continuous and rich markets

In this section we establish the following claim.

Claim 15 *Let E be a continuous and rich market. Then a matching μ is stable if and only if $S(\mu)$ has full μ-measure.*

Proof. It is clear that the stability of a matching μ implies that $S(\mu)$ has full μ-measure.

Conversely, assume that $S(\mu)$ has full μ-measure and that E is continuous and rich. Note that it suffices to show that $S_M(\mu) \cap IR(\mu)$ is closed.

We start by establishing a lower hemicontinuity property of the correspondence $z \mapsto T^s_z(\mu)$.

Claim 16 *For each $z \in Z$, $(z^*, c) \in T^s_z(\mu)$ and open neighborhood V_c of c, there exists an open neighborhood V_z of z such that $T^s_{\tilde{z}}(\mu) \cap (\{z^*\} \times V_c) \neq \emptyset$ for each $\tilde{z} \in V_z$.*

Proof. We have that $z^* = \emptyset$ and $c \in \mathbb{C}(z, \emptyset)$. By the continuity of the contract correspondence, let V_z be an open neighborhood of z such that $\mathbb{C}(\tilde{z}, \emptyset) \cap V_c \neq \emptyset$ for each $\tilde{z} \in V_z$. Thus, for each $\tilde{z} \in V_z$, $(\emptyset, \tilde{c}) \in T^s_{\tilde{z}}(\mu) \cap (\{\emptyset\} \times V_c)$. ■

We complete the proof by showing that $S_M(\mu) \cap IR(\mu)$ is closed. Let

$$\begin{aligned}
O_M = \quad & \{(z, \delta) \in Z \times X : \text{ there exists } (a, \delta') \in \Delta \text{ such that } a \in \{m, s\}, \\
& \operatorname{supp}(\delta') \subseteq T^m_z(\mu) \cup \operatorname{supp}(\delta) \text{ if } a = m, \ \operatorname{supp}(\delta') \subseteq T^s_z(\mu) \text{ if } a = s \text{ and} \\
& (a, \delta', \mu) \succ_z (m, \delta, \mu)\}, \\
O_S = \quad & \{(z, \delta) \in Z \times (X_\emptyset \setminus X) : \text{ there exists } (a, \delta') \in \Delta \text{ such that} \\
& a \in \{m, s\}, \operatorname{supp}(\delta') \subseteq T^a_z(\mu) \text{ and } (a, \delta', \mu) \succ_z (s, \delta, \mu)\} \text{ and} \\
O_W = \quad & \{(z, \delta) \in Z \times X : \text{ there exists } (z', c) \in \operatorname{supp}(\delta) \text{ and } (a, \delta') \in \Delta \\
& \text{such that } a \in \{m, s\}, \operatorname{supp}(\delta') \subseteq T^a_{z'}(\mu) \text{ and } (a, \delta', \mu) \succ_{z'} (w, 1_{(z,c)}, \mu)\};
\end{aligned}$$

then $(S_M(\mu) \cap IR(\mu))^c = O_M \cup O_S \cup O_W$ and it suffices to show that O_M, O_S and O_W are open.

We start by showing that O_M is open. Let $(z, \delta) \in O_M$ and $(a, \delta') \in \Delta$ such that $a \in \{m, s\}$, $\operatorname{supp}(\delta') \subseteq T^m_z(\mu) \cup \operatorname{supp}(\delta)$ if $a = m$, $\operatorname{supp}(\delta') \subseteq T^s_z(\mu)$ if $a = s$ and $(a, \delta', \mu) \succ_z (m, \delta, \mu)$.

Suppose first that $a = m$ and, thus, $\delta' \in X$. Then $\delta' \in \Lambda(z, \delta, \mu)$. Let, by the continuity of preferences, $V_{\delta'}$, V_z and V_δ be open neighborhoods of δ', z and δ, respectively, such that $(a, \bar{\delta}, \mu) \succ_{\tilde{z}} (m, \tilde{\delta}, \mu)$ for each $(\bar{\delta}, \tilde{z}, \tilde{\delta}) \in V_{\delta'} \times V_z \times V_\delta$. Let, by the richness of E, $\tilde{V}_z$ and $\tilde{V}_\delta$ be open neighborhoods of z and δ, respectively, such that $\Lambda(z, \delta, \mu) \cap V_{\delta'} \neq \emptyset$ for each $(\tilde{z}, \tilde{\delta}) \in V_z \times V_\delta$.

Thus, for each $(\tilde{z},\tilde{\delta}) \in (V_z \cap \tilde{V}_z) \times (V_\delta \cap \tilde{V}_\delta)$, there exists $\bar{\delta} \in V_{\delta'}$ such that $\text{supp}(\bar{\delta}) \subseteq T^m_{\tilde{z}}(\mu) \cup \text{supp}(\tilde{\delta})$. Since $(\bar{\delta},\tilde{z},\tilde{\delta}) \in V_{\delta'} \times V_z \times V_\delta$, it follows that $(a,\bar{\delta},\mu) \succ_{\tilde{z}} (m,\tilde{\delta},\mu)$ and, hence, $(\tilde{z},\tilde{\delta}) \in O_M$. Thus, $(V_z \cap \tilde{V}_z) \times (V_\delta \cap \tilde{V}_\delta) \subseteq O_M$, showing that O_M is open.

Consider next the case $a = s$ and, thus, $\delta' \in X_\emptyset \setminus X$. Then $\delta' = 1_{(\emptyset,c)}$ for some $c \in C$ and an analogous argument to the one above using Claim 16 shows that O_M is open in this case.

An analogous argument shows that O_S and O_W are open. ■

B.7 An example of a Rosen market

We further illustrate the implications of stability in the following example of a Rosen market. Let $\nu(z_1) = \nu(z_2) = 1/2$ for some $0 < z_1 < z_2$ and productivity levels satisfy $q(z_1) = r(z_1) = q(z_2) = 1$ and $r(z_2) = 2$. Moreover, let $g(r) = r$ and $\theta(r/nq) = (r/nq)^\alpha$ where $\alpha \in (0,1)$.

Note that $wq(z_1) = wq(z_2) = w$ and that $R(z_2,w) > R(z_1,w)$ for each $w > 0$. Thus, we can have three cases: (i) all individuals of type z_1 are workers and all individuals of type z_2 are managers, (ii) all individuals of type z_2 are managers, some individuals of type z_1 are workers and the remaining individuals of type z_1 are managers, and (iii) all individuals of type z_1 are workers, some individuals of type z_2 are workers and the remaining individuals of type z_2 are managers.

Consider the case where all individuals of type z_1 are workers and all individuals of type z_2 are managers. Since $\nu(z_1) = \nu(z_2) = 1/2$, this requires each manager to be matched with one worker. The wage w is such that this firm size is optimal: $w = (1-\alpha)2^{1+\alpha}$. Thus, let μ assign measure 1/2 to $(z_2, 1_{(z_1,w)})$ i.e. $\mu = \frac{1}{2}1_{(z_2,1_{(z_1,w)})}$. We then have that the managers' payoff is $R(z_2,w) = \alpha 2^{1+\alpha}$.

We claim that μ is stable if and only if $\alpha \in [1/2,\bar{\alpha}]$ where $(1-\bar{\alpha})2^{1+\bar{\alpha}} = \bar{\alpha}\left(2^{1+\bar{\alpha}}\right)^{1-\frac{1}{\bar{\alpha}}}$.[42] Indeed, μ is stable if and only if (a) $R(z_2,w) \geq w$ (i.e. those of type z_2 prefer to be a manager rather than a worker) and (b) $w \geq R(z_1,w)$ (i.e. those

[42]This equality is equivalent to $\left(\frac{\bar{\alpha}}{1-\bar{\alpha}}\right)^{\frac{\bar{\alpha}}{1+\bar{\alpha}}} = 2$; letting $f(\alpha) = \left(\frac{\alpha}{1-\alpha}\right)^{\frac{\alpha}{1+\alpha}}$, then $f(1/2) = 1/2$, f is strictly increasing in α on $(1/2,1)$ and $\lim_{\alpha\to 1} f(\alpha) = \infty$; hence $\bar{\alpha} \in (1/2,1)$ exists and is unique. It can be seen that $\bar{\alpha}$ is approximately equal to 0.82.

of type z_1 prefer to be a worker rather than a manager). We have that (a) holds if and only if $\alpha \geq 1/2$ and that (b) holds if and only if $\alpha \leq \bar{\alpha}$. Thus, μ is indeed stable if and only if $\alpha \in [1/2, \bar{\alpha}]$.

We consider now the case where $\alpha > \bar{\alpha}$. In this case, the above matching fails to be stable because those of type z_1 prefer to be a manager rather than a worker. To obtain a stable matching in this case, a measure $\beta > 0$ of individuals of type z_1 will be managers and the remaining ones will be workers, which requires a wage w such that $w = R(z_1, w)$. Let μ be such that, for some $n_1, n_2, \beta, w > 0$, $\mu(z_2, n_2 1_{(z_1,w)}) = 1/2$ and $\mu(z_1, n_1 1_{(z_1,w)}) = \beta$.

The values for n_1, n_2, β and w are set as follows. The firm size are set to be optimal:

$$n_1 = n(z_1, z_1, w) = \left(\frac{1-\alpha}{w}\right)^{1/\alpha} \text{ and } n_2 = n(z_2, z_1, w) = \left(\frac{(1-\alpha)2^{1+\alpha}}{w}\right)^{1/\alpha}.$$

We have that $\int_{Z\times X} \delta(\{z_1\} \times C)\mathrm{d}\mu(z, \delta) = \beta n_1 + \frac{n_2}{2}$ and, thus, for μ to be a matching, it must be that $\frac{1}{2} = \nu(z_1) = \beta + \beta n_1 + \frac{n_2}{2}$. Using the expression for n_1 and n_2, it follows that

$$w = (1-\alpha)\left(\frac{2^{\frac{1+\alpha}{\alpha}} + 2\beta}{1-2\beta}\right)^{\alpha}.$$

Finally, we set $w = R(z_1, w) = \alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}}$, yielding

$$\beta = \frac{\alpha}{2} - (1-\alpha)2^{1/\alpha}.$$

Note that $\beta = 0$ if and only if $\left(\frac{\alpha}{1-\alpha}\right)^{\frac{\alpha}{1+\alpha}} = 2$, which holds if and only if $\alpha = \bar{\alpha}$. Moreover, $\beta \to 1/2$ as $\alpha \to 1$. Hence, $0 < \beta < 1/2$ since $\bar{\alpha} < \alpha < 1$.

We conclude with a comment on the case $\alpha < 1/2$. In this case, the first matching we considered fails to be stable because those of type z_2 prefer to be workers rather than managers. To obtain a stable matching in this case, a positive measure of individuals of type z_2 will be workers and the remaining ones will be managers, which requires a wage w such that $w = R(z_2, w)$. A stable matching μ can be constructed along the lines of the case $\alpha > \bar{\alpha}$ such that, for some $n^* = n(z_2, z_1, w) = n(z_2, z_2, w), \beta_1, \beta_2, w > 0$, $\mu(z_2, n^* 1_{(z_1,w)}) = \beta_1$ and $\mu(z_2, n^* 1_{(z_2,w)}) = \beta_2$.

B.8 Garicano and Rossi-Hansberg (2004)

We consider now the setting in Garicano and Rossi-Hansberg (2004) and show how it can be mapped into our framework.[43]

This setting can be briefly described as follows. Individuals are characterized by their knowledge, with $Z = [0, \bar{z}]$ denoting the set of knowledge levels and where $\bar{z} \in \mathbb{R}_{++}$; the knowledge distribution is denoted by ν. Individuals can be workers, managers or self-employed.

A firm consists of one manager and several workers of the same type, i.e. there is many-to-one matching. Production happens when a worker solves a problem with which he is faced. The set of all possible problems is Z and problems are drawn according to a probability measure F on Z with a continuous and strictly decreasing density. Each worker is allowed to ask the manager for the solution of the problem he has drawn if he cannot solve it himself. Knowledge is cumulative: If someone has knowledge $z \in Z$, then he can solve all problems in $[0, z]$. Thus, a worker with knowledge z asks for help with probability $1 - F(z)$. Asking for help incurs a communication cost: The manager incurs a cost of $0 < h < 1$ units of time to attempt solving the problem regardless of whether or not he succeeds. Individuals have one unit of time, which will be entirely spent working in the case of workers and on helping workers in the case of managers. Thus, a firm with a manager with knowledge z and workers with knowledge z' can have a measure n of workers provided that

$$nh(1 - F(z')) = 1,$$

i.e. such that the manager exhausts his time helping the workers. Problems in $[0, \max\{z, z'\}]$ get to be solved, either by the workers or by the manager. Workers draw one problem per unit of time spent in production and output is 1 if the problem is solved and 0 otherwise. Expected production is then $F(\max\{z, z'\})n$ and the

[43] Unlike what we did regarding Rosen's (1982) setting, we do not provide a detailed analysis of this setting but rather leave it for a future paper, Carmona and Laohakunakorn (2023b), the characterization of stable matchings in a generalized version of Garicano and Rossi-Hansberg's (2004) model.

managers' rent is $(F(\max\{z, z'\}) - c)n$, where c is the wage paid to the workers.

A self-employed individual with knowledge z solves the problems that he can and, thus, produces $F(z)$.

The representation of this setting in the general framework of markets with occupational choice is as follows. First, let Z and ν be as above. Second, we let contracts be wages and set $C = \mathbb{R}_+$ and $\mathbb{C} \equiv C$. We incorporate the time constraint of managers in the set X of feasible matches for managers and let

$$X = \{n1_{(z,c)} : (z, c) \in Z \times C \text{ and } n \in \mathbb{R}_+ \text{ such that } nh(1 - F(z)) = 1\}$$

since managers can hire several workers all of the same type such that his time constraint is satisfied. Occupations are the same as in the general framework: $A = \{w, s, m\}$. Finally, preferences are defined by specifying payoff functions as follows:

$$\begin{aligned} &U_z(w, 1_{(z',c)}) = c \text{ for each } 1_{(z',c)} \in X_w, \\ &U_z(s, 1_{(\emptyset,c)}) = F(z) \text{ for each } 1_{(\emptyset,c)} \in X_s, \text{ and} \\ &U_z(m, n1_{(z',c)}) = (F(\max\{z, z'\}) - c)n \text{ for each } n1_{(z',c)} \in X_m. \end{aligned}$$

B.9 Garicano and Rossi-Hansberg (2006)

In this section we consider the setting in Garicano and Rossi-Hansberg (2006), which extends that of Garicano and Rossi-Hansberg (2004) as follows.

Individuals are characterized by their cognitive ability. The set of possible abilities is $[0, 1]$ and ν is its distribution. Knowledge is no longer a given characteristic of individuals but rather their choice. An individual's ability affects his cost of acquiring knowledge: if he learns how to solve problems in $[0, k]$, with $k \in \mathbb{R}_+$, and has ability $z \in [0, 1]$, then the cost is $(t - z)k$, where $t > 1$. The set of all possible problems is $\mathbb{R}_+$ and problems are drawn according to a probability measure F on $\mathbb{R}_+$ with a continuous and strictly decreasing density.

As in the previous section, a firm consists of a manager and several workers but these can now be organized in several layers as follows.[44] Workers that belong to

[44]We depart here from the terminology of Garicano and Rossi-Hansberg (2006) to facilitate the

the same layer have the same knowledge and only those in layer 1 spend time in production; in fact, workers in layer 1 spend all their time in production. Problems that they cannot solve are asked to layer 2; those that layer 2 cannot solve are then passed to layer 3 and so on. The last layer is that of the manager: If there are $L \in \mathbb{N}$ layers of workers, then the manager is in layer $L+1$ and attempts to solve only those problems that layers $1, \ldots, L$ cannot solve. If the knowledge of workers in layer $1 \leq l \leq L$ is k_l, then the number n_l of workers in layer l satisfies:

$$n_1 h(1 - F(\max\{k_1, \ldots, k_{l-1}\})) = n_l \text{ if } 1 < l \leq L \text{ and}$$
$$n_1 h(1 - F(\max\{k_1, \ldots, k_L\})) = 1.$$

This is the time constraint of layer l as it attempts to solve the problems draw by those in layer 1 that layers $1, \ldots, l-1$ have not been able to solve. Production is then $F(\max\{k_1, \ldots, k_L, k\})n_1$ if the knowledge of the manager is k. Profits equal production minus wages and learning costs: If the ability of the manager is z and that of those in layer l is z_l and their wage is v_l for each $l = 1, \ldots, L$, then the profit of the firm, or the manager's rent, is

$$F(\max\{k_1, \ldots, k_L, k\})n_1 - \sum_{l=1}^{L}((t - z_l)k_l + v_l)n_l - (t - z)k.$$

A self-employed individual with knowledge k and ability z has a profit of $F(k) - (t - z)k$.

The representation of this setting in the general framework of markets with occupational choice is as follows. First, let $Z = [0, 1]$ and ν be as above. The firm-specific variables, which are the knowledge of the manager and workers and the workers' wages, constitute the contracts. Thus, the set of contracts is $C = \{(k, k', v) : k, k', v \in \mathbb{R}_+\}$ and let $\mathbb{C} \equiv C$. The time constraints are incorporated in the set X of feasible

mapping of their setting to our framework of markets with occupational choice. While in their paper workers are only those in the lowest layer, in our terminology here, workers are all members of the firm except the highest layer. When there are only two layers, as in the previous section, no distinction arises between the two terminologies.

matches which, in addition, reflect the restriction that there are finitely many layers:

$$X = \Big\{ \sum_{l=1}^{L} n_l 1_{(z_l,k,k_l,v_l)} : L \in \mathbb{N}, z_l \in Z, k_l \in \mathbb{R}_+, v_l \in \mathbb{R}_+, n_l \in \mathbb{R}_+$$
$$\text{for each } 1 \leq l \leq L \text{ such that}$$
$$n_1 h(1 - F(\max\{k_1, \ldots, k_L\})) = 1 \text{ and}$$
$$n_1 h(1 - F(\max\{k_1, \ldots, k_{l-1}\})) = n_l \text{ for each } l = 2, \ldots, L \Big\}.$$

A possible match is then of the form $(z, \sum_{l=1}^{L} n_l 1_{(z_l,k,k_l,v_l)})$, thus specifying the ability z and knowledge k of the manager, the ability z_l, knowledge k_l and wage v_l of each worker, as well as the number of workers n_l, in layer l. Occupations are the same as in the general framework: $A = \{w, s, m\}$. Finally, preferences are defined by specifying payoff functions as follows: For each $z \in Z$,

$$U_z(w, 1_{(z',k,k',v)}) = v \text{ for each } 1_{(z',k,k',v)} \in X_w,$$
$$U_z(s, 1_{(\emptyset,k,k',v)}) = F(k) - (t - z)k \text{ for each } 1_{(\emptyset,k,k',v)} \in X_s, \text{ and}$$
$$U_z(m, \delta) = F(\max\{k_1, \ldots, k_L, k\}) n_1 - \sum_{l=1}^{L} \big(v_l - (t - z_l)k_l\big) n_l - (t - z)k$$

for each $\delta = \sum_{l=1}^{L} n_l 1_{(z_l,k,k_l,v_l)} \in X_m$.

B.10 Adding capital: Lucas (1978)

In this section we consider the setting in Lucas (1978). This is a setting where there is a capital market in addition to a labor market with occupational choice and this feature requires an extension to our framework of markets with occupational choice.

Lucas's (1978) setting is as follows. There is a workforce of size N and K units of capital. Individuals are characterized by their managerial talent $z \in \mathbb{R}_+$; $\nu \in \mathcal{M}(\mathbb{R}_+)$ is the managerial talent distribution and satisfies $\nu(\mathbb{R}_+) = N$. Production requires a manager who employs labor and capital: if the manager has talent x and manages n units of labor and k of capital, then output is $xg \circ f(n, k)$. The manager's rent is $xg(f(n, k)) - vn - rk$, where v is the wage and r the rental price of capital.

We can think of this setting as one where managers are being matched with workers and owners of capital, each of the latter having one unit of capital and preferences for higher rental prices of capital. The matching problem would be an hybrid between a marriage market and a labor market with occupation choice since there would be two distinct groups, capitalists vs non-capitalists, to match as well as matching subject to occupational choice within the non-capitalist group. A contract could specify a wage and a rental price of capital to be paid by the manager to the workers and capitalists, respectively.

An easier approach to represent this setting is as follows: Note that, for a fixed rental price of capital, it defines a labor market with occupational choice with the amount of capital hired by a firm being included in the contract between the manager and workers. In this representation, as in Lucas (1978), capitalists are not explicitly modeled and, consequentially, the rental price of capital can no longer be part of the contract since then there is no counterpart to the manager's desire to set it equal to zero. The notion of a stable matching can then be applied to the market defined by each rental price of capital. Thus, an equilibrium in this setting is a rental price of capital and a matching such that the matching is stable given the rental price of capital and the capital market clears.

In light of the above, let, for each $r \in \mathbb{R}_+$, $Z = \mathbb{R}_+$ be the set of possible managerial talent and ν be as above. The set of contracts is $C = \mathbb{R}^2_+$ with a generic contract $c = (v, k)$ specifying the wage and amount of capital and $\mathbb{C} \equiv C$. A manager is matched with only one type of workers and one contract, thus the set of feasible matches is $X = \{n1_{(z,c)} : (z, c) \in Z \times C \text{ and } n \in \mathbb{R}_+\}$. Occupations are the same as in the general framework: $A = \{w, s, m\}$. Finally, preferences are defined via payoff functions: For each $z \in Z$,

$$U_z(w, 1_{(z',(v,k))}) = v \text{ for each } 1_{(z',v,k)} \in X_w,$$

$$U_z(s, 1_{(\emptyset,c)}) = 0 \text{ for each } 1_{(\emptyset,c)} \in X_s, \text{ and}$$

$$U_z(m, n1_{(z',(v,k))}) = zg(f(n, k)) - vn - rk \text{ for each } n1_{(z',v,k)} \in X_m.$$

This defines a market E_r. We then say that (r, μ) is an *equilibrium* if μ is a stable matching of E_r and

$$\int_{Z\times X}\left(\frac{1}{\delta(Z\times C)}\int_{Z\times\mathbb{R}^2_+} k\mathrm{d}\delta(z,v,k)\right)\mathrm{d}\mu(z,\delta)=K.$$

When $\delta = n1_{(z,v,k)}$ for some (n,z,v,k), note that $\frac{1}{\delta(Z\times C)}\int_{Z\times\mathbb{R}^2_+} k'\mathrm{d}\delta(z,v,k')=k$, i.e. this is the amount of capital hired by a given manager; thus, writing $k(\delta)=k$, it follows that the total amount of capital hired is

$$\int_{Z\times X}\left(\frac{1}{\delta(Z\times C)}\int_{Z\times\mathbb{R}^2_+} k\mathrm{d}\delta(z,v,k)\right)\mathrm{d}\mu(z,\delta)=\int_{Z\times X} k(\delta)\mathrm{d}\mu(z,\delta)$$

which, in equilibrium, equals the total amount of capital available.

B.11 Relationship with CKK

CKK consider a many-to-one matching setting with a continuum of workers and finitely many managers. It can briefly described as follows. There is a finite set $\{1,\ldots,n\}$ of managers and a mass of workers. The set of workers' types is Θ, a compact metric space, and G is the type distribution with $G(\Theta)=1$.

Each worker has a strict preference P over $\{1,\ldots,n\}\cup\{\emptyset\}$. Let $\mathcal{P}$ denote the (finite) set of all possible workers' preferences and, for each $P\in\mathcal{P}$, let Θ_P denote the set of all worker types whose preference is given by P. It is assumed that the set Θ_P is measurable and that $G(\partial\Theta_P)=0$. Manager $m\in\{1,\ldots,n\}$ has preferences described by $\succeq_m$.

We represent the above setting as a two-sided market E without empty workers or externalities as follows.[45] Regarding the workers, set $W=\cup_{P\in\mathcal{P}}\mathrm{int}(\Theta_P)$, $\nu_W=G|_W$ and $\succ_w = P$ whenever $w\in\Theta_P$. Because $G(\partial\Theta_P)=0$, it follows that $G(\Theta\setminus W)=0$; thus, only a null set of types of workers are excluded in our representation. Note that W is a Polish space (see e.g. Aliprantis and Border (2006, Corollary 3.35, p. 89)).

Regarding the managers, set $M=\{1,\ldots,n\}$, $\nu_M(B)=|B|$ for each $B\subseteq M$ and $\succ_m$ be the asymmetric part of $\succeq_m$. Furthermore, let the feasible matches be $X=\mathcal{M}_1(W)$.

[45]See Section B.2 for a defintion of a two-sided market without empty workers or externalities.

CKK describe a matching by listing the workforce of each manager: $\boldsymbol{\delta} = (\delta_m)_{m \in M_\emptyset}$ is a matching if $\delta_m \in X$ for each $m \in M_\emptyset$ and $\sum_{m \in M_\emptyset} \delta_m = G$.[46] A matching $\boldsymbol{\delta}$ is stable if (i) $\delta_m(\Theta_P) = 0$ for each $P \in \mathcal{P}$ and $m \in M$ such that $m \prec_P \emptyset$, and (ii) there does not exist $m \in M$ and $\delta' \in X$ such that $\delta'(E) \leq D^{\preceq m}(\boldsymbol{\delta})(E)$ for each Borel $E \subseteq W$ and $\delta' \succ_m \delta_m$, where $D^{\preceq m}(\boldsymbol{\delta})$ is the measure of workers assigned to manager m or worse, i.e. $D^{\preceq m}(\boldsymbol{\delta})(E) = \sum_{P \in \mathcal{P}} \sum_{m' \in M_\emptyset : m' \preceq_P m} \delta_{m'}(\text{int}(\Theta_P) \cap E)$ for each Borel $E \subseteq W$.

Given a matching $\boldsymbol{\delta}$ in CKK, define $\mu_{\boldsymbol{\delta}}$ by setting

$$\mu_{\boldsymbol{\delta}} = \sum_{m \in M} 1_{(m, \delta_m)} + 1_\emptyset \otimes (\delta_\emptyset \circ f^{-1}),$$

where $f : W \to \{1_w : w \in W\}$ is defined by setting, for each $w \in W$, $f(w) = 1_w$.[47] The following result shows that every matching $\boldsymbol{\delta}$ such that $\mu_{\boldsymbol{\delta}}$ is stable in the two-sided market E without empty workers or externalities is stable in CKK's setting, i.e. stability with a continuum of managers implies stability with finitely many managers.

Theorem 6 *If $\boldsymbol{\delta} = (\delta_m)_{m \in M_\emptyset}$ is a matching in CKK, then:*

1. *$\mu_{\boldsymbol{\delta}}$ is a matching.*
2. $\text{supp}(D^{\preceq m}(\boldsymbol{\delta})) = \text{supp}(\delta_m) \cup T_m(\mu_{\boldsymbol{\delta}})$.
3. *If $\mu_{\boldsymbol{\delta}}$ is stable, then $\boldsymbol{\delta}$ is stable.*

Proof. Let $\boldsymbol{\delta}$ be a matching in CKK and write μ instead of $\mu_{\boldsymbol{\delta}}$.

Consider the first claim. Condition (TS3) of the definition of a matching is trivial and condition (TS4) holds by the definition of μ. Condition (TS1) holds since $\mu(B \times X) = |B| = \nu_M(B)$ for each $B \subseteq M$. Condition (TS2) holds since $\int_{M_\emptyset \times X} \delta(B) \mathrm{d}\mu(m, \delta) = \sum_{m \in M_\emptyset} \delta_m(B) = G(B) = \nu_W(B)$ for each Borel $B \subseteq W$.

[46] Note that δ_m, for each $m \in M_\emptyset$, and δ' belong to the more specific set $X = \{\delta \in \mathcal{M}(W) : \delta(B) \leq G(B) \text{ for each Borel } B \subseteq W\}$ that CKK consider.

[47] The function f is the standard homeomorphism between W and $\{1_w : w \in W\}$.

Consider next the second claim. Let $w \in \text{supp}(D^{\preceq m}(\boldsymbol{\delta}))$ and O be an open neighborhood of w. Then $O \cap \text{int}(\Theta_{P_w})$ is an open neighborhood of w and, hence,

$$0 < D^{\preceq m}(\boldsymbol{\delta})(O \cap \text{int}(\Theta_{P_w})) = \sum_{m' \in M_\emptyset : m \succeq_w m'} \delta_{m'}(O \cap \text{int}(\Theta_{P_w})) \leq \sum_{m' \in M_\emptyset : m \succeq_w m'} \delta_{m'}(O).$$

It then follows that $w \in \text{supp}(\sum_{m' \in M_\emptyset : m \succeq_w m'} \delta_{m'}) = \cup_{m' \in M_\emptyset : m \succeq_w m'} \text{supp}(\delta_{m'})$. Thus, if $w \notin \text{supp}(\delta_m)$, then $w \in \text{supp}(\delta_{m'})$ for some $m' \in M_\emptyset$ such that $m \succ_w m'$. If $m' \in M$, then $w \in T_m(\mu)$ since $(m', \delta_{m'}) \in \text{supp}(\mu)$, $w \in \text{supp}(\delta_{m'})$ and $m \succ_w m'$. If $m' = \emptyset$, then $(\emptyset, 1_w) \in \text{supp}(\mu)$, $w \in \text{supp}(1_w)$ and $m \succ_w \emptyset$, hence $w \in T_m(\mu)$.

Conversely, if $w \in T_m(\mu)$, then there is $(m', \delta') \in \text{supp}(\mu)$ such that $w \in \text{supp}(\delta')$ and $m \succ_w m'$. If $m' \in M$, then $\delta' = \delta_{m'}$ and, hence, $w \in \cup_{m' \in M_\emptyset : m \succ_w m'} \text{supp}(\delta_{m'})$. If $m' = \emptyset$, then $\delta' = 1_w$ and $w \in \text{supp}(\delta_\emptyset)$. In conclusion, $w \in \cup_{m' \in M_\emptyset : m \succ_w m'} \text{supp}(\delta_{m'})$.

It remains to show that $\cup_{m' \in M_\emptyset : m \succeq_w m'} \text{supp}(\delta_{m'}) = \text{supp}(\sum_{m' \in M_\emptyset : m \succeq_w m'} \delta_{m'})$ is contained in $\text{supp}(D^{\preceq m}(\boldsymbol{\delta}))$, which follows from an argument analogous to the one above.

Finally, we establish the last claim. Assume that μ is a stable matching. Let $P \in \mathcal{P}$ and $m \in M$ be such that $\emptyset \succ_P m$ and let $w \in \Theta_P$. Thus, $\emptyset \succ_w m$. If $w \in \text{supp}(\delta_m)$, then $(m, \delta_m) \in \text{supp}(\mu) \cap IR_W(\mu)^c$, a contradiction to the stability of μ. Thus, $w \notin \text{supp}(\delta_m)$ and, hence, $\Theta_P \subseteq \text{supp}(\delta_m)^c$. Thus, $\delta_m(\Theta_P) = 0$.

Suppose that there is $m \in M$ and $\delta' \in X$ such that $\delta' \sqsubset D^{\preceq m}(\boldsymbol{\delta})$ and $\delta' \succ_m \delta_m$. Since $(m, \delta_m) \in \text{supp}(\mu)$ and $\text{supp}(\delta') \subseteq \text{supp}(D^{\preceq m}(\boldsymbol{\delta})) \subseteq T_m(\mu) \cup \text{supp}(\delta_m)$, it follows that $(m, \delta_m) \in \text{supp}(\mu) \cap S_M(\mu)^c$, a contradiction to the stability of μ. ■

B.12 Relationship with Azevedo and Hatfield (2018)

We consider Azevedo and Hatfield's (2018) setting in its simplified version presented in Section S.10 in CKK. We show that this special case of their model is a particular case of our two-sided market without empty workers or externalities.[48]

This is a setting with a finite set M of manager types, a finite set W of worker types, non-null measures ν_M and ν_W and no contracts.[49] Each manager hires at most

[48]See Section B.2 for a definition of a two-sided market without empty workers or externalities.

[49]The latter aspect can be formalized by letting C be a singleton and $\mathbb{C} \equiv C$, but it is simpler to just omit contracts altogether.

one worker per each worker type, thus, the set X of feasible matches for a manager is

$$X = \left\{ \sum_{w \in A} 1_w : A \in 2^W \right\}.$$

In particular, the zero measure belongs to X since it equals $1_\emptyset = \sum_{w \in \emptyset} 1_w$. Together with the absence of contracts, this makes it unnecessary to introduce the empty worker type since unmatched managers are those matched with the zero measure.

Preferences do not depend on the matching. In this case, $\succ_m$ can be defined on X for each $m \in M$ and $\succ_w$ can be defined on $M_\emptyset$ for each $w \in W$.

For each $m \in M$ and $E \in 2^W$, let

$$c_m(E) = \{A \in 2^W : A \subseteq E \text{ and there is no } B \subseteq E \text{ such that } 1_B \succ_m 1_A\}.$$

A matching $\mu \in \mathcal{M}(M_\emptyset \times X)$ is stable according to definition S2 in CKK if

1. $\mu(m, \delta) = 0$ if there exists $w \in \text{supp}(\delta)$ such that $\emptyset \succ_w m$,

2. $c_m(E) = E$ for each $(m, 1_E) \in \text{supp}(\mu)$, and

3. There are no $m \in M$ and $E, E' \in 2^W$ with $E \cap E' = \emptyset$ and $E' \neq \emptyset$ such that (i) $E' \subseteq c_m(E \cup E')$, (ii) $(m, 1_E) \in \text{supp}(\mu)$ and (iii) for each $w \in E'$, there exists $\hat{m} \in M_\emptyset$ and $\hat{E} \in 2^W$ such that $(\hat{m}, 1_{\hat{E}}) \in \text{supp}(\mu)$, $w \in \text{supp}(1_{\hat{E}}) = \hat{E}$ and $m \succ_w \hat{m}$.

Note that condition 1 is equivalent to $\text{supp}(\mu) \subseteq IR_W(\mu)$. Furthermore, condition 2 is equivalent to the requirement that, for each $(m, 1_E) \in \text{supp}(\mu)$, there does not exist $\delta' \in X$ such that $\text{supp}(\delta') \subseteq \text{supp}(\delta)$ and $\delta' \succ_m 1_E$. Finally, condition 3 is equivalent to the requirement that, for each $(m, 1_E) \in \text{supp}(\mu)$, there does not exist $\delta^* = 1_{E^*}$ such that $\text{supp}(\delta^*) \nsubseteq \text{supp}(\delta)$, $\text{supp}(\delta^*) \subseteq \text{supp}(\delta) \cup T_m(\mu)$ and $\delta^* \succ_m 1_E$.[50] Thus, the above definition is equivalent to the notion of stability for the two-sided market without empty workers or externalities.

[50]Condition 3 (i) is equivalent to the existence of $E^* \in 2^W$ such that $1_{E^*} \succ_m 1_E$ and $E' \subseteq E^* \subseteq E \cup E'$, and condition 3 (iii) to $\text{supp}(1_{E'}) \subseteq T_m(\mu)$. Thus, $\text{supp}(1_{E^*}) \subseteq \text{supp}(1_E) \cup T_m(\mu)$.

B.13 An example of a roommate market with externalities and relationship to Wu (2021)

Consider the following roommate market where contracts are omitted for simplicity. Let $Z = \{a, b, c\}$ with $\nu(a) = \nu(b) = \nu(c) = 1$, and let preferences be given by:

- $u_a(b, \mu) = 1 - 2(\mu(b, c) + \mu(c, b))$ and $u_a(c, \mu) = \frac{1}{2}$ for each $\mu \in \mathcal{M}(Z \times Z_\emptyset)$,
- $u_b(c, \mu) = 1$ and $u_b(a, \mu) = \frac{1}{2}$ for all $\mu \in \mathcal{M}(Z \times Z_\emptyset)$,
- $u_c(a, \mu) = 1$ and $u_c(b, \mu) = \frac{1}{2}$ for all $\mu \in \mathcal{M}(Z \times Z_\emptyset)$,
- $u_z(\emptyset, \mu) = 0$ for each $z \in Z$ and $\mu \in \mathcal{M}(Z \times Z_\emptyset)$, and
- $u_z(z, \mu) = -1$ for each $z \in Z$ and $\mu \in \mathcal{M}(Z \times Z_\emptyset)$.

Note that a prefers to match with b rather than c if and only if the measure of b that is matched with c is less than $\frac{1}{4}$.

By Corollary 2, there exists a stable matching; indeed, the matching μ such that $\mu(a, c) = \frac{3}{4}$, $\mu(a, b) = \frac{1}{4}$, $\mu(b, c) = \frac{1}{4}$ and $\mu(b, \emptyset) = \frac{1}{2}$ is stable.

For the remainder of this section, we explain why Wu's (2021) result does not apply to this setting. First, note that given the assumptions of the roommate market, any two matchings μ and $\hat{\mu}$ are equivalent if $\mu(a, b) + \mu(b, a) = \hat{\mu}(a, b) + \hat{\mu}(b, a)$, $\mu(b, c) + \mu(c, b) = \hat{\mu}(b, c) + \hat{\mu}(c, b)$ and $\mu(a, c) + \mu(c, a) = \hat{\mu}(a, c) + \hat{\mu}(c, a)$. Thus, by identifying equivalent matchings, we can also represent a matching μ as an element of $\mathcal{M} = \{\mu \in [0, 1]^3 : \mu_1 + \mu_3 \leq \nu(a), \mu_1 + \mu_2 \leq \nu(b), \mu_2 + \mu_3 \leq \nu(c)\}$, i.e. we can write

$$\mu = (\mu(a, b) + \mu(b, a), \mu(b, c) + \mu(c, b), \mu(a, c) + \mu(c, a))$$

in line with Wu's (2021) notation. In what follows, we abuse notation and treat μ both as an element of $\mathcal{M}$ defined above and also as a measure over $Z \times Z_\emptyset$, whichever is more convenient (for example, we write $(a, b) \in \text{supp}(\mu)$ or $(b, a) \in \text{supp}(\mu)$ to mean $\mu_1 > 0$).

Wu's (2021) matching game consists of a set of players I, a set of matchings $\mathcal{M}$, a participation function $\phi : \mathcal{M} \to [0, 1]^I$, a complete and transitive preference relation

$\trianglerighteq_i$ defined on $\mathcal{M}_i = \{\mu \in \mathcal{M} : \phi_i(\mu) > 0\}$ for each player i, and a blocking relation $\sqsupset$.

We represent the above roommate market as a matching game as follows. Let $I = Z$, let $\mathcal{M} = \{\mu \in [0,1]^3 : \mu_1 + \mu_3 \leq 1, \mu_1 + \mu_2 \leq 1, \mu_2 + \mu_3 \leq 3\}$, let $\phi_i(\mu) = \mu(\{i\} \times Z) + \mu(Z \times \{i\})$, let $\trianglerighteq_a$ be any complete and transitive relation such that $(1,0,0) \trianglerighteq_a (0,0,1)$,[51] let $\trianglerighteq_b$ and $\trianglerighteq_c$ be any complete and transitive relations, and define the blocking notion $\sqsupset$ as follows: $\hat{\mu} \sqsupset \tilde{\mu}$ ($\hat{\mu}$ blocks $\tilde{\mu}$) if for each $(z, z') \in \text{supp}(\hat{\mu})$, there exists $\tilde{z}$ and $\tilde{z}'$ such that (i) $(z, \tilde{z}) \in \text{supp}(\tilde{\mu})$ or $(\tilde{z}, z) \in \text{supp}(\tilde{\mu})$, (ii) $(z', \tilde{z}') \in \text{supp}(\tilde{\mu})$ or $(\tilde{z}', z') \in \text{supp}(\tilde{\mu})$, (iii) $u_z(z', \tilde{\mu}) > u_z(\tilde{z}, \tilde{\mu})$ and (iv) $u_{z'}(z, \tilde{\mu}) > u_{z'}(\tilde{z}', \tilde{\mu})$. This blocking notion corresponds to the one we use in the current paper.

Now we argue that the above matching game is not a *convex matching game* as defined by Wu (2021), and so his Theorem 1 does not apply. Let $\mu^1 = (1,0,0)$, let $\mu^2 = (0,1,0)$, let $\mu^3 = (0,0,1)$ and let $\mu^* = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. The participation vectors are $\phi(\mu^1) = (1,1,0)$, $\phi(\mu^2) = (0,1,1)$ and $\phi(\mu^3) = (1,0,1)$. Let $w = (w^1, w^2, w^3) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and note that $\mu^* = \sum_{j=1}^3 w^j \mu^j$ and $\sum_{j=1}^3 w^j \phi(\mu^j) \leq \mathbf{1}$; thus, μ^* is a ϕ-convex combination of $\{\mu^j\}_{j=1}^3$.[52] Wu's (2021) definition of a convex matching game (Definition 2, part (ii)) then requires that $\mu^3 \not\sqsupset \mu^*$ because $\phi_a(\mu^3) > 0$, $\sum_{j=1}^3 w^j \phi_a(\mu^j) = 1$ and $\mu^j \trianglerighteq_a \mu^3$ for all μ^j with $\phi_a(\mu^j) > 0$.[53]

However, by the notion of blocking defined above, we have $\mu^3 \sqsupset \mu^*$ because $(a,b) \in \text{supp}(\mu^*)$ or $(b,a) \in \text{supp}(\mu^*)$, $(c,b) \in \text{supp}(\mu^*)$ or $(b,c) \in \text{supp}(\mu^*)$, $u_a(c, \mu^*) > u_a(b, \mu^*)$ and $u_c(a, \mu^*) > u_c(b, \mu^*)$.

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[51] This restriction reflects the fact that a prefers to matched with b rather than c, if the remaining agents remain unmatched.

[52] As defined by Wu (2021), $\sum_{j=1}^n w^j \mu^j$ is a ϕ-convex combination if $\sum_{j=1}^n w^j \phi(\mu^j) \leq \mathbf{1}$.

[53] We have $\mu^1 \trianglerighteq_a \mu^3$ by the restriction we imposed on $\trianglerighteq_a$ to reflect preferences in the roommate market and we have $\mu^3 \trianglerighteq_a \mu^3$ because $\trianglerighteq_a$ is complete.

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