Supplementary Material for "Improving the Organization of Knowledge in Production by Screening Problems"

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Contents

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1 Introduction

This paper contains supplementary material to our paper "Improving the Organization of Knowledge in Production by Screening Problems". It contains:

Section 2: A proof of Theorem 4 on the existence of η-optimal organizations.

Section 3: Sufficient conditions for optimal organizations to have at least two layers and nonexistence of lexicographically optimal organizations when ξ is sufficiently small.

Section 4: Codes for the computations in Section 5.2.

Section 5: Description of the simulations used in Section 5.3 and results for other parameter values.

Section 6: An example to illustrate Theorem 7.

Section 7: Proofs for the results with cumulative knowledge.

2 Existence of optimal organizations

This section contains the proof of Theorem 4. It requires strengthening the proofs of the results stated in the main body of the paper in a way that they shows that not only optimal organizations have certain properties but also that any organization that does not have them is dominated, according to the relation $>$ defined below, by another one that satisfies them.

2.1 Notation

Recall the following notation: For each $i, k \in L$,

$$\nu_i = c\mu(A_i) + \xi\mu(B_i \setminus A_i) \text{ and}$$

$$\alpha_{ik} = \begin{cases} hF\left(A_i \setminus \cup_{l \prec_k i} A_l\right) + \pi F\left(A_i^c \setminus \cup_{l \prec_k i} B_l\right) & \text{if } i \in l_k \text{ and } k \neq i, \\ 0 & \text{otherwise.} \end{cases}$$

In addition, sometimes we abuse notation and use L to denote the cardinality of the set L.

Given two organizations O and $\hat{O}$, we write $\hat{O} > O$ if (i) $\hat{y} > y$ and $\hat{L} \subseteq L$, or (ii) $\hat{y} = y$ and $\hat{L} \subset L$ or (iii) $\hat{y} = y$, $L = \hat{L}$, $\hat{l}_i \subseteq l_i$ for all $i \in L$ and $\hat{l}_j \neq l_j$ for some $j \in L$. Note that, while the relation $>$ is neither the relation in the definition of lexicographical optimality nor in the definition of η-optimality, we have that if $\hat{O} > O$, then O is not an optimal organization.

In addition, we write $\hat{O} \geq O$ if $\hat{O} = O$ or $\hat{O} > O$. Furthermore, we write $\hat{O} \gtrsim O$ if $\hat{O} > O$ or $\hat{y} = y$, $\hat{L} = L$ and $\hat{l}_i = l_i$ for all $i \in L$.

2.2 Allocation of labor

In this section we show that any optimal organization satisfies $t_i^p + t_i^h = 1$ for each $i \in L$ and $l_j = \{j\}$ for each $j \in L$ such that $t_j^p = 0$. More generally, any organization that fails to satisfy at least one of these condition is dominated by some organization that satisfies all of them.

Let $\mathcal{O}_1$ be the set of organizations O such that $y > 0$ and $l_i = \{i\}$ for each $i \in L$ with $t_i^p = 0$.

Lemma 2.1 *If $O \notin \mathcal{O}_1$ and $y > 0$, then there is $\hat{O} \in \mathcal{O}_1$ such that $\hat{O} > O$.*

Proof. Let $O \notin \mathcal{O}_1$ be such that $y > 0$ and $I = \{i \in L : t_i^p = 0 \text{ and } \{i\} \subset l_i\}$; then $I \neq \emptyset$. Define $\hat{O}$ as follows: $\hat{L} = L$; for each $j \in L$, set: $\hat{A}_j = A_j$, $\hat{B}_j = B_j$, $\hat{\beta}_j = \beta_j$, $\hat{t}_j^p = t_j^p$ and $\hat{t}_j^h = t_j^h$; finally, set $\hat{l}_j = l_j$ and $\hat{\prec}_j = \prec_j$ for each $j \in L \setminus I$ and $\hat{l}_j = \{j\}$ for each $j \in I$.

Since $\hat{t}_k^p = t_k^p = 0$ for each $k \in I$, we have that $\hat{\beta}_i \hat{t}_i^h = \beta_i t_i^h = \sum_{k \in L \setminus I} \beta_k t_k^p \alpha_{ik} = \sum_{k \in L \setminus I} \hat{\beta}_k \hat{t}_k^p \hat{\alpha}_{ik} = \sum_{k \in L} \hat{\beta}_k \hat{t}_k^p \hat{\alpha}_{ik}$ for each $i \in L$ and, hence, $\hat{O}$ is an organization. Using again $\hat{t}_i^p = t_i^p = 0$ for each $i \in I$, it follows that $\hat{y} = \sum_{i \in L \setminus I} \beta_i (t_i^p F(\cup_{l \in l_i} A_l) - c\mu(A_i) - \xi\mu(B_i \setminus A_i)) - \sum_{i \in I} \beta_i (c\mu(A_i) + \xi\mu(B_i \setminus A_i)) = y$. This, together with $\hat{L} = L$, $\hat{l}_j = l_j$ for all $j \in L \setminus I$ and $\hat{l}_j \subset l_j$ for each $j \in I \neq \emptyset$, show that $\hat{O} > O$. Since $y > 0$, it follows that $\hat{y} > 0$ and, by construction, $\hat{O} \in \mathcal{O}_1$. ■

Let $\mathcal{O}_2$ be the set of organizations O such that $y > 0$ and $t_i^p + t_i^h > 0$ for each $i \in L$.

Lemma 2.2 *If $O \in \mathcal{O}_1 \setminus \mathcal{O}_2$, then there is $\hat{O} \in \mathcal{O}_1 \cap \mathcal{O}_2$ such that $\hat{O} > O$.*

Proof. Let $O \in \mathcal{O}_1 \setminus \mathcal{O}_2$ and $I = \{i \in L : t_i^p + t_i^h = 0\}$; then $I \neq \emptyset$. Since $y > 0$, then $I \neq L$; hence $\lambda := 1/(\sum_{i \in L \setminus I} \beta_i) > 1$. Define $\hat{O}$ as follows: $\hat{L} = L \setminus I$; for each $j \in \hat{L}$, set: $\hat{A}_j = A_j$, $\hat{B}_j = B_j$, $\hat{\beta}_j = \lambda \beta_j$, $\hat{l}_j = l_j \setminus I$, $\hat{\prec}_j = \prec_j |_{\hat{l}_j}$ (i.e., for each $i, l \in \hat{l}_j$, $i \hat{\prec}_j l$ if and only if $i \prec_j l$), $\hat{t}_j^p = t_j^p$ and $\hat{t}_j^h = t_j^h$. Thus, provided that $\hat{y} > 0$ and that $\hat{O}$ is indeed an organization, it follows that $\hat{O} \in \mathcal{O}_1 \cap \mathcal{O}_2$.

Given the definition of λ, we have that $\sum_{i \in \hat{L}} \hat{\beta}_i = 1$. We claim that, to show that $\hat{O}$ is an organization, it suffices to show that $\hat{\alpha}_{ik} = \alpha_{ik}$ for each $i, k \in L \setminus I$ such that $i \neq k$ and $i \in l_k$. Indeed, if this is the case, then $\hat{t}_k^p = t_k^p = 0$ for each $k \in I$ implies that $\hat{\beta}_i \hat{t}_i^h = \lambda \beta_i t_i^h = \sum_{k \in L \setminus I} \lambda \beta_k t_k^p \alpha_{ik} = \sum_{k \in L \setminus I} \hat{\beta}_k \hat{t}_k^p \hat{\alpha}_{ik}$ for each $i \in \hat{L} = L \setminus I$ and, hence, $\hat{O}$ is an organization.

We now show that $\hat{\alpha}_{ik} = \alpha_{ik}$ for each $i, k \in L \setminus I$ such that $i \neq k$ and $i \in l_k$. Let $\mathcal{L}_{ik} = \{l \in l_k : l \prec_k i\}$ and $\hat{\mathcal{L}}_{ik} = \{l \in \hat{l}_k : l \hat{\prec}_k i\} = \{l \in \hat{l}_k : l \prec_k i\}$. We then have that $\hat{\mathcal{L}}_{ik} \subseteq \mathcal{L}_{ik}$ and that $\mathcal{L}_{ik} \setminus \hat{\mathcal{L}}_{ik} \subseteq I$. If $\mathcal{L}_{ik} = \hat{\mathcal{L}}_{ik}$, then $\hat{\alpha}_{ik} = \alpha_{ik}$; thus, assume that $\hat{\mathcal{L}}_{ik} \subset \mathcal{L}_{ik}$. Let $j \in \mathcal{L}_{ik} \setminus \hat{\mathcal{L}}_{ik}$. Then $j \in I$ and $t_j^h = 0$. Thus, $0 = \beta_j t_j^h = \sum_{l \in L} \beta_l t_l^p \alpha_{jl}$. In particular, it follows that $t_k^p = 0$ or $\alpha_{jk} = 0$. Because $O \in \mathcal{O}_1$, the former would imply that $l_k = \{k\}$, a contradiction to $i \in l_k$ and $i \neq k$. Thus, $\alpha_{jk} = 0$, implying that $F(A_j \setminus \cup_{l \prec_k j} A_l) = 0 = F(A_j^c \setminus \cup_{l \prec_k j} B_l)$. Since both $A_j \setminus \cup_{l \prec_k j} A_l \in \mathcal{I}$ and $A_j^c \setminus \cup_{l \prec_k j} B_l \in \mathcal{I}$ (i.e. both sets are finite unions of intervals), we have that $A_j \subseteq \cup_{l \prec_k j} A_l$ and $A_j^c \subseteq \cup_{l \prec_k j} B_l$ by footnote 9 in the main text. Since this holds for each $j \in \mathcal{L}_{ik} \setminus \hat{\mathcal{L}}_{ik}$, it follows that $\cup_{l \in \mathcal{L}_{ik}} A_l = \cup_{l \in \hat{\mathcal{L}}_{ik}} A_l$ and, thus,

$A_i \setminus \cup_{l\in\mathcal{L}_{ik}} A_l = A_i \setminus \cup_{l\in\hat{\mathcal{L}}_{ik}} A_l$. By taking the smallest $j \in \mathcal{L}_{ik} \setminus \hat{\mathcal{L}}_{ik}$ according to $\prec_k$, we see that $\Omega = A_j \cup A_j^c \subseteq \cup_{l\prec_k j} B_l$ and, hence, $A_i^c \setminus \cup_{l\in\hat{\mathcal{L}}_{ik}} B_l = \emptyset = A_i^c \setminus \cup_{l\in\mathcal{L}_{ik}} B_l$. Thus, $\hat{\alpha}_{ik} = \alpha_{ik}$ as claimed.

By taking, in the above argument, the greatest $i \in \hat{l}_k$ according to $\prec_k$, we obtain that $\cup_{l\in\hat{l}_k} A_l = A_i \cup (\cup_{l\in\hat{\mathcal{L}}_{ik}} A_l) = A_i \cup (\cup_{l\in\mathcal{L}_{ik}} A_l) = \cup_{l\in l_k} A_l$ for each $k \in L \setminus I$. Thus, since $\lambda > 1$ and $y > 0$,

$$\hat{y} = \lambda \sum_{k\in L\setminus I} \beta_k (t_k^p F(\cup_{l\in l_k} A_k) - c\mu(A_k) - \xi\mu(B_k \setminus A_k)) >$$
$$\sum_{k\in L\setminus I} \beta_k (t_k^p F(\cup_{l\in l_k} A_k) - c\mu(A_k) - \xi\mu(B_k \setminus A_k)) - \sum_{i\in I} \beta_i (c\mu(A_i) + \xi\mu(B_i \setminus A_i)) = y.$$

This shows that $\hat{y} > 0$ and, together with $\hat{L} \subset L$, shows that $\hat{O} > O$. ■

Let $\mathcal{O}_L$ be the set of organizations O such that $y > 0$ and $t_i^p + t_i^h = 1$ for each $i \in L$ (the subscript in $\mathcal{O}_L$ stands for "labor"). Note that $\mathcal{O}_L \subseteq \mathcal{O}_2$.

Lemma 2.3 *If $O \in (\mathcal{O}_1 \cap \mathcal{O}_2) \setminus \mathcal{O}_L$, then there is $\hat{O} \in \mathcal{O}_1 \cap \mathcal{O}_L$ such that $\hat{O} > O$.*

Proof. Let $O \in (\mathcal{O}_1 \cap \mathcal{O}_2) \setminus \mathcal{O}_L$ and $I = \{i \in L : t_i^p + t_i^h < 1\}$; thus, $I \neq \emptyset$. Since $O \in \mathcal{O}_2$, it follows that $y > 0$ and $t_i^p + t_i^h > 0$ for each $i \in L$.

We now show that there is $\tilde{O} \in \mathcal{O}_1 \cap \mathcal{O}_2$ such that $\tilde{O} > O$ and $|\tilde{I}| = |I| - 1$, i.e. the number of layers i of $\tilde{O}$ with $\tilde{t}_i^p + \tilde{t}_i^h < 1$ equals the cardinality of I minus one. Repeating this argument $|I|$ times, produces the desired $\hat{O}$.

Let $i \in I$. Set $\lambda = \frac{1}{t_i^p + t_i^h} > 1$ and $\gamma = \frac{1}{1-\beta_i+\frac{\beta_i}{\lambda}} > 1$. Define $\tilde{O}$ as follows: The layers are the same: $\tilde{L} = L$; for all $j \in L$: $\tilde{A}_j = A_j$, $\tilde{B}_j = B_j$, $\tilde{l}_j = l_j$, $\tilde{\prec}_j = \prec_j$; for all $j \neq i$, $\tilde{\beta}_j = \gamma\beta_j$, $\tilde{t}_j^p = t_j^p$, $\tilde{t}_j^h = t_j^h$; and, finally, $\tilde{\beta}_i = \frac{\gamma}{\lambda}\beta_i$, $\tilde{t}_i^p = \lambda t_i^p$ and $\tilde{t}_i^h = \lambda t_i^h$. Thus, $\tilde{t}_i^p + \tilde{t}_i^h = 1$ and, hence, $|\tilde{I}| = |I| - 1$.

We have that $\tilde{\beta}_j \tilde{t}_j^p = \gamma\beta_j t_j^p$ and $\tilde{\beta}_j \tilde{t}_j^h = \gamma\beta_j t_j^h$ for all $j \in L$. Since $\tilde{\alpha}_{jk} = \alpha_{jk}$ for each $k, j \in L$ (as $\tilde{L} = L$, $\tilde{A}_j = A_j$, $\tilde{B}_j = B_j$, $\tilde{l}_j = l_j$ and $\tilde{\prec}_j = \prec_j$ for all $j \in L$), it follows that $\tilde{\beta}_j \tilde{t}_j^h = \sum_{k\in L} \tilde{\alpha}_{jk} \tilde{\beta}_k \tilde{t}_k^p$ for each $j \in L$. Moreover, $\sum_{j\in L} \tilde{\beta}_j = \gamma \sum_{j\neq i} \beta_j + \frac{\gamma}{\lambda}\beta_i = \gamma\left(1 - \beta_i + \frac{\beta_i}{\lambda}\right) = 1$. Thus, $\tilde{O}$ satisfies all requirements of an organization. Moreover, provided that $\tilde{y} > 0$, we have that $\tilde{O} \in \mathcal{O}_1 \cap \mathcal{O}_2$ as $\tilde{l}_j = l_j$ and $\tilde{t}_j^p + \tilde{t}_j^h \geq t_j^p + t_j^h$ for all $j \in \tilde{L} = L$.

We have that $-\frac{\gamma}{\lambda}\beta_i\nu_i \geq -\gamma\beta_i\nu_i$ since $\lambda > 1$. Since $y > 0$ and $\gamma > 1$, it follows that

$$\begin{aligned}\tilde{y} &= \gamma\sum_{j\in L}\beta_j t_j^p F(\cup_{l\in l_j}A_l) - \gamma\sum_{j\neq i}\beta_j\nu_j - \frac{\gamma}{\lambda}\beta_i\nu_i \\ &\geq \gamma\sum_{j\in L}\beta_j t_j^p F(\cup_{l\in l_j}A_l) - \gamma\sum_{j\in L}\beta_j\nu_j = \gamma y > y.\end{aligned}$$

This shows that $\tilde{y} > 0$ and, together with $L = \tilde{L}$, shows that $\tilde{O} > O$ as claimed. ■

Corollary 2.1 *If $O \notin \mathcal{O}_L$ and $y > 0$, then there is $\hat{O} \in \mathcal{O}_1 \cap \mathcal{O}_L$ such that $\hat{O} > O$.*

Proof. Let $O \notin \mathcal{O}_L$ be such that $y > 0$. Let $O_1 = O$ if $O \in \mathcal{O}_1$, and $O_1 \in \mathcal{O}_1$ be given by Lemma 2.1 otherwise. Then $O_1 \geq O$. Proceeding by induction, we obtain $O_2 \in \mathcal{O}_1 \cap \mathcal{O}_2$ such that $O_2 \geq O$. If $O_2 \in \mathcal{O}_L$, then $O_2 > O$ and set $\hat{O} = O_2$; otherwise, by Lemma 2.3, there is $\hat{O} \in \mathcal{O}_1 \cap \mathcal{O}_L$ such that $\hat{O} > O_2$ and, hence, $\hat{O} > O$. ■

2.3 Specialization

In this section we show that any organization that does not have the properties stated in Theorem 1 is dominated by some organization that has them.

Given an organization O, let M be the set of $i \in L$ such that $\delta_i = \frac{1}{\gamma_i}$ and $\delta_j = 0$ for all $j \in L \setminus \{i\}$ is a solution to (19)–(21). It follows by Lemma A.6 that $M \neq \emptyset$.

Corollary 2.2 *If $O \in \mathcal{O}_L$, then $\frac{\theta_i}{\gamma_i} \geq y$.*

Proof. Indeed, y is (6) at (β, t^p, t^h) and (β, t^p, t^h) satisfies (7)–(12) as $O \in \mathcal{O}_L$. Thus, letting $(\bar{\beta}, \bar{t}^p, \bar{t}^h)$ be a solution to (6)–(12) and $\bar{y}$ be (6) at $(\bar{\beta}, \bar{t}^p, \bar{t}^h)$, then $\bar{y} \geq y$ and, by Lemmas A.4, A.5 and A.6 respectively, $\bar{y}$ is also (13) at $\bar{\delta} = (\bar{\beta}_1\bar{t}_1^p, \ldots, \bar{\beta}_L\bar{t}_L^p)$, (19) at $\bar{\delta}$ and (19) at $\hat{\delta}$, the latter being given in Lemma A.6. ■

The focus of this section is on the class of organizations such that the conclusion of Theorem 1 holds. Let $\mathcal{O}_S$ be the set of organizations O such that $y > 0$ and there is $i \in M$ such that $t_i^p = 1$, $t_i^h = 0$, $l_i = L$, $\beta_i = \frac{1}{\gamma_i}$, $y = \frac{\theta_i}{\gamma_i}$, $t_j^p = 0$, $t_j^h = 1$, $\alpha_{ji} > 0$, $l_j = \{j\}$ and $\beta_j = \frac{\alpha_{ji}}{\gamma_i}$ for each $j \in L \setminus \{i\}$.

However, it is convenient to first consider the class of organizations obtained from $\mathcal{O}_S$ by dropping the requirement that $i \in M$. Thus, we let Let $\mathcal{O}_A$ be the set of organizations O such that there is $i \in L$ such that $t_i^p = 1$, $t_i^h = 0$, $l_i = L$, $\beta_i = \frac{1}{\gamma_i}$, $y = \frac{\theta_i}{\gamma_i}$, $t_j^p = 0$, $t_j^h = 1$, $\alpha_{ji} > 0$, $l_j = \{j\}$ and $\beta_j = \frac{\alpha_{ji}}{\gamma_i}$ for each $j \in L \setminus \{i\}$. Here and above, the subscript in $\mathcal{O}_S$ stands for "specialization" and the one in $\mathcal{O}_A$ stands for "auxiliary".

We will build from organizations in $\mathcal{O}_L$ to organizations in $\mathcal{O}_S$. The first step is provided by the following class of organizations. Let $\mathcal{O}_3$ be the set of organizations O such that there is $i \in M$ such that $l_i = L$ and $\alpha_{ji} > 0$ for all $j \in L \setminus \{i\}$.

Lemma 2.4 *If $O \in \mathcal{O}_L \setminus \mathcal{O}_3$, then there is $\hat{O} \in \mathcal{O}_A$ such that $\hat{O} > O$.*

Proof. Let $O \in \mathcal{O}_L \setminus \mathcal{O}_3$, $i \in M$ and $\hat{L} = \{j \in l_i : \alpha_{ji} > 0\} \cup \{i\}$. Since $O \notin \mathcal{O}_3$, we have that $\hat{L} \subset L$. Note that $\hat{L} \neq \emptyset$ as $i \in \hat{L}$. Now remove layers $L \setminus \hat{L}$; formally, define $\hat{O}$ as follows: Layers are $\hat{L}$; for each $j \in \hat{L}$, set: $\hat{A}_j = A_j$ and $\hat{B}_j = B_j$; for each $j \in \hat{L} \setminus \{i\}$, set $\hat{l}_j = \{j\}$, $\hat{\beta}_j = \frac{\alpha_{ji}}{\gamma_i}$, $\hat{t}_j^p = 0$, and $\hat{t}_j^h = 1$; finally, set $\hat{l}_i = \hat{L}$, $\hat{\prec}_i = \prec_i |_{\hat{l}_i}$, $\hat{\beta}_i = \frac{1}{\gamma_i}$, $\hat{t}_i^p = 1$ and $\hat{t}_i^h = 0$.

We first claim that, for each $j \in \hat{L} \setminus \{i\}$, $\hat{\alpha}_{ji} = \alpha_{ji}$. Let $\mathcal{L}_{ji} = \{l \in l_i : l \prec_i j\}$ and $\hat{\mathcal{L}}_{ji} = \{l \in \hat{l}_i : l \hat{\prec}_i j\} = \{l \in \hat{l}_i : l \prec_i j\}$. We then have that $\hat{\mathcal{L}}_{ji} \subseteq \mathcal{L}_{ji}$ and that $\alpha_{ki} = 0$ for each $k \in \mathcal{L}_{ji} \setminus \hat{\mathcal{L}}_{ji}$.

If $\mathcal{L}_{ji} = \hat{\mathcal{L}}_{ji}$, then $\hat{\alpha}_{ji} = \alpha_{ji}$; thus, assume that $\hat{\mathcal{L}}_{ji} \subset \mathcal{L}_{ji}$. Let $k \in \mathcal{L}_{ji} \setminus \hat{\mathcal{L}}_{ji}$. Then $\alpha_{ki} = 0$, implying that $F(A_k \setminus \cup_{l \prec_i k} A_l) = 0 = F(A_k^c \setminus \cup_{l \prec_i k} B_l)$. Since both $A_k \setminus \cup_{l \prec_i k} A_l \in \mathcal{I}$ and $A_k^c \setminus \cup_{l \prec_i k} B_l \in \mathcal{I}$, we have that $A_k \subseteq \cup_{l \prec_i k} A_l$ and $A_k^c \subseteq \cup_{l \prec_i k} B_l$ by footnote 9 in the main text. Since this holds for each $k \in \mathcal{L}_{ji} \setminus \hat{\mathcal{L}}_{ji}$, it follows that $\cup_{l \in \mathcal{L}_{ji}} A_l = \cup_{l \in \hat{\mathcal{L}}_{ji}} A_l$ and, thus, $A_j \setminus \cup_{l \in \mathcal{L}_{ji}} A_l = A_j \setminus \cup_{l \in \hat{\mathcal{L}}_{ji}} A_l$. By taking the smallest $k \in \mathcal{L}_{ji} \setminus \hat{\mathcal{L}}_{ji}$ according to $\prec_i$, we see that $\Omega = A_k \cup A_k^c \subseteq \cup_{l \prec_i k} B_l$ and, hence, $A_j^c \setminus \cup_{l \in \hat{\mathcal{L}}_{ji}} B_l = \emptyset = A_j^c \setminus \cup_{l \in \mathcal{L}_{ji}} B_l$. Thus, $\hat{\alpha}_{ji} = \alpha_{ji}$ as claimed.

By taking, in the above argument, the greatest $j \in \hat{l}_i$ according to $\prec_i$, we obtain that $\cup_{l \in \hat{l}_i} A_l = A_j \cup (\cup_{l \in \hat{\mathcal{L}}_{ji}} A_l) = A_j \cup (\cup_{l \in \mathcal{L}_{ji}} A_l) = \cup_{l \in l_i} A_l$. Thus, as $\alpha_{ji} = 0$ for each $j \in L \setminus \hat{L}$,

$$\theta_i = F(\cup_{l \in l_i} A_l) - \nu_i - \sum_{j \in L} \alpha_{ji} \nu_j = F(\cup_{l \in \hat{l}_i} A_l) - \nu_i - \sum_{j \in \hat{L}} \hat{\alpha}_{ji} \nu_j = \hat{\theta}_i.$$

Furthermore, $\gamma_i = 1 + \sum_{j \in L} \alpha_{ji} = 1 + \sum_{j \in \hat{L}} \hat{\alpha}_{ji} = \hat{\gamma}_i$. It then follows that $\hat{O}$ is an organization and that $\hat{O} \in \mathcal{O}_A$. Moreover, it follows by Corollary 2.2 that $\hat{y} = \frac{\theta_i}{\gamma_i} \geq y > 0$ which, together with $\hat{L} \subset L$, implies that $\hat{O} > O$. ■

Let $\mathcal{O}_4$ be the set of organizations O such that there is $i \in M$ such that $l_j = \{j\}$ for all $j \in L \setminus \{i\}$.

Lemma 2.5 *If $O \in (\mathcal{O}_L \cap \mathcal{O}_3) \setminus \mathcal{O}_4$, then there is $\hat{O} \in \mathcal{O}_A$ such that $\hat{O} > O$.*

Proof. Let $O \in (\mathcal{O}_L \cap \mathcal{O}_3) \setminus \mathcal{O}_4$. Since $O \in \mathcal{O}_3$, there is $i \in M$ such that $l_i = L$ and $\alpha_{ji} > 0$ for all $j \in L \setminus \{i\}$. It follows by Corollary 2.2 that $\frac{\theta_i}{\gamma_i} \geq y > 0$, the latter because $O \in \mathcal{O}_L$.

Since $O \notin \mathcal{O}_4$, there is $k \neq i$ such that $\{k\} \subset l_k$. Define $\hat{O}$ as follows: Layers are L; for each $j \in L$, set: $\hat{A}_j = A_j$ and $\hat{B}_j = B_j$; for each $j \neq i$, set $\hat{l}_j = \{j\}$, $\hat{\beta}_j = \frac{\alpha_{ji}}{\gamma_i}$, $\hat{t}_j^p = 0$ and $\hat{t}_j^h = 1$; finally, set $\hat{l}_i = l_i$, $\hat{\prec}_i = \prec_i$, $\hat{\beta}_i = \frac{1}{\gamma_i}$, $\hat{t}_i^p = 1$ and $\hat{t}_i^h = 0$.

Since $\hat{l}_i = l_i$, $\hat{\prec}_i = \prec_i$, $\hat{A}_j = A_j$ and $\hat{B}_j = B_j$ for each $j \in L$, it follows that $\hat{\alpha}_{ij} = \alpha_{ij}$ for all $j \in L$ and, hence, $\hat{\gamma}_i = \gamma_i$ and $\hat{\theta}_i = \theta_i$. This, together with $\alpha_{ji} > 0$ for all $j \neq i$, implies that $\hat{O}$ is an organization, that $\hat{y} = \frac{\theta_i}{\gamma_i}$ and, thus, that $O \in \mathcal{O}_A$.

Finally, $\hat{y} = \frac{\theta_i}{\gamma_i} \geq y$, $\hat{L} = L$ and $\hat{l}_j \subseteq l_j$ for all $j \in L$ and $\hat{l}_j \subset l_j$ for some $j \in L$ shows that $\hat{O} > O$. ■

Lemma 2.6 *If $O \in (\mathcal{O}_L \cap \mathcal{O}_3 \cap \mathcal{O}_4) \setminus \mathcal{O}_A$, then there is $\hat{O} \in \mathcal{O}_S$ such that $\hat{O} > O$.*

Proof. Let $O \in (\mathcal{O}_L \cap \mathcal{O}_3 \cap \mathcal{O}_4) \setminus \mathcal{O}_A$; then there is $i \in M$ such that $l_i = L$, $\alpha_{ji} > 0$ and $l_j = \{j\}$ for each $j \in L \setminus \{i\}$ (indeed, by $O \in \mathcal{O}_3$, there is $i_1 \in M$ such that $l_{i_1} = L$ and $\alpha_{ji_1} > 0$ for each $j \in L \setminus \{i_1\}$ and, by $O \in \mathcal{O}_4$, there is $i_2 \in M$ such that $l_j = \{j\}$ for all $j \in L \setminus \{i_2\}$. But $i_1 = i_2$ since otherwise $l_{i_1} = \{i_1\} \subset \{i_1, i_2\} \subseteq L$). We consider first the case where (β, t^p, t^h) does not solve (6)–(12). Since $O \in \mathcal{O}_L$, it follows that $t_i^p + t_i^h = 1$ for all $i \in L$ and, hence, (β, t^p, t^h) satisfies (7)–(12). Thus, by Lemma A.5 and A.6, it follows that $\frac{\theta_i}{\gamma_i} > y > 0$, the latter because $O \in \mathcal{O}_L$.

Let $\hat{O}$ be equal to O except that, for each $j \neq i$, $\hat{\beta}_j = \frac{\alpha_{ji}}{\gamma_i}$, $\hat{t}_j^p = 0$ and $\hat{t}_j^h = 1$, and $\hat{\beta}_i = \frac{1}{\gamma_i}$, $\hat{t}_i^p = 1$ and $\hat{t}_i^h = 0$. Clearly, $\hat{\alpha}_{jk} = \alpha_{jk}$, $\hat{\gamma}_j = \gamma_j$ and $\hat{\theta}_j = \theta_j$ for all $k, j \in L$.

Since $O \in \mathcal{O}_3 \cap \mathcal{O}_4$ and, in particular, $i \in M = \hat{M}$, it follows that $\hat{O} \in \mathcal{O}_S$. We also have that $\hat{O} > O$ because $\hat{y} = \frac{\theta_i}{\gamma_i} > y$ and $\hat{L} = L$.

By the above, we may therefore assume that (β, t^p, t^h) solves (6)–(12). Since $O \notin \mathcal{O}_A$, then $(\beta, t^p, t^h) \neq (\hat{\beta}, \hat{t}^p, \hat{t}^h)$ (where the latter is as defined in the previous paragraph). Letting $\delta = (\beta_1 t_1^p, \ldots, \beta_L t_L^p)$ and $\hat{\delta} = (\hat{\beta}_1 \hat{t}_1^p, \ldots, \hat{\beta}_L \hat{t}_L^p)$, it follows that $\delta \neq \hat{\delta}$: Indeed, if $\delta = \hat{\delta}$, then $\beta_j t_j^p = \hat{\beta}_j \hat{t}_j^p$ for each $j \in L$. Thus, for each $j \neq i$, $\beta_j t_j^p = 0$ and, hence, $t_j^p = 0 = \hat{t}_j^p$ by (8). Moreover, $t_j^h = 1 = \hat{t}_j^h$ by (11) for each $j \neq i$ and $t_i^h = 0 = \hat{t}_i^h$ by (12) and (8). Hence, $t_i^p = 1 = \hat{t}_i^p$ by (11) and $\beta_i = \beta_i t_i^p = \hat{\beta}_i \hat{t}_i^p = \hat{\beta}_i$. Finally, for each $j \neq i$, $\beta_j = \beta_j t_j^h = \alpha_{ji} \beta_i t_i^p = \alpha_{ji} \beta_i = \alpha_{ji} \hat{\beta}_i = \hat{\beta}_j$. In conclusion, if $\delta = \hat{\delta}$, then $(\beta, t^p, t^h) = (\hat{\beta}, \hat{t}^p, \hat{t}^h)$.

Since $\delta \neq \hat{\delta}$, there exists $j \neq i$ such that $\delta_j > 0$; in particular, it follows that $|L| \geq 2$. Since the convex combination of two solutions of a linear programming problem is also a solution, there exists a solution $\tilde{\delta}$ to (19)–(21) such that $\tilde{\delta}_i > 0$ and $\tilde{\delta}_j > 0$. Put $z_i = 1$, $z_j = -\frac{\gamma_i}{\gamma_j}$ and $z_l = 0$ for all $l \notin \{i, j\}$; then, for all ε in a neighborhood of zero (in $\mathbb{R}$), $\tilde{\delta} - \varepsilon z$ satisfies (20)–(21). Optimality of $\tilde{\delta}$ then implies that

$$\frac{\theta_i}{\gamma_i} = \frac{\theta_j}{\gamma_j}.$$

We have that $y = \theta_i / \gamma_i$ by Lemmas A.5 and A.6. Moreover, $l_j = \{j\}$ implies that $\cup_{l \in l_j} A_l = A_j$ and that $\alpha_{lj} = 0$ for each $l \in L$; hence, $\theta_j = F(\cup_{l \in l_j} A_l) - (\nu_j + \sum_{l=1}^{L} \nu_l \alpha_{lj}) = F(A_j) - \nu_j$ and $\gamma_j = 1 + \sum_{l=1}^{L} \nu_l \alpha_{lj} = 1$. Thus,

$$y = \frac{\theta_i}{\gamma_i} = \frac{\theta_j}{\gamma_j} = F(A_j) - \nu_j.$$

Therefore, the organization $\hat{O}$ with just layer j, i.e. $\hat{L} = \{j\}$, and $\hat{B}_j = B_j$, $\hat{A}_j = A_j$, $\hat{\beta}_j = 1$, $\hat{t}_j^p = 1$ and $\hat{t}_j^h = 0$ belongs to $\mathcal{O}_S$ (note that $\hat{M} = \{j\}$ since $\hat{M}$ is a nonempty subset of $\hat{L}$) and obtains as much output as O. Since $|L| > 1$, it follows that $\hat{O} > O$. ∎

Lemma 2.7 *If $O \in \mathcal{O}_A \setminus O_S$, then there is $\hat{O} \in \mathcal{O}_S$ such that $\hat{O} > O$.*

Proof. Let $O \in \mathcal{O}_A \setminus O_S$. Then there is $i \in L$ such that $t_i^p = 1$, $t_i^h = 0$, $l_i = L$, $\beta_i = \frac{1}{\gamma_i}$, $y = \frac{\theta_i}{\gamma_i}$, $t_j^p = 0$, $t_j^h = 1$, $\alpha_{ji} > 0$, $l_j = \{j\}$ and $\beta_j = \frac{\alpha_{ji}}{\gamma_i}$ for each $j \in L \setminus \{i\}$.

Since $O \notin \mathcal{O}_S$, it follows that $i \notin M$. Thus, letting $k \in M$, which, in particular, implies that $|L| \geq 2$, we have that $y = \frac{\theta_i}{\gamma_i} < \frac{\theta_k}{\gamma_k} = F(A_k) - \nu_k$, the latter equality because $l_k = \{k\}$.

Therefore, the organization $\hat{O}$ with just layer k, i.e. $\hat{L} = \{k\}$, and $\hat{B}_k = B_k$, $\hat{A}_k = A_k$, $\hat{\beta}_k = 1$, $\hat{t}_k^p = 1$ and $\hat{t}_k^h = 0$ belongs to $\mathcal{O}_S$ (note that $\hat{M} = \{k\}$ since $\hat{M}$ is a nonempty subset of $\hat{L}$) and satisfies $\hat{y} = F(A_k) - \nu_k > y$ and $\hat{L} \subset L$. Thus, $\hat{O} > O$. ■

Any organization in $\mathcal{O}_S$ has all the properties considered so far:

$$\mathcal{O}_S = \mathcal{O}_1 \cap \mathcal{O}_2 \cap \mathcal{O}_3 \cap \mathcal{O}_4 \cap \mathcal{O}_L \cap \mathcal{O}_A. \tag{2.1}$$

The following result summarizes this section.

Corollary 2.3 *If $O \notin \mathcal{O}_S$ and $y > 0$, then there is $\hat{O} \in \mathcal{O}_S$ such that $\hat{O} > O$.*

Proof. Note that, by (2.1), $\mathcal{O}_L \subseteq \mathcal{O}_2$ and $(\mathcal{O}_L \cap \mathcal{O}_3 \cap \mathcal{O}_4 \cap \mathcal{O}_A) \setminus \mathcal{O}_1 = \emptyset$,

$$\mathcal{O}_L \setminus \mathcal{O}_S = (\mathcal{O}_L \setminus \mathcal{O}_3) \cup ((\mathcal{O}_L \cap \mathcal{O}_3) \setminus \mathcal{O}_4) \cup ((\mathcal{O}_L \cap \mathcal{O}_3 \cap \mathcal{O}_4) \setminus \mathcal{O}_A).$$

Let $O \notin \mathcal{O}_S$ be such that $y > 0$. Let $O_L = O$ if $O \in \mathcal{O}_L$, and $O_L \in \mathcal{O}_L$ be given by Corollary 2.1 otherwise. Then $O_L \geq O$. If $O_L > O$ and $O_L \in \mathcal{O}_S$, then set $\hat{O} = O_L$; if, otherwise, $O_L \notin \mathcal{O}_S$, then $O_L \in \mathcal{O}_L \setminus \mathcal{O}_3$ or $O_L \in (\mathcal{O}_L \cap \mathcal{O}_3) \setminus \mathcal{O}_4$ or $O_L \in (\mathcal{O}_L \cap \mathcal{O}_3 \cap \mathcal{O}_4) \setminus \mathcal{O}_A$. In either case, by Lemmas 2.4, 2.5 and 2.6 respectively, there is $\tilde{O} \in O_A$ such that $\tilde{O} > O_L$ and, hence, $\tilde{O} > O$. If $\tilde{O} \in \mathcal{O}_S$, then let $\hat{O} = \tilde{O}$; otherwise, let $\hat{O} \in \mathcal{O}_S$ be given by Lemma 2.7 so that $\hat{O} > \tilde{O}$. In either case, we have that $\hat{O} \in \mathcal{O}_S$ and $\hat{O} > O$. ■

We conclude this section with sufficient conditions for an organization to belong to $\mathcal{O}_S$.

Lemma 2.8 *Let O be an organization such that $y > 0$.*

(a) If $L = 1$ and $t_1^p = 1$, then $O \in O_S$.

(b) Let $l_1 = L$, $\beta_1 = \frac{1}{\gamma_1}$, $t_1^p = 1$ and, for each $j \in L \setminus \{1\}$, $l_j = \{j\}$, $\beta_j = \frac{\alpha_{j1}}{\gamma_1}$ and $t_j^h = 1$. If $\alpha_{j1} > 0$ for all $j \in L \setminus \{1\}$ and $1 \in M$, then $O \in \mathcal{O}_S$.

Proof. If $L = \{1\}$, then all the conditions defining $\mathcal{O}_S$ are satisfied since $l_1 = \{1\} = L$, $M = \{1\}$ (recall that M is a nonempty subset of L), $\beta_1 = 1$, $t_1^h = 0$, $\gamma_1 = 1$ and $\theta_1 = F(A_1) - \nu_1$. This shows (a). Part (b) is immediate. ■

2.4 No overlap

In this section we show that any organization that does not have the properties stated in Lemma 1 is dominated by some organization that has them, as well as those in Theorem 1.

Let $\mathcal{O}_{D1}$ be the set of organizations such that $A_l \cap A_k = \emptyset$ and $(B_l \setminus A_l) \cap B_k = \emptyset$ for each $k, l \in L$ such that $k < l$.

Lemma 2.9 *If $O \in \mathcal{O}_S \setminus \mathcal{O}_{D1}$, then there is $\hat{O} \in \mathcal{O}_S \cap \mathcal{O}_{D1}$ such that $\hat{O} > O$.*

Proof. Let $O \in \mathcal{O}_S \setminus \mathcal{O}_{D1}$ and let P be the set of pairs $(k, l) \in L^2$ with $k < l$ such that $A_l \cap A_k \neq \emptyset$ or $(B_l \setminus A_l) \cap B_k \neq \emptyset$. We show that there is $\dot{O} \in \mathcal{O}_S$ such that $\dot{O} > O$ and (i) $\dot{O} \in \mathcal{O}_S \cap \mathcal{O}_{D1}$ or (ii) $|\dot{P}| = |P| - 1$. Repeating this argument at most $|P|$ times produces the desired $\hat{O}$.

Let $(k, l) \in P$. Define an organization $\tilde{O}$ to be equal to O except that $\tilde{A}_l = A_l \setminus A_k$ and $\tilde{B}_l = (B_l \setminus B_k) \cup \tilde{A}_l$. We have that $\tilde{A}_j \setminus \cup_{i<j} \tilde{A}_i = A_j \setminus \cup_{i<j} A_i$ and $\tilde{A}_j^c \setminus \cup_{i<j} \tilde{B}_i = A_j^c \setminus \cup_{i<j} B_i$ for each $j \in L$. Indeed, this is clear for all $j < l$. When $j = l$, we have that $\tilde{A}_l \setminus \cup_{i<l} \tilde{A}_i = (A_l \cap A_k^c) \cap (\cap_{i<l} A_i^c) = A_l \setminus \cup_{i<l} A_i$ and $\tilde{A}_l^c \setminus \cup_{i<l} \tilde{B}_i = (A_l^c \cup A_k) \cap (\cap_{i<l} B_i^c) = A_l^c \cap (\cap_{i<l} B_i^c) = A_l^c \setminus \cup_{i<l} B_i$ since $A_k \cap (\cap_{i<l} B_i^c) \subseteq A_k \cap B_k^c = \emptyset$. Finally, if $j > l$, $\cup_{i<j} \tilde{A}_i = \cup_{i<j} A_i$ and $\cup_{i<j} \tilde{B}_i = \cup_{i<j} B_i$ and the result follows. We then have that $\tilde{\alpha}_j = \alpha_j$ for each $j \in L$. Hence, under the standard normalization that layer 1 consists of the workers (i.e. $t_1^p = 1$), $\tilde{O}$ is an organization and belongs to $\mathcal{O}_S$ by Lemma 2.8 provided $1 \in \tilde{M}$ since, for each $j \in L$, $\tilde{\beta}_j = \beta_j$, $\tilde{t}_j^p = t_j^p$, $\tilde{t}_j^h = t_j^h$, $\tilde{l}_j = l_j$ and $\tilde{\prec}_j = \prec_j$.

Assume first that $1 \in \tilde{M}$. As $A_k \subseteq B_k$, we have that

$$\begin{aligned}
\tilde{B}_l \setminus \tilde{A}_l &= (B_l \setminus B_k) \setminus (A_l \setminus A_k) = (B_l \cap B_k^c \cap A_l^c) \cup (B_l \cap B_k^c \cap A_k) \\
&= B_l \cap B_k^c \cap A_l^c = (B_l \setminus A_l) \cap B_k^c, \\
(\tilde{B}_l \setminus \tilde{A}_l) \cap B_k &= (B_l \setminus A_l) \cap B_k^c \cap B_k = \emptyset, \\
\tilde{A}_l \cap A_k &= A_l \cap A_k^c \cap A_k = \emptyset \text{ and} \\
\tilde{y} - y &= \beta_l \left[c(\mu(A_l) - \mu(A_l \setminus A_k)) + \xi(\mu(B_l \setminus A_l) - \mu((B_l \setminus A_l) \cap B_k^c))\right] \\
&= \beta_l \left[c\mu(A_k \cap A_l) + \xi\mu((B_l \setminus A_l) \cap B_k)\right].
\end{aligned}$$

Since $(k, j) \in P$ and A_k, A_l, $B_l \setminus A_l$ and B_k belong to $\mathcal{I}$, it follows from $A_k \cap A_l \neq \emptyset$ or $(B_l \setminus A_l) \cap B_k \neq \emptyset$ that $\mu(A_k \cap A_l) > 0$ or $\mu((B_l \setminus A_l) \cap B_k) > 0$. In either case, $\tilde{y} > y$. Since $\tilde{L} = L$, this implies that $\tilde{O} > O$. Thus, in the case where $1 \in \tilde{M}$, set $\dot{O} = \tilde{O}$.

If $1 \notin \tilde{M}$, then take $i \in \tilde{M}$ and define $\dot{O}$ by $\dot{L} = \{i\}$, $\dot{\beta}_i = 1$, $\dot{t}_i^p = 1$, $\dot{A}_i = \tilde{A}_i$ and $\dot{B}_i = \tilde{B}_i$. Then $\dot{y} > \tilde{y} > y$, $\dot{L} \leq \tilde{L} = L$ and, thus, $\dot{O} > O$. Moreover, $\dot{O} \in \mathcal{O}_S$ by Lemma 2.8 and $\dot{O} \in \mathcal{O}_{D1}$ trivially. ■

Lemma 2.10 *If $O \in \mathcal{O}_{D1}$, then*

(a) $B_l \cap A_k = \emptyset$ and $(B_l \setminus A_l) \cap B_k = \emptyset$ for each $k, l \in L$ such that $k < l$, and

(b) $A_l \cap A_k = \emptyset$ and $(B_l \setminus A_l) \cap (B_k \setminus A_k) = \emptyset$ for each $k, l \in L$ such that $k \neq l$.

Proof. Let $O \in \mathcal{O}_{D1}$ and $k, l \in L$ be such that $k < l$. Then $(B_l \setminus A_l) \cap B_k = \emptyset$ and $A_l \cap A_k = \emptyset$. Thus, $B_l \cap A_k = B_l \cap A_k \cap A_l^c = (B_l \setminus A_l) \cap A_k \subseteq (B_l \setminus A_l) \cap B_k = \emptyset$. This shows (a).

As for (b), let $k, l \in L$ be such that $k \neq l$. Then either $k < l$ or $l < k$; for concreteness, let $k < l$. Then $A_l \cap A_k = \emptyset$ and $(B_l \setminus A_l) \cap (B_k \setminus A_k) \subseteq (B_l \setminus A_l) \cap B_k = \emptyset$ from (a). This shows (b). ■

Let $\mathcal{O}_{D2}$ be the set of organizations such that $B_L = A_L$.

Lemma 2.11 *If $O \in (\mathcal{O}_S \cap \mathcal{O}_{D1}) \setminus \mathcal{O}_{D2}$, then there is $\hat{O} \in \mathcal{O}_S \cap \mathcal{O}_{D1} \cap \mathcal{O}_{D2}$ such that $\hat{O} > O$.*

Proof. Let $O \in (\mathcal{O}_S \cap \mathcal{O}_{D1}) \setminus \mathcal{O}_{D2}$; then $B_L \setminus A_L \neq \emptyset$. Define an organization $\tilde{O}$ to be equal to O except that $\tilde{B}_L = A_L$. We have, clearly, that $\tilde{A}_j \setminus \cup_{i<j} \tilde{A}_i = A_j \setminus \cup_{i<j} A_i$ and $\tilde{A}_j^c \setminus \cup_{i<j} \tilde{B}_i = A_j^c \setminus \cup_{i<j} B_i$ for each $j \in L$. Hence, $\tilde{\alpha}_j = \alpha_j$ for all $j \in L$. It then follows that $\tilde{O}$ is an organization and belongs to $\mathcal{O}_S$ by Lemma 2.8 provided that $1 \in \tilde{M}$. It is also clear that $\tilde{O} \in \mathcal{O}_{D1} \cap \mathcal{O}_{D2}$, the former because $\tilde{A}_j \subseteq A_j$, $\tilde{B}_j \subseteq B_j$ and $\tilde{B}_j \setminus \tilde{A}_j \subseteq B_j \setminus A_j$ for each $j \in L$ and the latter by construction.

We have that $\tilde{\nu}_L < \nu_L$ which, together with $\tilde{\alpha}_j = \alpha_j$ for all $j \in L$, implies that $\tilde{y} > y$. Since $\tilde{L} = L$, this implies that $\tilde{O} > O$. Thus, set $\hat{O} = \tilde{O}$ if $1 \in \tilde{M}$.

If $1 \notin \tilde{M}$, then let $i \in \tilde{M}$, i.e. $F(A_i) - \tilde{\nu}_i > \tilde{y}$. Since $\tilde{\nu}_j = \nu_j$ for each $1 < j < L$, it follows that $i = L$. Thus, define $\hat{O}$ by $\hat{L} = \{L\}$, $\hat{\beta}_L = 1$, $\hat{t}_L^p = 1$, $\hat{A}_L = \tilde{A}_L = A_L$ and $\hat{B}_L = \tilde{B}_L = A_L$. Then $\hat{y} > \tilde{y} > y$, $\hat{L} \leq \tilde{L} = L$ and, thus, $\hat{O} > O$. Moreover, $\hat{O} \in \mathcal{O}_S$ by Lemma 2.8, $\hat{O} \in \mathcal{O}_{D1}$ trivially and $\hat{O} \in \mathcal{O}_{D2}$ since $\hat{B}_L = \hat{A}_L$. ■

Let $\mathcal{O}_{D3}$ be the set of organizations such that $B_i \cap A_{i+1} = \emptyset$ for each $1 \leq i < L$.

Lemma 2.12 *If $O \in (\mathcal{O}_S \cap \mathcal{O}_{D1} \cap \mathcal{O}_{D2}) \setminus \mathcal{O}_{D3}$, then there is $\hat{O} \in \mathcal{O}_S \cap \mathcal{O}_{D1} \cap \mathcal{O}_{D2} \cap \mathcal{O}_{D3}$ such that $\hat{O} > O$.*

Proof. Let $O \in (\mathcal{O}_S \cap \mathcal{O}_{D1} \cap \mathcal{O}_{D2}) \setminus \mathcal{O}_{D3}$ and $I = \{i \in \{1, \ldots, L-1\} : B_i \cap A_{i+1} \neq \emptyset\}$; then $I \neq \emptyset$. We show that there is $\dot{O} \in \mathcal{O}_S \cap \mathcal{O}_{D1} \cap \mathcal{O}_{D2}$ such that $\dot{O} > O$ and (i) $\dot{O} \in \mathcal{O}_S \cap \mathcal{O}_{D1} \cap \mathcal{O}_{D2} \cap \mathcal{O}_{D3}$ or (ii) $|\dot{I}| = |I| - 1$. Repeating this argument at most $|I|$ times produces the desired $\hat{O}$.

Let $i \in I$; then $B_i \cap A_{i+1} \neq \emptyset$. Hence, $(B_i \setminus A_i) \cap A_{i+1} \neq \emptyset$ since $A_i \cap A_{i+1} = \emptyset$ as $O \in \mathcal{O}_{D1}$. Define an organization $\tilde{O}$ to be equal to O except that $\tilde{B}_i = B_i \setminus A_{i+1}$. We clearly have that $\tilde{\alpha}_j = \alpha_j$ for all $j \leq i$. Since $\cup_{l<j} \tilde{B}_l = \cup_{l<j} B_l$ for each $j > i$, it follows that $\tilde{\alpha}_j = \alpha_j$ for all $j > i$ as well. Hence, $\tilde{O}$ is an organization and belongs to $\mathcal{O}_S$ by Lemma 2.8 provided that $1 \in \tilde{M}$. It is also clear that $\tilde{O} \in \mathcal{O}_{D1} \cap \mathcal{O}_{D2}$, the former because $\tilde{A}_j \subseteq A_j$, $\tilde{B}_j \subseteq B_j$ and $\tilde{B}_j \setminus \tilde{A}_j \subseteq B_j \setminus A_j$ for each $j \in L$ and the latter because $i \neq L$. Moreover, $|\tilde{I}| = |I| - 1$.

We have that $\tilde{B}_i \setminus \tilde{A}_i = (B_i \setminus A_i) \setminus A_{i+1}$. Hence, $\mu(\tilde{B}_i \setminus \tilde{A}_i) = \mu(B_i \setminus A_i) - \mu((B_i \setminus A_i) \cap A_{i+1}) < \mu(B_i \setminus A_i)$ since $(B_i \setminus A_i) \cap A_{i+1} \neq \emptyset$ and $(B_i \setminus A_i) \cap A_{i+1} \in \mathcal{I}$. Thus, $\tilde{\nu}_i < \nu_i$ which, together with $\tilde{\alpha}_j = \alpha_j$ for all $j \in L$, implies that $\tilde{y} > y$. Since $\tilde{L} = L$, this implies that $\tilde{O} > O$. Thus, in the case where $1 \in \tilde{M}$, set $\dot{O} = \tilde{O}$.

If $1 \notin \tilde{M}$, then take $i \in \tilde{M}$ and define $\dot{O}$ by $\dot{L} = \{i\}$, $\dot{\beta}_i = 1$, $\dot{t}_i^p = 1$, $\dot{A}_i = \tilde{A}_i$ and $\dot{B}_i = \tilde{A}_i$. Then $\dot{y} = F(\tilde{A}_i) - c\mu(\tilde{A}_i) \geq F(\tilde{A}_i) - c\mu(\tilde{A}_i) - \xi\mu(\tilde{B}_i \setminus \tilde{A}_i) > \tilde{y} > y$, $\dot{L} \leq \tilde{L} = L$ and, thus, $\dot{O} > O$. Moreover, $\dot{O} \in \mathcal{O}_S$ by Lemma 2.8, $\dot{O} \in \mathcal{O}_{D1} \cap \mathcal{O}_{D3}$ trivially and $\dot{O} \in \mathcal{O}_{D2}$ since $\dot{B}_i = \dot{A}_i$. ■

Let

$$\mathcal{O}_D = \mathcal{O}_{D1} \cap \mathcal{O}_{D2} \cap \mathcal{O}_{D3}.$$

Summing up this section:

Corollary 2.4 *If $O \notin (\mathcal{O}_S \cap \mathcal{O}_D)$ and $y > 0$, then there is $\hat{O} \in \mathcal{O}_S \cap \mathcal{O}_D$ such that $\hat{O} > O$.*

Proof. Let $O \notin (\mathcal{O}_S \cap \mathcal{O}_D)$ be such that $y > 0$. Let $O_1 = O$ if $O \in \mathcal{O}_S$, and $O_1 \in \mathcal{O}_S$ be given by Corollary 2.3 otherwise. Then $O_1 \geq O$.

Let $O_2 = O_1$ if $O_1 \in \mathcal{O}_S \cap \mathcal{O}_{D1}$, and $O_2 \in \mathcal{O}_S \cap \mathcal{O}_{D1}$ be given by Lemma 2.9 otherwise. Then $O_2 \geq O$.

Let $O_3 = O_2$ if $O_2 \in \mathcal{O}_S \cap O_{D1} \cap O_{D2}$, and $O_3 \in \mathcal{O}_S \cap \mathcal{O}_{D1} \cap \mathcal{O}_{D2}$ be given by Lemma 2.11 otherwise. Then $O_3 \geq O$.

Finally, Let $\hat{O} = O_3$ if $O_3 \in \mathcal{O}_S \cap \mathcal{O}_D$, and $\hat{O} \in \mathcal{O}_S \cap \mathcal{O}_D$ be given by Lemma 2.12 otherwise. Then $\hat{O} \geq O$ and, since $O \notin (\mathcal{O}_S \cap \mathcal{O}_D)$, we have $\hat{O} \neq O$ and, hence, $\hat{O} > O$. ■

For each $O \in \mathcal{O}_D$, let

$$\begin{aligned} \mathcal{C} &= \{A_l \cap (B_j \setminus A_j) : l, j \in L \text{ and } j < l-1\} \\ &\cup \{A_l \cap (\cap_{j<l-1}(B_j \setminus A_j)^c) : l \in L\} \\ &\cup \{(B_l \setminus A_l) \cap (\cap_{j>l+1} A_j^c) : 1 \leq l < L\} \end{aligned}$$

with the usual convention that the intersection of an empty family of subsets of Ω is Ω itself. As discussed in Section 4, we have that $\mathcal{C}$ is a partition of $\cup_{l\in L} B_l$.

Recall that

$$\mathcal{C}(A_l) = \{A_l \cap (B_j \setminus A_j) : j < l-1\} \cup \{A_l \cap (\cap_{j<l-1}(B_j \setminus A_j)^c)\}$$

for each $l \in L$ and

$$\mathcal{C}(B_l \setminus A_l) = \{A_j \cap (B_l \setminus A_l) : j > l+1\} \cup \{(B_l \setminus A_l) \cap (\cap_{j>l+1} A_j^c)\}$$

for each $1 \le l < L$. Thus,

$$A_l = \cup_{C \in \mathcal{C}(A_l)} C \text{ for each } l \in L \text{ and} \tag{2.2}$$

$$B_l \setminus A_l = \cup_{C \in \mathcal{C}(B_l \setminus A_l)} C \text{ for each } 1 \le l < L; \tag{2.3}$$

thus, $\mathcal{C}(A_l)$ (resp. $\mathcal{C}(B_l \setminus A_l)$) is a partition of A_l (resp. $B_l \setminus A_l$). Moreover, as noted in Section 4, we can obtain $\{A_l, B_l\}_{l \in L}$ from $\mathcal{C}$ by using (2.2) and (2.3) together with

$$B_L = A_L \text{ and} \tag{2.4}$$

$$B_l = A_l \cup (B_l \setminus A_l) \text{ for each } 1 \le l < L. \tag{2.5}$$

Note that (2.2)–(2.5) simply reproduce the formulas in Footnote 13 in the main text.

We have that $\{\mathcal{C}(A_l), \mathcal{C}(B_l \setminus A_l) : l \in L\}$ is a collection of subsets of Ω such that

$$\mathcal{C}(A_l) \cap \mathcal{C}(A_k) = \emptyset \text{ for each } k, l \in L \text{ with } k \ne l, \tag{2.6}$$

$$\mathcal{C}(B_l \setminus A_l) \cap \mathcal{C}(B_k \setminus A_k) = \emptyset \text{ for each } k, l \in L \text{ with } k \ne l, \text{ and} \tag{2.7}$$

$$\mathcal{C}(A_k) \cap \mathcal{C}(B_l \setminus A_l) = \emptyset \text{ for each } k, l \in L \text{ with } k \le l+1. \tag{2.8}$$

It then follows that if $\mathcal{C}$ is pairwise disjoint and $\{A_l, B_l\}_{l \in L}$ are defined from $\mathcal{C}$ via (2.2)–(2.5), then O automatically belongs to $\mathcal{O}_D$ as the following lemma shows.

Lemma 2.13 *Let O be an organization such that $y > 0$. If $\mathcal{C}$ is a pairwise disjoint collection of subsets of Ω such that $\{\mathcal{C}(A_l), \mathcal{C}(B_l \setminus A_l) : l \in L\}$ satisfies* (2.6)–(2.8), *and $\{A_l, B_l\}_{l \in L}$ satisfies* (2.2)–(2.5), *then $O \in \mathcal{O}_D$.*

Proof. It follows immediately from (2.4) that $O \in \mathcal{O}_{D2}$. Note that $B_i \cap A_{i+1} = ((B_i \setminus A_i) \cap A_{i+1}) \cup (A_i \cap A_{i+1})$ for each $1 \le i < L$ and $(B_l \setminus A_l) \cap B_k = ((B_l \setminus A_l) \cap (B_k \setminus A_k)) \cup ((B_l \setminus A_l) \cap A_k)$ whenever $k < l$. Thus, it is enough to show that $A_l \cap A_k = \emptyset$ and $(B_l \setminus A_l) \cap (B_k \setminus A_k) = \emptyset$ for each $k, l \in L$ with $k \ne l$, and $(B_l \setminus A_l) \cap A_k = \emptyset$ for each $k, l \in L$ with $k < l$ and $k = l + 1$.

Let $k, l \in L$ be such that $k < l$ or $k = l + 1$. In either case, $k \leq l + 1$ and, hence, $\mathcal{C}(B_l \setminus A_l) \cap \mathcal{C}(A_k) = \emptyset$. Thus,

$$(B_l \setminus A_l) \cap A_k = \cup_{C \in \mathcal{C}(B_l \setminus A_l)} \left(\cup_{C \in \mathcal{C}(A_k)} (C \cap C') \right) = \emptyset$$

since $C \cap C' = \emptyset$ if $C \neq C'$ and $\mathcal{C}(B_l \setminus A_l) \cap \mathcal{C}(A_k) = \emptyset$. The argument for the remaining intersections is analogous, thus we obtain that $O \in \mathcal{O}_D$. ■

2.5 No gaps

In this section we show that any organization that does not have the properties stated in Theorem 2 is dominated by some organization that has them, as well as those in the previous results.

Let $\mathcal{O}_{G1}$ be the set of organizations O such that $\min_{1 \leq i \leq L} a_i = 0$, where, recall, $a_i = \min B_i$ for each $i \in L$ with the standard convention that $\min \emptyset = \infty$.

Lemma 2.14 *If $O \in (\mathcal{O}_S \cap \mathcal{O}_D) \setminus \mathcal{O}_{G1}$, then there is $\hat{O} \in \mathcal{O}_S \cap \mathcal{O}_D \cap \mathcal{O}_{G1}$ such that $\hat{O} > O$.*

Proof. Let $O \in (\mathcal{O}_S \cap \mathcal{O}_D) \setminus \mathcal{O}_{G1}$. For each $C \in \mathcal{C}$, let $a_C = \min C$; then $\min_{1 \leq i \leq L} a_i = \min_{C \in \mathcal{C}} a_C$. Thus, letting $C \in \mathcal{C}$ be such that $a_C = \min_{1 \leq i \leq L} a_i$, the fact that $O \notin \mathcal{O}_{G1}$ implies that $a_C > 0$. Thus, $[0, a_C) \subseteq (\cup_{l=1}^{L} B_l)^c$.

Let $\varepsilon > 0$ be such that $[a_C, a_C + \varepsilon) \subseteq C$ and let $0 < \varepsilon' < \varepsilon$ be such that $F([0, \varepsilon')) = F([a_C, a_C + \varepsilon))$; the existence of ε' follows by Lemma A.12.

Define an organization $\tilde{O}$ equal to O except that $\tilde{C} = [0, \varepsilon') \cup (C \setminus [a_C, a_C + \varepsilon))$ and $\{\tilde{A}_l, \tilde{B}_l\}_{l=1}^{L}$ are defined from $\{\tilde{D} : D \in \mathcal{C}\}$ via (2.2)–(2.5). Note that $\tilde{C} \cap \tilde{D} = \emptyset$ whenever $D \in \mathcal{C}$ is such that $C \neq D$ because $[0, \varepsilon) \subseteq (\cup_{l=1}^{L} B_l)^c$.

We have that $F(\tilde{D}) = F(D)$ for each $D \in \mathcal{C}$. Thus, $F(\tilde{A}_j) = \sum_{D \in \mathcal{C}(A_j)} F(\tilde{D}) = \sum_{D \in \mathcal{C}(A_j)} F(D) = F(A_j)$ for each $j \in L$. In addition, for each $j \in L$, let $\mathcal{C}_j = (\cup_{l \leq j} \mathcal{C}(A_l)) \cup (\cup_{l < j} \mathcal{C}(B_l \setminus A_l))$. Then,

$$F(A_j \cup (\cup_{l<j} B_l)) = F((\cup_{l \leq j} A_l) \cup (\cup_{l<j} (B_l \setminus A_l))) = F(\cup_{D \in \mathcal{C}_j} D) = \sum_{D \in \mathcal{C}_j} F(D)$$

$$= \sum_{D \in \mathcal{C}_j} F(\tilde{D}) = F(\tilde{A}_j \cup (\cup_{l<j} \tilde{B}_l)).$$

It then follows from Lemma A.11 that $\tilde{\alpha}_j = \alpha_j$ for all $j \in L$. Thus, $\tilde{O} \in \mathcal{O}_S$ by Lemma 2.8 provided that $1 \in \tilde{M}$. In addition, it follows that $\tilde{O} \in \mathcal{O}_D$ by Lemma 2.13 and that $\tilde{O} \in \mathcal{O}_{G1}$ by construction.

We have that $\mu(\tilde{C}) < \mu(C)$ and $\mu(\tilde{D}) = \mu(D)$ for each $D \in \mathcal{C} \setminus \{C\}$. Moreover, since $\tilde{O} \in \mathcal{O}_D$, $F(\cup_{l\in L}\tilde{A}_l) = \sum_{l\in L} F(\tilde{A}_l) = \sum_{l\in L} F(A_l) = F(\cup_{l\in L} A_l)$ since $F(\tilde{A}_l) = F(A_l)$ for each $l \in L$. It then follows that $\tilde{y} > y$. This, together with $\tilde{L} = L$, shows that $\tilde{O} > O$. Thus, in the case where $1 \in \tilde{M}$, set $\hat{O} = \tilde{O}$.

If $1 \notin \tilde{M}$, then take $i \in \tilde{M}$ and, therefore, $F(\tilde{A}_i) - (c\mu(\tilde{A}_i) + \xi\mu(\tilde{B}_i \setminus \tilde{A}_i)) > \tilde{y} > y$. Define $\hat{O}$ by $\hat{L} = \{i\}$, $\hat{\beta}_i = 1$, $\hat{t}_i^p = 1$,

$$\hat{A}_i = \begin{cases} \tilde{A}_i & \text{if } \min \tilde{A}_i = 0, \\ [0, \varepsilon') \cup (\tilde{A}_i \setminus [\min A_i, \min A_i + \varepsilon)) & \text{otherwise,} \end{cases}$$

where $0 < \varepsilon < \max A_i$ and $0 < \varepsilon' < \varepsilon$ is such that $F([0, \varepsilon')) = F([\min A_i, \min A_i + \varepsilon))$, and $\hat{B}_i = \hat{A}_i$. Then, $F(\hat{A}_i) = F(\tilde{A}_i)$ and $\mu(\hat{A}_i) \leq \mu(\tilde{A}_i)$. Hence, $\hat{y} = F(\hat{A}_i) - c\mu(\hat{A}_i) \geq F(\tilde{A}_i) - c\mu(\tilde{A}_i) - \xi\mu(\tilde{B}_i \setminus \tilde{A}_i) > \tilde{y} > y$, $\hat{L} \leq \tilde{L} = L$ and, thus, $\hat{O} > O$. Moreover, $\hat{O} \in \mathcal{O}_S$ by Lemma 2.8, $\hat{O} \in \mathcal{O}_{D1} \cap \mathcal{O}_{D3}$ trivially, $\hat{O} \in \mathcal{O}_{D2}$ since $\hat{B}_i = \hat{A}_i$ and $\hat{O} \in \mathcal{O}_{G1}$ by construction. ■

Let $\mathcal{O}_G$ be the set of organizations $O \in \mathcal{O}_{G1}$ such that $\cup_{i=1}^{L} B_i = [0, \max_{1\leq i \leq L} b_i)$ where, recall, $b_i = \max B_i$ with the convention that $\max \emptyset = -\infty$.

Lemma 2.15 *If $O \in (\mathcal{O}_S \cap \mathcal{O}_D \cap \mathcal{O}_{G1}) \setminus \mathcal{O}_G$, then there is $\hat{O} \in \mathcal{O}_S \cap \mathcal{O}_D \cap \mathcal{O}_G$ such that $\hat{O} > O$.*

Proof. Let $O \in (\mathcal{O}_S \cap \mathcal{O}_D \cap \mathcal{O}_{G1}) \setminus \mathcal{O}_G$. Since $\cup_{l\in L} B_l = \cup_{C\in\mathcal{C}} C$ and $C \in \mathcal{I}$ for each $C \in \mathcal{C}$, we write $C = \cup_{r=1}^{m_C} [a_{Cr}, b_{Cr})$ where $[a_{Cr}, b_{Cr}) \cap [a_{Cr'}, b_{Cr'}) = \emptyset$ whenever $r \neq r'$. We then order the set $\{a_{Cr}, b_{Cr} : C \in \mathcal{C}, 1 \leq r \leq m_C\}$ and write it as $\{a_1, b_1, \ldots, a_m, b_m\}$ with $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m$, so that $\cup_{l\in L} B_l = \cup_{r=1}^{m} [a_r, b_r)$. Let $G = \{i \in \{2, \ldots, m\} : a_i > b_{i-1}\}$ be the set of "gaps". Since $O \in \mathcal{O}_{G1} \setminus \mathcal{O}_G$, $G \neq \emptyset$.

We will define an organization $\dot{O} \in \mathcal{O}_S \cap \mathcal{O}_D \cap \mathcal{O}_{G1}$ such that $\dot{O} > O$ and (i) $\dot{O} \in \mathcal{O}_S \cap \mathcal{O}_D \cap \mathcal{O}_G$ or (ii) $\dot{\mathcal{C}} = \{\dot{C} : C \in \mathcal{C}\}$, $\dot{C} = \cup_{r=1}^{m_C} [\dot{a}_{Cr}, \dot{b}_{Cr})$, with $[\dot{a}_{Cr}, \dot{b}_{Cr}) \cap$

$[\dot{a}_{Cr'}, \dot{b}_{Cr'}) = \emptyset$ whenever $r \neq r'$, for each $C \in \mathcal{C}$ (and, hence, $|\{\dot{a}_{Cr} : C \in \mathcal{C}, 1 \leq r \leq m_C\}| = m$) and $\min \dot{G} = \min G + 1$. Repeating this argument at most m times produces the desired $\hat{O}$ since then $|\{\hat{a}_{Cr} : C \in \mathcal{C}, 1 \leq r \leq m_C\}| = m$ and $\min \hat{G} > m$ implies that $\hat{G} = \emptyset$.

We have that $[b_{\min G-1}, a_{\min G}) \subseteq (\cup_{l=1}^{L} B_l)^c$. Let $C \in \mathcal{C}$ and $1 \leq r' \leq m_C$ be such that $a_{Cr'} = a_{\min G}$. By Lemma A.12, let $\varepsilon > 0$ be such that $F([b_{\min G-1}, b_{Cr'} - \varepsilon)) = F([a_{Cr'}, b_{Cr'}))$ and $\mu([b_{\min G-1}, b_{Cr'} - \varepsilon)) < \mu([a_{Cr'}, b_{Cr'}))$.

Define an organization $\tilde{O}$ equal to O except that

$$\tilde{C} = (C \setminus [a_{Cr'}, b_{Cr'})) \cup [b_{\min G-1}, b_{Cr'} - \varepsilon)$$

and $\{\tilde{A}_l, \tilde{B}_l\}_{l=1}^{L}$ are defined from $\{\tilde{D} : D \in \mathcal{C}\}$ via (2.2)–(2.5). Note that $\tilde{C} \cap \tilde{D} = \emptyset$ whenever $D \in \mathcal{C}$ is such that $C \neq D$ because $[b_{\min G-1}, a_{\min G}) \subseteq (\cup_{l=1}^{L} B_l)^c$.

We clearly have that $\tilde{\mathcal{C}} = \{\tilde{D} : D \in \mathcal{C}\}$ and that $\tilde{D} = \cup_{r=1}^{m_D}[\tilde{a}_{Dr}, \tilde{b}_{Dr})$ for each $D \in \mathcal{C}$ (with $\tilde{a}_{Cr'} = b_{\min G-1}$, $\tilde{b}_{Cr'} = b_{Cr'} - \varepsilon$, and $\tilde{a}_{Dr} = a_{Dr}$ and $\tilde{b}_{Dr} = b_{Dr}$ whenever $(D, r) \neq (C, r')$). In addition, $\min \tilde{G} = \min G + 1$ since $\tilde{a}_i = a_i$ and $\tilde{b}_i = b_i$ for each $i \neq \min G$, $\tilde{a}_{\min G} = b_{\min G-1} = \tilde{b}_{\min G-1}$ and $\tilde{b}_{\min G} < b_{\min G} \leq a_{\min G+1}$.

Furthermore, we have that $a_1 = 0$ since $O \in \mathcal{O}_{G1}$ and $\min G > 1$ by definition. Hence, $\tilde{a}_1 = a_1 = 0$ and, thus, $\tilde{O} \in \mathcal{O}_{G1}$.

In the case where $1 \in \tilde{M}$, an argument completely analogous to the proof of Lemma 2.14 shows that setting $\dot{O} = \tilde{O}$ gives the desired conclusions.

If $1 \notin \tilde{M}$, then take $i \in \tilde{M}$ and define

$$\dot{A}_i = [0, \dot{a})$$

where $\dot{a} > 0$ is such that $F([0, \dot{a})) = F(\tilde{A}_i)$. If $F(\tilde{A}_i) = 1$, then set $\dot{a} = \sup \Omega$; otherwise, the existence of $\dot{a}$ follows by an argument analogous to that of Lemma A.12: Consider $g : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $g(a) = F([0, a))$ for each $a \in \mathbb{R}_+$. Then g is continuous, $g(0) = 0 < F(\tilde{A}_i)$ and $\lim_{a \to \sup \Omega} g(a) = 1 > F(\tilde{A}_i)$. Hence, $\dot{a}$ exists by the intermediate value theorem. We then have that $\mu([0, \dot{a})) \leq \mu(\tilde{A}_i)$. To see this, first note that $\tilde{A}_i \in \mathcal{I}$ and, hence, we write $\tilde{A}_i = \cup_{r=1}^{m}[a_r, b_r)$ where $[a_r, b_r) \cap [a_{r'}, b_{r'}) = \emptyset$ whenever $r \neq r'$. If $\mu([0, \dot{a})) > \mu(\tilde{A}_i)$, then $\dot{a} > \sum_{r=1}^{m}(b_r - a_r)$.

Since $\sum_{r=1}^{j-1}(b_r - a_r) \leq a_j$ for each $j = 1, \dots, m$ with $\sum_{r=1}^{0}(b_r - a_r) = 0$ (which can easily be established by induction), it follows by Lemma A.12 that

$$F([0, \sum_{r=1}^{m}(b_r - a_r))) = \sum_{j=1}^{m} F([\sum_{r=1}^{j-1}(b_r - a_r), \sum_{r=1}^{j}(b_r - a_r))) \geq \sum_{j=1}^{m} F([a_j, b_j)) = F(\tilde{A}_i)$$

and, hence, $F([0, \dot{a})) > F([0, \sum_{r=1}^{m}(b_r - a_r))) \geq F(\tilde{A}_i)$, a contradiction. Thus, $\mu([0, \dot{a})) \leq \mu(\tilde{A}_i)$.

Define $\dot{O}$ by $\dot{L} = \{i\}$, $\dot{\beta}_i = 1$, $\dot{t}_i^p = 1$, $\dot{A}_i = [0, \dot{a})$ as above and $\dot{B}_i = \dot{A}_i$. We then have that $\dot{O} \in \mathcal{O}_G$ by construction. An argument completely analogous to the proof of Lemma 2.14 shows that $\dot{O} > O$ and $\dot{O} \in \mathcal{O}_S \cap \mathcal{O}_D \cap \mathcal{O}_G$. ∎

Summing up this section:

Corollary 2.5 *If $O \notin \mathcal{O}_S \cap \mathcal{O}_D \cap \mathcal{O}_G$ and $y > 0$, then there is $\hat{O} \in \mathcal{O}_S \cap \mathcal{O}_D \cap \mathcal{O}_G$ such that $\hat{O} > O$.*

Proof. Let $O \notin \mathcal{O}_S \cap \mathcal{O}_D \cap \mathcal{O}_G$ be such that $y > 0$. Let $O_1 = O$ if $O \in \mathcal{O}_S \cap \mathcal{O}_D$, and $O_1 \in \mathcal{O}_S \cap \mathcal{O}_D$ be given by Corollary 2.4 otherwise. Then $O_1 \geq O$.

Let $O_2 = O_1$ if $O_1 \in \mathcal{O}_S \cap \mathcal{O}_D \cap \mathcal{O}_{G1}$, and $O_2 \in \mathcal{O}_S \cap \mathcal{O}_D \cap \mathcal{O}_{G1}$ be given by Lemma 2.14 otherwise. Then $O_2 \geq O$.

Finally, let $\hat{O} = O_2$ if $O_2 \in \mathcal{O}_S \cap \mathcal{O}_D \cap \mathcal{O}_G$, and $\hat{O} \in \mathcal{O}_S \cap \mathcal{O}_D \cap \mathcal{O}_G$ be given by Lemma 2.15 otherwise. Then $\hat{O} \geq O$ and, since $O \notin \mathcal{O}_S \cap \mathcal{O}_D \cap \mathcal{O}_G$, we have $\hat{O} \neq O$ and, hence, $\hat{O} > O$. ∎

2.6 Order of sets

In this section we establish Theorem 3. For convenience, let

$$\hat{\mathcal{O}} = \mathcal{O}_S \cap \mathcal{O}_D \cap \mathcal{O}_G.$$

Let $\mathcal{O}_{<1}$ be the set of organizations O such that $C < C'$ for all $C, C' \in \mathcal{C}$ with $c_C > c_{C'}$.

Lemma 2.16 *If $O \in \hat{\mathcal{O}} \setminus \mathcal{O}_{<1}$, then there is $\hat{O} \in \hat{\mathcal{O}} \cap \mathcal{O}_{<1}$ such that $\hat{O} > O$.*

Proof. For each organization O and each $C \in \mathcal{C}$, let I_C be the set of $C' \in \mathcal{C} \setminus \{C\}$ such that $c_C > c_{C'}$ but $C < C'$ does not hold. Let $I = \{C \in \mathcal{C} : I_C \neq \emptyset\}$. It follows by definition that if $I = \emptyset$ then $O \in \mathcal{O}_{<1}$.

Let $O \in \hat{\mathcal{O}} \setminus \mathcal{O}_{<1}$; then $I \neq \emptyset$. Order $\mathcal{C}$ and write $\mathcal{C} = \{C_1, \ldots, C_n\}$ such that $c_{C_l} \geq c_{C_{l+1}}$ for each $l = 1, \ldots, n$. Let $i \in \{1, \ldots, n\}$ be the smallest $l \in \{1, \ldots, n\}$ such that $I_{C_l} \neq \emptyset$ and let $j \in \{1, \ldots, n\}$ be the smallest $l \in \{1, \ldots, n\}$ such that $C_l \in I_{C_i}$. We will define an organization $\tilde{O} \in \hat{\mathcal{O}}$ such that $\tilde{O} > O$ and either (i) $\tilde{O} \in \hat{\mathcal{O}} \cap \mathcal{O}_{<1}$ or (ii) $\tilde{\mathcal{C}} = \{\tilde{C} : C \in \mathcal{C}\}$, $|\tilde{I}_{\tilde{C}_i}| \leq |I_{C_i}| - 1$ and $|\tilde{I}_{\tilde{C}_l}| = \emptyset$ for each $l < i$. By repeating this argument at most $\sum_{C \in I} |I_C|$ times, we obtain the desired $\hat{O}$.

Since $C_i, C_j \in \mathcal{I}$, $C_i = \cup_{r=1}^{m} E_r$ and $C_j = \cup_{r=1}^{m'} E'_r$ where $\{E_r : r = 1, \ldots, m\}$ is a collection of pairwise disjoint intervals and so is $\{E'_r : r = 1, \ldots, m'\}$. Since $C_i \cap C_j = \emptyset$, $E_r \cap E'_s = \emptyset$ for each $1 \leq r \leq m$ and $1 \leq s \leq m'$. Since $C_i < C_j$ does not hold, there is $1 \leq r \leq m$ and $1 \leq s \leq m'$ such that $E'_s < E_r$.

We will define an organization $\dot{O} \in \hat{\mathcal{O}}$ by just changing E_r and E'_s such that

(a) $\dot{O} > O$,

(b) $|\{(r, s) \in \{1, \ldots, m\} \times \{1, \ldots, m'\} : \dot{E}'_s < \dot{E}_r\}| \leq |\{(r, s) \in \{1, \ldots, m\} \times \{1, \ldots, m'\} : E'_s < E_r\}| - 1$,

(c) $\dot{\mathcal{C}} = \{\dot{C} : C \in \mathcal{C}\}$,

(d) $\dot{D} = D$ for each $D \in \mathcal{C} \setminus \{C_i, C_j\}$,

(e) $\dot{C}_i \cup \dot{C}_j = C_i \cup C_j$,

(f) $\dot{\alpha}_l = \alpha_l$ for each $l \in L$ and, hence, $\dot{c}_{\dot{D}} = c_D$ for each $D \in \mathcal{C}$,

(g) $\max \dot{C}_i \leq \max C_i$ and

(h) $\min \dot{C}_j \geq \min C_j$.

By repeating this argument at most $|\{(r, s) : E'_s < E_r\}|$ times, we obtain the desired $\tilde{O}$. Indeed, we then have that $\tilde{C}_i < \tilde{C}_j$ by (b), and that all the properties (a) and (c)–(h) hold in $\tilde{O}$. Since $\tilde{C}_i < \tilde{C}_j$, it follows that $\tilde{C}_j \notin \tilde{I}_{\tilde{C}_i}$. In addition, we claim that if $D \notin I_{C_i}$, then $\tilde{D} \notin \tilde{I}_{\tilde{C}_i}$. To see this, note that $D \notin I_{C_i}$ implies $c_D \geq c_{C_i}$ or $c_D < c_{C_i}$ and $C_i < D$. We have that $\tilde{D} = D$ by (d); moreover, $\tilde{c}_{\tilde{C}_i} = c_{C_i}$ and $\tilde{c}_{\tilde{D}} = c_D$ by (f). Hence, $\tilde{c}_{\tilde{D}} \geq \tilde{c}_{\tilde{C}_i}$ if $c_D \geq c_{C_i}$ and, thus, $\tilde{D} \notin \tilde{I}_{\tilde{C}_i}$. If $c_D < c_{C_i}$ and $C_i < D$, then $\max \tilde{C}_i \leq \max C_i \leq \min D = \min \tilde{D}$ by (g) and (d). Thus, $\tilde{C}_i < \tilde{D}$ and $\tilde{D} \notin \tilde{I}_{\tilde{C}_i}$.

By combining $\tilde{C}_j \notin \tilde{I}_{\tilde{C}_i}$ with the fact that $D \notin I_{C_i}$ implies $\tilde{D} \notin \tilde{I}_{\tilde{C}_i}$, we obtain that $|\tilde{I}_{\tilde{C}_i}| \leq |I_{C_i}| - 1$.

We finally show that $\tilde{I}_{\tilde{C}_l} = \emptyset$ for all $l < i$. Let $l < i$ be given. By the definition of i, we have that $I_{C_l} = \emptyset$ and, by (d), $\tilde{C}_l = C_l$. Let $D \in \mathcal{C}$ be such that $\tilde{c}_{\tilde{D}} < \tilde{c}_{\tilde{C}_l}$. Then $c_D < c_{C_l}$ by (f); since $I_{C_l} = \emptyset$, $C_l < D$ and, hence, $\max C_l \leq \min D$. If $D \notin \{C_i, C_j\}$, then $\max \tilde{C}_l = \max C_l \leq \min D = \min \tilde{D}$ by (d). If $D = C_i$, then $c_{C_j} < c_{C_i} < c_{C_l}$ and, due to $I_{C_l} = \emptyset$, $C_l < C_i$ and $C_l < C_j$. Because $\tilde{C}_i \cup \tilde{C}_j = C_i \cup C_j$ and $\tilde{C}_i < \tilde{C}_j$, it follows that

$$\min \tilde{C}_i = \min(\tilde{C}_i \cup \tilde{C}_j) = \min(C_i \cup C_j) \geq \max C_l = \max \tilde{C}_l$$

and, thus, $\tilde{C}_l < \tilde{C}_i = \tilde{D}$. Finally, if $D = C_j$, then $I_{C_l} = \emptyset$ implies $C_l < C_j$ and, hence, $\max \tilde{C}_l = \max C_l \leq \min C_j \leq \min \tilde{C}_j$ by (h). Thus, $\tilde{C}_l < \tilde{C}_j$. This shows that $\tilde{I}_{\tilde{C}_l} = \emptyset$ and completes the argument to show that $\tilde{O}$ has all the desired properties.

We turn now to the properties of $\dot{O}$. Let $E_r = [a, b)$ and $E'_s = [a', b')$; then $a' < b' \leq a < b$. Let $\Omega' = E_r \cup E'_s$ and let $\hat{a} \in \Omega$ be such that $F(\Omega' \cap [0, \hat{a})) = F(E_r)$; the existence and uniqueness of $\hat{a}$ follows by an argument analogous to that of Lemma A.12, i.e. use the continuity and monotonicity of the function g defined by $g(a) = F(\Omega' \cap [0, a))$ for each $a \in \Omega$, $g(0) = 0 < F(E_r)$ and $\lim_{a \to \sup \Omega} g(a) = F(E_r) + F(E'_s) > F(E_r)$.

In addition, $\mu(\Omega' \cap [0, \hat{a})) < \mu(E_r)$. To see this, note that $\Omega' \cap [0, \hat{a}) = [a', \hat{a})$ or $\Omega' \cap [0, \hat{a}) = [a', b') \cup [a, \hat{a})$. In the first case, the conclusion follows from Lemma A.12. In the second case, it follows that $F([a', b')) + F([a, \hat{a})) = F([a, b))$ and, hence, $F([a', b')) = F([\hat{a}, b))$. Lemma A.12 implies that $b' - a' < b - \hat{a}$; thus, $\mu(\Omega' \cap [0, \hat{a})) = b' - a' + \hat{a} - a < b - a = \mu(E_r)$.

Let $\dot{O}$ be equal to O except that $\dot{E}_r = \Omega' \cap [0, \hat{a})$ and $\dot{E}'_s = \Omega' \setminus [0, \hat{a})$. Thus, $F(\dot{D}) = F(D)$ for each $D \in \mathcal{C}$ and, hence, $\dot{\alpha}_l = \alpha_l$ for each $l \in L$; in particular (f) holds. Thus, $\dot{O} \in \mathcal{O}_S$ by Lemma 2.8 provided that $1 \in \dot{M}$. It is clear that $\dot{O} \in \mathcal{O}_G$ since $\cup_{l\in L}\dot{B}_l = \cup_{l\in L}B_l$. Furthermore, it follows by Lemma 2.13 that $\dot{O} \in \mathcal{O}_D$. Thus, $\dot{O} \in \hat{\mathcal{O}}$.

Properties (b)–(e) are satisfied by construction and we have already pointed out that (f) holds. We will show (a) below. As for (g), we have that $\max \dot{C}_i = \max C_i$ if $b < \max C_i$ and $\max \dot{C}_i < \max C_i$ if $b = \max C_i$; thus, $\max \dot{C}_i \leq \max C_i$. Similarly, we have that $\min \dot{C}_j = \min C_j$ if $a' > \min C_j$ and $\min \dot{C}_j > \min C_j$ if $a' = \min C_j$; thus, $\min \dot{C}_j \geq \min C_j$ and (h) holds.

Furthermore, $\mu(\dot{D}) = \mu(D)$ for each $D \in \mathcal{C} \setminus \{C_i, C_j\}$, $\mu(\dot{C}_i) < \mu(C_i)$ and $\mu(\dot{C}_j) + \mu(\dot{C}_i) = \mu(C_j) + \mu(C_i)$. We have that $F(\cup_{l\in L}A_l) = F(\cup_{l\in L}\dot{A}_l)$ since both O and $\dot{O}$ belong to $\mathcal{O}_D$ and $F(D) = F(\dot{D})$ for each $D \in \mathcal{C}$. Using $O, \dot{O} \in \mathcal{O}_S$ and Lemma A.10,

$$y = \frac{F(\cup_{l\in L}A_l) - \sum_{D\in\mathcal{C}} \mu(D)c_D}{\gamma} \text{ and } \dot{y} = \frac{F(\cup_{l\in L}A_l) - \sum_{D\in\mathcal{C}} \mu(\dot{D})c_D}{\gamma}.$$

Consequently, letting $\rho = \mu(C_i) - \mu(\dot{C}_i)$, we have that $\dot{y} - y = \frac{(c_{C_i} - c_{C_j})\rho}{\gamma} > 0$. This, together with $\dot{L} = L$, implies that $\dot{O} > O$. This completes the proof when $1 \in \dot{M}$.

If $1 \notin \dot{M}$, then define instead $\tilde{O}$ as in the proof of Lemma 2.15 to obtain $\tilde{O} \in \hat{\mathcal{O}}$ and $\tilde{O} > O$. Since $L = \{i\}$, it follows that $\tilde{\mathcal{C}} = \{\tilde{A}_i\}$ and, thus, $\tilde{O} \in \mathcal{O}_{<1}$ trivially. ∎

Let $\mathcal{O}_<$ be the set of organizations $O \in \mathcal{O}_{<1}$ such that C is an interval for each $C \in \mathcal{C}$.

Lemma 2.17 *If $O \in \hat{\mathcal{O}} \cap \mathcal{O}_{<1}$ and $\mathcal{C} = \{C_1, \ldots, C_{|\mathcal{C}|}\}$ is such that $c_{C_1} \geq \cdots \geq c_{C_{|\mathcal{C}|}}$, then there is $\hat{O} \in \hat{\mathcal{O}} \cap \mathcal{O}_<$ such that $\hat{y} \geq y$ and $\hat{L} \leq L$. Moreover, when $\hat{y} = y$ and $\hat{L} = L$, then $\hat{l}_i = l_i$ for each $i \in L$, $\hat{\mathcal{C}} = \{\hat{C} : C \in \mathcal{C}\}$, $\hat{C}_1 < \cdots < \hat{C}_{|\mathcal{C}|}$ and $F(\hat{C}) = F(C)$ for each $C \in \mathcal{C}$.*

Proof. Let $O \in \hat{\mathcal{O}} \cap \mathcal{O}_{<1}$. Define $\{c_1, \ldots, c_n\} = \{c_C : C \in \mathcal{C}\}$ with $c_1 > \ldots > c_n$ and $\mathcal{C}_i = \{C \in \mathcal{C} : c_C = c_i\}$ for each $1 \leq i \leq n$. For each $1 \leq i \leq n$, there is $k_i \in \{1, \ldots, |\mathcal{C}|\}$ and $r_i \in \{0, \ldots, |\mathcal{C}| - 1\}$ such that $\mathcal{C}_i = \{C_{k_i}, \ldots, C_{k_i+r_i}\}$. Let I be the set of $i \in \{1, \ldots, n\}$ such that $C_{k_i} < \cdots < C_{k_i+r_i}$ does not hold; we may assume that $I \neq \emptyset$ since, otherwise, just set $\hat{O} = O$. We will define an organization $\tilde{O} \in \hat{\mathcal{O}} \cap \mathcal{O}_{<1}$ such that $\tilde{y} \geq y$, $\tilde{L} \leq L$ such that (a) if $\tilde{y} = y$ and $\tilde{L} = L$, then $\tilde{l}_i = l_i$

for each $i \in L$, $\tilde{\mathcal{C}} = \{\tilde{C} : C \in \mathcal{C}\}$, $\tilde{C}_{k_i} < \cdots < \tilde{C}_{k_i+r_i}$ for each $i \notin \tilde{I}$, $F(\tilde{C}) = F(C)$ for each $C \in \mathcal{C}$ and $|\tilde{I}| = |I| - 1$, and (b) otherwise, $\tilde{O} \in \hat{\mathcal{O}} \cap \mathcal{O}_<$. By repeating this argument at most $|I|$ times, we obtain the desired $\hat{O}$.

Since $O \in \mathcal{O}_{<1}$, $\cup_{C\in\mathcal{C}_i} C < \cup_{C\in\mathcal{C}_{i+1}} C$ for each $i = 1, \ldots, n-1$. Since $O \in \mathcal{O}_G$, $\cup_{l\in L} B_l = \cup_{i=1}^{n} \cup_{C\in\mathcal{C}_i} C$ is an interval. Thus, it follows that $\cup_{C\in\mathcal{C}_i} C$ is an interval for each $i = 1, \ldots, n$.

Let $i \in I$ and $\cup_{C\in\mathcal{C}_i} C = [a, b)$. Then obtain $\{\tilde{C} : C \in \mathcal{C}_i\}$ such that $\tilde{C}$ is an interval and $F(\tilde{C}) = F(C)$ for each $C \in \mathcal{C}_i$ as follows. Write $\mathcal{C}_i = \{C_{k_i}, \ldots, C_{k_i+r_i}\} = \{C_{i_1}, \ldots, C_{i_{r_i+1}}\}$. Let b_1 be such that $F([a, b_1)) = F(C_{i_1})$ and set $\tilde{C}_{i_1} = [a_1, b_1)$ with $a_1 = a$; assuming that $\tilde{C}_{i_1}, \ldots, \tilde{C}_{i_{j-1}}$ are such that, for each $1 \leq l \leq j-1$, $F(\tilde{C}_{i_l}) = F(C_{i_l})$ and $\tilde{C}_{i_l} = [a_l, b_l)$ with $a = a_1 < b_1 = a_2 < b_2 = \cdots = a_{j-1} < b_{j-1}$, let $a_j = b_{j-1}$ and b_j such that $F([a_j, b_j)) = F(C_{i_j})$. The existence and uniqueness of b_j follows by an argument analogous to that of Lemma A.12, i.e. use the continuity and monotonicity of the function g defined by $g(x) = F([a_j, x))$ for each $x \in \Omega$, $g(a_j) = 0 < F(C_{i_j})$ and $\lim_{x\to b} g(x) \geq F(C_{i_j})$.

Let $\tilde{O}$ be equal to O except that, for each $C \in \mathcal{C}_i$, C is replaced with $\tilde{C}$. We have that $\tilde{L} = L$ with $\tilde{l}_i = l_i$ for each $i \in L$, $\tilde{\mathcal{C}} = \{\tilde{C} : C \in \mathcal{C}\}$, $\tilde{C}_{k_i} < \cdots < \tilde{C}_{k_i+r_i}$, $F(\tilde{C}) = F(C)$ for each $C \in \mathcal{C}$ and $|\tilde{I}| = |I| - 1$ by construction. Also, $\cup_{C\in\mathcal{C}_i} C = \cup_{C\in\mathcal{C}_i} \tilde{C}$ so that $\sum_{C\in\mathcal{C}_i} \mu(C) = \sum_{C\in\mathcal{C}_i} \mu(\tilde{C})$. It then follows by Lemma A.10 that

$$\tilde{y} - y = c_i \left(\sum_{C\in\mathcal{C}_i} \mu(C) - \sum_{C\in\mathcal{C}_i} \mu(\tilde{C}) \right) = 0.$$

This completes the proof when $1 \in \tilde{M}$.

If $1 \notin \tilde{M}$, then define instead $\tilde{O}$ as in the case $1 \notin \tilde{M}$ in proof of Lemma 2.15. In particular, $\tilde{L} = 1 \leq L$ and $\tilde{B}_1 = \tilde{A}_1 = [0, \tilde{a})$ and $\tilde{y} > y$, thus, we obtain $\tilde{O} \in \hat{\mathcal{O}} \cap \mathcal{O}_<$.

■

Let

$$\mathcal{O}^* = \mathcal{O}_S \cap \mathcal{O}_D \cap \mathcal{O}_G \cap \mathcal{O}_<.$$

Recall that, given two organizations O and O', we write $\hat{O} \gtrsim O$ if $\hat{O} > O$ or $\hat{y} = y$, $\hat{L} = L$ and $\hat{l}_i = l_i$ for all $i \in L$.

Summing up this section:

Corollary 2.6 *If $O \notin \mathcal{O}^*$ and $y > 0$, then there is $\hat{O} \in \mathcal{O}^*$ such that $\hat{O} \gtrsim O$.*

Proof. Let $O \notin \mathcal{O}^*$ be such that $y > 0$. Let $O_1 = O$ if $O \in \hat{\mathcal{O}}$, and $O_1 \in \hat{\mathcal{O}}$ be given by Corollary 2.5 otherwise. Then $O_1 \geq O$.

Let $O_2 = O_1$ if $O_1 \in \hat{\mathcal{O}} \cap \mathcal{O}_{<1}$, and $O_2 \in \hat{\mathcal{O}} \cap \mathcal{O}_{<1}$ be given by Lemma 2.16 otherwise. Then $O_2 \geq O$.

Finally, let $\hat{O} = O_2$ if $O_2 \in \mathcal{O}^*$, and $\hat{O} \in \mathcal{O}^*$ be given by Lemma 2.17 otherwise. Then $\hat{O} \gtrsim O$. ■

2.7 Existence of optimal organizations

Recall from Section A.10 that $O \in \mathcal{O}^*$ is fully specified by (L, ψ, μ) such that $L \in \mathbb{N}$, ψ is a bijection from $\mathcal{C}$ to $\{1, \ldots, m\}$ where $m = |\mathcal{C}|$, and $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{R}_+^m$ is such that $\sum_{j=1}^m \mu_j \leq \mu(\Omega)$.

We start with the following lemma. Let $\mathcal{O}_Q$ be the set of quasi-organization defined by (L, ψ, μ) as in Section A.10. Specifically, if $O \in \mathcal{O}_Q$ is actually an organization, then $O \in \mathcal{O}^*$. For further use, we say that O is a *specialized quasi-organization* if $t_1^p = 1$, $t_1^h = 0$, $\beta_1 = \frac{1}{\gamma}$, $l_1 = L$, $\prec_1 = <$ and, for each $i \neq 1$, $t_i^p = 0$, $t_i^h = 1$, $\beta_1 = \frac{\alpha_i}{\gamma}$ and $l_i = \{i\}$. Note that if O is a specialized quasi-organization, then O need not belong to $\mathcal{O}_Q$; for that $\mathcal{C}$ must be ordered.

Lemma 2.18 *Let O be a specialized quasi-organization. If $\hat{O}$ is such that $\hat{L} = L \setminus \{i \in L : \alpha_i = 0\}$ but otherwise equal to O, then $\hat{\alpha}_i = \alpha_i$ for each $i \in \hat{L}$ and $\cup_{l \in \hat{L}} A_l = \cup_{l \in L} A_l$. Consequently, $\hat{O}$ is an organization, $\hat{y} = y$, $\hat{L} \leq L$ and $\hat{L} = L$ if and only if O is an organization. In addition, if $O \in \mathcal{O}_Q$, then $\hat{O} \in \mathcal{O}^*$.*

Proof. Let O be a specialized quasi-organization and let $i \in L$ be such that $\alpha_i = 0$. Then $A_i \subseteq \cup_{l<i} A_l \subseteq \cup_{l<i} B_l$ and $A_i^c \subseteq \cup_{l<i} B_l$. Since $B_i \setminus A_i \subseteq A_i^c$, we have that $B_i = A_i \cup (B_i \setminus A_i) \subseteq \cup_{l<i} B_l$.

Let $\hat{O}$ be such that $\hat{L} = L \setminus \{i \in L : \alpha_i = 0\}$ and $\hat{l}_1 = \hat{L}$ but otherwise equal to O. We start by showing that, for each $j \in L$, $\cup_{l<j} A_l = \cup_{l \in \hat{L} : l<j} A_l$. Let $\mathcal{L}_j = \{l \in \hat{L} : l < j\}$ and proceed by induction. The conclusion trivially holds for $j = 1$. Assuming that $\cup_{l<j-1}A_l = \cup_{l\in\mathcal{L}_{j-1}}A_l$, we have that, if $j-1\in\hat{L}$, $\cup_{l<j}A_l = (\cup_{l<j-1}A_l)\cup A_{j-1} = (\cup_{l\in\mathcal{L}_{j-1}}A_l)\cup A_{j-1} = \cup_{\mathcal{L}_j}A_l$; if $j-1\notin\hat{L}$, then $\cup_{l<j}A_l = (\cup_{l<j-1}A_l)\cup A_{j-1} = \cup_{l<j-1}A_l = \cup_{l\in\mathcal{L}_{j-1}}A_l = \cup_{l\in\mathcal{L}_j}A_l$ since $A_{j-1}\subseteq\cup_{l<j-1}A_l$.

An analogous argument, now using $B_i\subseteq\cup_{l<i}B_l$ for each $i\notin\hat{L}$, shows that, for each $j\in L$, $\cup_{l<j}B_l = \cup_{l\in\hat{L}:l<j}B_l$. It then follows that $\hat{\alpha}_i = \alpha_i$ for each $i\in\hat{L}$.

Furthermore, $\cup_{l\in\hat{L}}A_l = \cup_{l\in L}A_l$ since if $L\in\hat{L}$, $\cup_{l\in L}A_l = (\cup_{l<L}A_l)\cup A_L = (\cup_{l\in\mathcal{L}_L}A_l)\cup A_L = \cup_{l\in\hat{L}}A_l$; if $L\notin\hat{L}$, then $\cup_{l\in L}A_l = (\cup_{l<L}A_l)\cup A_L = \cup_{l<L}A_l = \cup_{l\in\mathcal{L}_L}A_l = \cup_{l\in\hat{L}}A_l$ since $A_L\subseteq\cup_{l<L}A_l$.

The remaining properties are now clear. ■

Let $X_L = \{(\mu_1,\ldots,\mu_m)\in\mathbb{R}^m_+ : \sum_{j=1}^m \mu_j \leq \mu(\Omega)\}$ and recall that

$$\max_L\Big(\max_\psi\Big(\max_{\mu\in X_L} y_{L,\psi}(\mu)\Big)\Big)$$

exits when Ω is bounded.

Let $O^*\in\mathcal{O}_Q$ be defined by setting (L^*,ψ^*,μ^*) according to (24)–(26). We then have that, for each $O\in\mathcal{O}^*$, there is (L,ψ,μ) such that $Y = y_{L,\psi}(\mu)-(L-1)\eta \leq y_{L,\psi}-(L-1)\eta \leq \max_\psi y_{L,\psi}-(L-1)\eta \leq \max_L\big(\max_\psi y_{L,\psi}-(L-1)\eta\big) = Y^*$. Hence,

$$Y\leq Y^*. \tag{2.9}$$

Also note that if $O\in\mathcal{O}^*$ is such that $L=L^*$, then $l_1 = L = L^* = l_1^*$ and $l_i = \{i\} = l_i^*$ for each $i\in L\setminus\{1\}$; hence,

$$l_i = l_i^* \text{ for each } i\in L. \tag{2.10}$$

We can now turn to the proof of Theorem 4 where we show that actually $O^*\in\mathcal{O}^*$ and that O^* is an η-optimal organization.

Proof of Theorem 4. We have that O^* is an organization. Suppose not; then $\hat{O}\in\mathcal{O}^*$ given by Lemma 2.18 is such that $\hat{Y} = \hat{y}-(\hat{L}-1)\eta = y^*-(\hat{L}-1)\eta > y^*-(L^*-1)\eta = Y^*$, a contradiction to (2.9).

It suffices to show that O^* is an η-optimal organization. Suppose not; then there exists an organization O such that (i) $Y>Y^*$ or (ii) $y=y^*$, $L=L^*$, $l_i\subseteq l_i^*$ for each $i\in L^*$ and $l_j\neq l_j^*$ for some $j\in L^*$. It is clear from (2.9) and (2.10) that $O\notin\mathcal{O}^*$.

Thus, it follows from Corollary 2.6 that there exists $\hat{O} \in \mathcal{O}^*$ such that (iii) $\hat{O} > O$ or (iv) $\hat{y} = y$, $\hat{L} = L$ and $\hat{l}_i = l_i$ for all $i \in L$.

If (iii) holds or if (iv) holds together with (i) we obtain that $\hat{Y} > Y^*$. But this is a contradiction since $\hat{O} \in \mathcal{O}^*$ together with (2.9) imply $\hat{Y} \leq Y^*$.

If, instead, (iv) and (ii) hold, then $\hat{L} = L^*$, $\hat{l}_i \subseteq l_i^*$ for each $i \in L^*$ and $\hat{l}_j \neq l_j^*$ for some $j \in \hat{L}$. But this is a contradiction since $\hat{O} \in \mathcal{O}^*$ together with (2.10) imply that $\hat{l}_i = l_i^*$ for each $i \in L^*$. This contradiction shows that O^* is an η-optimal organization and completes the proof of Theorem 4. ■

3 Optimal organizations with at least two layers and nonexistence of lexicographically optimal organizations

In this section we provide sufficient conditions for optimal organizations to have at least two layers when ξ is sufficiently small. We then use this result to show that no lexicographically optimal organization exists when ξ is sufficiently small.

The sufficient conditions for optimal organizations to have at least two layers require that communication is not too costly and that organizations with one layer do not have full knowledge:

(A3) $h < 1$.

(A4) $f(\bar{\omega}) < c$.

Indeed, if (A4) holds, then the solution μ_1 to the maximization problem defining y_1 is less than $\bar{\omega}$. More importantly, if we add (A3) and (A4) to (A1), we obtain that any lexicographically optimal organization has $L \geq 2$ when ξ is sufficiently small.

To see the above and to see what happens in the case of η-optimal organizations, consider the organization with two layers described in Theorem 5. It satisfies $L = 2$, $B_1 \setminus A_1 = (A_1 \cup A_2)^c$, $A_1 = [0, \mu_1)$ and $A_2 = [\mu_1, \mu_1 + \mu_2)$ for some $(\mu_1, \mu_2) \in X_2 =$

$\{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 + x_2 \leq \bar{\omega}\}$. Let

$$\bar{y}_2 = \max_{(\mu_1,\mu_2)\in X_2} \frac{F(\mu_1+\mu_2) - c\mu_1 - ch(F(\mu_1+\mu_2) - F(\mu_1))\mu_2}{1 + h(F(\mu_1+\mu_2) - F(\mu_1))}$$

be its output when μ_1 and μ_2 are chosen optimally and when $\xi = 0$ (as we assume $\xi > 0$ throughout the paper, $\bar{y}_2$ is merely an auxiliary concept).[1] As we show in Appendix A, we then have that $\bar{y}_2 > y_1$. Thus, any lexicographically optimal organization has $L \geq 2$ when ξ is sufficiently small and, if $\eta < \bar{y}_2 - y_1$, then the same holds for any η-optimal organization.

We now show formally that if ξ and η are sufficiently small and (A1), (A3) and (A4) hold, then any η-optimal organization has at least two layers (in fact, only the requirement that Ω be bounded in (A1) is needed). Lemma 3.1 shows that $\bar{y}_2$ is strictly above the output of the best organization with one layer.

Lemma 3.1 *If (A1), (A3) and (A4) hold, then* $\bar{y}_2 > y_1 \geq 0$.

Proof. Let $\mu_1 \in \Omega$ be such that $y_1 = F(\mu_1) - c\mu_1$. It is clear that $0 \leq y_1 < 1$ and that $\mu_1 < \bar{\omega}$, the latter by (A4). Letting $\bar{y}_2(x_1, x_2) = \frac{F(x_1+x_2) - cx_1 - ch(F(x_1+x_2) - F(x_1))x_2}{1+h(F(x_1+x_2)-F(x_1))}$ for each $(x_1, x_2) \in X_2$, it follows that $\bar{y}_2(\mu_1, 0) = y_1$ and $\frac{\partial \bar{y}_2(\mu_1,0)}{\partial x_2} = f(\mu_1)(1 - y_1 h)$. Since $h < 1$ by (A3) and $y_1 < 1$, it follows that there is $\mu_2 > 0$ such that $\bar{y}_2(\mu_1, \mu_2) > \bar{y}_2(\mu_1, 0) = y_1$. ■

Lemma 3.2 states and proves the main conclusion of this section.

Lemma 3.2 *If (A1), (A3) and (A4) hold and* $0 < \eta < \bar{y}_2 - y_1$*, then there is* $\xi' > 0$ *such that, for each* $0 < \xi < \xi'$, $L \geq 2$ *in any* η*-optimal organization.*

Proof. Let $\xi' = \frac{\bar{y}_2 - y_1 - \eta}{\bar{\omega}}$; then $\xi' > 0$ by Lemma 3.1. Let O be a lexicographically optimal organization and suppose that $L < 2$. Then $L = 1$ and $y = y_1$. Consider $\hat{O}$ with $\hat{L} = 2$, $\hat{B}_1 = (\hat{A}_1 \cup \hat{A}_2)^c$, $\hat{A}_1 = [0, \mu_1)$ and $\hat{A}_2 = [\mu_1, \mu_1 + \mu_2)$ for some $(\mu_1, \mu_2) \in X_2$ such that $\bar{y}_2(\mu_1, \mu_2) = \bar{y}_2$. Then $\hat{y} = \bar{y}_2 - \frac{\xi(\bar{\omega} - \mu_1 - \mu_2)}{1+h(F(\mu_1+\mu_2)-F(\mu_1))} \geq \bar{y}_2 - \xi\bar{\omega}$.

[1] To obtain the output of this organization, we need to subtract $\xi(1 - \mu_1 - \mu_2)$ to the numerator of $\bar{y}_2$.

Hence, $\hat{Y} \geq \bar{y}_2 - \xi\bar{\omega} - \eta > y_1 = Y$ since $\xi < \xi'$. But this contradicts the optimality of O. ■

Existence of η-optimal organizations follows from Theorem 4. As the next result shows, under (A1)–(A4), there is no lexicographically optimal organization when ξ is sufficiently small.

Theorem 3.1 *If (A1)–(A4) hold, then there exists $\tilde{\xi} > 0$ such that there is no lexicographically optimal organization for each $0 < \xi < \tilde{\xi}$.*

The intuition for Theorem 3.1 is as follows. Under (A1)–(A4), if a lexicographically optimal organization exists, then $L \geq 2$ and it has the structure described in Theorem 5 (the latter is shown in its proof). To be concrete, consider the case where $L = 3$ (the proof of Theorem 3.1 considers the general case). Since $\alpha_2 = hF(A_2)$, $\alpha_3 = hF(A_3)$ and $B_1 \setminus A_1 = \Omega \setminus (A_1 \cup A_2)$, its output is

$$y = \frac{\sum_{i=1}^{3} F(A_i) - c\mu(A_1) - ch\sum_{i=2}^{3} F(A_i)\mu(A_i) - \xi(\bar{\omega} - \mu(A_1) - \mu(A_2))}{1 + h\sum_{i=2}^{3} F(A_i)}.$$

Output can now be increased by adding a forth layer and by splitting A_3 in half, so that if $A_3 = [a, b)$ and $\hat{A}_3$ and $\hat{A}_4$ denote the knowledge sets of layers 3 and 4 in the new organization, then $\hat{A}_3 = [a, (a+b)/2)$ and $\hat{A}_4 = [(a+b)/2, b)$. The only change to output is that $-chF(A_3)\mu(A_3)$ is replaced with $-ch\big(F(\hat{A}_3)\mu(\hat{A}_3) + F(\hat{A}_4)\mu(\hat{A}_4)\big) = -\frac{chF(A_3)\mu(A_3)}{2}$. Thus, the learning costs of A_3 are cut in half and output increases.

The proof of Theorem 3.1 requires the following technical lemma showing that if $L = 2$ is optimal, then the size $\mu(A_2)$ of layer 2's knowledge set would necessarily be bounded below. Let $\bar{\xi} > 0$ be given by Theorem 5 and $\xi' > 0$ be given by Lemma 3.2.

Lemma 3.3 *If (A1)–(A4) hold, then there is $\varepsilon > 0$ such that, for each $\xi \in (0, \min\{\bar{\xi}, \xi'\})$, if O is a lexicographically optimal organization with $L = 2$, then $\mu(A_2) > \varepsilon$.*

Proof. Suppose not; then, there are sequences $\{\xi_k\}_{k=1}^{\infty}$, $\{O_k\}_{k=1}^{\infty}$ and $\{\mu_{2,k}\}_{k=1}^{\infty}$ such that, for each $k \in \mathbb{N}$, $0 < \xi_k < \min\{\bar{\xi}, \xi'\}$, O_k is a lexicographically optimal organization with $L_k = 2$ and $\mu_{2,k} = \mu(A_{2,k}) \to 0$. Let $\mu_{1,k} = \mu(A_{1,k})$ for each $k \in \mathbb{N}$;

since $\{\xi_k\}_{k=1}^{\infty}$ and $\{\mu_{1,k}\}_{k=1}^{\infty}$ are bounded, we may assume that they converge; let $\xi = \lim_k \xi_k$ and $\mu_1 = \lim_k \mu_{1,k}$.

For each $k \in \mathbb{N}$, it follows by Lemma A.21 that

$$y_k = \frac{F(\mu_{1,k}+\mu_{2,k}) - c\mu_{1,k} - ch(F(\mu_{1,k}+\mu_{2,k}) - F(\mu_{1,k}))\mu_{2,k} - \xi_k(\bar{\omega} - \mu_{1,k} - \mu_{2,k})}{1 + h(F(\mu_{1,k}+\mu_{2,k}) - F(\mu_{1,k}))}.$$

We then have that $\mu_{2,k} > 0$ for all $k \in \mathbb{N}$. Indeed, if $\mu_{2,k} = 0$ for some k, then $y_k = F(\mu_{1,k}) - c\mu_{1,k} - \xi_k(\bar{\omega} - \mu_{1,k}) \leq y_1$ and this contradicts the optimality of O_k. Furthermore, $\mu_1 > 0$; indeed, otherwise, $y_k \to 0$ and, hence, $y_k < y_1$ for all k sufficiently large, contradicting the optimality of O_k (recall that (A2) implies that $y_1 > 0$). Thus, $\mu_{1,k} > 0$ and $\mu_{2,k} > 0$ for all k sufficiently large. In addition, $\xi = 0$ since, otherwise, $\lim_k y_k = F(\mu_1) - c\mu_1 - \xi(\bar{\omega} - \mu_1) < y_1$ if $\mu_1 < \bar{\omega}$ and $\lim_k y_k = F(\bar{\omega}) - c\bar{\omega} < y_1$ by (A4) if $\mu_1 = \bar{\omega}$; hence, $y_k < y_1$ for all k sufficiently large, contradicting the optimality of O_k.

Suppose first that $\mu_{1,k} + \mu_{2,k} < \bar{\omega}$ for all k sufficiently large and fix such k. For each $i = 1, 2$, $\mu_{i,k}$ satisfies the first-order condition

$$\frac{\partial y_k(\mu_{1,k}, \mu_{2,k})}{\partial \mu_{i,k}} = 0 \Leftrightarrow y_k = \frac{\frac{\partial \theta_k(\mu_{1,k},\mu_{2,k})}{\partial \mu_{i,k}}}{\frac{\partial \gamma_k(\mu_{1,k},\mu_{2,k})}{\partial \mu_{i,k}}}.$$

In particular, for $i = 2$, we obtain

$$y_k = \frac{f(\mu_{1,k}+\mu_{2,k}) - ch(F(\mu_{1,k}+\mu_{2,k}) - F(\mu_{1,k})) - chf(\mu_{1,k}+\mu_{2,k})\mu_{2,k} + \xi_k}{hf(\mu_{1,k}+\mu_{2,k})}. \quad (3.1)$$

Thus, (3.1), together with $f(\bar{\omega}) > 0$, implies that $\lim_k y_k = \frac{f(\mu_1)}{hf(\mu_1)} = \frac{1}{h} > 1$. But this is a contradiction, since $y_k \leq 1$ for all $k \in \mathbb{N}$.

Hence, $\mu_{1,k} + \mu_{2,k} = \bar{\omega}$ for infinitely many k; taking a subsequence if necessary, we may assume that $\mu_{1,k} + \mu_{2,k} = \bar{\omega}$ for all k. Hence, $\mu_1 = \bar{\omega}$ and $\lim_k y_k = 1 - c\bar{\omega} = F(\bar{\omega}) - c\bar{\omega} < y_1$ since $f(\bar{\omega}) < c$ by (A4). Hence, $y_k < y_1$, contradicting the optimality of O_k. This contradiction establishes our claim and concludes the proof. ■

We now prove Theorem 3.1. Let $\bar{\xi} > 0$ be given by Theorem 5, $\xi' > 0$ be given by Lemma 3.2 and $\varepsilon > 0$ be given by Lemma 3.3. Define $\tilde{\xi} = \min\{\bar{\xi}, \xi', chf(\bar{\omega})\varepsilon\}$ (note that $f(\bar{\omega}) > 0$ by (A1) and, hence, $\tilde{\xi} > 0$) and let $0 < \xi < \tilde{\xi}$. Suppose that a lexicographically optimal organization O exists. Then $L \geq 2$ by Lemma 3.2 and

$$y = \frac{\sum_{i=1}^{L} F(A_i) - c\mu(A_1) - ch\sum_{i=2}^{L} F(A_i)\mu(A_i) - \xi(\bar{\omega} - \mu(A_1) - \mu(A_2))}{1 + h\sum_{i=2}^{L} F(A_i)}$$

by Lemma A.21.

Consider first the case where $L > 2$. In this case, write $A_L = [a, b)$ with $a < b$, $m = (a + b)/2$ and consider an organization $\hat{O}$ equal to O except that it has $L + 1$ layers, $\hat{A}_L = [a, m)$ and $\hat{A}_{L+1} = [m, b)$. Then

$$F(\hat{A}_L)\mu(\hat{A}_L) + F(\hat{A}_{L+1})\mu(\hat{A}_{L+1}) = \frac{\mu(A_L)}{2}(F(\hat{A}_L) + F(\hat{A}_{L+1})) = \frac{F(A_L)\mu(A_L)}{2}$$

and, hence, $\hat{y} > y$. But this contradicts the optimality of O. This contradiction shows that no lexicographically optimal organization exists when $L > 2$.

We finally consider the case where $L = 2$. In this case, $\mu(A_2) > \varepsilon$ by Lemma 3.3 and, hence, $F(A_2) > f(\bar{\omega})\varepsilon$. Write $A_2 = [a, b)$ with $a < b$, $m = (a+b)/2$ and consider an organization $\hat{O}$ equal to O except that it has 3 layers, $\hat{A}_1 = A_1$, $\hat{A}_2 = [a, m)$, $\hat{A}_3 = [m, b)$ and $\hat{B}_1 \setminus \hat{A}_1 = \Omega \setminus (\hat{A}_1 \cup \hat{A}_2)$. Since $F(\hat{A}_2) + F(\hat{A}_3) = F(A_2)$ and $\mu(\hat{A}_2) = \mu(\hat{A}_3) = \mu(A_2)/2$,

$$\hat{y} - y = \frac{-chF(\hat{A}_2)\mu(\hat{A}_2) - chF(\hat{A}_3)\mu(\hat{A}_3) + chF(A_2)\mu(A_2) + \xi\mu(\hat{A}_2) - \xi\mu(A_2)}{1 + hF(A_2)}.$$

Since $chF(A_2) > chf(\bar{\omega})\varepsilon > \xi$, it follows that

$$\hat{y} - y = \frac{-chF(A_2)\frac{\mu(A_2)}{2} + chF(A_2)\mu(A_2) - \xi\frac{\mu(A_2)}{2}}{1 + hF(A_2)} = \frac{(chF(A_2) - \xi)\frac{\mu(A_2)}{2}}{1 + hF(A_2)} > 0.$$

But this contradicts the optimality of O. This contradiction shows that no lexicographically optimal organization exists.

4 Codes for the small ξ case

In this section we include the codes used for the computations in Section 5.2. The codes are written in python and executed in spyder 3.3.6. The following three codes are used: First, org.py computes the optimal organization as a function of the parameters and hier.py computes the best hierarchy, also as a function of the parameters.[2] Then diff.py makes all the reported computations for the chosen parameter values.

4.1 org.py

```
"""
Computes the optimal organization when the density is affine,
f(x)=a-bx,
as a function of the parameters c, h, b and barL
"""


def sol(c,h,b,barL):


    import numpy as np
    from scipy.optimize import minimize, LinearConstraint


    a=(2+b)/2 #so that f is indeed a density
    def F(x):
        return (x*(2*a-b*x))/2
    #this is the cumulative distribution


#bound on parameters to compute xi
    cl=0.1
    cu=1.49
    bl=1
    bu=1.99
    al=(2+bl)/2
    au=(2+bu)/2
    hl=0.1
```

[2] Both of these codes use the built-in function `minimize`. We have tried replacing it in org.py with `basinhopping` and the results are virtually the same but the program takes far longer to run.

```
    hu=1
    barxi1=(cl*hl*(2-bu))/(2*(au**2)*(barL**2))
    barxi2=(hl*(2-bu)*(al**2-cu**2))/4
    xi=min(barxi1,barxi2)/2

#Initial guess is x0=(0,...,0)
#The variable cons=(1,...,1) is used to define the constraint
    x0=[]
    cons=[]
    for i in range(barL):
        x0.append(0)
        cons.append(1)
#y(x) is output as a function of (mu_1,...,mu_{\bar L})
    def y(x):
        ss=[F(sum(x))-F(x[0]+sum(x[2:len(x)]))]
        s=[ss[0]*x[1]]
        for i in range(3,len(x)+1):
            ss.append(F(x[0]+sum(x[2:i]))-F(x[0]+sum(x[2:i-1])))
            s.append(ss[i-2]*x[i-1])
        return (F(sum(x))-c*x[0]-c*h*sum(s)-xi*(1-x[0]-x[1]))/(1+h*sum(ss))
    def g(x):
        return -y(x)
#The built-in function minimize is used to maximize y, hence to minimize g
#variable bigmu contains the solution, which is a vector of dimension barL,
#for each L=1,...,bar L
    bigmu=[]
    con = LinearConstraint([cons], [-np.inf], [1])
    #con is the constraint mu_1+...+mu_{\bar L}\leq 1

    for j in range(1,barL+1):
```

```
        #this is the numner of layers; if i\leq j, then mu_i is between 0 and 1
        #otherwise, it must be zero - beta captures this
        print('org',j)
        beta=[]
        for i in range(barL):
            if i+1<=j:
                beta.append((0,1))
            else:
                beta.append((0,0))
        res=minimize(g,x0,bounds=beta,constraints=con)
        x0=res.x #initial guess of next iteration is the solution to this one
        bigmu.append(res.x)

#Next we find the optimal L
    mu=bigmu[0]
    L=1
    for i in range(2,barL+1):
        if y(bigmu[i-1])-(i-1)/barL>y(mu)-(L-1)/barL:
            mu=bigmu[i-1]
            L=i

#Next compute alpha and beta (called here size)
    alpha=[F(sum(mu[0:L]))-F(mu[0]+sum(mu[2:L]))]
    for i in range(3,L+1):
        alpha.append(F(mu[0]+sum(mu[2:i]))-F(mu[0]+sum(mu[2:i-1])))

    size=[1/(1+sum(alpha))]
    for i in range(0,L-1):
        size.append(alpha[i]/(1+sum(alpha)))
```

```
    return [L,mu,y(mu)-(L-1)/barL,size]
```

4.2 hier.py

```
def sol(c,h,b,barL):

    import numpy as np
    from scipy.optimize import minimize, LinearConstraint

    a=(2+b)/2
    def F(x):
        return (x*(2*a-b*x))/2

    x0=[]
    cons=[]
    for i in range(barL):
        x0.append(0)
        cons.append(1)

    def y(l,x):
        s=[]
        ss=[]
        for i in range(2,len(x)+1):
            ss.append(1-F(sum(x[0:i-1])))
            s.append(ss[i-2]*x[i-1])
        return (F(sum(x[0:l]))-c*x[0]-c*h*sum(s[0:l-1]))/(1+h*sum(ss[0:l-1]))

    bigmu=[]
    con = LinearConstraint([cons], [-np.inf], [1])

    for j in range(1,barL+1):
```

```
    print('hier',j)
    beta=[]
    for i in range(barL):
        if i+1<=j:
            beta.append((0,1))
        else:
            beta.append((0,0))
    def g(x):
        return -y(j,x)
    res=minimize(g,x0,bounds=beta,constraints=con)
    x0=res.x
    bigmu.append(res.x)

mu=bigmu[0]
L=1
for i in range(2,barL+1):
    if y(i,bigmu[i-1])-(i-1)/barL>y(L,mu)-(L-1)/barL:
        mu=bigmu[i-1]
        L=i

alpha=[]
for i in range(2,L+1):
    alpha.append(1-F(sum(mu[0:i-1])))

size=[1/(1+sum(alpha))]
for i in range(0,L-1):
    size.append(alpha[i]/(1+sum(alpha)))



return [L,mu,y(L,mu)-(L-1)/barL,size]
```

4.3 diff.py

```
import org, hier


c=1.4
h=0.5
b=1
barL=100

a=(2+b)/2


cl=0.1
cu=1.49
bl=1
bu=1.99
al=(2+bl)/2
au=(2+bu)/2
hl=0.1
hu=1
barxi1=(cl*hl*(2-bu))/(2*(au**2)*(barL**2))
barxi2=(hl*(2-bu)*(al**2-cu**2))/4
xi=min(barxi1,barxi2)/2



x=org.sol(c,h,b,barL)
y=hier.sol(c,h,b,barL)
d=(x[2]+(x[0]-1)/barL-y[2]-(y[0]-1)/barL)/(y[2]+(y[0]-1)/barL)
dn=(x[2]-y[2])/y[2]
print('a',a,'b',b,'xi',xi,'eta',1/barL,'c',c,'h',h)
```

```
print('Increase in net output is',100*dn)
print('Increase in output is',100*d)

print('Optimal organization')
print('L',x[0])
print('net output', x[2])
print('output',x[2]+(x[0]-1)/barL)
print('sum of mu',sum(x[1]))
for i in range(0,x[0]):
    print('mu',i,'is',x[1][i])
    print('beta',i,'is',x[3][i])

print('Best hierarchy')
print('L',y[0])
print('net output', y[2])
print('output',y[2]+(y[0]-1)/barL)
print('sum of mu',sum(x[1]))
for i in range(0,y[0]):
    print('mu',i,'is',y[1][i])
    print('beta',i,'is',y[3][i])

def F(x):
    return (x*(2*a-b*x))/2
w=y[1]
ss=[F(sum(w))-F(w[0]+sum(w[2:len(w)]))]
s=[ss[0]*w[1]]
for i in range(3,len(w)+1):
    ss.append(F(w[0]+sum(w[2:i]))-F(w[0]+sum(w[2:i-1])))
    s.append(ss[i-2]*w[i-1])
o=(F(sum(w))-c*w[0]-c*h*sum(s)-xi*(1-w[0]-w[1]))/(1+h*sum(ss))
```

```
g1=100*(o-y[2])/y[2]
g2=100*(x[2]-o)/y[2]
print('gain with same L',g1,'%',g1/dn)
print('gain from L',g2,'%',g2/dn)
```

5 Simulations for intermediate values of ξ

In this section, we describe the computational approach used in Section 5.3, and we report simulation results for other configurations of parameter values.

5.1 Computations

We assume that $\pi = h$ and that $h < 1$. The former simplifies the expression for α_i for each $i \in L \setminus \{1\}$ since, by Lemma A.11, $\alpha_i = h(1 - F((\cup_{j<i} B_j) \setminus A_i))$. The latter then implies that $\alpha_i < 1$ and Lemma A.16 implies that $\alpha_i > 0$.

We use the approach described in Section A.10 to compute optimal organizations. In what follows, we describe the candidates for optimal organizations when the number of layers is L and $L \in \{1, 2, 3, 4\}$.

5.1.1 $L = 1$

The best organization with one layer does not depend on ξ, i.e. $B_1 \setminus A_1 = \emptyset$ always. In this case, $\mu_1 = \min\left\{\max\left\{\frac{a-c}{b}, 0\right\}, 1\right\}$ and $y_1 = F(\mu_1) - c\mu_1$.

5.1.2 $L = 2$

In this case, $\mathcal{C} = \{A_1, A_2, B_1 \setminus A_1\}$. Since $A_1 < C$ for each $C \in \mathcal{C}$, there are two possible orders:

1. $A_1 < A_2 < B_1 \setminus A_1$, and

2. $A_1 < B_1 \setminus A_1 < A_2$.

We let $\mu_0 = \mu(A_1)$, $\mu_1 = \mu(A_2)$ and $\mu_2 = \mu(B_1 \setminus A_1)$. In order 1, $A_1 = [0, \mu_0)$, $A_2 = [\mu_0, \mu_0 + \mu_1)$ and $B_1 \setminus A_1 = [\mu_0 + \mu_1, \mu_0 + \mu_1 + \mu_2)$. Hence, $\alpha_2 = h(1 - F(\mu_0) - F(\sum_{i=0}^{2} \mu_i) + F(\mu_0 + \mu_1))$ and $y = (F(\mu_0 + \mu_1) - c\mu_0 - c\alpha_2\mu_1 - \xi\mu_2)/(1 + \alpha_2)$.

In order 2, $A_1 = [0, \mu_0)$, $B_1 \setminus A_1 = [\mu_0, \mu_0 + \mu_2)$ and $A_2 = [\mu_0 + \mu_2, \mu_0 + \mu_1 + \mu_2)$. Hence, $\alpha_2 = h(1 - F(\mu_0 + \mu_2))$ and $y = (F(\mu_0) + F(\sum_{i=0}^{2} \mu_i) - F(\mu_0 + \mu_2) - c\mu_0 - c\alpha_2\mu_1 - \xi\mu_2)/(1 + \alpha_2)$.

5.1.3 $L = 3$

In this case, $\mathcal{C} = \{A_3 \cap (B_1 \setminus A_1), A_1, A_2, A_3 \cap (B_1 \setminus A_1)^c, (B_1 \setminus A_1) \cap A_3^c, B_2 \setminus A_2\}$. We have that $A_1 < C$ for each $C \in \mathcal{C} \setminus \{A_1\}$ and $A_2 < A_3 \cap (B_1 \setminus A_1)^c$ by Corollary 2. Moreover, Corollary 1 implies that $A_3 \cap (B_1 \setminus A_1) < A_3 \cap (B_1 \setminus A_1)^c$, $A_3 \cap (B_1 \setminus A_1) < (B_1 \setminus A_1) \cap A_3^c$, $(B_1 \setminus A_1) \cap A_3^c < B_2 \setminus A_2$ and $A_2 < B_2 \setminus A_2$.

When $L = 3$, we have that

$$B_1 \setminus A_2 = A_1 \cup (A_3 \cap (B_1 \setminus A_1)) \cup ((B_1 \setminus A_1) \cap A_3^c) \text{ and}$$
$$(B_1 \cup B_2) \setminus A_3 = A_1 \cup ((B_1 \setminus A_1) \cap A_3^c) \cup A_2 \cup (B_2 \setminus A_2).$$

Then:

$$\begin{aligned}
\alpha_2 = {} & h(1 - F(A_1) - F(A_3 \cap (B_1 \setminus A_1)) - F((B_1 \setminus A_1) \cap A_3^c)), \\
\alpha_3 = {} & h(1 - F(A_1) - F((B_1 \setminus A_1) \cap A_3^c) - F(A_2) - F(B_2 \setminus A_2)), \\
\gamma = {} & 1 + \alpha_2 + \alpha_3 \text{ and} \\
\theta = {} & F(A_1) + F(A_2) + F(A_3 \cap (B_1 \setminus A_1)) + F(A_3 \cap (B_1 \setminus A_1)^c) \\
& - c\mu(A_1) - \xi\mu((B_1 \setminus A_1) \cap A_3^c) - c\alpha_2\mu(A_2) - \xi\alpha_2\mu(B_2 \setminus A_2) \\
& - c\alpha_3\mu(A_3 \cap (B_1 \setminus A_1)^c) - (c\alpha_3 + \xi)\mu(A_3 \cap (B_1 \setminus A_1)).
\end{aligned}$$

The following lemma uses $A_3 \cap (B_1 \setminus A_1) < (B_1 \setminus A_1) \cap A_3^c$ to obtain an inequality via Theorem A.1 that will be used to order additional members of $\mathcal{C}$. The idea is that when swapping part of $A_3 \cap (B_1 \setminus A_1)$ with part of $(B_1 \setminus A_1) \cap A_3^c$, keeping their Lebesgue measures constant, $F(A_3 \cap (B_1 \setminus A_1))$ decreases and $F((B_1 \setminus A_1) \cap A_3^c)$ increases since $A_3 \cap (B_1 \setminus A_1) < (B_1 \setminus A_1) \cap A_3^c$. Consequently, there is a trade-off between production and fraction of workers, because a decrease in $F(A_3 \cap (B_1 \setminus A_1))$ makes production decline while an increase in $F((B_1 \setminus A_1) \cap A_3^c)$, by reducing α_3, increases the fraction of workers. But this change cannot be beneficial and, therefore, the first effect must dominate.

Lemma 5.1 *If $A_3 \cap (B_1 \setminus A_1) \neq \emptyset$ and $(B_1 \setminus A_1) \cap A_3^c \neq \emptyset$, then $\frac{1-ch\mu(A_3)}{h} > y$.*

Proof. For each $0 < \varepsilon < \min\{F(A_3 \cap (B_1 \setminus A_1)), F((B_1 \setminus A_1) \cap A_3^c)\}$, we have that

$$y_{F((B_1 \setminus A_1) \cap A_3^c), F(A_3 \cap (B_1 \setminus A_1))}(\varepsilon) = \frac{\theta - \varepsilon + ch\mu(A_3)\varepsilon}{\gamma - h\varepsilon}.$$

Since $A_3 \cap (B_1 \setminus A_1) < (B_1 \setminus A_1) \cap A_3^c$ by Theorem 3, then there is $\varepsilon \in (0, F(A_3 \cap (B_1 \setminus A_1)))$ such that $y_{F((B_1 \setminus A_1) \cap A_3^c), F(A_3 \cap (B_1 \setminus A_1))}(\varepsilon) \leq y$ by Theorem A.1. In fact, it must be that $y_{F((B_1 \setminus A_1) \cap A_3^c), F(A_3 \cap (B_1 \setminus A_1))}(\varepsilon) < y$ since, otherwise, there is an optimal organization where $A_3 \cap (B_1 \setminus A_1) < (B_1 \setminus A_1) \cap A_3^c$ does not hold, contradicting Theorem 3. Indeed, let $A_3 \cap (B_1 \setminus A_1) = [a, b)$ with $\varepsilon < b - a$ and consider an organization $\hat{O}$ equal to O except that $(\hat{B}_1 \setminus \hat{A}_1) \cap \hat{A}_3^c) = (B_1 \setminus A_1) \cap A_3^c) \cup [a, \varepsilon)$ and $\hat{A}_3 \cap (\hat{B}_1 \setminus \hat{A}_1) = A_3 \cap (B_1 \setminus A_1) \setminus [a, \varepsilon)$. Then $\hat{y} = y$, showing that $\hat{O}$ is also optimal.

It then follows from $y_{F((B_1 \setminus A_1) \cap A_3^c), F(A_3 \cap (B_1 \setminus A_1))}(\varepsilon) < y$ that $\frac{1-ch\mu(A_3)}{h} > y$. ■

The following ordering follows from Lemma 5.1 together with Theorem A.1.

Lemma 5.2 *If $A_3 \cap (B_1 \setminus A_1) \neq \emptyset$ and $(B_1 \setminus A_1) \cap A_3^c \neq \emptyset$, then $A_3 \cap (B_1 \setminus A_1)^c < (B_2 \setminus A_2)$.*

Proof. This is trivial if $A_3 \cap (B_1 \setminus A_1)^c = \emptyset$ or $B_2 \setminus A_2 = \emptyset$ and it follows from Theorem A.1 otherwise. Indeed, for each $0 < \varepsilon < \min\{F(A_3 \cap (B_1 \setminus A_1)^c), F(B_2 \setminus A_2)\}$,

$$y_{F(A_3 \cap (B_1 \setminus A_1)^c), F(B_2 \setminus A_2)}(\varepsilon) = \frac{\theta + \varepsilon - ch\mu(A_3)\varepsilon}{\gamma + h\varepsilon} > y$$

since $\frac{1-ch\mu(A_3)}{h} > y$ by Lemma 5.1. ■

The above discussion, together with the conclusion of Lemma 5.2, gives us the following orders:

1. $A_1 < A_3 \cap (B_1 \setminus A_1) < A_2 < A_3 \cap (B_1 \setminus A_1)^c < (B_1 \setminus A_1) \cap A_3^c < B_2 \setminus A_2$.

2. $A_1 < A_3 \cap (B_1 \setminus A_1) < A_2 < (B_1 \setminus A_1) \cap A_3^c < A_3 \cap (B_1 \setminus A_1)^c < B_2 \setminus A_2$.

3. $A_1 < A_3 \cap (B_1 \setminus A_1) < (B_1 \setminus A_1) \cap A_3^c < A_2 < A_3 \cap (B_1 \setminus A_1)^c < B_2 \setminus A_2$.

4. $A_1 < A_2 < A_3 \cap (B_1 \setminus A_1) < A_3 \cap (B_1 \setminus A_1)^c < (B_1 \setminus A_1) \cap A_3^c < B_2 \setminus A_2$.

5. $A_1 < A_2 < A_3 \cap (B_1 \setminus A_1) < (B_1 \setminus A_1) \cap A_3^c < A_3 \cap (B_1 \setminus A_1)^c < B_2 \setminus A_2$.

However, since Lemma 5.2 has assumptions that may fail to hold, we also consider the following additional three orders:

6. $A_1 < A_3 \cap (B_1 \setminus A_1) < A_2 < (B_1 \setminus A_1) \cap A_3^c < B_2 \setminus A_2 < A_3 \cap (B_1 \setminus A_1)^c$.

7. $A_1 < A_3 \cap (B_1 \setminus A_1) < (B_1 \setminus A_1) \cap A_3^c < A_2 < B_2 \setminus A_2 < A_3 \cap (B_1 \setminus A_1)^c$.

8. $A_1 < A_2 < A_3 \cap (B_1 \setminus A_1) < (B_1 \setminus A_1) \cap A_3^c < B_2 \setminus A_2 < A_3 \cap (B_1 \setminus A_1)^c$.

It turns out that neither of these three additional orders is optimal in any of our simulations; in fact, order 1 is the optimal one in all of them.

5.1.4 $L = 4$

In this case, $\mathcal{C} = \{A_1, A_3 \cap (B_1 \setminus A_1), A_4 \cap (B_1 \setminus A_1), A_2, A_4 \cap (B_2 \setminus A_2), A_3 \cap (B_1 \setminus A_1)^c, A_4 \cap (B_1 \setminus A_1)^c \cap (B_2 \setminus A_2)^c, (B_1 \setminus A_1) \cap A_3^c \cap A_4^c, (B_2 \setminus A_2) \cap A_4^c, B_3 \setminus A_3\}$.

First, we rule out as many orders as we can. Corollaries 1 and 2 imply that:

1. $A_1 < C$ for each $C \in \mathcal{C} \setminus \{A_1\}$ (Corollary 2).

2. $A_3 \cap (B_1 \setminus A_1) < A_3 \cap (B_1 \setminus A_1)^c$ (Corollary 1).

3. $A_3 \cap (B_1 \setminus A_1) < (B_1 \setminus A_1) \cap A_3^c \cap A_4^c$ (Corollary 1).

4. $A_4 \cap (B_1 \setminus A_1) < A_4 \cap (B_2 \setminus A_2)$ (Corollary 2).

5. $A_4 \cap (B_1 \setminus A_1) < A_4 \cap (B_1 \setminus A_1)^c \cap (B_2 \setminus A_2)^c$ (Corollary 1).

6. $A_4 \cap (B_1 \setminus A_1) < (B_1 \setminus A_1) \cap A_3^c \cap A_4^c$ (Corollary 1).

7. $A_4 \cap (B_2 \setminus A_2) < A_4 \cap (B_1 \setminus A_1)^c \cap (B_2 \setminus A_2)^c$ (Corollary 1).

8. $A_4 \cap (B_2 \setminus A_2) < (B_2 \setminus A_2) \cap A_4^c$ (Corollary 1).

9. $A_2 < (B_2 \setminus A_2) \cap A_4^c$ (Corollary 1).

10. $A_3 \cap (B_1 \setminus A_1)^c < B_3 \setminus A_3$ (Corollary 1).

11. $(B_1 \setminus A_1) \cap A_3^c \cap A_4^c < (B_2 \setminus A_2) \cap A_4^c$ (Corollary 2).

12. $(B_1 \setminus A_1) \cap A_3^c \cap A_4^c < B_3 \setminus A_3$ (Corollary 2).

13. $A_2 < A_3 \cap (B_1 \setminus A_1)^c$ (Corollary 2).

14. $A_3 \cap (B_1 \setminus A_1)^c < A_4 \cap (B_1 \setminus A_1)^c \cap (B_2 \setminus A_2)^c$ (Corollary 2).

15. $(B_2 \setminus A_2) \cap A_4^c < B_3 \setminus A_3$ (Corollary 2).

16. $A_2 < A_4 \cap (B_2 \setminus A_2)$ (Corollary 2).

When $L = 4$, we have that

$$\begin{aligned}
B_1 \setminus A_2 &= (A_3 \cap (B_1 \setminus A_1)) \cup (A_4 \cap (B_1 \setminus A_1)) \\
&\cup((B_1 \setminus A_1) \cap (A_3^c \cap A_4^c)) \cup A_1, \\
(B_1 \cup B_2) \setminus A_3 &= ((B_1 \setminus A_1) \cap A_4) \cup ((B_1 \setminus A_1) \cap (A_3^c \cap A_4^c)) \cup A_1 \\
&\cup(A_4 \cap (B_2 \setminus A_2)) \cup ((B_2 \setminus A_2) \cap A_4^c) \cup A_2, \text{ and} \\
(B_1 \cup B_2 \cup B_3) \setminus A_4 &= (A_3 \cap (B_1 \setminus A_1)) \cup ((B_1 \setminus A_1) \cap (A_3^c \cap A_4^c)) \cup A_1 \\
&\cup((B_2 \setminus A_2) \cap A_4^c) \cup A_2 \cup (B_3 \setminus A_3) \cup (A_3 \cap (B_1 \setminus A_1)^c).
\end{aligned}$$

Hence,

$$\begin{aligned}
\alpha_2 &= h(1 - F(A_3 \cap (B_1 \setminus A_1)) - F(A_4 \cap (B_1 \setminus A_1)) \\
&-F((B_1 \setminus A_1) \cap (A_3^c \cap A_4^c)) - F(A_1)), \\
\alpha_3 &= h(1 - F((B_1 \setminus A_1) \cap A_4) - F((B_1 \setminus A_1) \cap (A_3^c \cap A_4^c)) - F(A_1) \\
&-F(A_4 \cap (B_2 \setminus A_2)) - F((B_2 \setminus A_2) \cap A_4^c) - F(A_2)), \text{ and} \\
\alpha_4 &= h(1 - F(A_3 \cap (B_1 \setminus A_1)) - F((B_1 \setminus A_1) \cap (A_3^c \cap A_4^c)) - F(A_1) \\
&-F((B_2 \setminus A_2) \cap A_4^c) - F(A_2) - F(B_3 \setminus A_3) - F(A_3 \cap (B_1 \setminus A_1)^c)).
\end{aligned}$$

Also,

$$
\begin{aligned}
\gamma \quad &= 1 + \alpha_2 + \alpha_3 + \alpha_4 \text{ and} \\
\theta \quad &= F(A_1) + F(A_3 \cap (B_1 \setminus A_1)) + F(A_4 \cap (B_1 \setminus A_1)) + F(A_2) \\
&+F(A_4 \cap (B_2 \setminus A_2)) + F(A_3 \cap (B_1 \setminus A_1)^c) + F(A_4 \cap (B_1 \setminus A_1)^c \cap (B_2 \setminus A_2)^c) \\
&-c\mu(A_1) - (c\alpha_3 + \xi)\mu(A_3 \cap (B_1 \setminus A_1)) - (c\alpha_4 + \xi)\mu(A_4 \cap (B_1 \setminus A_1)) \\
&-c\alpha_2\mu(A_2) - (c\alpha_4 + \xi\alpha_2)\mu(A_4 \cap (B_2 \setminus A_2)) - c\alpha_3\mu(A_3 \cap (B_1 \setminus A_1)^c) \\
&-c\alpha_4\mu(A_4 \cap (B_1 \setminus A_1)^c \cap (B_2 \setminus A_2)^c) - \xi\mu((B_1 \setminus A_1) \cap A_3^c \cap A_4^c) \\
&-\xi\alpha_2\mu((B_2 \setminus A_2) \cap A_4^c) - \xi\alpha_3\mu(B_3 \setminus A_3).
\end{aligned}
$$

We use orders.py to find all orders consistent with the above results; there are 192 in total which are listed in the code.

We note that as it was the case where $L = 3$, order 1, which is now

$$
\begin{aligned}
&A_1 < A_4 \cap (B_1 \setminus A_1) < A_3 \cap (B_1 \setminus A_1) < A_2 < A_3 \cap (B_1 \setminus A_1)^c < A_4 \cap (B_2 \setminus A_2) \\
&< A_4 \cap (B_1 \setminus A_1)^c \cap (B_2 \setminus A_2)^c < (B_1 \setminus A_1) \cap A_3^c \cap A_4^c < (B_2 \setminus A_2) \cap A_4^c < B_3 \setminus A_3,
\end{aligned}
$$

is optimal in all our simulations.

5.2 Codes

We briefly describe the codes used in our simulations. The codes were written in Python 3, and are available here.

The starting point are the codes orgL2.py, orgL3.py and orgL4.py, each of which computes the optimal organization for the corresponding number of layers. In each of these codes, each possible optimal ordering of $\mathcal{C}$ is considered and the built-in function `minimize` is used to find the size of each element of $\mathcal{C}$ and corresponding output.[3] Then the order that leads to the highest such output is selected; the code returns the order of $\mathcal{C}$, the size of each element of $\mathcal{C}$, the output of the optimal organization, the size β_i of each layer and the costs of learning ν_i of each layer.

[3]We use the solution to the optimal organization with $L = 1$ as the initial guess except when it features $\mu(A_1) = 0$. In this case, we use `basinhopping` instead of `minimize`.

One aspect of the above codes which is worth discussing concerns the choice of the ordering of $\mathcal{C}$, which we illustrate in the case where $L = 2$. In this case there are two possible orderings: $\psi_1 = (A_1 < A_2 < B_1 \setminus A_1)$ and $\psi_2 = (A_1 < B_1 \setminus A_1 < A_2)$. These two orders are the same if $B_1 \setminus A_1 = \emptyset$, namely $A_1 < A_2$. Hence, in orgL2.py, the order $A_1 < B_1 \setminus A_1 < A_2$ is the optimal one only if $y_{L,\psi_2} > y_{L,\psi_1}$ and $\mu(B_1 \setminus A_1) > 1/100000$, i.e. $\mu(B_1 \setminus A_1)$ is significantly above 0.[4]

The next step is performed by orgLaux.py, which solves $\max_{L \in \{1,2,3,4\}} y_L$. One issue with this maximization problem is that often $y_{L+1} \geq y_L$ (and then possibly $y_{L+1} > y_L$ due to approximation errors) by simply taking the organization that yields y_L and adding layer $L + 1$ with $B_{L+1} = A_{L+1} = \emptyset$. To avoid this, for e.g. $L = 3$ to be better than $L = 2$, we require not only that $y_3 > y_2$ but also that $\mu(A_3) = \mu(A_3 \cap (B_1 \setminus A_1)) + \mu(A_3 \cap (B_1 \setminus A_1)^c) > 1/100000$.

Finally, the computations and graphs reported in Section 5.3 are produced using orgL.py, orgLoptimal.py and orgLh.py.

5.3 Simulations for other configurations of paramters

In the main text we consider the baseline case $c = 1$, $h = 0.5$, and $b = 1$, where the optimal hierarchy when $\xi = c$ has 3 layers. Here we start by considering alternative values of c, chosen so that the optimal hierarchy has 4 layers ($c = 1.2$), 2 layers ($c = 0.6$) and 1 layer ($c = 0.2$). Then we consider changes to the density by varying b relative to the baseline case. Next we consider changes to the relative and absolute values of c and h. Finally, we reconsider the effects of a 10% fall in h for different values of c.

[4]This approach requires checking that the relevant sets that distinguish between certain orders are (significantly) nonempty. An alternative approach is to require that $y_{L,\psi_2} > y_{L,\psi_1} + 1/1000000$ for ψ_2 to be considered better than ψ_1, which also ensures that differences between the orders are not just the result of the approximate nature of the `minimize` algorithm. We take the latter approach in orgL4.py, where it is more convenient because of the large number of possible orders.

5.3.1 $c = 1.2$, $h = 0.5$ and $b = 1$

The case where $c = 1.2$ is illustrated in Figure 1.[5] We have an analogous pattern to the baseline case except that in the high range of ξ, the optimal organization has 4 layers (by design) and in the middle range A_1 is already empty. It is also the case that $A_4 \subseteq B_1 \setminus A_1$ in the middle range, hence the optimal organization has the same structure as in the case where $c = 1$.

5.3.2 $c = 0.6$, $h = 0.5$ and $b = 1$

The case where $c = 0.6$ is illustrated in Figure 2. In this case there is no middle range and A_1 is never empty. The latter fact increases the learning costs of layer 1, which is always above those of layer 4 and also of those of layer 3, except when $\xi = 0.06$ where ν_3 is slightly above ν_1.

5.3.3 $c = 0.2$, $h = 0.5$ and $b = 1$

The case where $c = 0.2$ is illustrated in Figure 3. Here the optimal organization has just one layer because $h = 0.5$ is sufficiently high to prevent adding additional layers which could benefit from screening.

5.3.4 $c = 1$, $h = 0.5$ and $b = 0.1$

The case where the density is flatter than the baseline case is illustrated in Figure 4. This case is analogous to the baseline case: The optimal organization is an hierarchy when $\xi \in \{0.9, 1\}$, it is fully screening when $\xi \leq 0.4$ and there is a middle range with $A_3 \subseteq (B_1 \setminus A_1)^c$ and $A_4 \subseteq B_1 \setminus A_1$. The differences are: (i) the transition from

[5] For the optimal organization, figure "Pure knowledge sets" gives the sizes of A_1 (red), A_2 (blue), $A_3 \cap (B_1 \setminus A_1)^c$ (yellow) and $A_4 \cap (B_1 \setminus A_1)^c \cap (B_2 \setminus A_2)^c$ (black); figure "$B1 \setminus A1$" gives the sizes of $A_3 \cap (B_1 \setminus A_1)$ (red), $A_4 \cap (B_1 \setminus A_1)$ (blue) and $(B_1 \setminus A_1) \cap A_3^c \cap A_4^c$ (yellow); figure "beta" gives β_1 (red), β_2 (blue), β_3 (yellow) and β_4 (black); and figure "Costs of learning" gives ν_1 (red), ν_2 (blue), ν_3 (yellow) and ν_4 (black). The sizes of the elements of $\mathcal{C}$ are given as a percentage of the total measure of Ω.

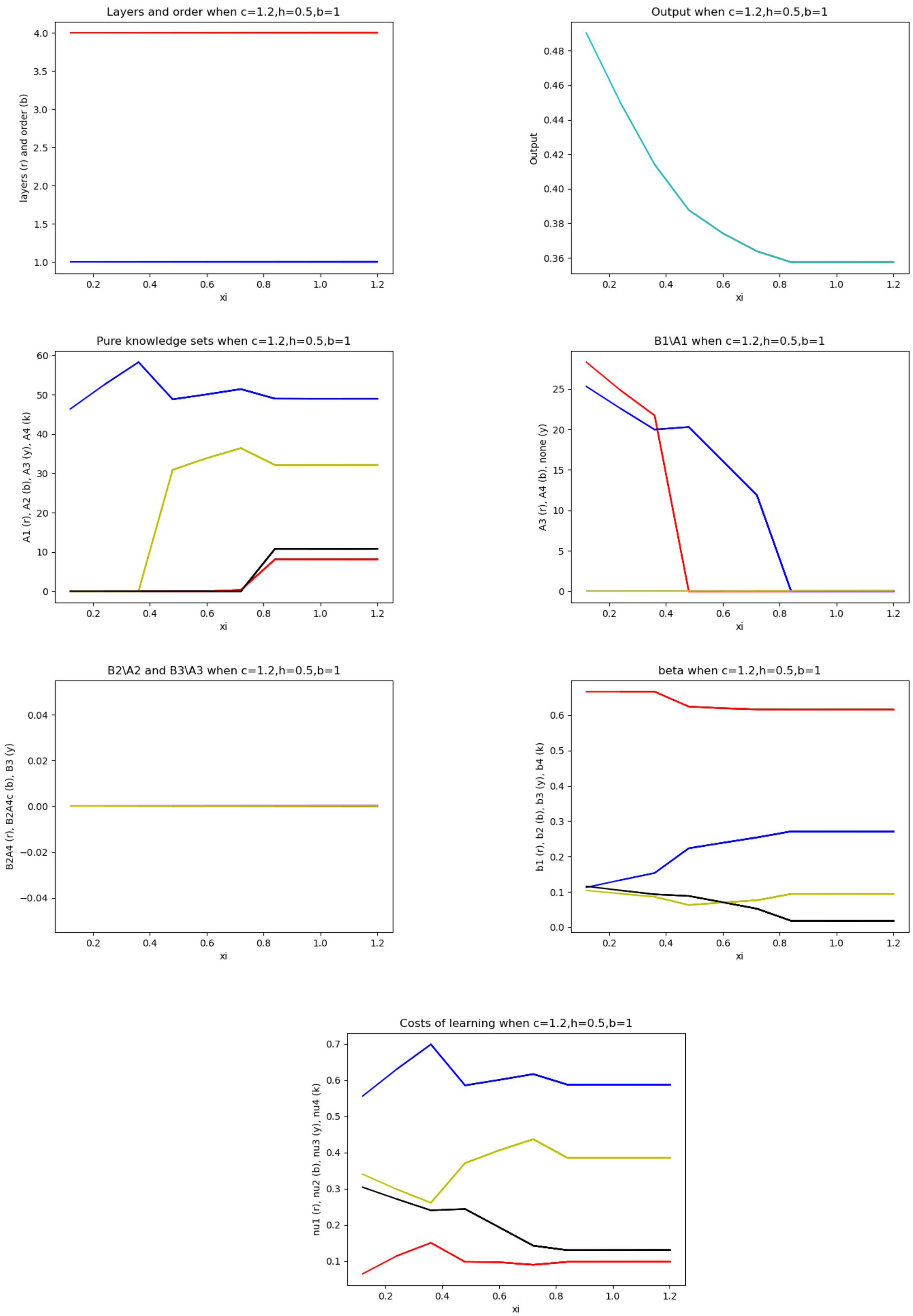

Figure 1: Optimal organization at different levels of ξ when $c = 1.2$, $h = 0.5$ and $b = 1$

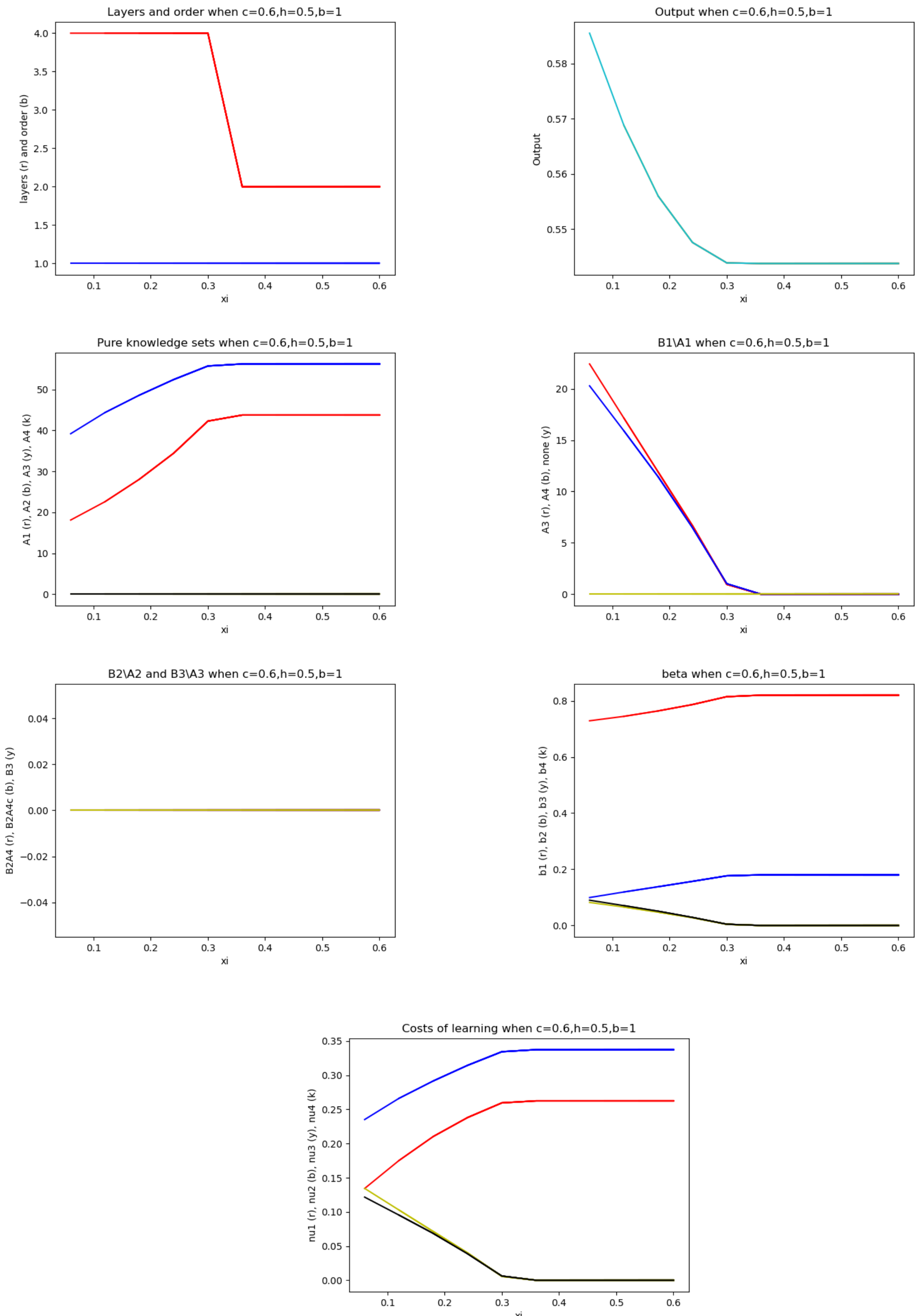

Figure 2: Optimal organization at different levels of ξ when $c = 0.6$, $h = 0.5$ and $b = 1$

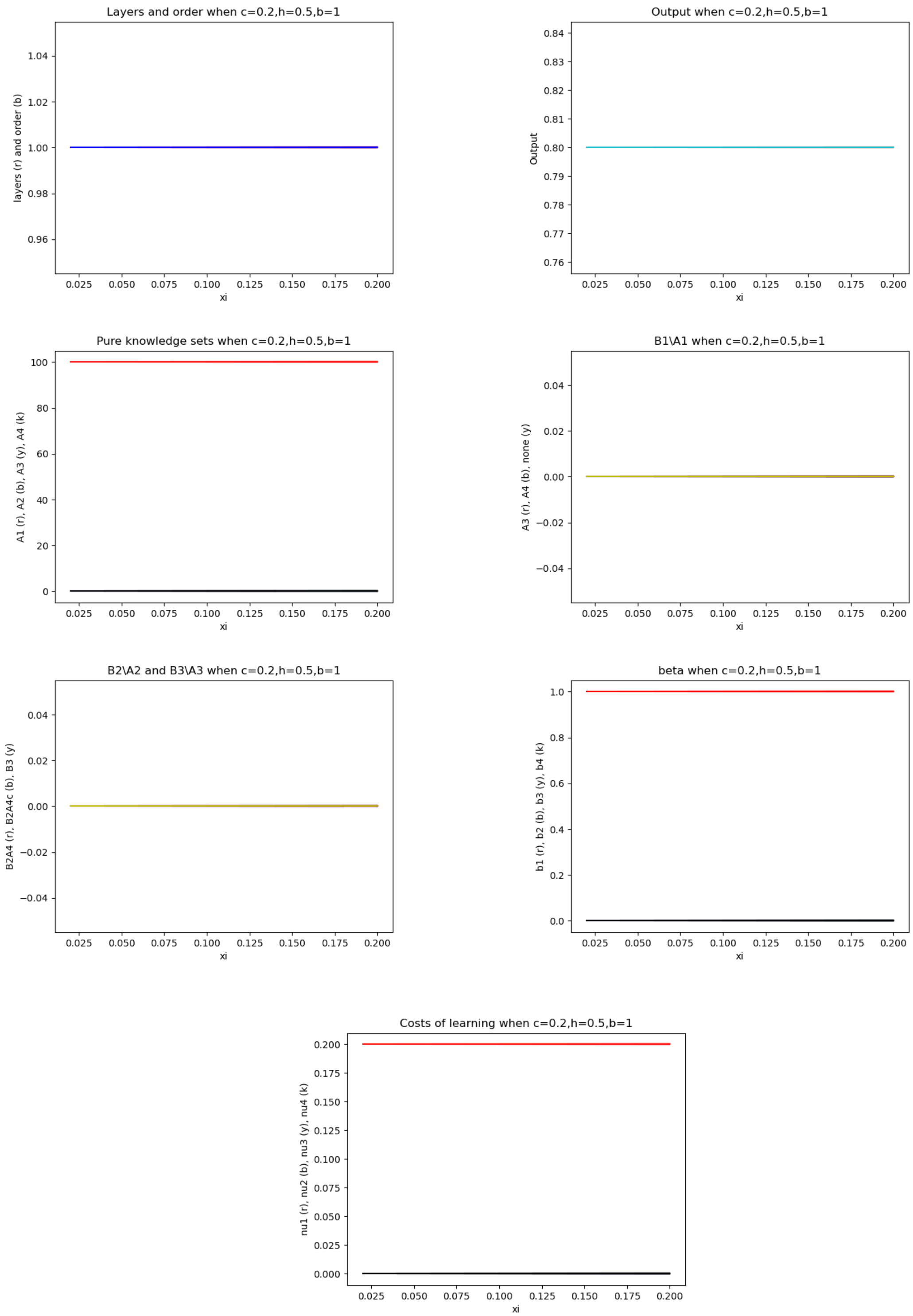

Figure 3: Optimal organization at different levels of ξ when $c = 0.2$, $h = 0.5$ and $b = 1$

hierarchy to the middle range happens before ($\xi = 0.8$ vs $\xi = 0.7$ in the baseline case) and (ii) A_1 becomes empty earlier too (at $\xi = 0.7$ vs $\xi = 0.4$ in the baseline case).

5.3.5 $c = 1$, $h = 0.5$ and $b = 1.9$

The case where the density is steeper than the baseline case is illustrated in Figure 5. The pattern is similar to the baseline case: The optimal organization, which has always 4 layers, is an hierarchy when $\xi \geq 0.6$. It is followed by a middle range, when $\xi \in \{0.3, 0.4, 0.5\}$, where $\emptyset \neq A_4 \subseteq B_1 \setminus A_1$ and $A_3 \cap (B_1 \setminus A_1)^c$ is nonempty. When $\xi \leq 0.2$, $A_3 \cup A_4 \subseteq B_1 \setminus A_1$. There are, however, the following interesting differences:

1. The first time $A_3 \cap (B_1 \setminus A_1)$ is nonempty is at $\xi = 0.3$ and it is also the case that $A_3 \cap (B_1 \setminus A_1)^c$ is nonempty.

2. When $\xi \leq 0.2$, $(B_2 \setminus A_2) \cap A_4^c$ is nonempty.

5.3.6 $c = 0.5$, $h = 0.25$ and $b = 1$

Same density as in the baseline case, costs are both low, with c still bigger than h. This case, illustrated in Figure 6, is analogous to the baseline case except that there is no middle range. The optimal organization is a hierarchy with 3 layers whenever $\xi \geq 0.3$ (hence, for a larger set of ξs) and is fully screening when $\xi < 0.3$.

5.3.7 $c = 0.25$, $h = 0.5$ and $b = 1$

Same density as in the baseline case, costs are both low, with h now bigger than c. This case, illustrated in Figure 7, favors small organizations: The optimal organization is, for each $\xi \in \{0.1, \ldots, 1\}$, a hierarchy with 2 layers. The workers know about 85% of Ω and the managers know the remaining 15%.

5.3.8 Response to a 10% fall in h

Finally we consider the response of an optimal organization to a fall in h when $c \in \{0.2, 0, 6, 1.2\}$ in addition to the baseline case considered in the main text where $c = 1$.

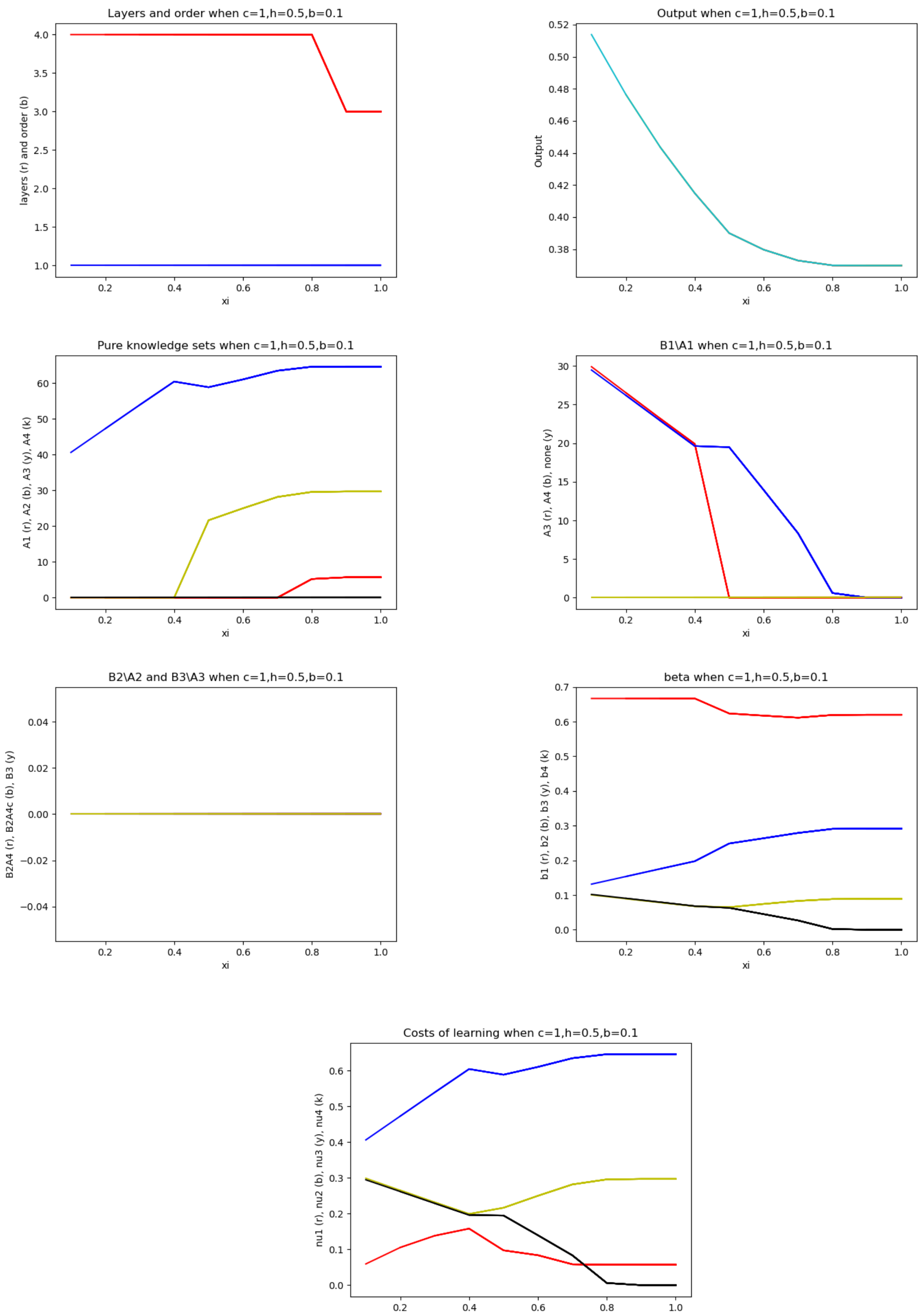

Figure 4: Optimal organization at different levels of ξ when $c = 1$, $h = 0.5$ and $b = 0.1$

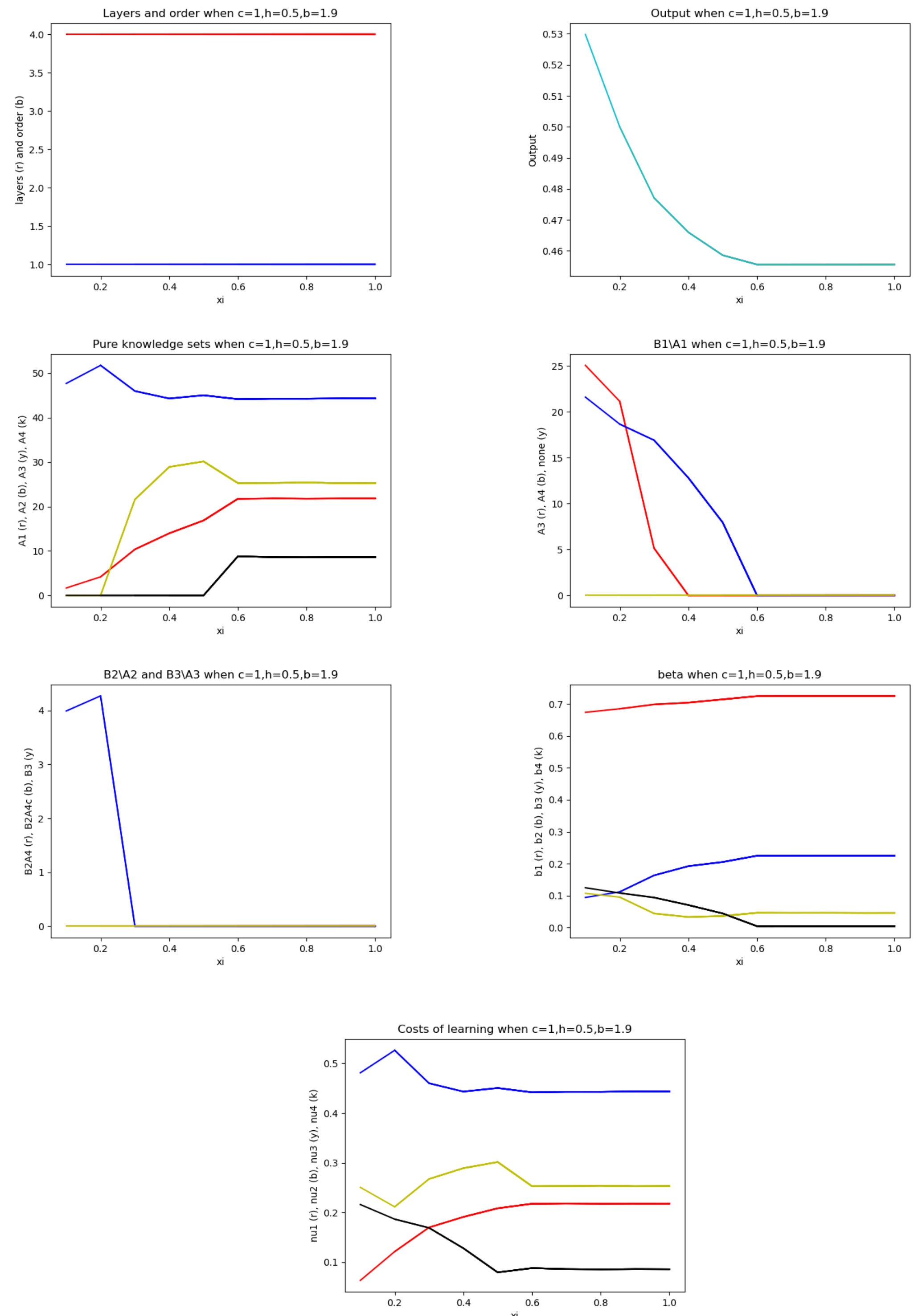

Figure 5: Optimal organization at different levels of ξ when $c = 1$, $h = 0.5$ and $b = 1.9$

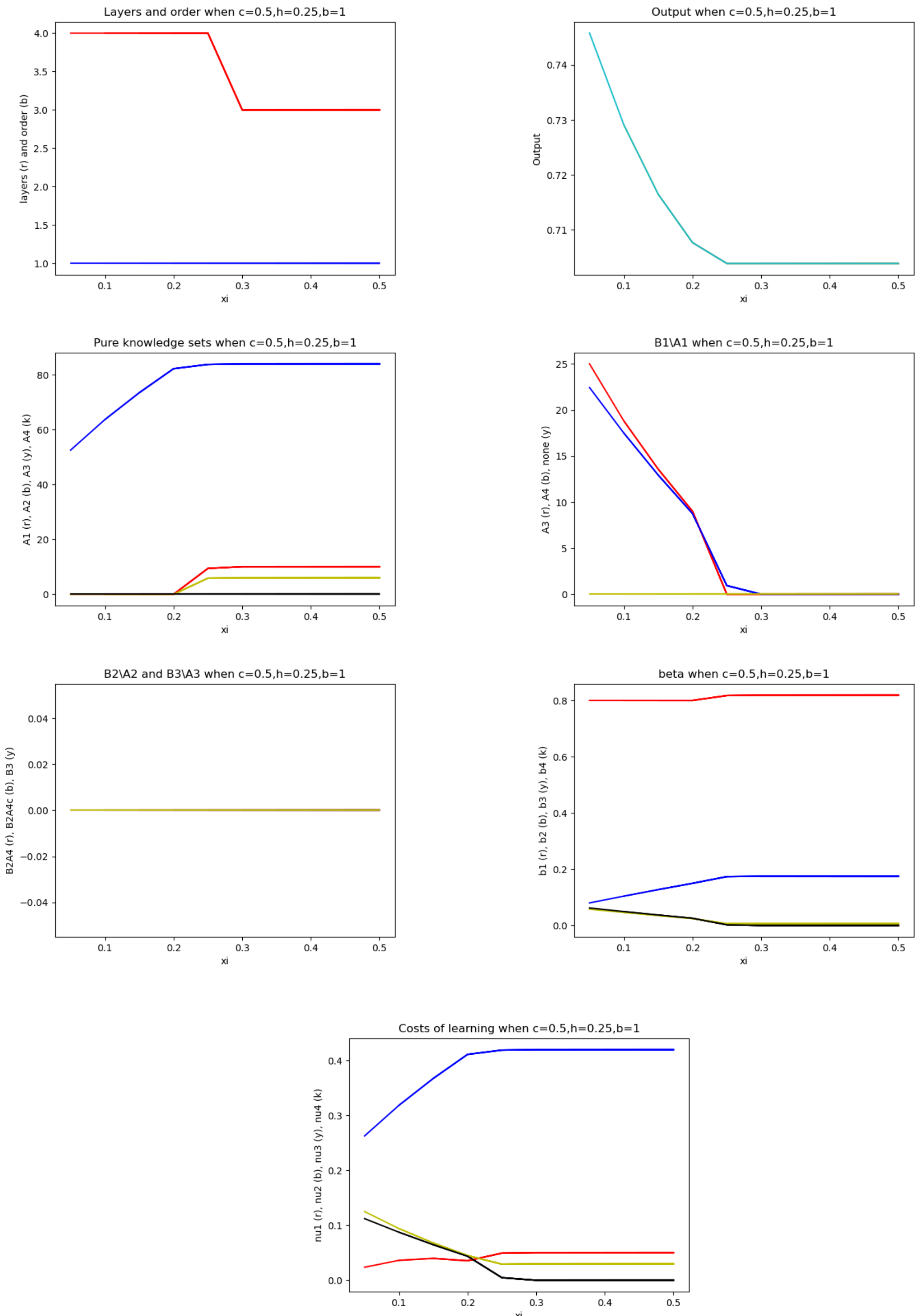

Figure 6: Optimal organization at different levels of ξ when $c = 0.5$, $h = 0.25$ and $b = 1$

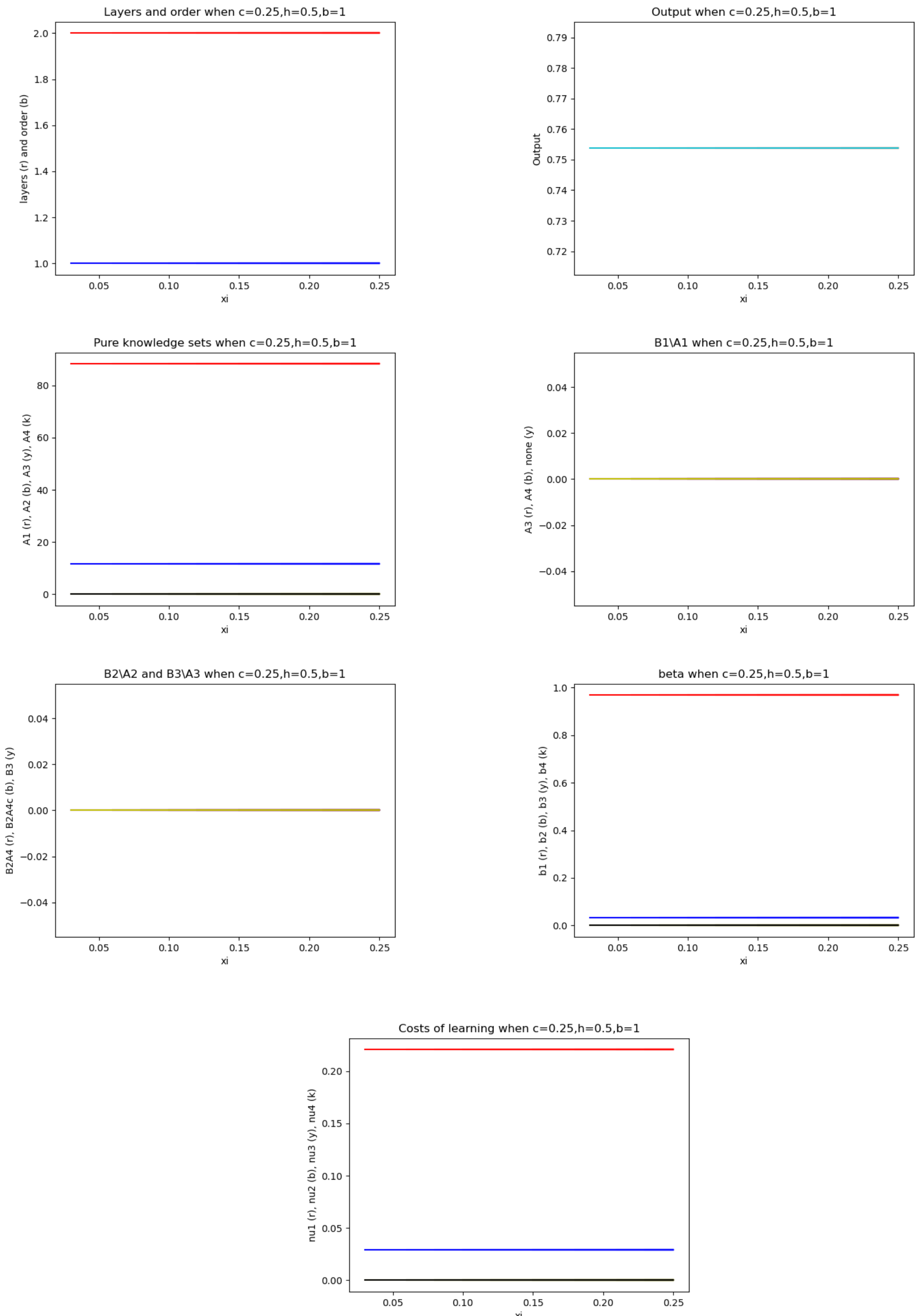

Figure 7: Optimal organization at different levels of ξ when $c = 0.25$, $h = 0.5$ and $b = 1$

The case where $c = 1.2$ is described in Figure 8.[6] It is analogous to the case where $c = 1$, except in the middle range of ξ. When $c = 1$, $\mu(A_1)$ declines but here $A_1 = \emptyset$ before the fall in h; hence, the increase in A_2 and $A_3 \cap (B_1 \setminus A_1)^c$ is done at the expense of $A_4 \cap (B_1 \setminus A_1)$ — the organization substitutes screening for knowledge. Consequently, ν_4 falls while it remained constant when $c = 1$.

The case where $c = 0.6$ is described in Figure 9. Recall that in this case there is no middle range for ξ when $h = 0.5$ and this continues to the be case when h is 10% lower. In the range of ξ where the optimal organization is a hierarchy, the pattern is similar to the case of $c = 1$. It is, however, different when the optimal organization is fully screening. In both cases, the size of A_2 increases but, whereas when $c = 1$, $A_3 = A_3 \cap (B_1 \setminus A_1)$ and $A_4 = A_4 \cap (B_1 \setminus A_1)$ declined, now they increase. The increase in the sizes of A_2, A_3 and A_4 is compensated by a decrease in the size of A_1, which is possible because, unlike when $c = 1$, A_1 is nonempty when $c = 0.6$. Thus, when $c = 1$, the drop in h forces a substitution from screening to knowledge which does not happen when $c = 0.6$; in this case, there is a substitution between knowledge that does not require communication to one that does. As a result, the changes to personnel are opposite and so are the changes to the learning costs of the middle layers.

The case where $c = 0.2$ is described in Figure 10. The 10% drop in h is large enough to lead to an increase in the number of layers, from 1 to 2. Consequently, there is an increase in $\mu(A_2)$, β_2 and ν_2 and a decline in $\mu(A_1)$, β_1 and ν_1.

6 An example on the optimal number of layers when h is small

Let $\Omega = [0, 1]$ (i.e. $\bar{\omega} = 1$), $f(z) = a - 2(a - 1)z$ for each $z \in \Omega$ with $1 < a < 2$ and $a > ch$. Thus, f is strictly decreasing since $a > 1$, Ω is bounded and $f(\bar{\omega}) = 2 - a > 0$.

[6] See Footnote 5 for the meaning of the variables in this figure. The change in output is given as a percentage change. However, changes in β, ν, and in the sizes of the knowledge and screening sets are absolute changes, since these variables are initially zero in some cases.

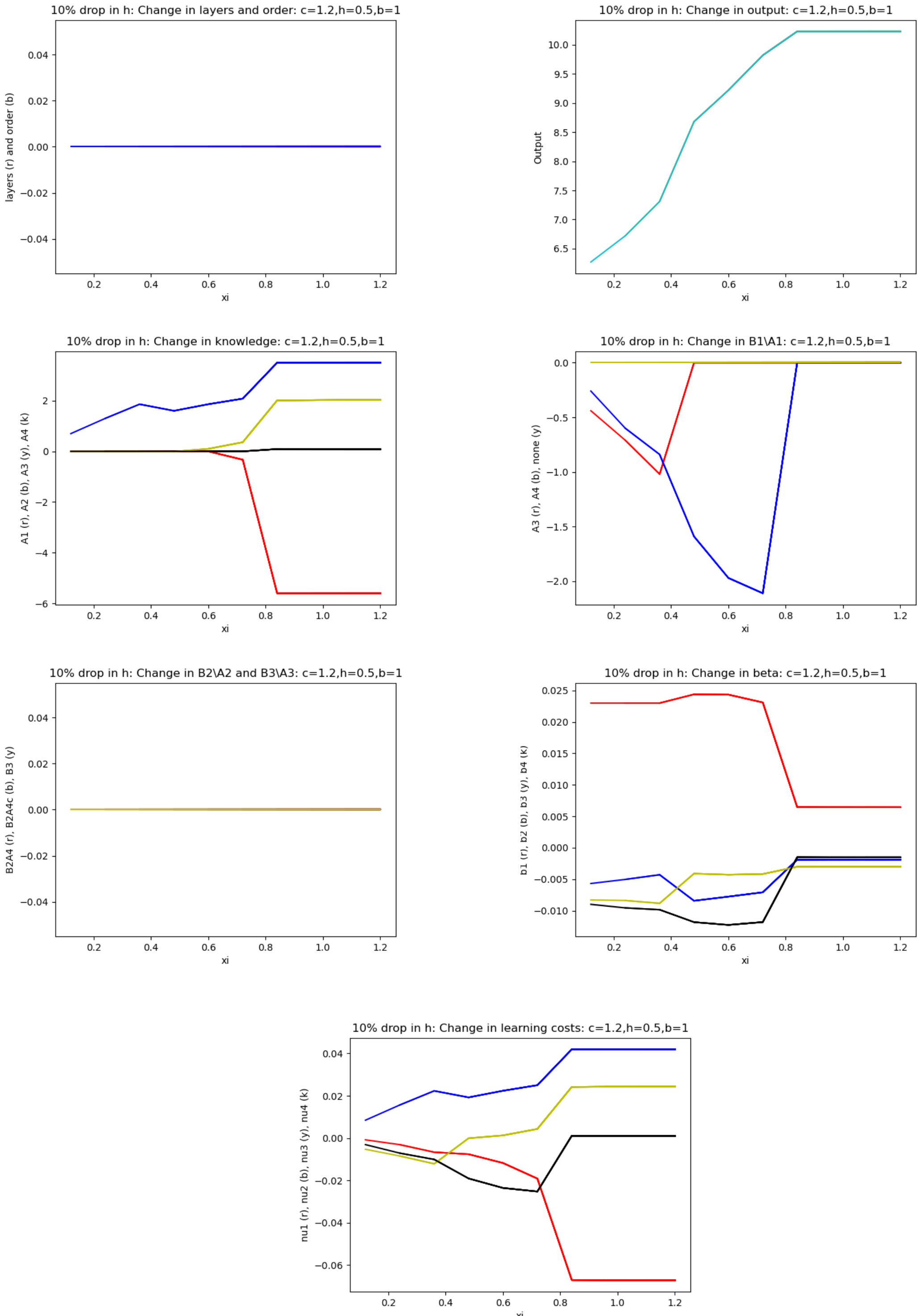

Figure 8: Effects of a 10% drop in h on the optimal organization when $c = 1.2$, $h = 0.5$ and $b = 1$

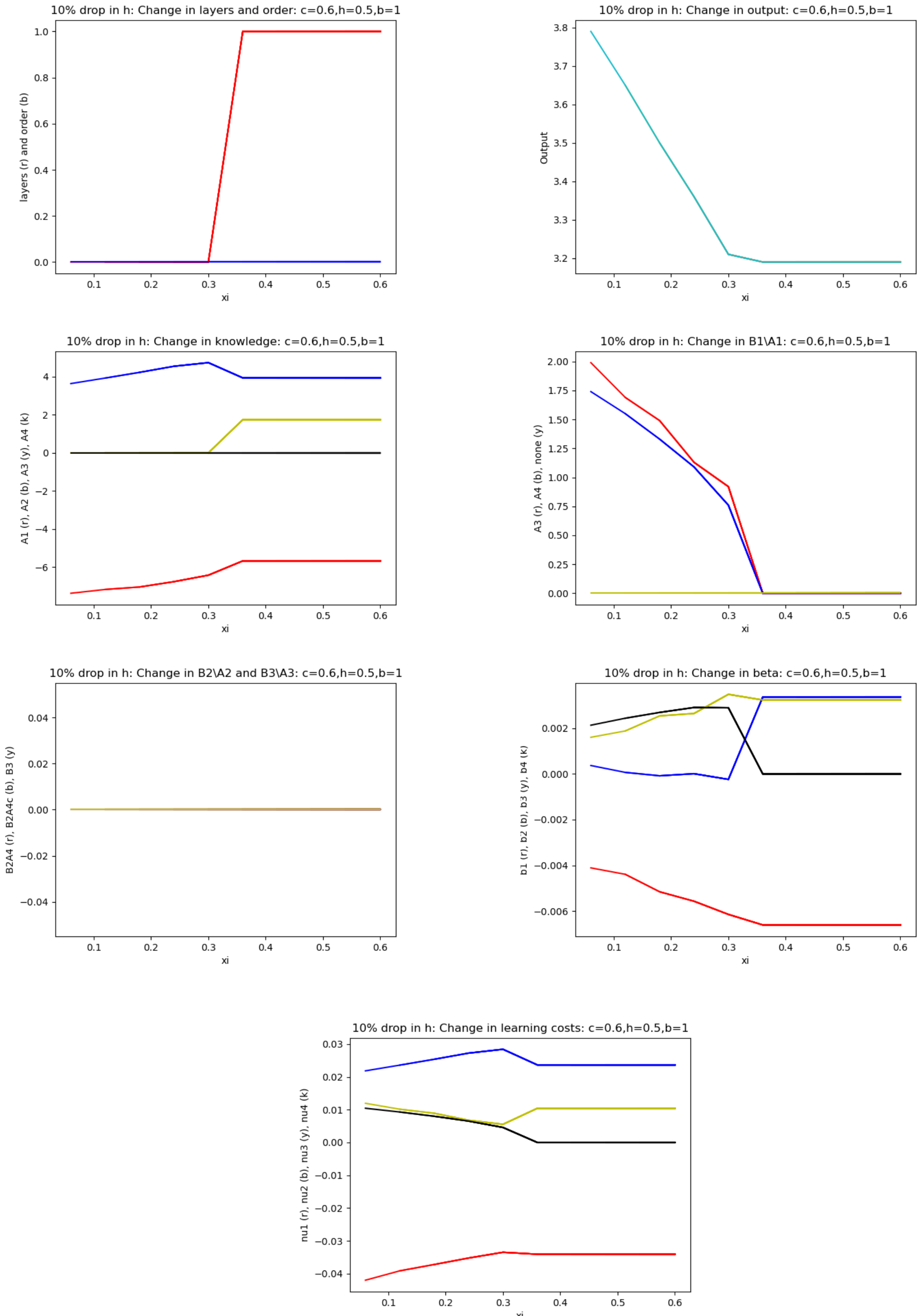

Figure 9: Effects of a 10% drop in h on the optimal organization when $c = 0.6$, $h = 0.5$ and $b = 1$

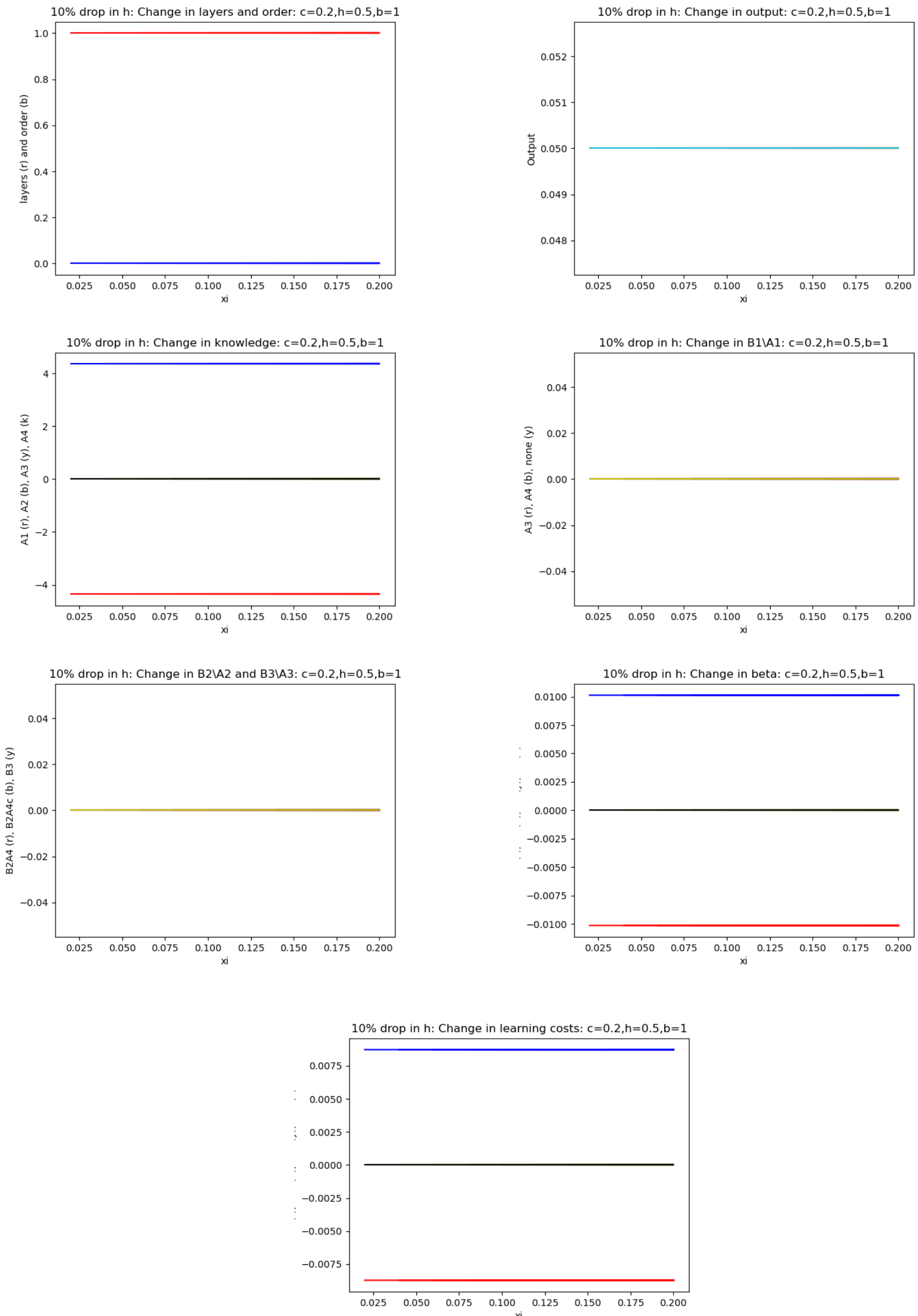

Figure 10: Effects of a 10% drop in h on the optimal organization when $c = 0.2$, $h = 0.5$ and $b = 1$

Suppose that $0 < c < 1$ and $1 < a < 2 - c$. Then $f(\bar{\omega}) > c$ and, hence, $y_1 = 1 - c$. Since $\frac{c}{f(\bar{\omega})} = \frac{c}{2-a}$ and $y_2 = \frac{1-ch}{1+h}$, (3) holds if and only if $h \leq \frac{2-a-c}{c+c(2-a)}$, in which case we can conclude that O_2^* is the optimal organization. For example, if $a = 1.5$ and $c = 0.25$, then the latter inequality requires $h \leq \frac{2}{3}$; thus, when h is sufficiently small, O_2^* is the optimal organization.

If, instead, $a \geq 2 - c$, then $\frac{c}{f(\bar{\omega})} = \frac{c}{2-a} \geq 1 > \sup_L y_L$ and O_2^* is not an optimal organization. In fact, any optimal organization has $L > 2$ layers. For example, if $c = 1$ (which immediately implies that $a \geq 2 - c$ since $a > 1$) and $a > 5/3$, then $y_3(0, 1/2, 1/2) > y_2(0, 1)$ for all h sufficiently small, which shows more explicitly that O_2^* is not optimal.

Claim 1 *If $c = 1$ and $a > 5/3$, then $y_3(0, 1/2, 1/2) > y_2(0, 1)$ for all h sufficiently small.*

Proof. Indeed, $F(1/2) = \frac{1+a}{4}$, $y_2(0, 1) = \frac{1-h}{1+h}$, $y_3(0, 1/2, 1/2) = \frac{1-\frac{h}{2}-\frac{h}{2}(1-F(1/2))}{1+h+h(1-F(1/2))}$ and

$$y_3(0, 1/2, 1/2) - y_2(0, 1) = \frac{h}{\gamma_2\gamma_3}\frac{\theta_3\gamma_2 - \theta_2\gamma_3}{h},$$

writing $y_3(0, 1/2, 1/2) = \frac{\theta_3}{\gamma_3}$ and $y_2(0, 1) = \frac{\theta_2}{\gamma_2}$. Since

$$\frac{\theta_3\gamma_2 - \theta_2\gamma_3}{h} = -1 + \frac{3}{2}F(1/2) + h - \frac{h}{2}F(1/2) > -1 + \frac{3}{2}F(1/2)$$

and, since $a > 5/3$,

$$-1 + \frac{3}{2}F(1/2) = \frac{3a-5}{8} > 0,$$

it follows that $y_3(0, 1/2, 1/2) > y_2(0, 1)$ for all h sufficiently small. ■

As we have shown, $L > 2$ in the optimal organization when $c = 1$. Theorem 7 implies that, for each $a < 13/7$, O_3^* is the optimal organization whenever h is sufficiently close to zero.

Claim 2 *If $c = 1$ and $a < 13/7$, then O_3^* is the optimal organization for all h sufficiently small.*

Proof. First note that μ_2^*, which we write simply as μ_2, satisfies

$$\mu_2 + \frac{a\mu_2 - (a-1)\mu_2^2}{a - 2(a-1)\mu_2} = y_3 + 1,$$

which after some manipulation yields:

$$3(a-1)\mu_2^2 - [2a + 2(a-1)(y_3+1)]\mu_2 + (y_3+1)a = 0.$$

The quadratic formula then implies that either

$$\mu_2 = \frac{2a + 2(a-1)(y_3+1) + \sqrt{[2a+2(a-1)(y_3+1)]^2 - 12a(a-1)(y_3+1)}}{6(a-1)} \text{ or}$$
$$\mu_2 = \frac{2a + 2(a-1)(y_3+1) - \sqrt{[2a+2(a-1)(y_3+1)]^2 - 12a(a-1)(y_3+1)}}{6(a-1)}.$$

We have that $y_3 \to 1$ as $h \to 0$; thus, approximating y_3 with 1, we get:

$$\bar{\mu}_2 = \frac{6a - 4 + \sqrt{(6a-4)^2 - 24a(a-1)}}{6(a-1)} \text{ or } \bar{\mu}_2 = \frac{6a - 4 - \sqrt{(6a-4)^2 - 24a(a-1)}}{6(a-1)}.$$

Since $(6a-4)^2 - 24a(a-1) = 12a(a-2) + 16 \in (4, 16)$, the latter since $1 < a < 2$, and $\frac{6a-4+\sqrt{(6a-4)^2-24a(a-1)}}{6(a-1)} > 1$, it follows that

$$\bar{\mu}_2 = \frac{6a - 4 - \sqrt{(6a-4)^2 - 24a(a-1)}}{6(a-1)} \text{ and}$$
$$\mu_2 = \frac{2a + 2(a-1)(y_3+1) - \sqrt{[2a+2(a-1)(y_3+1)]^2 - 12a(a-1)(y_3+1)}}{6(a-1)};$$

in particular, $\mu_2 \to \bar{\mu}_2$ as $h \to 0$. Thus, to conclude from Theorem 7 that O_3^* is the optimal organization for all h sufficiently close to zero, it suffices to show that (recall that $c = 1$ and that $\sup_L y_L \to 1$ as $h \to 0$)

$$\frac{1 - F(\bar{\mu}_2)}{f(\bar{\omega})} < 1. \tag{6.1}$$

We now write $\bar{\mu}_2(a)$ to explicitly denote that $\bar{\mu}_2$ depends on a. We have that

$$\frac{d\bar{\mu}_2(a)}{da} = -12 + 6\left(\sqrt{16 + 12a(a-2)} - \frac{12(a-1)^2}{\sqrt{16+12a(a-2)}}\right) < 0$$

since

$$\sqrt{16 + 12a(a-2)} - \frac{12(a-1)^2}{\sqrt{16+12a(a-2)}} < 2 \Leftrightarrow 4 < 2\sqrt{16 + 12a(a-2)}$$

and $\sqrt{16+12a(a-2)} \in (2,4)$ (recall that $16+12a(a-2) \in (4,16)$). Moreover,

$$\bar{\mu}_2(2) = \frac{2}{3} \text{ and } \bar{\mu}_2(1) = 1,$$

the latter using L'Hôpital's rule. Thus, $\bar{\mu}_2 > 2/3$ and

$$\frac{1-F(\bar{\mu}_2)}{f(\bar{\omega})} < \frac{1-F(2/3)}{f(\bar{\omega})} = \frac{5-2a}{9(2-a)} < 1$$

since $F(x) = ax-(a-1)x^2$ for each $x \in \Omega$ and $a < 13/7$. ■

7 Cumulative knowledge

This section contains the proofs of our results for the case of cumulative knowledge. It starts with an outline of the argument in Section 7.1. The proofs themselves follow this introductory section.

7.1 Road map

The structure of the proofs is the same as in Section 2 and we use the same arguments used there whenever this is possible. However, the presence of cumulative knowledge introduces an extra condition to the notion of an organization which needs to be checked. This is sometimes an easy task, illustrated by Lemma 7.1 below, in which case the proofs in Section 2 can be used.

Some results in the non-cumulative knowledge case do not extend, namely Lemma 1 of the main paper that states, in particular, that knowledge sets of different layers are disjoint. This creates the need of obtaining a partition of the union of the screening sets different from the one obtained in main paper, which has been one of the major difficulties that this proof had to overcome. Once such partition has been obtained, the remainder of the argument for the general characterization results, Theorem 7.2, is the same as in Section 2, requiring only small adjustment in some cases and new versions of some lemmas.

Part of the complexity of the argument of the general characterization results is that we actually establish a more general statement. In fact, Theorems 7.1 and 7.2

state that any optimal organizations have a certain property and, to prove them, all one needs is to show that if O is an organization with cumulative knowledge that fails to satisfy that property, then there is another organization $\hat{O}$ with cumulative knowledge which is better than O (i.e. yields an higher output, or has less layers, or smaller lists). Instead, we show that $\hat{O}$ is not only better than O but also that it has the property in question.

The reason why the stronger conclusion we establish is useful concerns existence of optimal organizations. The difficulty with existence is that the "space of organizations" fails to have enough mathematical structure to allow for standard techniques to be used. For instance, we need to specify the knowledge set of layer 1, which is, a priori, just the union of disjoint intervals; the collection of the sets that are a union of disjoint intervals do not form a space over which the maximization of the organization's output can easily be done. In contrast, our approach allow us to restrict attention to organizations that satisfy the properties in Theorems 7.1 and 7.2 and, as we show, can be described by the elements of the product of a finite set and a compact subset of an Euclidean space, thus a compact set overall. Since an organization's output is a continuous function, the well-known Weierstrass' Theorem is then all we need to obtain a solution of the maximization problem consisting of choosing an element of such compact set to maximize the output of the organization.

After establishing the general characterization results, Theorems 7.1 and 7.2, and the existence of optimal organizations, Theorem 7.3, we prove Theorem 7.4 on the order of cumulative knowledge, Theorem 7.5 on hierarchies and Theorem 8 in the main paper on the case of small ξ. The first of these results has no counterpart in the case of non-cumulative knowledge; the other two, while exploring some ideas similar to analogous results in the case of non-cumulative knowledge, have to deal with several new issues that arise due to the cumulative nature of knowledge.

7.2 Specialization

The main result of this section is as follows.

Theorem 7.1 *If O is an optimal organization with cumulative knowledge, then there is $i \in L$ such that $t_i^p = 1$, $t_i^h = 0$, $l_i = L$, $\beta_i = \frac{1}{1+\sum_{l\in L\setminus\{i\}} \alpha_{li}}$ and $y = \frac{F(\cup_{l\in L} A_l) - \nu_i - \sum_{l\in L\setminus\{i\}} \alpha_{li}\nu_l}{1+\sum_{l\in L\setminus\{i\}} \alpha_{li}}$. Furthermore, for each $j \neq i$, $t_j^p = 0$, $t_j^h = 1$, $\alpha_{ji} > 0$, $l_j = \{j\}$ and $\beta_j = \frac{\alpha_{ji}}{1+\sum_{l\in L\setminus\{i\}} \alpha_{li}}$.*

As noted above, we use the same notation as in Section 2 despite focusing on organizations with cumulative knowledge. Thus, let $\mathcal{O}_S$ now denotes the set of organizations O with cumulative knowledge such that $y > 0$ and there is $i \in M$ such that $t_i^p = 1$, $t_i^h = 0$, $l_i = L$, $\beta_i = \frac{1}{\gamma_i}$, $y = \frac{\theta_i}{\gamma_i}$, $t_j^p = 0$, $t_j^h = 1$, $\alpha_{ji} > 0$, $l_j = \{j\}$ and $\beta_j = \frac{\alpha_{ji}}{\gamma_i}$ for each $j \in L \setminus \{i\}$. The set M is as in the main paper, i.e. M is the set of $i \in L$ such that $\delta_i = \frac{1}{\gamma_i}$ and $\delta_j = 0$ for all $j \in L \setminus \{i\}$ is a solution to the maximization problem defined by (19)–(21) in the main paper.

The results in Sections 2.2 and 2.3 are easily seen to extend to the case of cumulative knowledge using the following lemma.

Lemma 7.1 *If O is an organization with cumulative knowledge and $\hat{O}$ is an organization such that $\hat{L} \subseteq L$ and $\hat{A}_i = A_i$ and $\hat{B}_i = B_i$ for all $i \in \hat{L}$, then there exists a cumulative knowledge order $\hat{\prec}$ such that $(\hat{O}, \hat{\prec})$ is an organization with cumulative knowledge.*

Proof. Let $\hat{\prec} = \prec|_{\hat{L}}$, i.e., for each $i, j \in \hat{L}$, $i \hat{\prec} j$ if and only if $i \prec j$. Then $(\hat{O}, \hat{\prec})$ is an organization with cumulative knowledge. ■

The organizations $\hat{O}$ in the proofs of the results in Sections 2.2 and 2.3 are such that the set of layers is reduced and, hence, satisfy the conditions of Lemma 7.1. Thus, Corollary 2.3 holds as stated for the case of cumulative knowledge, from which Theorem 7.1 follows.

Corollary 7.1 *If $O \notin \mathcal{O}_S$ and $y > 0$, then there is $\hat{O} \in \mathcal{O}_S$ such that $\hat{O} > O$.*

7.3 Partition

Consider the partition $\mathcal{C}$ of the union of the screening sets of an optimal organization with cumulative knowledge introduced in Section 6 of the main paper. It is easy to see that $C \cap C' = \emptyset$ if $C, C' \in \mathcal{C}$ and $C \neq C'$; moreover, $\cup_{C \in \mathcal{C}} C = \cup_{l \in L} B_l$ follows by the following lemma and, thus, $\mathcal{C}$ is indeed a partition of $\cup_{l \in L} B_l$.

Lemma 7.2 *For each $l \in L$,*

$$B_{i_l} \backslash B_{i_{l-1}} = \cap_{j<l}(B_{i_j} \backslash A_{i_j})^c \cap [(A_{i_l} \backslash A_{i_{l-1}}) \cup ((B_{i_l} \backslash A_{i_l}) \cap (\cup_{k=l+1}^{L}(A_{i_k} \backslash A_{i_{k-1}}) \cup (B_{i_L} \backslash A_{i_L})))].$$

Proof. For each $l \in L$, we have that

$$\begin{aligned}
& B_{i_l} \setminus B_{i_{l-1}} = B_{i_l} \setminus \cup_{j<l} B_{i_j} = (A_{i_l} \cup (B_{i_l} \setminus A_{i_l})) \setminus (A_{i_{l-1}} \cup (\cup_{j<l} B_{i_j} \setminus A_{i_j})) \\
& = \cap_{j<l}(B_{i_j} \setminus A_{i_j})^c \cap [(A_{i_l} \cap A_{i_{l-1}}^c) \cup ((B_{i_l} \setminus A_{i_l}) \cap A_{i_{l-1}}^c)] \\
& = \cap_{j<l}(B_{i_j} \setminus A_{i_j})^c \cap [(A_{i_l} \setminus A_{i_{l-1}}) \cup ((B_{i_l} \setminus A_{i_l}) \cap (\cup_{k=l}(A_{i_k} \setminus A_{i_{k-1}}) \cup (B_{i_L} \setminus A_{i_L})))] \\
& = \cap_{j<l}(B_{i_j} \setminus A_{i_j})^c \cap [(A_{i_l} \setminus A_{i_{l-1}}) \cup ((B_{i_l} \setminus A_{i_l}) \cap (\cup_{k=l+1}(A_{i_k} \setminus A_{i_{k-1}}) \cup B_{i_L} \setminus A_{i_L}))].
\end{aligned}$$

■

The sets in $\mathcal{C}$ can be used to describe an organization in place of the class of the knowledge and screening sets $\{A_l, B_l\}_{l \in L}$, since the latter can be obtained from the former. To see this, consider first the following lemma that does that for the differences $A_{i_l} \setminus A_{i_{l-1}}$.

Lemma 7.3 *For each $l \in L$,*

$$\begin{aligned}
A_{i_l} \setminus A_{i_{l-1}} \quad & = (\cap_{j<l}(B_{i_j} \setminus A_{i_j})^c \cap (A_{i_l} \setminus A_{i_{l-1}})) \cup \\
& \cup \left(\bigcup_{k=1}^{l-1} (\cap_{j<k}(B_{i_j} \setminus A_{i_j})^c \cap (B_{i_k} \setminus A_{i_k}) \cap (A_{i_l} \setminus A_{i_{l-1}})) \right).
\end{aligned}$$

Proof. Let $l \in L$ and note that the set in the right-hand side of the equation in the statement of the lemma is

$$(A_{i_l} \setminus A_{i_{l-1}}) \cap (\cap_{j<l}(B_{i_j} \setminus A_{i_j})^c \cup \left(\bigcup_{k=1}^{l-1} (\cap_{j<k}(B_{i_j} \setminus A_{i_j})^c \cap (B_{i_k} \setminus A_{i_k}) \right) = A_{i_l} \setminus A_{i_{l-1}}$$

since $\cap_{j<l}(B_{i_j} \setminus A_{i_j})^c = (\cup_{j<l}(B_{i_j} \setminus A_{i_j}))^c$ and $\bigcup_{k=1}^{l-1}(\cap_{j<k}(B_{i_j} \setminus A_{i_j})^c \cap (B_{i_k} \setminus A_{i_k}) = \cup_{j<l}(B_{i_j} \setminus A_{i_j})$. ■

We then have that

$$A_{i_l} = \cup_{j=1}^{l}(A_{i_j} \setminus A_{i_{j-1}}), \tag{7.1}$$

by noting that $A_{i_1} \in \mathcal{C}$. In addition,

$$B_{i_l} = \cup_{j=1}^{l}(B_{i_j} \setminus B_{i_{j-1}}), \tag{7.2}$$

where $B_{i_j} \setminus B_{i_{j-1}}$ is as in Lemma 7.2.

The argument to establish some of our remaining results will consist in improving a given organization O by changing some set $C \in \mathcal{C}$ and to argue that the output of the organization increases through a decrease in its learning costs. For this reason, we write the learning costs of an organization using the sets in $\mathcal{C}$ as follows. First, we need to express the sets $B_{i_l} \setminus A_{i_l}$, with $l \in L$, using the elements of the partition $\mathcal{C}$, which is done in the following lemma.

Lemma 7.4 *For each $l \in L$,*

$$B_{i_l} \setminus A_{i_l} = \cup_{m=1}^{l} \left[\cap_{j<m}(B_{i_j} \setminus A_{i_j})^c \cap (B_{i_m} \setminus A_{i_m}) \cap (\cup_{k>l}(A_{i_k} \setminus A_{i_{k-1}}) \cup (B_{i_L} \setminus A_{i_L})) \right].$$

Proof. Let $l \in L$ and $1 \leq m \leq l$. We have that

$$(B_{i_m} \setminus A_{i_m}) \cap (\cup_{k>l}(A_{i_k} \setminus A_{i_{k-1}}) \cup (B_{i_L} \setminus A_{i_L})) = B_{i_m} \setminus A_{i_m} \cap A_{i_l}^c = B_{i_m} \setminus A_{i_l}.$$

Hence, for each m, the term in square brackets is contained in $B_{i_m} \setminus A_{i_l} \subseteq B_{i_l} \setminus A_{i_l}$. Thus, the set in the right-hand side of the equation in the statement of the lemma is contained in $B_{i_l} \setminus A_{i_l}$.

Conversely, let $\omega \in B_{i_l} \setminus A_{i_l}$ and let m be the first $1 \leq m' \leq l$ such that $\omega \in B_{i_{m'}}$; thus, $\omega \in B_{i_m}$ and $\omega \notin B_{i_{m'}}$ for all $m' < m$. Hence,

$$\omega \in \cap_{j<m}(B_{i_j} \setminus A_{i_j})^c \cap (B_{i_m} \setminus A_{i_l}) =$$
$$\cap_{j<m}(B_{i_j} \setminus A_{i_j})^c \cap (B_{i_m} \setminus A_{i_m}) \cap (\cup_{k>l}(A_{i_k} \setminus A_{i_{k-1}}) \cup (B_{i_L} \setminus A_{i_L})).$$

Thus, ω belongs to the set in the right-hand side of the equation in the statement of the lemma. ■

Recall that, by Theorem 7.1, the output of an optimal organization is

$$y = \frac{F(\cup_{l \in L} A_l) - \sum_{l \in L} \alpha_l \nu_l}{\sum_{l \in L} \alpha_l}$$

where $\nu_l = c\mu(A_l) + \xi\mu(B_l \setminus A_l)$, layer 1 is such that $t_1^p = 1$, $\alpha_1 = 1$ and $\alpha_l = \alpha_{l1}$. Using Lemmas 7.3 and 7.4 together with (7.1), we can write

$$\sum_{l\in L} \alpha_l \nu_l = \sum_{l\in L} \alpha_{i_l} \nu_{i_l} = \sum_{C\in\mathcal{C}} c_C \mu(C) \tag{7.3}$$

by defining, for each $C \in \mathcal{C}$, the cost c_C of learning C as follows:

$$c_{\cap_{j<l}(B_{i_j}\setminus A_{i_j})^c \cap (A_{i_l}\setminus A_{i_{l-1}})} = c\sum_{j=l}^{L} \alpha_{i_j},$$

$$c_{\cap_{j<l}(B_{i_j}\setminus A_{i_j})^c \cap (B_{i_l}\setminus A_{i_l}) \cap (A_{i_k}\setminus A_{i_{k-1}})} = c\sum_{j=k}^{L} \alpha_{i_j} + \xi \sum_{j=l}^{L} \alpha_{i_j} \text{ and}$$

$$c_{\cap_{j<l}(B_{i_j}\setminus A_{i_j})^c \cap (B_{i_l}\setminus A_{i_l}) \cap (B_{i_L}\setminus A_{i_L})} = \xi \sum_{j=l}^{L} \alpha_{i_j}.$$

Based on the above partition and costs, we obtain the following characterization result.

Theorem 7.2 *If O is an optimal organization with cumulative knowledge, then:*

1. $\cup_{i=1}^{L} B_i = [\min_{1\le i\le L} a_i, \max_{1\le i\le L} b_i)$ *and* $\min_{1\le i\le L} a_i = 0$.

2. *For each $C, C' \in \mathcal{C}$, if $c_C > c_{C'}$, then $C < C'$.*

3. *If $\mathcal{C} = \{C_1, \ldots, C_{|\mathcal{C}|}\}$ is such that $c_{C_1} \ge \cdots \ge c_{C_{|\mathcal{C}|}}$, then there exists an optimal organization with cumulative knowledge $\hat{O}$ such that $\hat{\mathcal{C}} = \{\hat{C} : C \in \mathcal{C}\}$, $\hat{C}_1 < \cdots < \hat{C}_{|\mathcal{C}|}$, $\hat{C}$ is an interval and $F(\hat{C}) = F(C)$ for each $C \in \mathcal{C}$, $\hat{y} = y$, $\hat{L} = L$ and $\hat{l}_i = l_i$ for each $i \in L$.*

7.4 No gaps

In this section, we proof part 1 of Theorem 7.2. The proof is analogous to the proof used in Section 2 although several adjustments are needed, mostly due to the fact that a different partition of $\cup_{l\in L} B_l$ is used. The following lemma provides a first of such adjustments, namely on the formula for α_j.

Lemma 7.5 *For each $j \in L$, let $l, k \in L$ be such that $j = i_l$ and $k = \max\{m : i_m < i_l\}$. Then*

$$\alpha_j = \begin{cases} h \sum_{m=k+1}^{l} F(A_{i_m} \setminus A_{i_{m-1}}) + \pi(1 - F(A_{i_l} \cup B_{i_k})) & \text{if } k < l, \\ \pi(1 - F(B_{i_k})) & \text{if } k > l. \end{cases}$$

Proof. We have that $\cup_{m<j} A_m = A_{i_k}$ and $\cup_{m<j} B_j = B_{i_k}$. If $k > l$, then $A_{i_l} \subseteq A_{i_k} \subseteq B_{i_k}$ and, hence, $A_{i_l} \setminus A_{i_k} = \emptyset$ and $A_{i_l}^c \cap B_{i_k}^c = B_{i_k}^c$. Thus, $\alpha_j = hF(\emptyset) + \pi F(B_{i_k}^c) = \pi(1 - F(B_{i_k}))$.

Suppose next that $k < l$. Then, $A_j \setminus \cup_{m<j} A_m = A_{i_l} \setminus A_{i_k}$ and $A_j^c \setminus \cup_{m<j} B_m = A_{i_l}^c \cap B_{i_k}^c$. Since $A_{i_l} \setminus A_{i_k} = \cup_{m=k+1}^{l} (A_{i_m} \setminus A_{i_{m-1}})$, it follows that $F(A_{i_l} \setminus A_{i_k}) = \sum_{m=k+1}^{l} F(A_{i_m} \setminus A_{i_{m-1}})$. Thus, $\alpha_j = h \sum_{m=k+1}^{l} F(A_{i_m} \setminus A_{i_{m-1}}) + \pi(1 - F(A_{i_l} \cup B_{i_k}))$. ■

From Lemma 7.3, we obtain a partition $\mathcal{C}(A_{i_l} \setminus A_{i_{l-1}})$ of $A_{i_l} \setminus A_{i_{l-1}}$ for each $l \in L$:

$$\begin{aligned} \mathcal{C}(A_{i_l} \setminus A_{i_{l-1}}) &= \{\cap_{j<l} (B_{i_j} \setminus A_{i_j})^c \cap (A_{i_l} \setminus A_{i_{l-1}})\} \\ &\quad \cup \{\cap_{j<k} (B_{i_j} \setminus A_{i_j})^c \cap (B_{i_k} \setminus A_{i_k}) \cap (A_{i_l} \setminus A_{i_{l-1}}) : k < l\}. \end{aligned}$$

Similarly, we obtain partitions $\mathcal{C}(A_{i_l})$ and $\mathcal{C}(B_{i_l})$ of A_{i_l} and B_{i_l}, respectively, by using Lemmas 7.2 and 7.3 together with (7.1) and (7.2):

$$\begin{aligned} \mathcal{C}(A_{i_l}) &= \{\cap_{j<m} (B_{i_j} \setminus A_{i_j})^c \cap (A_{i_m} \setminus A_{i_{m-1}}) : m \le l\} \\ &\quad \cup \{\cap_{j<k} (B_{i_j} \setminus A_{i_j})^c \cap (B_{i_k} \setminus A_{i_k}) \cap (A_{i_m} \setminus A_{i_{m-1}}) : k < m \le l\}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}(B_{i_l}) &= \{\cap_{j<m} (B_{i_j} \setminus A_{i_j})^c \cap (A_{i_m} \setminus A_{i_{m-1}}) : m \le l\} \\ &\quad \cup \{\cap_{j<k} (B_{i_j} \setminus A_{i_j})^c \cap (B_{i_k} \setminus A_{i_k}) \cap (A_{i_m} \setminus A_{i_{m-1}}) : m \le l \text{ and } m < k\} \\ &\quad \cup \{\cap_{j<k} (B_{i_j} \setminus A_{i_j})^c \cap (B_{i_m} \setminus A_{i_m}) \cap (B_{i_L} \setminus A_{i_L}) : m \le l\}. \end{aligned}$$

Thus, from Lemma 7.5, we can write

$$\alpha_j = \begin{cases} h \sum_{m=k+1}^{l} \sum_{C \in \mathcal{C}(A_{i_m} \setminus A_{i_{m-1}})} F(C) + \pi(1 - \sum_{C \in \mathcal{C}(A_{i_l}) \cup \mathcal{C}(B_{i_k})} F(C)) & \text{if } k < l, \\ \pi(1 - \sum_{C \in \mathcal{C}(B_{i_k})} F(C)) & \text{if } k > l. \end{cases} \tag{7.4}$$

It follows from the above that the knowledge and screening sets can be defined from the elements of the partition $\mathcal{C}$. The question that we address now is the following. Given an organization O, suppose that we change some sets $C \in \mathcal{C}$ to obtain sets $\{\hat{C} : C \in \mathcal{C}\}$ but leave all the remaining elements of the organization O intact. Do we obtain a new organization $\hat{O}$ with cumulative knowledge? This question is addressed in the following lemma.

Lemma 7.6 *Let O be an organization with cumulative knowledge and $\mathcal{C}$ be the partition of $\cup_{l \in L} B_l$. If $\{\tilde{C} : C \in \mathcal{C}\}$ is a pairwise disjoint collection of subsets of Ω and $\{\tilde{A}_l, \tilde{B}_l\}_{l \in L}$ are defined via (7.1) and (7.2), then $\tilde{A}_{i_l} \subseteq A_{i_{l+1}}$ and $\tilde{B}_{i_l} \subseteq B_{i_{l+1}}$ for each $l \in \{1, \ldots, L-1\}$.*

Proof. We have that, for each $l \in \{1, \ldots, L-1\}$, $A_{i_l} = \cup_{m=1}^{l}(A_{i_m} \setminus A_{i_{m-1}}) \subseteq \cup_{m=1}^{l+1}(A_{i_m} \setminus A_{i_{m-1}}) = A_{i_{l+1}}$ and, analogously, $B_{i_l} = \cup_{m=1}^{l}(B_{i_m} \setminus B_{i_{m-1}}) \subseteq \cup_{m=1}^{l+1}(B_{i_m} \setminus B_{i_{m-1}}) = B_{i_{l+1}}$. ■

Let $\mathcal{O}_{G1}$ be the set of organizations O with cumulative knowledge such that $\min_{1 \leq i \leq L} a_i = 0$, where, recall, $a_i = \min B_i$ for each $i \in L$ with the standard convention that $\min \emptyset = \infty$.

Lemma 7.7 *If $O \in \mathcal{O}_S \setminus \mathcal{O}_{G1}$, then there is $\hat{O} \in \mathcal{O}_S \cap \mathcal{O}_{G1}$ such that $\hat{O} > O$.*

Proof. Let $O \in \mathcal{O}_S \setminus \mathcal{O}_{G1}$. For each $C \in \mathcal{C}$, let $a_C = \min C$; then $\min_{1 \leq i \leq L} a_i = \min_{C \in \mathcal{C}} a_C$. Thus, let $C \in \mathcal{C}$ be such that $a_C = \min_{1 \leq i \leq L} a_i$, the fact that $O \notin \mathcal{O}_{G1}$ implies that $a_C > 0$. Thus, $[0, a_C) \subseteq (\cup_{l=1}^{L} B_l)^c$.

Let $0 < \varepsilon < a_C$ be such that $[a_C, a_C + \varepsilon) \subseteq C$ and let $0 < \varepsilon' < 0$ be such that $F([0, \varepsilon')) = F([a_C, a_C + \varepsilon))$; the existence of ε' follows by Lemma A.12.

Define an organization $\tilde{O}$ equal to O except that $\tilde{C} = [0, \varepsilon') \cup (C \setminus [a_C, a_C + \varepsilon))$ and $\{\tilde{A}_l, \tilde{B}_l\}_{l=1}^{L}$ are defined from $\{\tilde{C} : C \in \mathcal{C}\}$ via (7.1) and (7.2). Note that $\tilde{C} \cap \tilde{D} = \emptyset$ whenever $D \in \mathcal{C}$ is such that $C \neq D$ because $[0, \varepsilon) \subseteq (\cup_{l=1}^{L} B_l)^c$.

We have that $F(\tilde{D}) = F(D)$ for each $D \in \mathcal{C}$. Thus, $F(\tilde{A}_{i_l}) = \sum_{D \in \mathcal{C}(A_{i_l})} F(\tilde{D}) = \sum_{D \in \mathcal{C}(A_{i_l})} F(D) = F(A_{i_l})$ for each $l \in L$. In addition, it follows from (7.4) that

$\tilde{\alpha}_j = \alpha_j$ for all $j \in L$. Thus, $\tilde{O} \in \mathcal{O}_S$ by Lemmas 2.8 and 7.6 provided that $1 \in \tilde{M}$. In addition, it follows that $\tilde{O} \in \mathcal{O}_{G1}$ by construction.

We have that $\mu(\tilde{C}) < \mu(C)$ and $\mu(\tilde{D}) = \mu(D)$ for each $D \in \mathcal{C} \setminus \{C\}$. Moreover, $F(\cup_{l \in L} \tilde{A}_l) = F(\tilde{A}_{i_L}) = F(A_{i_L}) = F(\cup_{l \in L} A_l)$ since $F(\tilde{A}_l) = F(A_l)$ for each $l \in L$. It then follows that $\tilde{y} > y$. This, together with $\tilde{L} = L$, shows that $\tilde{O} > O$. Thus, in the case where $1 \in \tilde{M}$, set $\hat{O} = \tilde{O}$.

If $1 \notin \tilde{M}$, then take $i \in \tilde{M}$ and, therefore, $F(\tilde{A}_i) - (c\mu(\tilde{A}_i) + \xi\mu(\tilde{B}_i \setminus \tilde{A}_i)) > \tilde{y} > y$. Define $\hat{O}$ by $\hat{L} = \{i\}$, $\hat{\beta}_i = 1$, $\hat{t}_i^p = 1$,

$$\hat{A}_i = \begin{cases} \tilde{A}_i & \text{if } \min \tilde{A}_i = 0, \\ [0, \varepsilon') \cup (\tilde{A}_i \setminus [\min A_i, \min A_i + \varepsilon)) & \text{otherwise,} \end{cases}$$

where $0 < \varepsilon < \max A_i$ and $0 < \varepsilon' < \varepsilon$ is such that $F([0, \varepsilon')) = F([\min A_i, \min A_i + \varepsilon))$, and $\hat{B}_i = \hat{A}_i$. Then, $F(\hat{A}_i) = F(\tilde{A}_i)$ and $\mu(\hat{A}_i) \leq \mu(\tilde{A}_i)$. Hence, $\hat{y} = F(\hat{A}_i) - c\mu(\hat{A}_i) \geq F(\tilde{A}_i) - c\mu(\tilde{A}_i) - \xi\mu(\tilde{B}_i \setminus \tilde{A}_i) > \tilde{y} > y$, $\hat{L} \leq \tilde{L} = L$ and, thus, $\hat{O} > O$. Moreover, $\hat{O} \in \mathcal{O}_S$ by Lemma 2.8 and $\hat{O} \in \mathcal{O}_{G1}$ by construction. ■

Let $\mathcal{O}_G$ be the set of organizations $O \in \mathcal{O}_{G1}$ such that $\cup_{i=1}^{L} B_i = [0, \max_{1 \leq i \leq L} b_i)$ where, recall, $b_i = \max B_i$ with the convention that $\max \emptyset = -\infty$. The argument in the proof of Lemma 2.15 implies the following result.

Lemma 7.8 *If $O \in (\mathcal{O}_S \cap \mathcal{O}_{G1}) \setminus \mathcal{O}_G$, then there is $\hat{O} \in \mathcal{O}_S \cap \mathcal{O}_G$ such that $\hat{O} > O$.*

7.5 Order of sets

Parts 2 and 3 of Theorem 7.2 follows from exactly the same arguments used in the analogous result in Section 2. Let $\mathcal{O}_{<1}$ be the set of organizations O such that $C < C'$ for all $C, C' \in \mathcal{C}$ with $c_C > c_{C'}$. Furthermore, let $\mathcal{O}_<$ be the set of organizations $O \in \mathcal{O}_{<1}$ such that C is an interval for each $C \in \mathcal{C}$. Then let

$$\mathcal{O}^* = \mathcal{O}_S \cap \mathcal{O}_G \cap \mathcal{O}_<.$$

Recall that, given two organizations O and O', we write $\hat{O} \gtrsim O$ if $\hat{O} > O$ or $\hat{y} = y$, $\hat{L} = L$ and $\hat{l}_i = l_i$ for all $i \in L$.

Summing up this section:

Corollary 7.2 *If $O \notin \mathcal{O}^*$ and $y > 0$, then there is $\hat{O} \in \mathcal{O}^*$ such that $\hat{O} \gtrsim O$.*

7.6 Existence

In this section we consider the existence of η-optimal organizations with cumulative knowledge. Given our previous results, Theorems 7.1 and 7.2, all that is left to determine is the number L of layers, the order $\prec$ of L, an ordering of $\mathcal{C}$, i.e. to write $\mathcal{C} = \{C_1, \ldots, C_m\}$ with $C_1 < \cdots < C_m$ and $m = |\mathcal{C}|$, and the size $\mu(C)$ of each $C \in \mathcal{C}$. Letting $\mu_j = \mu(C_j)$ for each $j = 1, \ldots, m$, we then have $C_1 = [0, \mu_1)$, $C_2 = [\mu_1, \mu_1 + \mu_2)$ and so on, so that, for each $j = 1, \ldots, m$,

$$C_j = \left[\sum_{i=1}^{j-1} \mu_i, \sum_{i=1}^{j} \mu_i\right).$$

Then we obtain $\{A_1, B_1, \ldots, A_L, B_L\}$ via (7.1)–(7.2).

Note, however, that fixing the number $L \in \mathbb{N}$ of layers, an ordering $\prec$ of L, an ordering ψ of $\mathcal{C}$ (formally, ψ is a bijection from $\mathcal{C}$ onto $\{1, \ldots, m\}$) and $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{R}^m_+$ such that $\sum_{j=1}^m \mu_j \leq \mu(\Omega)$ may fail to define an organization because the requirement that $\beta_l > 0$ for each $l \in L$ may fail. To allow for this case, we say that O is a *quasi-organization with cumulative knowledge* if it satisfies $\beta_l \geq 0$ for each $l \in L$ and all the conditions of the definition of an organization with cumulative knowledge except possibly the one requiring $\beta_l > 0$ for each $l \in L$. For each $(L, \prec, \psi, \mu)$, let

$$y_{L,\prec,\psi}(\mu_1, \ldots, \mu_m)$$

be the output of the resulting quasi-organization with cumulative knowledge (computed using (1)) and

$$y_{L,\prec,\psi} = \max_{(\mu_1,\ldots,\mu_m)\in\mathbb{R}^m_+} y_{L,\prec,\psi}(\mu_1, \ldots, \mu_m) \tag{7.5}$$

$$\text{subject to } \sum_{j=1}^{m} \mu_j \leq \mu(\Omega). \tag{7.6}$$

Then an optimal organization with cumulative knowledge O^* is obtained by letting L^* be such that

$$\max_{\prec,\psi} \big(y_{L^*,\prec,\psi} - \eta(L^* - 1)\big) = \max_{L} \big(\max_{\prec,\psi} y_{L,\prec,\psi} - \eta(L - 1)\big), \tag{7.7}$$

$\prec^*$ be such that

$$\max_{\psi} y_{L^*,\prec^*,\psi} = \max_{\prec,\psi} y_{L^*,\prec,\psi}, \tag{7.8}$$

ψ^* be such that

$$y_{L^*,\prec^*,\psi^*} = \max_{\psi} y_{L^*,\prec^*,\psi} \tag{7.9}$$

and $(\mu_1^*, \ldots, \mu_m^*) \in \mathbb{R}_+^m$ be such that $\sum_{j=1}^m \mu_j^* \leq \mu(\Omega)$ and

$$y_{L^*,\prec^*,\psi^*}(\mu_1^*, \ldots, \mu_m^*) = y_{L^*,\prec^*,\psi^*}. \tag{7.10}$$

It turns out that O^* is actually an organization with cumulative knowledge which is η-optimal since it maximizes $Y = y - (L-1)\eta$.

Theorem 7.3 *If Ω is bounded, then an η-optimal organization with cumulative knowledge exists.*

We state the following lemma for further use, which is analogous and can be established by the same argument used for Lemma 2.18. Let $\mathcal{O}_Q$ be the set of quasi-organizations with cumulative knowledge defined by $(L, \prec, \psi, \mu)$. Specifically, if $O \in \mathcal{O}_Q$ is actually an organization with cumulative knowledge, then $O \in \mathcal{O}^*$. Furthermore, we say that O is a *specialized quasi-organization with cumulative knowledge* if $t_1^p = 1$, $t_1^h = 0$, $\beta_1 = \frac{1}{\gamma}$, $l_1 = L$, $\prec_1 = <$ and, for each $i \neq 1$, $t_i^p = 0$, $t_i^h = 1$, $\beta_1 = \frac{\alpha_i}{\gamma}$ and $l_i = \{i\}$.

Lemma 7.9 *Let O be a specialized quasi-organization with cumulative knowledge. If $\hat{O}$ is such that $\hat{L} = L \setminus \{i \in L : \alpha_i = 0\}$ but otherwise equal to O, then $\hat{\alpha}_i = \alpha_i$ for each $i \in \hat{L}$ and $\cup_{l \in \hat{L}} A_l = \cup_{l \in L} A_l$. Consequently, $\hat{O}$ is an organization with cumulative knowledge, $\hat{y} = y$, $\hat{L} \leq L$ and $\hat{L} = L$ if and only if O is an organization with cumulative knowledge. In addition, if $O \in \mathcal{O}_Q$, then $\hat{O} \in \mathcal{O}^*$.*

The remainder of the proof of Theorem 7.3 is analogous to that of Theorem 4.

7.7 The cumulative knowledge order

Theorem 7.4 *If O is an optimal organization with cumulative knowledge, then $\prec = <$.*

Proof. The proof is by induction. We first claim that $1 \prec i$ for each $i \in L \setminus \{1\}$. Suppose not; then $i \prec 1$ for some $i \in L \setminus \{i\}$. Define an organization $\hat{O}$ equal to O except that $\hat{L} = L \setminus \{i\}$ and, consequently, $\hat{<} = < |_{\hat{L}}$ and $\hat{\prec} = \prec |_{\hat{L}}$. For each $j \in \hat{L}$, let $\mathcal{L}_j = \{l \in \hat{L} : l < j\}$. We then have that $\cup_{l \in \mathcal{L}_j} A_l = \cup_{l<j} A_l$ since $A_i \subseteq A_1$; this also implies that $\cup_{l \in \hat{L}} A_l = \cup_{l \in L} A_l$. In addition, $\cup_{l \in \mathcal{L}_j} B_l = \cup_{l<j} B_l$ since $B_i \subseteq B_1$. Thus, $\hat{\alpha}_j = \alpha_j$ for each $j \in \hat{L}$ and, hence, $\hat{\gamma} = \sum_{j \in \hat{L}} \alpha_j < \sum_{j \in L} \alpha_j = \gamma$ since $\alpha_i > 0$ by Theorem 7.1. Also, $\hat{\theta} \geq \theta$. Thus, $\hat{y} > y$ and $\hat{L} \subset L$, contradicting the optimality of O. Thus, $1 \prec i$ for each $i \in L \setminus \{1\}$.

Assume that $1 \prec \cdots \prec k \prec i$ for each $i \in L \setminus \{1, \ldots, k\}$; we now claim that $k+1 \prec i$ for each $i \in L \setminus \{1, \ldots, k+1\}$. If not, then $i \prec k+1$ for some $i \in L \setminus \{1, \ldots, k+1\}$ and, hence, $A_i \subseteq A_{k+1}$ and $B_i \subseteq B_{k+1}$. The argument is now analogous to the above.

Then, by induction, it follows that $1 \prec \cdots \prec L$. ■

7.8 Hierarchies

In this section we establish the following result.

Theorem 7.5 *If O is an optimal organization with cumulative knowledge, $\pi = h$ and $\xi = c$, then (a) $B_l = A_l$ for each $l \in L$, (b) $A_1 < A_2 \setminus A_1 < \cdots < A_L \setminus A_{L-1}$, (c) $\beta_2 > \cdots > \beta_L$, and (d) $\beta_1 > \beta_2$ if $h < 1$.*

The proof of the above result starts with the following lemma, which follows from the arguments used to prove Lemma A.16 together with Lemma 7.1.

Lemma 7.10 *If O is an optimal organization with cumulative knowledge, then $\mu(B_j) > 0$ for all $j \in L \setminus \{1\}$.*

We turn to the proof of Theorem 7.5. While the argument is analogous to the proof of Corollary 3, there are some difficulties that arise due to the requirement imposed by the cumulative nature of knowledge.

Proof of Theorem 7.5. The proof of (a) is by induction and, thus, we start by showing that $B_1 = A_1$. Suppose not; then $B_1 \setminus A_1 \neq \emptyset$. Define an organization $\hat{O}$ equal to O except that $\hat{A}_1 = B_1$, $\hat{A}_j = A_j \cup (B_1 \setminus A_1)$ for each $j \in L$ (recall that $1 \prec j$

for each $j \neq 1$ by part 4 of Theorem 7.2) and $\hat{L} = L \setminus \{j \in L : \hat{\alpha}_j = 0\}$. Lemma 7.9 implies that $\hat{O}$ is an organization and that, for each $j \in \hat{L}$,

$$\hat{\alpha}_j = hF(\hat{A}_j \setminus \cup_{l<j}\hat{A}_l) + \pi F(\hat{A}_j^c \setminus \cup_{l<j}\hat{B}_l).$$

Then $\hat{\nu}_j = c\mu(\hat{A}_j) + \xi\mu(\hat{B}_j \setminus A_j) = c\mu(\hat{B}_j) = c\mu(B_j) = \nu_j$ for each $j \in L$ since $\xi = c$. Furthermore, we clearly have that, for each $j \in L$, $\cup_{l<j}B_l = \cup_{l<j}\hat{B}_l$, $\cup_{l<j}A_l \subseteq \cup_{l<j}\hat{A}_l$ and $\cup_{l\in L}A_l \subseteq \cup_{l\in L}\hat{A}_l$. In addition, we have that $\hat{A}_j \setminus \cup_{l<j}\hat{A}_l = (A_j \setminus \cup_{l<j}A_l) \setminus (B_1 \setminus A_1)$ for each $j > 1$. Indeed, $\hat{A}_j = A_j \cup (B_1 \setminus A_1)$ and $\hat{A}_j \setminus \cup_{l<j}\hat{A}_l = (A_j \cup (B_1 \setminus A_1)) \cap (\cup_{l<j}A_l)^c \cap (B_1 \setminus A_1)^c = [A_j \cap (\cup_{l<j}A_l)^c \cap (B_1 \setminus A_1)^c] \cup [(B_1 \setminus A_1) \cap (\cup_{l<j}A_l)^c \cap (B_1 \setminus A_1)^c] = (A_j \setminus \cup_{l<j}A_l) \setminus (B_1 \setminus A_1)$. In particular, this implies that $\hat{\alpha}_j \leq \alpha_j$ for each $j \in L$.

Suppose first that $\cup_{l\in L}A_l \subset \cup_{l\in L}\hat{A}_l$. Since $(\cup_{l\in L}\hat{A}_l) \setminus (\cup_{l\in L}A_l) \in \mathcal{I}$, it follows that $F(\cup_{l\in L}\hat{A}_l) > F(\cup_{l\in L}A_l)$. This, together with $\hat{\alpha}_j \leq \alpha_j$ for each $j \in L$, implies that $\hat{\theta} > \theta$ and $\hat{\gamma} \leq \gamma$; hence, $\hat{y} > y$. Since $\hat{L} \leq L$, this contradicts the optimality of O.

Thus, assume that $\cup_{l\in L}A_l = \cup_{l\in L}\hat{A}_l$. Hence, $B_1 \setminus A_1 \subseteq \cup_{l\in L}A_l$ and, in fact, $B_1 \setminus A_1 \subseteq \cup_{l>1}A_l$ since $(B_1 \setminus A_1) \cap A_1 = \emptyset$. Let $j \in L$ be the smallest $j' \in L$ such that $(B_1 \setminus A_1) \cap A_{j'} \neq \emptyset$. Then $(B_1 \setminus A_1) \cap (A_j \setminus \cup_{l<j}A_l) \neq \emptyset$ and, thus, $\hat{A}_j \setminus (\cup_{l<j}\hat{A}_l) = (A_j \setminus \cup_{l<j}A_l) \setminus (B_1 \setminus A_1) \subset A_j \setminus \cup_{l<j}A_l$. Hence, $F(\hat{A}_j \setminus (\cup_{l<j}\hat{A}_l)) < F(A_j \setminus (\cup_{l<j}A_l))$. This, together with $\hat{A}_j^c \setminus (\cup_{l<j}\hat{B}_l) \subseteq A_j^c \setminus (\cup_{l<j}B_l)$, implies that $\hat{\alpha}_j < \alpha_j$. This, together with $\cup_{l\in L}A_l = \cup_{l\in L}\hat{A}_l$ and $\hat{\alpha}_l \leq \alpha_l$ for each $l \in L$, implies that $\hat{\theta} \geq \theta$ and $\hat{\gamma} < \gamma$; hence, $\hat{y} > y$. Since $\hat{L} \leq L$, this contradicts the optimality of O. This contradiction shows that $B_1 = A_1$.

Let $1 < i \leq L$ and assume that $B_j = A_j$ for all $j < i$. Suppose that $B_i \neq A_i$; then $B_i \setminus A_i \neq \emptyset$. We first claim that $(B_i \setminus A_i) \cap \cup_{l=1}^{i-1}A_l = \emptyset$. Suppose not; then $(B_i \setminus A_i) \cap \cup_{l=1}^{i-1}A_l \neq \emptyset$. Let $E = (B_i \setminus A_i) \cap (\cup_{l=1}^{i-1}A_l)^c$ and define an organization $\tilde{O}$ equal to O except that $\tilde{B}_i = A_i \cup E$. Note that when $j \prec i$, then $j < i$ by part 4 of Theorem 7.2 and $\tilde{B}_j = B_j = A_j \subseteq A_i \subseteq \tilde{B}_i$; thus, $\tilde{O}$ is an organization with cumulative knowledge.

We have that $\tilde{\nu}_i = c(\mu(B_i) - \mu((B_i \setminus A_i) \cap \cup_{l=1}^{i-1}A_l)) < c\mu(B_i) = \nu_i$ since $(B_i \setminus A_i) \cap \cup_{l=1}^{i-1}A_l$ is nonempty and belongs to $\mathcal{I}$. In addition, $\cup_{l<j}\tilde{B}_j = \cup_{l<j}B_j$ for all $j \in L$.

It then follows from $\cup_{l<j}\tilde{B}_j = \cup_{l<j}B_j$ and $\tilde{A}_j = A_j$ for all $j \in L$ that $\tilde{\alpha}_j = \alpha_j$ for

each $j \in L$ and that $\cup_{l\in L}\tilde{A}_l = \cup_{l\in L}A_l$. This, together with $\tilde{\nu}_i < \nu_i$ and $\alpha_j > 0$ for all $j \in L$ (by Theorem 7.1) implies that $\tilde{y} > y$. Since $\tilde{L} = L$, this contradicts the optimality of O. This shows that $B_i \setminus A_i \cap \cup_{l=1}^{i-1} A_l = \emptyset$.

Define an organization $\hat{O}$ equal to O except that $\hat{A}_i = B_i$, $\hat{A}_j = A_j \cup (B_i \setminus A_i)$ for all $j \in L$ such that $i < j$, and $\hat{L} = L \setminus \{j \in L : \hat{\alpha}_j = 0\}$. Note that if $j \in L$ is such that $j < i$, then $\hat{A}_i = B_i \subseteq B_j = A_j = \hat{A}_j$; hence, $\hat{O}$ is an organization with cumulative knowledge.

Then, for each $j \in L$, $\nu_j = \hat{\nu}_j$, $\cup_{l<j}B_l = \cup_{l<j}\hat{B}_l$, $\cup_{l<j}A_l \subseteq \cup_{l<j}\hat{A}_l$, $\cup_{l\in L}A_l \subseteq \cup_{l\in L}\hat{A}_l$ and $\hat{A}_j \setminus \cup_{l<j}\hat{A}_l = (A_j \setminus \cup_{l<j}A_l) \setminus (B_i \setminus A_i)$ for each $j > i$ as above. Since $\hat{A}_j = A_j$ for each $j < i$, we obtain that $\hat{\alpha}_j \leq \alpha_j$ for each $j \neq i$. Furthermore, since $B_j = A_j$ for all $j < i$ and $h = \pi$, $\alpha_i = h(F(A_i \setminus \cup_{l<i}A_l) + F(A_i^c \setminus \cup_{l<i}A_l)) = h(1 - F(\cup_{l<i}A_l)) = h(F(\hat{A}_i \setminus \cup_{l<i}A_l) + F(\hat{A}_i^c \setminus \cup_{l<i}A_l)) = \hat{\alpha}_i$. Hence, $\hat{\alpha}_j \leq \alpha_j$ for each $j \in L$.

Suppose first that $\cup_{l=i}^L A_l \subset \cup_{l=i}^L \hat{A}_l$. Since $\cup_{l=i}^L \hat{A}_l = (\cup_{l=i}^L A_l) \cup (B_i \setminus A_i)$, it follows that $(\cup_{l=i}^L A_l)^c \cap (B_i \setminus A_i) \neq \emptyset$. By $(B_i \setminus A_i) \cap (\cup_{l=1}^{i-1} A_l) = \emptyset$ we obtain that $(\cup_{l=1}^L A_l)^c \cap (B_i \setminus A_i) = (\cup_{l=i}^L A_l)^c \cap (B_i \setminus A_i) \cap (\cup_{l=1}^{i-1} A_l)^c = (\cup_{l=i}^L A_l)^c \cap (B_i \setminus A_i) \neq \emptyset$. Hence, $\cup_{l\in L}A_l \subset \cup_{l\in L}\hat{A}_l$, and $\hat{y} > y$ as above. But this, together with $\hat{L} \leq L$, contradicts the optimality of O.

Thus, we may assume that $\cup_{l=i}^L A_l = \cup_{l=i}^L \hat{A}_l$. Hence, $B_i \setminus A_i \subseteq \cup_{l=i}^L A_l$ and, in fact, $B_i \setminus A_i \subseteq \cup_{l>i} A_l$ since $(B_i \setminus A_i) \cap A_i = \emptyset$ obviously. Thus, as above, there is $j > i$ such that $\hat{\alpha}_j < \alpha_j$ and $\hat{y} > y$. Since $\hat{L} \leq L$, this contradicts the optimality of O. This contradiction shows that $B_i = A_i$.

The above shows that $B_l = A_l$ for each $l \in L$; thus, (a) follows.

It follows from $B_l = A_l$ for each $l \in L$ that $\mathcal{C} = \{A_1, A_2 \setminus A_1, \ldots, A_L \setminus A_{L-1}\}$. Since $c_{A_l} = c \sum_{j=l}^L \alpha_j$ and $\alpha_j > 0$ for all $j \in L$ by Theorem 7.1, it follows by part 2 of Theorem 7.2 that $A_1 < A_2 \setminus A_1 < \cdots < A_L \setminus A_{L-1}$. Thus, (b) follows.

By (a), we have that $B_l = A_l$ and, hence, $F(A_l) > 0$ for each $l \in L$ by Lemma 7.10. Thus, for each $1 < i < L$,

$$\alpha_i = h(1 - \sum_{l=1}^{i-1} F(A_l)) > h(1 - \sum_{l=1}^{i-1} F(A_l) - F(A_i)) = \alpha_{i+1}.$$

Moreover, $\beta_i = \frac{\alpha_i}{\gamma} > \frac{\alpha_{i+1}}{\gamma} = \beta_{i+1}$, proving (c).

Since $\alpha_2 = h(1 - F(A_1))$, we have that $\alpha_2 < 1$ if $h < 1$. Thus, $\beta_1 = \frac{1}{\gamma} > \frac{\alpha_2}{\gamma} = \beta_2$ and (d) holds. ■

7.9 Optimal organizations when ξ is small

We establish Theorem 8 in this section using a series of lemmas. While some results carry through from the proof of the analogous Theorem 5, there are many differences which have already been noted regarding the statements of the theorems themselves.

Throughout this section, we assume that Assumptions (A1) and (A2) hold as well as the assumptions made in Section 3. Let $y_1 = \max_{0 \le \mu_1 \le \bar{\omega}}(F(\mu_1) - c\mu_1)$ and define

$$\xi_1 = \frac{\pi f(\bar{\omega}) y_1}{1 + 2h};$$

then $y_1 > 0$ by (A2) and, thus, $\xi_1 > 0$ by (A1). Define also

$$\xi_2 = \frac{ch\eta}{1 + h\eta} > 0.$$

Then define

$$\bar{\xi} = \min\{\xi_1, \xi_2\}.$$

Suppose that O is an η-optimal organization with cumulative knowledge, that $0 < \xi < \bar{\xi}$ and that $L \ge 2$.

Lemma 7.11 $B_1 \cup A_2 = \Omega$.

Proof. Suppose not; then there is $a \in \Omega$ such that $[a, a + \varepsilon) \subseteq (B_1 \cup A_2)^c$ for all $\varepsilon > 0$ sufficiently small. Consider an organization $\hat{O}$ equal to O except that $\hat{B}_1 = B_1 \cup [a, a+\varepsilon)$ and $\hat{B}_i = B_i \cup [a, a+\varepsilon)$ for each $i > 1$. Since $[a, a+\varepsilon) \subseteq A_2^c \cap B_1^c$, we have that $\hat{A}_2^c \setminus \hat{B}_1 = (A_2^c \setminus B_1) \setminus [a, a+\varepsilon)$. Thus, $\hat{\alpha}_2(\varepsilon) = \alpha_2 - \pi F([a, a+\varepsilon))$ and $\hat{\alpha}_2'(0) = -\pi f(a)$. Since $\hat{A}_i^c \setminus \cup_{j<i} \hat{B}_j \subseteq A_i^c \setminus \cup_{j<i} B_j$, it follows that $\hat{\alpha}_i(\varepsilon) \le \alpha_i$ and, thus, $\hat{\alpha}_i'(0) = \lim_{\varepsilon \to 0} \frac{\hat{\alpha}_i(\varepsilon) - \alpha_i}{\varepsilon} \le 0$ for each $i > 2$. Hence,

$$\hat{\theta}(\varepsilon) = \theta - \xi\varepsilon - \xi \sum_{i=2}^{L} \hat{\alpha}_i(\mu(\hat{B}_i(\varepsilon)) - \mu(B_i)) - \sum_{i=2}^{L} (\hat{\alpha}_i(\varepsilon) - \alpha_i)\nu_i,$$

where, recall, $\nu_i = c\mu(A_i) + \xi\mu(B_i \setminus A_i)$ for each $i \in L$. The above equation uses $\mu(\hat{B}_1(\varepsilon)) - \mu(B_1) = \mu([a, a+\varepsilon)) = \varepsilon$; in addition, we have that $\mu(\hat{B}_i(\varepsilon)) - \mu(B_i) \leq \mu([a, a+\varepsilon)) = \varepsilon$ and, hence, letting $b_i(\varepsilon) = \mu(\hat{B}_i(\varepsilon)) - \mu(B_i)$,

$$\frac{db_i(0)}{d\varepsilon} \leq 1.$$

It follows that

$$\begin{aligned}\gamma\hat{y}'(0) &= -\xi - \xi\sum_{i=2}^{L}\alpha_i\frac{db_i(0)}{d\varepsilon} + \pi f(a)\nu_2 + \sum_{i>2}(-\hat{\alpha}_i'(0))\nu_i + \pi f(a)y + y\sum_{i>2}(-\hat{\alpha}_i'(0)) \\ &> \pi f(\bar{\omega})y - \xi\gamma > \pi f(\bar{\omega})y - \xi(1+2h)\end{aligned}$$

since $f(a) > f(\bar{\omega})$ and $\gamma = 1 + h\sum_{i=2}^{L} F(A_i \setminus A_{i-1}) + \pi\sum_{i=2}^{L} F(A_i^c \setminus B_{i-1}) \leq 1 + hF(A_L \setminus A_1) + \pi F(A_1^c) \leq 1 + 2h$. We then have that $y \geq y_1$ and $\pi f(\bar{\omega})y - \xi \geq \pi f(\bar{\omega})y_1 - \xi = \xi_2(1+2h) - \xi(1+2h) > 0$ since $\xi < \bar{\xi}$. It then follows that there is $\varepsilon > 0$ such that $\hat{y}(\varepsilon) > y$, a contradiction to the optimality of O. This contradiction shows that $B_1 \cup A_2 = \Omega$. ■

Lemma 7.12 $(B_1 \setminus A_1) \cap A_2 = \emptyset$.

Proof. Suppose not; then $(B_1 \setminus A_1) \cap A_2 \neq \emptyset$. Consider an organization $\hat{O}$ equal to O except that $\hat{B}_1 = B_1 \setminus A_2$; since $1 \prec i$ for each $i \in L$, $\hat{O}$ is an organization with cumulative knowledge. Then $\hat{A}_2^c \setminus \hat{B}_1 = A_2^c \cap (B_1^c \cup A_2) = A_2^c \setminus B_1$ and, hence $\hat{\alpha}_2 = \alpha_2$. Moreover, for each $i > 2$, $\cup_{l<i}\hat{B}_l = \cup_{l<i}B_l$ and, hence, $\hat{\alpha}_i = \alpha_i$. Thus, $\hat{\gamma} = \gamma$ and $\hat{\theta} > \theta$ since $\cup_{l\in L}\hat{A}_l = \cup_{l\in L}A_l$ and $\hat{\nu}_1 = \nu_1 - \xi\mu((B_1 \setminus A_1) \cap A_2) < \nu_1$. It follows that $\hat{y} > y$, a contradiction to the optimality of O. This contradiction shows that $(B_1 \setminus A_1) \cap A_2 = \emptyset$. ■

It follows from Lemmas 7.11 and 7.12 that

$$B_1 \setminus A_1 = \Omega \setminus (A_1 \cup A_2) = A_2^c.$$

Hence, for each $l > 1$,

$$B_l = \Omega.$$

Lemma 7.13 *If O is an organization with cumulative knowledge such that $\prec = <$ and $B_1 \setminus A_1 = A_2^c$, then $\alpha_l = hF(A_l \setminus A_{l-1})$ for each $l \geq 2$.*

Proof. Let $l \geq 2$. We have that $A_l \setminus \cup_{j<l} A_j = A_l \setminus A_{l-1}$ and that $A_l^c \setminus \cup_{j<l} B_j = \emptyset$. Hence, $\alpha_l = hF(A_l \setminus \cup_{j<l} A_j) + \pi F(A_l^c \setminus \cup_{j<l} B_j) = hF(A_l \setminus A_{l-1})$. ■

Lemma 7.14 *The output of the an optimal organization with cumulative knowledge is* $y = \frac{\theta}{\gamma}$ *where*

$$\theta = \sum_{i=1}^{L} F(A_i \setminus A_{i-1}) - \mu(A_1)(c-\xi)(1 + h\sum_{i=2}^{L} F(A_i \setminus A_{i-1}))$$
$$-\mu(A_2 \setminus A_1)((c-\xi)h\sum_{i=2}^{L} F(A_i \setminus A_{i-1}) - \xi)$$
$$-\sum_{j=3}^{L} \mu(A_j \setminus A_{j-1})((c-\xi)h\sum_{i=j}^{L} F(A_i \setminus A_{i-1})) - \xi\bar{\omega}(1 + h\sum_{i=2}^{L} F(A_i \setminus A_{i-1})) \text{ and}$$
$$\gamma = 1 + h\sum_{i=2}^{L} F(A_i \setminus A_{i-1}).$$

Proof. The conclusion follows by Lemmas 7.11–7.13 and from $\mu(A_l) = \sum_{i=1}^{l} \mu(A_i \setminus A_{i-1})$ for each $l \in L$, where, recall, $A_0 = \emptyset$. ■

Lemma 7.15 *The following holds:* $\mathcal{C} = \{A_1, A_2 \setminus A_1, \ldots, A_L \setminus A_{L-1}, A_L^c\}$, $c_{A_1} = c\sum_{j=1}^{L} \alpha_j$, $c_{A_2 \setminus A_1} = c\sum_{j=2}^{L} \alpha_j$, $c_{A_k \setminus A_{k-1}} = c\sum_{j=k}^{L} \alpha_j + \xi\sum_{j=1}^{L} \alpha_j$ *for each* $k = 3, \ldots, L$ *and* $c_{A_L^c} = \xi\sum_{j=1}^{L} \alpha_j$. *Thus,* $A_1 < A_2 \setminus A_1$, $A_3 \setminus A_2 < \cdots < A_L \setminus A_{L-1} < A_L^c$ *and* $A_1 < A_L^c$.

Proof. Recall that $\prec = <$ by part 4 of Theorem 7.2 and, hence, $i_1 = 1, \ldots, i_L = L$. Thus, the partition $\mathcal{C}$ of $\cup_{l=1}^{L} B_l = \Omega$ is

$$\begin{aligned} \mathcal{C} &= \{\cap_{j<l}(B_j \setminus A_j)^c \cap (A_l \setminus A_{l-1}) : l \in L\} \\ &\cup \{\cap_{j<l}(B_j \setminus A_j)^c \cap (B_l \setminus A_l) \cap (A_k \setminus A_{k-1}) : l, k \in L \text{ and } k > l\} \\ &\cup \{\cap_{j<l}(B_j \setminus A_j)^c \cap (B_l \setminus A_l) \cap (B_L \setminus A_L) : l \in L\}. \end{aligned}$$

Consider each of the sets $C \in \mathcal{C}$, first noting that $B_1 \setminus A_1 = A_2^c$ (hence, $(B_1 \setminus A_1)^c = A_2$) and, for each $l > 1$, $B_l \setminus A_l = A_l^c$ (hence, $(B_l \setminus A_l)^c = A_l$) and recalling that $B_0 = A_0 = \emptyset$.

The set $\cap_{j<l}(B_j \setminus A_j)^c \cap (A_l \setminus A_{l-1})$ equals A_1 when $l = 1$ and $A_2 \cap (A_2 \setminus A_1) = A_2 \setminus A_1$ when $l = 2$. Thus, $A_1, A_2 \setminus A_1 \in \mathcal{C}$, $c_{A_1} = c \sum_{j=1}^{L} \alpha_j$ and $c_{A_2 \setminus A_1} = c \sum_{j=2}^{L} \alpha_j$.

The set $\cap_{j<l}(B_j \setminus A_j)^c \cap (B_l \setminus A_l) \cap (A_k \setminus A_{k-1})$ equals $A_2^c \cap (A_k \setminus A_{k-1})$ when $l = 1$ and $k > 1$; thus, it equals $\emptyset$ if $k = 2$ and $A_k \setminus A_{k-1}$ if $k > 2$. Thus, $A_k \setminus A_{k-1} \in \mathcal{C}$ and $c_{A_k \setminus A_{k-1}} = c \sum_{j=k}^{L} \alpha_j + \xi \sum_{j=1}^{L} \alpha_j$ for each $k > 2$. Moreover, when $l > 1$ and $k > l$, it equals $A_2 \cap \cdots \cap A_{l-1} \cap A_l^c \cap (A_k \setminus A_{k-1}) = A_2 \setminus A_{k-1} = \emptyset$.

Finally, the set $\cap_{j<l}(B_j \setminus A_j)^c \cap (B_l \setminus A_l) \cap (B_L \setminus B_{L-1})$ equals $A_2^c \cap (B_L \setminus A_L) = A_2^c \cap \Omega \cap A_L^c = A_L^c$ when $l = 1$ Thus, $A_L^c \in \mathcal{C}$ and $c_{A_L^c} = \xi \sum_{j=1}^{L} \alpha_j$. Moreover, when $l > 1$, it equals $A_2 \cap \cdots \cap A_{l-1} \cap A_l^c \cap \Omega \cap A_L^c = A_2 \setminus A_L = \emptyset$.

We have that $\alpha_j > 0$ for each $j \in L$ by Theorem 7.1 and, thus, $A_1 < A_2 \setminus A_1$, $A_3 \setminus A_2 < \cdots < A_L \setminus A_{L-1} < A_L^c$ and $A_1 < A_L^c$ follows by part 2 of Theorem 7.2. ■

Lemma 7.16 *$A_l \setminus A_{l-1} \neq \emptyset$ for each $l > 1$.*

Proof. This follows because $\alpha_l = hF(A_l \setminus A_{l-1}) > 0$ by Lemma 7.13 and Theorem 7.1 and, hence, $F(A_l \setminus A_{l-1}) > 0$ for each $l > 1$. ■

Lemma 7.17 *If $L \geq 3$, then $A_2 \setminus A_1 < A_3 \setminus A_2$.*

Proof. Suppose that $A_2 \setminus A_1 < A_3 \setminus A_2$ does not hold; then, $c_{A_3 \setminus A_2} \geq c_{A_2 \setminus A_1}$ by part 2 of Theorem 7.2 and, thus, we may assume, by part 3 of Theorem 7.2, that $A_3 \setminus A_2 < A_2 \setminus A_1$. Let $a, a' \in \Omega$ and $\varepsilon > 0$ be such that $[a, a + \varepsilon) \subseteq A_3 \setminus A_2$, $[a', a' + \varepsilon) \subseteq A_2 \setminus A_1$ and $a + \varepsilon < a'$. Consider an organization $\hat{O}$ equal to O except that $\hat{A}_2 \setminus \hat{A}_1 = ((A_2 \setminus A_1) \setminus [a', a' + \varepsilon)) \cup [a, a + \varepsilon)$, $\hat{A}_3 \setminus \hat{A}_2 = ((A_3 \setminus A_2) \setminus [a, a + \varepsilon)) \cup [a', a' + \varepsilon)$ (so that $\hat{A}_2 = A_1 \cup (\hat{A}_2 \setminus A_1)$ and $\hat{A}_3 = \hat{A}_2 \cup (\hat{A}_3 \setminus \hat{A}_2)$) and $\hat{B}_1 \setminus \hat{A}_1 = \hat{A}_2^c$.

We clearly have that $\mu(\hat{C}) = \mu(C)$ for each $C \in \mathcal{C}$, $F(\hat{C}) = F(C)$ for each $C \notin \{A_2 \setminus A_1, A_3 \setminus A_2\}$, $F(\hat{A}_2 \setminus \hat{A}_1) > F(A_2 \setminus A_1)$ since $F([a, a + \varepsilon)) > F([a', a' + \varepsilon))$ by Lemma A.12 and $F(\hat{A}_3 \setminus \hat{A}_2) + F(\hat{A}_2 \setminus \hat{A}_1) = F(A_3 \setminus A_2) + F(A_2 \setminus A_1)$ since $(\hat{A}_3 \setminus \hat{A}_2) \cup (\hat{A}_2 \setminus \hat{A}_1) = (A_3 \setminus A_2) \cup (A_2 \setminus A_1)$. Thus, $F(\hat{A}_3 \setminus \hat{A}_2) < F(A_3 \setminus A_2)$. By choosing $\varepsilon > 0$ sufficiently small, $\hat{\alpha}_3 = hF(\hat{A}_3 \setminus \hat{A}_2) > 0$ and $\hat{O}$ is an organization with cumulative knowledge.

It then follows that $\sum_{i=j}^{L} F(\hat{A}_i \setminus \hat{A}_{i-1}) = \sum_{i=j}^{L} F(\hat{A}_i \setminus \hat{A}_{i-1})$ for each $j \neq 3$ and $\sum_{i=3}^{L} F(\hat{A}_i \setminus \hat{A}_{i-1}) < \sum_{i=3}^{L} F(\hat{A}_i \setminus \hat{A}_{i-1})$. Lemma 7.14 then implies that $\hat{y} > y$, a contradiction to the optimality of O. This contradiction shows that $A_2 \setminus A_1 < A_3 \setminus A_2$. ■

Lemma 7.18 *If $L = 2$, then $A_2 \setminus A_1 < A_2^c$.*

Proof. We first show that $F(A_2 \setminus A_1) \geq \eta$. Since $L = 2$, it follows that $y = \frac{\theta}{\gamma} \geq y_1 + \eta$, i.e. $\theta \geq (y_1 + \eta)\gamma$. Thus, $\theta > y_1 + \eta$ since $\gamma = 1 + \alpha_2 > 1$. We have that

$$\begin{aligned}
\theta &= F(A_1 \cup A_2) - \sum_{i=1}^{2} \alpha_i (c\mu(A_i) + \xi\mu(B_i \setminus A_i)) \\
&= F(A_1) + F(A_2 \setminus A_1) - c\mu(A_1) - \xi\mu(B_1 \setminus A_1) - c\alpha_2\mu(A_2) - \xi\alpha_2\mu(B_2 \setminus A_2) \\
&\leq F(A_1) - c\mu(A_1) + F(A_2 \setminus A_1) \\
&\leq y_1 + F(A_2 \setminus A_1).
\end{aligned}$$

Hence, $y_1 + F(A_2 \setminus A_1) \geq \theta \geq y_1 + \eta$ and, thus, $F(A_2 \setminus A_1) \geq \eta$.

Since $\xi < \bar{\xi} \leq \xi_2 = \frac{ch\eta}{1+h\eta} < c$, it follows that $\xi < (c - \xi)h\eta \leq (c - \xi)hF(A_2 \setminus A_1)$, i.e. $\xi(1 + hF(A_2 \setminus A_1)) < chF(A_2 \setminus A_1)$. Hence,

$$c_{A_2^c} = \xi(\alpha_1 + \alpha_2) = \xi(1 + hF(A_2 \setminus A_1)) < chF(A_2 \setminus A_1) = c\alpha_2 = c_{A_2 \setminus A_1}.$$

Therefore, it follows by part 2 of Theorem 7.2 that $A_2 \setminus A_1 < A_2^c$. ■

It follows from Lemmas 7.15, 7.17 and 7.18 that $A_1 < A_2 \setminus A_1 < A_3 \setminus A_2 < \cdots < A_L \setminus A_{L-1} < A_L^c$. Let $\mu_i = \mu(A_i \setminus A_{i-1})$ (with $\mu_1 = \mu(A_1)$); we then have that $A_1 = [0, \mu_1)$, $A_2 = [\mu_1, \mu_1 + \mu_2)$ and, in general, $A_i \setminus A_{i-1} = [\sum_{j=1}^{j-1} \mu_j, \sum_{j=1}^{j} \mu_j)$ for each $j \in L$.

Lemma 7.19 $F(A_2 \setminus A_1) = \frac{\xi}{(c-\xi)h} + f(\mu_1 + \mu_2)\mu_3$.

Proof. Let $\varepsilon > 0$ and let $\hat{O}$ be an organization with cumulative knowledge equal to O except that $\hat{A}_2 \setminus \hat{A}_1 = [\mu_1, \mu_1 + \mu_2 + \varepsilon)$, $\hat{A}_3 \setminus \hat{A}_2 = [\mu_1 + \mu_2 + \varepsilon, \mu_1 + \mu_2 + \mu_3)$ and

$\hat{B}_1 \setminus \hat{A}_1 = \hat{A}_2^c$. Then, letting $a = \mu_1 + \mu_2$, it follows by Lemma 7.14 that $\hat{\gamma}(\varepsilon) = \gamma$ and

$$\begin{aligned}\hat{\theta}(\varepsilon) &= \theta - \left[(\mu_2+\varepsilon)((c-\xi)h\sum_{i=2}^{L} F(A_i \setminus A_{i-1}) - \xi) - \mu_2((c-\xi)h\sum_{i=2}^{L} F(A_i \setminus A_{i-1}) - \xi)\right] \\ &\quad - \left[(\mu_3-\varepsilon)(c-\xi)h(\sum_{i=3}^{L} F(A_i \setminus A_{i-1}) - F([a, a+\varepsilon))) - \mu_3(c-\xi)h\sum_{i=3}^{L} F(A_i \setminus A_{i-1})\right] \\ &= \theta - \xi\varepsilon - \varepsilon(c-\xi)hF(A_2 \setminus A_1) + \mu_3(c-\xi)hF([a,a+\varepsilon)) - \varepsilon(c-\xi)hF([a,a+\varepsilon)).\end{aligned}$$

Hence,

$$\gamma\hat{y}'(0) = \xi - (c-\xi)hF(A_2 \setminus A_1) + f(a)\mu_3(c-\xi)h.$$

Since O is optimal, it follows that $\gamma\hat{y}'(0) \leq 0$ and, hence, using $a = \mu_1 + \mu_2$,

$$F(A_2 \setminus A_1) \geq \frac{\xi}{(c-\xi)h} + f(\mu_1+\mu_2)\mu_3.$$

Considering next an organization $\hat{O}$ with cumulative knowledge equal to O except that $\hat{A}_2 \setminus \hat{A}_1 = [\mu_1, \mu_1+\mu_2-\varepsilon)$, $\hat{A}_3 \setminus \hat{A}_2 = [\mu_1+\mu_2-\varepsilon, \mu_1+\mu_2+\mu_3)$ and $\hat{B}_1 \setminus \hat{A}_1 = \hat{A}_2^c$ and arguing as above yields

$$F(A_2 \setminus A_1) \leq \frac{\xi}{(c-\xi)h} + f(\mu_1+\mu_2)\mu_3.$$

The result then follows. ■

Using an argument analogous to the above, we obtain:

Lemma 7.20 *$F(A_i \setminus A_{i-1}) = f(\sum_{j=1}^{i} \mu_j)\mu_{i+1}$ for each $i > 2$.*

Lemma 7.21 *$\alpha_2 > \alpha_3 > \cdots > \alpha_L$.*

Proof. We have that $\alpha_l = hF(A_l \setminus A_{l-1})$ for each $l \geq 2$, hence, the statement of the lemma is equivalent to $F(A_2 \setminus A_1) > F(A_3 \setminus A_2) > \cdots > F(A_L \setminus A_{L-1})$.

For each $i > 1$, $F(A_i \setminus A_{i-1}) = \int_{\sum_{j=1}^{i-1}\mu_j}^{\sum_{j=1}^{i}\mu_j} f\mathrm{d}\mu < f(\sum_{j=1}^{i-1}\mu_j)\mu_i$ since f is strictly decreasing. Thus, by Lemma 7.19,

$$F(A_3 \setminus A_2) < f(\mu_1+\mu_2)\mu_3 = F(A_2 \setminus A_1) - \frac{\xi}{(c-\xi)h} < F(A_2 \setminus A_1).$$

Moreover, by Lemma 7.20,

$$F(A_{i+1} \setminus A_i) < f(\sum_{j=1}^{i}\mu_j)\mu_{i+1} = F(A_i \setminus A_{i-1})$$

for each $i = 3, \ldots, L-1$. Hence, the lemma follows. ■

Lemma 7.22 *If $h < 1$, then $\beta_1 > \sum_{l=2}^{L} \beta_l$.*

Proof. We have that $\beta_l = \frac{\alpha_l}{\gamma}$ for each $l \in L$, hence $\beta_1 > \sum_{l=2}^{L} \beta_l$ is equivalent to $\alpha_1 > \sum_{l=2}^{L} \alpha_l$. This inequality holds because $\sum_{l=2}^{L} \alpha_l = h \sum_{l=2}^{L} F(A_l \setminus A_{l-1}) = hF(A_L \setminus A_1) \leq h < 1 = \alpha_1$. ■