Supplementary Material for "The Folk Theorem for the Prisoner's Dilemma with Endogenous Private Monitoring"

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A Introduction

This paper contains supplementary material to our paper "The Folk Theorem for the Prisoner's Dilemma with Endogenous Private Monitoring". It provides, for the prisoner's dilemma on the left-hand side of Figure 1, an alternative proof for the claim that cooperation in each period is a sequential equilibrium outcome of the repeated prisoner's dilemma with endogenous private monitoring.

The interest of this alternative proof is that it uses a strategy which is both pure and explicitly specified. Our approach is related to the one we use to proof our folk theorem since, for each player i, the two strategies coincide at histories in $H_i^* \cup H_i^{*0}$. It is also related to the approach in Sekiguchi (1997) and Bhaskar and Obara (2002), who both use strategies such that the continuation strategy of each player at each

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of his private history is either the grim-trigger strategy or the strategy that always plays D. Our strategy uses, in reduced form, three continuation strategies; this is clearly seen in the exact automaton representation in Figure A.1 of the strategy we use.[1] Briefly, our strategy $\sigma = (\sigma_1, \sigma_2)$ is specified as follows: for each $i \in \{1, 2\}$ and history h_i of length t,

$$\sigma_i(h_i) = \begin{cases} (r_i^*, C) & \text{if } h_i \in H_i \setminus H_i^{*0} \text{ and } \mu(h_{-i}^{*,t}|h_i) \geq \mu_i^*, \\ (r_i^*, D) & \text{if } h_i \in H_i \setminus H_i^{*0} \text{ and } 0 < \mu(h_{-i}^{*,t}|h_i) < \mu_i^*, \\ (1_{(c,d)}, D) & \text{if } h_i \in H_i^{*0}, \end{cases}$$

where $\mu_i^* \in (0, 1)$. To emphasize the dependence of $\mu^* = (\mu_1^*, \mu_2^*)$, we write this strategy as σ^{μ^*}. As a function of $y_i \in Y_i$ only, $\sigma_i^{\mu_i^*}$ has the following representation as an automaton, whose initial state is not shown and depends on the history h_i via $\mu(h_{-i}^{*,t}|h_i)$:[2]

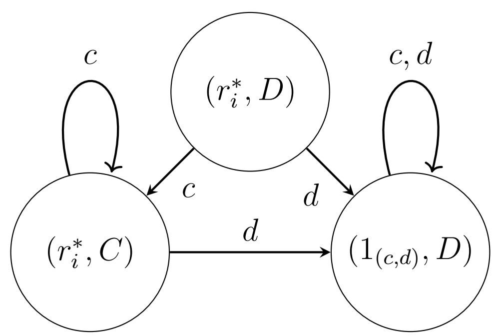

Figure A.1: Reduced strategy as an automaton

The strategy σ^{μ^*} is such that (C, C) is played in each period. This happens for the same reason as in the proof of Theorem 1, namely that $\mu(h_{-i}^{*,t}|h_i^{*,t}) = 1$ for each $i \in \{1, 2\}$ and $t \in \mathbb{N}_0$. Thus, if σ^{μ^*} is part of a sequential equilibrium, then cooperation in each period is a feature of a sequential equilibrium outcome. The former condition is established, under a stronger form of responsiveness of the aggregation function α, in the following result.

[1]The notion of a reduced strategy and an exact automaton can be found in e.g. Osborne and Rubinstein (1994) and Kalai (1990) respectively.

[2]This follows from Claim A.2 in the proof of Theorem A.1 below.

Theorem A.1 *If α is strongly responsive, then there exists $\delta^* \in (0,1)$ such that, for each $\delta \geq \delta^*$ and $X \in \mathcal{X}$, there exists $\mu^* \in (0,1)^2$ and a system of beliefs μ such that (σ^{μ^*}, μ) is a sequential equilibrium when $R_i = X^2$ for each $i \in \{1,2\}$.*

Besides focusing only on cooperation, Theorem A.1 requires a stronger form of responsiveness. We say that α is *strongly responsive* if it responsive and satisfies:

3. $1_{(c,c)} \in \arg\max_r \alpha_{i,Y_{-i}}(r, 1_{(c,d)})[c]$.

4. $1_{(c,c)} \in \arg\max_r \frac{\alpha_i(r,1_{(c,d)})[c,c]}{\alpha_i(r,1_{(c,d)})[c,c]+\alpha_i(r,1_{(c,d)})[c,d]}$.

In property 3, $\alpha_{Y_{-i}}(r,r')[y_{-i}] = \sum_{y_i} \alpha(r,r')[y_i, y_{-i}]$ for each $y_{-i} \in Y_{-i}$; throughout this supplementary material, we shall also use $\alpha_{Y_i}(r,r')[y_i] = \sum_{y_{-i}} \alpha(r,r')[y_i, y_{-i}]$ for each $y_i \in Y_i$.

Responsive aggregation functions reflect the choices of both players. Strong responsiveness make this dependence be more specific:

3. If a player proposes signal d for himself and c for his opponent, then the probability of the player observing c is maximized when his opponent chooses signal (c,c) with probability 1.

4. If a player proposes signal d for himself and c for his opponent, then the probability of signal (c,c) conditional on the opponent observing c is maximized when his opponent chooses signal (c,c) with probability 1.

To understand properties 3 and 4, let player 1 be the opponent and player 2 the original player; in addition, assume momentarily that player 1 is restricted to choosing degenerate distributions on Y. Note that, by property 1, if player 1 chooses signal (c,d) with probability 1, then (c,d) occurs with probability 1 and player 2 observes c with zero probability. The same conclusion holds if player 1 chooses (d,d). Thus, player 2 can observe c only if player 1 chooses (c,c) or (d,c) and property 3 requires, in particular, that the corresponding probability in the former case is no less than that of the latter case. Since player 1 is restricted to choosing a degenerate distribution, property 4 holds since, by property 1, the probability of (c,c) is strictly positive only when player 1 chooses (c, c) with probability 1 given that player 2 is choosing (c, d) with probability 1. Thus, the requirement of properties 3 and 4 is that its conclusion holds for all distributions and not just for degenerate ones.

The aggregation function in the motivating example is strongly responsive. Furthermore, if α is a mixed extension, then α is strongly responsive if

(a) $\alpha(1_y, 1_{y'})[\tilde{y}] = 0$ for each $y, y' \in Y$ and $\tilde{y} \notin \{y, y'\}$.

(b) $\alpha_i(1_y, 1_{y'})[y] > 0$ for each $y, y' \in Y$ and $i \in \{1, 2\}$.

(c) $\alpha_i(1_{(c,c)}, 1_{(c,d)})[c, c] \geq \alpha_i(1_{(d,c)}, 1_{(c,d)})[d, c]$.[3]

A.1 Proof of Theorem A.1

A.1.1 Parametrization

For each $i \in \{1, 2\}$, let

$$\hat{\mu}_i = \alpha_i(1_{(c,c)}, 1_{(c,d)})[(c, c)].$$

Then $\hat{\mu}_i \in (0, 1)$ by property 2. Let μ_i^* solve:

$$\mu(3(1-\delta) + \delta(2\hat{\mu}_i - (1-\hat{\mu}_i)(1-\delta))) = 2\mu - (1-\mu)(1-\delta), \tag{A.1}$$

i.e.

$$\mu_i^* = \frac{1-\delta}{\delta(1-\hat{\mu}_i)(3-\delta)}.$$

We have that $\mu(3(1-\delta) + \delta(2\hat{\mu}_i - (1-\hat{\mu}_i)(1-\delta))) \leq 2\mu - (1-\mu)(1-\delta)$ if and only if $\mu \geq \mu_i^*$. Moreover, $\mu_i^* > 0$ and $\mu_i^* \to 0$ as $\delta \to 1$. Therefore,

Claim A.1 *There exists $\delta_1 \in (0, 1)$ such that $\mu_i^* \in (0, \hat{\mu}_i)$ for each $\delta \geq \delta_1$ and $i \in \{1, 2\}$.*

Let $\underline{\mu} > 0$ be such that

$$\frac{\underline{\mu}}{(1-\underline{\mu}) \min_{i,r} \alpha_{i,Y_i}(r, 1_{(d,c)})[d]} < \min_i \hat{\mu}_i$$

[3] See Section A.2 for a proof of this claim.

for each $\mu < \underline{\mu}$. Such $\underline{\mu}$ exists since $\min_r \alpha_{i,Y_i}(r, 1_{(d,c)})[d] > 0$ by property 2 and, hence, $\lim_{\mu\to 0} \frac{\mu}{(1-\mu)\min_{i,r}\alpha_{i,Y_i}(r,1_{(d,c)})[d]} = 0$.

Let $\delta^* \in (0,1)$ be such that $\delta^* \geq \delta_1$ and, for each $\delta \geq \delta^*$ and $i \in \{1,2\}$,

$$\mu_i^* < \underline{\mu} \text{ and} \tag{A.2}$$

$$-1 + \delta(1 + 2\underline{\mu}(1 - \hat{\mu}_i)) > 0. \tag{A.3}$$

It follows that δ^* exists since, for each i, $\lim_{\delta\to 1}\mu_i^* = 0$ and $\lim_{\delta\to 1}\big(-1+\delta(1+2\underline{\mu}(1-\hat{\mu}_i))\big) = 2\underline{\mu}(1-\hat{\mu}_i) > 0$.

Let $\delta \geq \delta^*$, $X \in \mathcal{X}$ and $R_i = X^2$ for each $i \in \{1,2\}$.

A.1.2 The assessment

Let

$$h_i^{*,t} = \big(\overbrace{(r_i^*, C, c), \ldots, (r_i^*, C, c)}^{t \text{ periods}}\big) \text{ for each } t \geq 0.$$

$$H_i^B = \{h_i \in H_i : h_i = (h_i^{*,t} \cdot (r_i^*, C, d) \cdot h_i') \text{ for some } t \geq 0 \text{ and } h_i' \in H_i\}.$$

$$H_i^{*0} = \{h_i \in H_i : \prod_{t=1}^{\ell(h_i)} \alpha_i(r_i^{t,C}, 1_{(c,s_i^t)})[(y_i^t, c)] = 0\}.$$

Note that $H_i^B \subseteq H_i^{*0}$. Indeed, $r_i^{t+1,C} = 1_{(c,c)}$, $s_i^{t+1} = C$ and $y_i^{t+1} = d$ imply that $\alpha_i(r_i^{t+1,C}, 1_{(c,s_i^{t+1})})[(y_i^{t+1}, c)] = \alpha_i(1_{(c,c)}, 1_{(c,c)})[(d,c)] = 1_{(c,c)}(d,c) = 0$ by property 1.

The **strategy** is as follows. Let $r_i^{*,s_{-i}} = 1_{(s_{-i},c)}$ for each $i \in \{1,2\}$ and $s_{-i} \in \{C, D\}$, and define, for each $h_i \in H_i$,

$$\sigma_i(h_i) = \begin{cases} (1_{(c,d)}, D) & \text{if } h_i \in H_i^{*0}, \\ (r_i^*, D) & \text{if } h_i \in H_i \setminus H_i^{*0} \text{ and } \mu(h_{-i}^{*,\ell(h_i)}|h_i) < \mu_i^*, \\ (r_i^*, C) & \text{if } h_i \in H_i \setminus H_i^{*0} \text{ and } \mu(h_{-i}^{*,\ell(h_i)}|h_i) \geq \mu_i^*. \end{cases}$$

We also write $\sigma_i(r_i, s_i|h_i)$ for the probability that $\sigma_i(h_i)$ assigns to (r_i, s_i).

The **beliefs** are as follows. Let

$$H_i^C = \{h_i \in H_i : r_i^t = 1_{(c,d)} \text{ and } s_i^t = D \text{ for all } 1 \leq t \leq \ell(h_i)\},$$

$$H_i^D = \{h_i \in H_i^B : h_i = (h_i^{*,t} \cdot (r_i^*, C, d) \cdot h_i') \text{ for some } t \geq 0 \text{ and } h_i' \in H_i^C\}.$$

For any $h_i \in H_i$,

1. $\text{supp}(\mu(\cdot|h_i)) \subseteq H^{*0}_{-i}$ if $h_i \in H^{*0}_i$,

2. $h^{*,\ell(h_i)}_{-i} \in \text{supp}(\mu(\cdot|h_i)) \subseteq \{h^{*,\ell(h_i)}_{-i}\} \cup H^D_{-i}$ and

$$\mu(h^*_{-i}|h_i) = \frac{\prod_{t=1}^{\ell(h_i)} \alpha_i(r^{t,C}_i, 1_{(c,s^t_i)})[(y^t_i, c)]}{\prod_{t=1}^{\ell(h_i)} \alpha_i(r^{t,C}_i, 1_{(c,s^t_i)})[(y^t_i, c)] + \sum_{h_{-i} \in H^D_{-i} \cap H^{\ell(h_i)}_{-i}} \prod_{t=1}^{\ell(h_i)} \alpha_i(r^{t,s^t_{-i}}_i, r^{t,s^t_i}_{-i})[(y^t_i, y^t_{-i})]}$$

   if $h_i \in H_i \setminus H^{*0}_i$, with $\mu(h^{*,0}_{-i}|h_i) = 1$ if $h_i = \emptyset$.

We also write $\mu(h^*_{-i}|h_i)$ for $\mu(h^{*,\ell(h_i)}_{-i}|h_i)$. In A.1.5, we show that there exists consistent beliefs satisfying the above two properties.

A.1.3 Preliminary results

From the specification of beliefs, we obtain the following claim.

Claim A.2 *For each $i \in \{1, 2\}$:*

1. $\mu(h^*_{-i}|h^*_i) = 1$.

2. *If $h_i \in H_i \setminus H^{*0}_i$, then*

$$\mu(h^*_{-i}|h_i \cdot (r^*_i, s_i, c)) = \begin{cases} 1 & \text{if } s_i = C, \\ \hat{\mu}_i & \text{if } s_i = D. \end{cases}$$

3. *If $h_i \in H_i \setminus H^{*0}_i$ and (r_i, s_i, y_i) is such that $\alpha_i(r^C_i, 1_{(c,s_i)})[(y_i, c)] > 0$, then*

$$\mu(h^*_{-i}|h_i \cdot (r_i, s_i, y_i)) = \frac{\mu\alpha_i(r^C_i, 1_{(c,s_i)})[(y_i, c)]}{\mu\alpha_{i,Y_i}(r^C_i, 1_{(c,s_i)})[y_i] + (1-\mu)\alpha_{i,Y_i}(r^D_i, 1_{(d,c)})[y_i]}$$

   where $\mu = \mu(h^*_{-i}|h_i)$.

Proof. Part 1 follows since $h_i = h^*_i$ implies that

$$\prod_{t=1}^{\ell(h_i)} \alpha_i(r^{t,C}_i, 1_{(c,s^t_i)})[(y^t_i, c)] = \prod_{t=1}^{\ell(h^*_i)} \alpha_i(1_{(c,c)}, 1_{(c,c)})[(c, c)] = 1$$

by property 1 and, for each $h_{-i} \in H^D_{-i} \cap H^{\ell(h_i)}_{-i}$ with $h_{-i} = h^{*,k}_{-i} \cdot (r^*_{-i}, C, d) \cdot h'_{-i}$, we have that $\alpha_i(r^{k+1,s^{k+1}_{-i}}_i, r^{k+1,s^{k+1}_i}_{-i})[(y^{k+1}_i, y^{k+1}_{-i})] = \alpha_i(r^{*,C}_i, r^{*,C}_{-i})[(c, d)] = 1_{(c,c)}[(c, d)] = 0$.

For part 2, write $\theta(h_{-i}) = \prod_{t=1}^{\ell(h_i)} \alpha_i(r_i^{t,s_{-i}^t}, r_{-i}^{t,s_i^t})[(y_i^t, y_{-i}^t)]$ for each $h_{-i} \in H_{-i}^D \cap H_{-i}^{\ell(h_i)}$ and analogously for $h_{-i} \in H_{-i}^D \cap H_{-i}^{\ell(h_i)+1}$. It then follows that

$$\begin{aligned}\sum_{h_{-i}\in H_{-i}^D\cap H_{-i}^{\ell(h_i)+1}} \theta(h_{-i}) &= \theta(h_{-i}^{*,\ell(h_i)})\alpha_i(1_{(c,c)}, 1_{(c,s_i)})[(c,d)] \\ &\quad + \sum_{y_{-i}} \sum_{h_{-i}\in H_{-i}^D\cap H_{-i}^{\ell(h_i)}} \theta(h_{-i})\alpha_i(1_{(d,c)}, 1_{(d,c)})[(c, y_{-i})] \qquad \text{(A.4)} \\ &= \theta(h_{-i}^{*,\ell(h_i)})\alpha_i(1_{(c,c)}, 1_{(c,s_i)})[(c,d)]\end{aligned}$$

since, for each $h_{-i} \in \left(H_{-i}^D \cap H_{-i}^{\ell(h_i)+1}\right) \setminus \{h^{*,\ell(h_i)} \cdot (r_{-i}^*, C, d)\}$, $r_{-i}^{\ell(h_i)+1} = 1_{(c,d)}$, $s_{-i}^{\ell(h_i)+1} = D$, $r_i^{*,D} = 1_{(d,c)}$ and $\alpha_i(1_{(d,c)}, 1_{(d,c)})[(c, y_{-i})] = 0$ for each $y_{-i} \in \{c, d\}$ by property 1. Hence,

$$\begin{aligned}\mu(h_{-i}^*|h_i \cdot (r_i^*, s_i, c)) &= \frac{\theta(h_{-i}^*)\alpha_i(1_{(c,c)}, 1_{(c,s_i)})[(c,c)]}{\theta(h_{-i}^*)\alpha_i(1_{(c,c)}, 1_{(c,s_i)})[(c,c)] + \theta(h_{-i}^*)\alpha_i(1_{(c,c)}, 1_{(c,s_i)})[(c,d)]} \\ &= \frac{\alpha_i(1_{(c,c)}, 1_{(c,s_i)})[(c,c)]}{\alpha_i(1_{(c,c)}, 1_{(c,s_i)})[(c,c)] + \alpha_i(1_{(c,c)}, 1_{(c,s_i)})[(c,d)]}.\end{aligned}$$

For part 3, write $\theta(H_{-i}^D \cap H_{-i}^{\ell(h_i)}) = \sum_{h_{-i}\in H_{-i}^D\cap H_{-i}^{\ell(h_i)}} \theta(h_{-i})$. Then (A.4) becomes

$$\begin{aligned}\textstyle\sum_{h_{-i}\in H_{-i}^D\cap H_{-i}^{\ell(h_i)+1}} \theta(h_{-i}) &= \theta(h_{-i}^{*,\ell(h_i)})\alpha_i(r_i^C, 1_{(c,s_i)})[(y_i, d)] \\ &\quad + \theta(H_{-i}^D \cap H_{-i}^{\ell(h_i)})\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i]\end{aligned}$$

and

$$\mu = \frac{\theta(h_{-i}^{*,\ell(h_i)})}{\theta(h_{-i}^{*,\ell(h_i)}) + \theta(H_{-i}^D \cap H_{-i}^{\ell(h_i)})}.$$

Hence,

$$\begin{aligned}\mu(h_{-i}^*|h_i \cdot (r_i, s_i, y_i)) &= \frac{\theta(h_{-i}^{*,\ell(h_i)})\alpha_i(r_i^C, 1_{(c,s_i)})[(y_i, c)]}{\theta(h_{-i}^{*,\ell(h_i)})\alpha_{i,Y_i}(r_i^C, 1_{(c,s_i)})[y_i] + \theta(H_{-i}^D \cap H_{-i}^{\ell(h_i)})\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i]} \\ &= \frac{\mu\alpha_i(r_i^C, 1_{(c,s_i)})[(y_i, c)]}{\mu\alpha_{i,Y_i}(r_i^C, 1_{(c,s_i)})[y_i] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i]}.\end{aligned}$$

■

For each $h \in H$, let $\pi(h)$ be the outcome path following history h: $\pi(h) = (\pi^1(h), \pi^2(h), \ldots)$ with $\pi^t(h) = (\pi_1^t(h), \pi_2^t(h))$ for each $t \in \mathbb{N}$. The outcome $\pi(h)$ is a (measurable) function from Y^∞ to $(R^2 \times S^2)^\infty$ such that $\pi^t(h) : Y^{t-1} \to R^2 \times S^2$ for each $t \in \mathbb{N}$. The space Y^∞ is endowed with the probability measure $\gamma(h)$ defined from the strategy. Specifically, set $\pi^1(h) = \sigma(h)$ and $\gamma^1(h) = \gamma(\cdot|\pi^1(h))$; assuming that $\pi^1(h), \ldots, \pi^{t-1}(h)$ and $\gamma^1(h), \ldots, \gamma^t(h)$ have been defined, set, for each $(y^1, \ldots, y^t) \in Y^t$,

$$\begin{aligned}
&h_i^t = ((\pi_i^1(h), y_i^1), \cdots, (\pi_i^t(h), y_i^t)),\\
&h_{-i}^t = ((\pi_{-i}^1(h), y_{-i}^1), \cdots, (\pi_{-i}^t(h), y_{-i}^t)),\\
&\pi^{t+1}(h)(y^1, \ldots, y^t) = \sigma(h \cdot h^t) \text{ and}\\
&\gamma^{t+1}(h)(y^1, \ldots, y^t) = \gamma(\cdot|\pi^{t+1}(h)).
\end{aligned}$$

Then set

$$\gamma(h)(\{(y^1, \ldots, y^t)\} \times Y^\infty) = \prod_{k=1}^{t} \gamma^k(h)(y^1, \ldots, y^{k-1})[y^k]$$

for each $t \geq 1$ and $(y^1, \ldots, y^t) \in Y^t$. In the following claims, we do not distinguish between "for all $y^\infty \in Y^\infty$" and "for $\gamma(h)$-a.e. $y^\infty \in Y^\infty$".

Claim A.3 *For each $i \in \{1, 2\}$:*

1. *If $h_i \in H_i^{*0}$, then $\pi_i^t(h_i, h_{-i}) = (1_{(c,d)}, D)$ for each $t \in \mathbb{N}$ and $h_{-i} \in H_{-i}$.*
2. *If $h_i \in H_i \setminus H_i^{*0}$ and $h_{-i} \in H_{-i}^{*0}$, then $\pi^t(h_i, h_{-i}) = ((1_{(c,d)}, 1_{(c,d)}), (D, D))$ for each $t > 1$ and*

$$\pi^1(h_i, h_{-i}) = \begin{cases} ((r_i^*, 1_{(c,d)}), (C, D)) & \text{if } \mu(h_{-i}^*|h_i) \geq \mu_i^*, \\ ((r_i^*, 1_{(c,d)}), (D, D)) & \text{if } \mu(h_{-i}^*|h_i) < \mu_i^*. \end{cases}$$

Proof. Let $h_i \in H_i^{*0}$; then $\alpha_i(r_i^{t,C}, 1_{(c,s_i^t)})[(y_i^t, c)] = 0$ for some $1 \leq t \leq \ell(h_i)$. Thus, for each $h_i' \in H_i$, $h_i \cdot h_i' \in H_i^{*0}$ and, hence, $\sigma_i(h_i \cdot h_i') = (1_{(c,d)}, D)$. This establishes part 1.

For part 2, let $h_i \in H_i \setminus H_i^{*0}$ and $h_{-i} \in H_{-i}^{*0}$. Then $\pi_{-i}^t(h) = (1_{(c,d)}, D)$ for each $t \in \mathbb{N}$ by part 1, $\sigma(h_i) = (r_i^*, C)$ if $\mu(h_{-i}^*|h_i) \geq \mu_i^*$ and $\sigma(h_i) = (r_i^*, D)$ if $\mu(h_{-i}^*|h_i) < \mu_i^*$. In either case, since $r_i^{*,D} = 1_{(d,c)}$, $\gamma(\cdot|\pi^1(h)) = \alpha_i(1_{(d,c)}, 1_{(d,c)}) = 1_{(d,c)}$ by property 1. Thus, $y^1 = (d, c)$. Since $\alpha_i(1_{(c,c)}, 1_{(c,s_i)})[(d, c)] = 0$ by property 1 for each $s_i \in S_i$, it follows that $h_i \cdot (\pi_i^1(h), y_i^1) = h_i \cdot (r_i^*, s_i, d) \in H_i^{*0}$ for each $s_i \in S_i$. The conclusion now follows by part 1. ■

Claim A.4 *For each $i \in \{1,2\}$: If $h_i \in H_i \backslash H_i^{*0}$ and $\mu(h^*_{-i}|h_i) \geq \mu^*_i$, then $\pi^t(h_i, h^*_{-i}) = (r^*, (C,C))$ for each $t \in \mathbb{N}$.*

Proof. Let $h_i \in H_i \setminus H_i^{*0}$ be such that $\mu(h^*_{-i}|h_i) \geq \mu^*_i$. Then $\sigma_i(h_i) = (r^*_i, C)$ and, since $\mu(h^*_i|h^*_{-i}) = 1$ by Claim A.2, $\sigma_{-i}(h^*_{-i}) = (r^*_{-i}, C)$; hence $\pi^1(h_i, h^*_{-i}) = (r^*, (C,C))$ and $y^1 = (c,c)$ since $\gamma^1(h_i, h^*_{-i}) = \alpha_i(1_{(c,c)}, 1_{(c,c)}) = 1_{(c,c)}$ by property 1. Thus, $\mu(h^*_{-i}|h_i \cdot (\pi^1_i(h_i, h^*_{-i}), y^1_i)) = 1$ by Claim A.2.

Assume that $\pi^1(h_i, h^*_{-i}) = \cdots = \pi^k(h_i, h^*_{-i}) = (r^*, (C,C))$, $y^1 = \cdots = y^k = (c,c)$ and $\mu(h^*_{-i}|h_i \cdot h^1_i) = \cdots = \mu(h^*_{-i}|h_i \cdot h^k_i) = 1$. Then $\sigma_i(h_i \cdot h^k_i) = (r^*_i, C)$. Since $h^k_{-i} = h^{*,k}$, $\mu(h^*_i|h^*_{-i} \cdot h^k_{-i}) = 1$ by Claim A.2 and, hence, $\sigma_{-i}(h^*_{-i}) = (r^*_{-i}, C)$. Thus $\pi^{k+1}(h_i, h^*_{-i}) = (r^*, (C,C))$ and $y^{k+1} = (c,c)$ since $\gamma^{k+1}(h_i, h^*_{-i}) = \alpha_i(1_{(c,c)}, 1_{(c,c)}) = 1_{(c,c)}$ by property 1. Thus, $\mu(h^*_{-i}|h_i \cdot h^{k+1}_i) = 1$ by Claim A.2.

The above inductive argument shows, in particular, that $\pi^t(h_i, h^*_{-i}) = (r^*, (C,C))$ and establishes the claim. ■

Let $U_i(h)$ be player i's expected payoff following history $h \in H$:

$$\begin{aligned} U_i(h) &= (1-\delta) \int_{Y^\infty} \sum_{t=1}^{\infty} \delta^{t-1} u_i(\pi^t(h)(y^\infty)) \mathrm{d}\gamma(h)(y^\infty) \\ &= (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \sum_{(y^1,\ldots,y^{t-1})} u_i(\pi^t(h)(y^1,\ldots,y^{t-1}))\gamma(h)(y^1,\ldots,y^{t-1}), \end{aligned}$$

where $\gamma(h)(y^1,\ldots,y^{t-1}) = \gamma(h)(\{y^1,\ldots,y^{t-1}\} \times Y^\infty)$. Let $U_i(h_i)$ be player i's expected payoff following history $h_i \in H_i$:

$$U_i(h_i) = \sum_{h_{-i} \in H_{-i}} \mu(h_{-i}|h_i) U_i(h_i, h_{-i}).$$

Claim A.5 *For each $\delta \geq \delta_1$, $i \in \{1,2\}$ and $h_i \in H_i$,*

$$U_i(h_i) = \begin{cases} 2\mu - (1-\mu)(1-\delta) & \text{if } \mu(h^*_{-i}|h_i) \geq \mu^*_i, \\ \mu\big(3(1-\delta) + \delta(2\hat{\mu}_i - (1-\hat{\mu}_i)(1-\delta))\big) & \text{if } \mu(h^*_{-i}|h_i) < \mu^*_i, \end{cases}$$

where $\mu = \mu(h^*_{-i}|h_i)$.

Proof. Let $h_i \in H_i$. Consider first the case where $h_i \in H_i \setminus H_i^{*0}$ and recall that $\mathrm{supp}(\mu(\cdot|h_i)) \subseteq \{h^*_{-i}\} \cup H^D_{-i}$. Suppose, in addition, that $\mu \geq \mu^*_i$. If $h_{-i} = h^*_{-i}$, then

Claim A.4 implies that $\pi^t(h_i, h^*_{-i}) = (r^*, (C, C))$ for each $t \in \mathbb{N}$; if $h_{-i} \in H^D_{-i} \subseteq H^B_{-i} \subseteq H^{*0}_{-i}$, Claim A.3 implies that $\pi^t(h_i, h_{-i}) = ((1_{(c,d)}, 1_{(c,d)}), (D, D))$ for each $t > 1$ and $\pi^1(h_i, h_{-i}) = ((r^*_i, 1_{(c,d)}), (C, D))$. Thus,

$$U_i(h_i) = 2\mu - (1 - \delta)(1 - \mu).$$

Suppose next that $\mu < \mu^*_i$. Then $\sigma_i(h_i) = (r^*_i, D)$. If $h_{-i} \in H^D_{-i} \subseteq H^{*0}_{-i}$, Claim A.3 implies that $\pi^t(h_i, h_{-i}) = ((1_{(c,d)}, 1_{(c,d)}), (D, D))$ for each $t > 1$ and $\pi^1(h_i, h_{-i}) = ((r^*_i, 1_{(c,d)}), (D, D))$.

If $h_{-i} = h^*_{-i}$, then $\pi^1(h_i, h^*_{-i}) = (r^*, (D, C))$ and $\gamma^1(h_i, h^*_{-i}) = \alpha_i(1_{(c,c)}, 1_{(c,d)})$, which is supported on $\{(c, c), (c, d)\}$ by properties 1 and 2.

If $y^1 = (c, c)$, then the resulting histories for player i and $-i$ are, respectively, $h_i \cdot (r^*_i, D, c)$ and $h^*_{-i} \cdot (r^*_{-i}, C, c) = h^{*,\ell(h_i)+1}_{-i}$. The history $h_i \cdot (r^*_i, D, c)$ belongs to $H_i \backslash H^{*0}_i$ since $\alpha_i(1_{(c,c)}, 1_{(c,d)})[(c, c)] > 0$ and is such that $\mu(h^*_{-i}|h_i \cdot (r^*_i, D, c)) = \hat{\mu}_i > \mu^*_i$ by Claim A.2. Thus, $\pi^t(h_i \cdot (r^*_i, D, c), h^{*,\ell(h_i)+1}_{-i}) = (r^*, (C, C))$ for each $t \geq 1$ by Claim A.4.

If $y^1 = (c, d)$, then the resulting history for player i is $h_i \cdot (r^*_i, D, c)$ as before and that of player $-i$ is $h^*_{-i} \cdot (r^*_{-i}, C, d) \in H^B_{-i} \subseteq H^{*0}_{-i}$. Since $h_i \cdot (r^*_i, D, c) \in H_i \setminus H^{*0}_i$ and $\mu(h^*_{-i}|h_i \cdot (r^*_i, D, c)) = \hat{\mu}_i > \mu^*_i$, Claim A.3 implies that

$$\pi^t(h_i \cdot (r^*_i, D, c), h_{-i} \cdot (r^*_{-i}, C, d)) = ((1_{(c,d)}, 1_{(c,d)}), (D, D)) \text{ for each } t > 1 \text{ and}$$
$$\pi^1(h_i \cdot (r^*_i, D, c), h_{-i} \cdot (r^*_{-i}, C, d)) = ((1_{(d,c)}, 1_{(c,d)}), (C, D)).$$

Thus, recalling that $\hat{\mu}_i = \alpha_i(1_{(c,c)}, 1_{(c,d)})[(c, c)]$ and $\alpha_i(1_{(c,c)}, 1_{(c,d)})[(c, d)] = 1 - \hat{\mu}_i$,

$$U_i(h_i) = \mu\big(3(1 - \delta) + \delta(2\hat{\mu}_i - (1 - \delta)(1 - \hat{\mu}_i))\big).$$

Finally, consider $h_i \in H^{*0}_i$. Since $\text{supp}(\mu(\cdot|h_i)) \subseteq H^{*0}_{-i}$, it follows that $\mu = 0$ and $\pi^t(h_i, h_{-i}) = ((1_{(c,d)}, 1_{(c,d)}), (D, D))$ for each $t \in \mathbb{N}$ by Claim A.3. Thus, $U_i(h_i) = 0 = \mu\big(3(1 - \delta) + \delta(2\hat{\mu}_i - (1 - \hat{\mu}_i)(1 - \delta))\big)$. ■

Define $V_i : [0, 1] \to \mathbb{R}$ by setting, for each $\mu \in [0, 1]$,

$$V_i(\mu) = \begin{cases} 2\mu - (1 - \mu)(1 - \delta) & \text{if } \mu \geq \mu^*_i, \\ \mu(3(1 - \delta) + \delta V_i(\hat{\mu}_i)) = \mu\big(3(1 - \delta) + \delta(2\hat{\mu}_i - (1 - \hat{\mu}_i)(1 - \delta))\big) & \text{if } \mu < \mu^*_i. \end{cases}$$

It follows by Claim A.5 that $U_i(h_i) = V_i(\mu(h^*_{-i}|h_i))$ for each $\delta \geq \delta_1$, $i \in \{1,2\}$ and $h_i \in H_i$.

Claim A.6 *The function V_i is strictly increasing, piecewise linear, continuous, and convex for each $i \in \{1,2\}$.*

Proof. It is clear that V_i is strictly increasing, affine on $[0,\mu_i^*)$ and on $[\mu_i^*, 1]$. The continuity of V_i follows because $V_i(\mu_i^*) = 2\mu_i^* - (1-\delta)(1-\mu_i^*) = \mu_i^*\big(3(1-\delta) + \delta(2\hat{\mu}_i - (1-\hat{\mu}_i)(1-\delta))\big) = \lim_{\mu\to\mu^*-} V_i(\mu)$ due to the definition of μ_i^*.

The slope of V_i in the range $[0,\mu_i^*)$ is $3(1-\delta) + \delta(2\hat{\mu}_i - (1-\hat{\mu}_i)(1-\delta)) = 2 + 1 - \delta - \frac{1-\delta}{\mu_i^*} < 2 + 1 - \delta$, the latter being the slope of V_i in the range $[\mu_i^*, 1]$. Thus, V_i is convex. ■

For each $i \in \{1,2\}$, $h_i \in H_i$ and $(r_i, s_i) \in R_i \times S_i$, let $U_i^{r_i,s_i}(h_i)$ be player i's expected payoff of an one-shot deviation from σ_i to (r_i, s_i); formally, $U_i^{r_i,s_i}(h_i)$ is defined in the same way as $U_i(h_i)$ by changing only $\pi^1(h)$ to $((r_i,s_i), \sigma_{-i}(h_{-i}))$ for each $h_{-i} \in H_{-i}$.

Claim A.7 *For each $i \in \{1,2\}$, $h_i \in H_i$ and $(r_i, s_i) \in R_i \times S_i$,*

$$\begin{aligned}
U_i^{r_i,s_i}(h_i) \;&= (1-\delta)\big(\mu u_i(s_i, C) + (1-\mu)u_i(s_i, D)\big) + \\
&\delta\mu \sum_{y_i} \alpha_{i,Y_i}(r_i^C, 1_{(c,s_i)})[y_i] V_i(\mu(h^*_{-i}|h_i\cdot(r_i,s_i,y_i))) + \\
&\delta(1-\mu) \sum_{y_i} \alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i] V_i(\mu(h^*_{-i}|h_i\cdot(r_i,s_i,y_i))),
\end{aligned}$$

where $\mu = \mu(h^*_{-i}|h_i)$.

Proof. If $h_{-i} = h^*_{-i}$, then $\sigma_{-i}(h^*_{-i}) = (r^*_{-i}, C)$ and player $-i$'s next period history is $h^*_{-i}\cdot(r^*_{-i}, C, y^1_{-i})$, hence equal to $h_{-i}^{*,\ell(h_i)+1}$ if $y^1_{-i} = c$ and an element of $H^B_{-i} \subseteq H^{*0}_{-i}$ if $y^1_{-i} = d$. If $h_{-i} \in H^{*0}_{-i}$, then $\sigma_{-i}(h^*_{-i}) = (1_{(c,d)}, D)$ and player $-i$'s next period history is an element of H^{*0}_{-i}.

It follows by Claim A.3 that $U_i(h_i\cdot(r_i,s_i,y_i), \hat{h}_{-i}) = U_i(h_i\cdot(r_i,s_i,y_i), \bar{h}_{-i})$ for each $\hat{h}_{-i}, \bar{h}_{-i} \in H^{*0}_{-i} \cap H^{\ell(h_i)+1}_{-i}$. Let then $U_i(h_i\cdot(r_i,s_i,y_i), H^{*0}_{-i})$ denotes this common value.

Since $\text{supp}(\mu(\cdot|h_i)) \subseteq \{h^*_{-i}\} \cup H^{*0}_{-i}$, it follows that $\mu(H^{*0}_{-i}|h_i) = 1 - \mu$ and

$$\begin{aligned} U_i^{r_i,s_i}(h_i) \;&= (1-\delta)\big(\mu u_i(s_i, C) + (1-\mu)u_i(s_i, D)\big) + \\ &\delta \sum_{y_i} \Big(\mu\alpha_i(r_i^C, 1_{(c,s_i)})[(y_i, c)]U_i(h_i \cdot (r_i, s_i, y_i), h^{*,\ell(h_i)+1}_{-i}) + \\ &\big(\mu\alpha_i(r_i^C, 1_{(c,s_i)})[(y_i, d)] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i]\big)U_i(h_i \cdot (r_i, s_i, y_i), H^{*0}_{-i})\Big). \end{aligned}$$

Furthermore, it follows by Claim A.2 that

$$\begin{aligned} &\sum_{y_i} \Big(\mu\alpha_i(r_i^C, 1_{(c,s_i)})[(y_i, c)]U_i(h_i \cdot (r_i, s_i, y_i), h^{*,\ell(h_i)+1}_{-i}) + \\ &\big(\mu\alpha_i(r_i^C, 1_{(c,s_i)})[(y_i, d)] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i]\big)U_i(h_i \cdot (r_i, s_i, y_i), H^{*0}_{-i})\Big) = \\ &\sum_{y_i} \Big(\mu\alpha_{i,Y_i}(r_i^C, 1_{(c,s_i)})[y_i] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i]\Big) \times \\ &\Big(\mu(h^{*,\ell(h_i)+1}_{-i}|h_i \cdot (r_i, s_i, y_i))U_i(h_i \cdot (r_i, s_i, y_i), h^{*,\ell(h_i)+1}_{-i}) + \\ &(1 - \mu(h^{*,\ell(h_i)+1}_{-i}|h_i \cdot (r_i, s_i, y_i))U_i(h_i \cdot (r_i, s_i, y_i), H^{*0}_{-i})\Big) = \\ &\sum_{y_i} \Big(\mu\alpha_{i,Y_i}(r_i^C, 1_{(c,s_i)})[y_i] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i]\Big)U_i(h_i \cdot (r_i, s_i, y_i)) = \\ &\sum_{y_i} \Big(\mu\alpha_{i,Y_i}(r_i^C, 1_{(c,s_i)})[y_i] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i]\Big)V_i(\mu(h^*_{-i}|h_i \cdot (r_i, s_i, y_i))) = \\ &\mu \sum_{y_i} \alpha_{i,Y_i}(r_i^C, 1_{(c,s_i)})[y_i]V_i(\mu(h^*_{-i}|h_i \cdot (r_i, s_i, y_i))) + \\ &(1-\mu) \sum_{y_i} \alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i]V_i(\mu(h^*_{-i}|h_i \cdot (r_i, s_i, y_i))). \end{aligned}$$

This completes the proof of the claim. ■

Claim A.8 *Let $i \in \{1, 2\}$, $h_i \in H_i \setminus H^{*0}_i$, $\mu = \mu(h^*_{-i}|h_i)$ and $(r_i, s_i) \in R_i \times S_i$. Then:*

$$\begin{aligned} &\sum_{y_i}(\mu\alpha_{i,Y_i}(r_i^C, 1_{(c,s_i)})[y_i] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i])V_i(\mu(h^*_{-i}|h_i \cdot (r_i, s_i, y_i))) \leq \\ &\begin{cases} \mu V_i(1) + (1-\mu)V_i(0) = 2\mu & \textit{if } s_i = C, \\ \mu\hat{\mu}_i V_i(1) + (1-\mu\hat{\mu}_i)V_i(0) = 2\mu\hat{\mu}_i & \textit{if } s_i = D. \end{cases} \end{aligned}$$

Proof. Let $i \in \{1, 2\}$, $h_i \in H_i \setminus H^{*0}_i$ and $(r_i, s_i) \in R_i \times S_i$ be given. First, we argue that when $s_i = C$:

$$\sum_{y_i}(\mu\alpha_{i,Y_i}(r_i^C, 1_{(c,c)})[y_i] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i])\mu(h^*_{-i}|h_i \cdot (r_i, C, y_i)) \leq \mu.$$

Indeed, for each y_i, this is trivial if $\alpha_i(r_i^C, 1_{(c,s_i)})[(y_i, c)] = 0$ and follows by Claim A.2 otherwise.

The value of the problem:

$$\sup_{r_i} \sum_{y_i} (\mu\alpha_{i,Y_i}(r_i^C, 1_{(c,c)})[y_i] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i]) V_i(\mu(h_{-i}^* | h_i \cdot (r_i, C, y_i))) \tag{A.5}$$

is at most the value of the problem:

$$\sup_{(p_0,p_1,\mu_0,\mu_1)\in[0,1]^4} \sum_{j=0}^{1} p_j V_i(\mu_j) \text{ subject to } \sum_{j=0}^{1} p_j\mu_j \leq \mu \text{ and } \sum_{j=0}^{1} p_j = 1. \tag{A.6}$$

This is because for any $r_i \in (\Delta Y)^2$:

1. $p_0 = \mu\alpha_{i,Y_i}(r_i^C, 1_{(c,c)})[d] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[d] \in [0,1]$,

2. $p_1 = \mu\alpha_{i,Y_i}(r_i^C, 1_{(c,c)})[c] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[c] \in [0,1]$,

3. $\mu_0 = \mu(h_{-i}^* | h_i \cdot (r_i, C, d)) \in [0,1]$,

4. $\mu_1 = \mu(h_{-i}^* | h_i \cdot (r_i, C, c)) \in [0,1]$,

5. $\sum_{j=0}^{1} p_j\mu_j = \sum_{y_i} (\mu\alpha_{i,Y_i}(r_i^C, 1_{(c,c)})[y_i] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i]) \mu(h_{-i}^* | h_i \cdot (r_i, C, y_i)) \leq \mu$, and

6. $\sum_{j=0}^{1} p_j = \sum_{y_i} (\mu\alpha_{i,Y_i}(r_i^C, 1_{(c,c)})[y_i] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i]) = 1$.

Thus, p_0, p_1, μ_0, and μ_1 satisfy the constraints of (A.6) and

$$\sum_{j=0}^{1} p_j V_i(\mu_j) = \sum_{y_i} (\mu\alpha_{i,Y_i}(r_i^C, 1_{(c,c)})[y_i] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i]) V_i(\mu(h_{-i}^* | h_i \cdot (r_i, C, y_i))).$$

Any solution to (A.6) must satisfy the constraint with equality since V_i is strictly increasing. Then, for any $(p_j,\, \mu_j)_{j=0,1}$ such that $\sum_{j=0}^{1} p_j\mu_j = \mu$, the convexity of V_i

implies that:

$$\begin{aligned}\sum_{j=0}^{1} p_j V_i(\mu_j) &= \sum_{j=0}^{1} p_j V_i(\mu_j(1) + (1-\mu_j)(0)) \\ &\leq \sum_{j=0}^{1} p_j(\mu_j V_i(1) + (1-\mu_j)V_i(0)) \\ &\leq \sum_{j=0}^{1} p_j \mu_j V_i(1) \\ &= \mu V_i(1).\end{aligned}$$

Indeed, the first line is because $\mu_j = \mu_j(1) + (1-\mu_j)(0)$, the second line is by Jensen's inequality, the third line is because $V_i(0) = 0$, and the last line is by the constraint $\sum_{j=0}^{1} p_j \mu_j = \mu$. Thus, the value of (A.6) is at most $\mu V_i(1) = 2\mu$.

Now we argue that when $s_i = D$:

$$\sum_{y_i} (\mu \alpha_{i,Y_i}(r_i^C, 1_{(c,d)})[y_i] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i])\mu(h_{-i}^* | h_i \cdot (r_i, D, y_i)) \leq \mu \hat{\mu}_i.$$

Note that, by Claim A.2:

$$\begin{aligned}&\sum_{y_i} (\mu \alpha_{i,Y_i}(r_i^C, 1_{(c,d)})[y_i] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i])\mu(h_{-i}^* | h_i \cdot (r_i, D, y_i)) \\ &\leq \mu(\alpha_i(r_i^C, 1_{(c,d)})[c,c] + \alpha_i(r_i^C, 1_{(c,d)})[d,c]) \\ &\leq \mu(\alpha_i(1_{(c,c)}, 1_{(c,d)})[c,c] + \alpha_i(1_{(c,c)}, 1_{(c,d)})[d,c]) = \mu \hat{\mu}_i,\end{aligned}$$

where the second inequality is by property 3 and the last equality follows because $\alpha_i(1_{(c,c)}, 1_{(c,d)})[d,c]) = 0$ by property 1.

Thus, the value of the problem:

$$\sup_{r_i} \sum_{y_i} (\mu \alpha_{i,Y_i}(r_i^C, 1_{(c,d)})[y_i] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i]) V_i(\mu(h_{-i}^* | h_i \cdot (r_i, D, y_i)))$$

is at most the value of the problem:

$$\sup_{(p_0, p_1, \mu_0, \mu_1) \in [0,1]^4} \sum_{j=0}^{1} p_j V_i(\mu_j) \text{ subject to } \sum_{j=0}^{1} p_j \mu_i \leq \mu \hat{\mu}_i \text{ and } \sum_{j=0}^{1} p_j = 1. \tag{A.7}$$

The solution to (A.7) is $\mu_1 = 1$, $\mu_0 = 0$, $p_1 = \mu \hat{\mu}_i$, and $p_0 = 1 - \mu \hat{\mu}_i$ (by the same argument as in the previous case), and the value is $\mu \hat{\mu}_i V(1) = 2\mu \hat{\mu}_i$. ■

Claim A.9 *For each* $i \in \{1,2\}$*,* $r_i \in R_i$*,* $y_i \in Y_i$ *and* $h_i \in H_i \setminus H_i^{*0}$ *such that* $\mu(h^*_{-i}|h_i) < \underline{\mu}$*,*

$$\mu(h^*_{-i}|h_i \cdot (r_i, D, y_i)) \leq \hat{\mu}_i.$$

Proof. Let $\underline{\mu} > 0$ be such that

$$\frac{\mu}{(1-\mu)\min_{i,r}\alpha_{i,Y_i}(r, 1_{(d,c)})[d]} < \min_i \hat{\mu}_i$$

for each $\mu < \underline{\mu}$. Such $\underline{\mu}$ exists since $\min_r \alpha_{i,Y_i}(r, 1_{(d,c)})[d] > 0$ by property 2 and, hence, $\lim_{\mu\to 0} \frac{\mu}{(1-\mu)\min_{i,r}\alpha_{i,Y_i}(r,1_{(d,c)})[d]} = 0$.

Let $i \in \{1,2\}$, $r_i \in R_i$, $y_i \in Y_i$ and $h_i \in H_i \setminus H_i^{*0}$ be such that $\mu(h^*_{-i}|h_i) < \underline{\mu}$. When $y_i = c$, we have that $\mu(h^*_{-i}|h_i \cdot (r_i, D, c)) \leq \hat{\mu}_i$ since, by Claim A.2:

$$\mu(h^*_{-i}|(h_i,(r_i,D,c))) \leq \frac{\mu\alpha_i(r_i^C, 1_{(c,d)})[c,c]}{\mu\alpha_{i,Y_i}(r_i^C, 1_{(c,d)})[c] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[c]}.$$

Property 4 implies that $1_{(c,c)} \in \arg\max_r \frac{\mu\alpha_i(r,1_{(c,d)})[c,c]}{\mu\alpha_{i,Y_i}(r,1_{(c,d)})[c]}$; since $\alpha_{i,Y_i}(1_{(d,c)}, 1_{(d,c)})[c] = 0$, this implies that $r_i^* \in \arg\max_{r_i} \mu(h^*_{-i}|(h_i,(r_i,D,c)))$, and $\mu(h^*_{-i}|h_i \cdot (r_i^*, D, c)) = \hat{\mu}_i$.

Consider next $y_i = d$. We also have that $\mu(h^*_{-i}|h_i \cdot (r_i, D, d)) \leq \hat{\mu}_i$ since, by Claim A.2:

$$\begin{aligned}\mu(h^*_{-i}|(h_i \cdot (r_i,D,d)) &\leq \frac{\mu\alpha_i(r_i^C, 1_{(c,d)})[d,c]}{\mu\alpha_{i,Y_i}(r_i^C, 1_{(c,d)})[d] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[d]} \\ &\leq \frac{\mu}{(1-\mu)\min_{i,r}\alpha_{i,Y_i}(r, 1_{(d,c)})[d]} < \hat{\mu}_i.\end{aligned}$$

■

Claim A.10 *Let* $i \in \{1,2\}$*,* $h_i \in H_i \setminus H_i^{*0}$*,* $\mu = \mu(h^*_{-i}|h_i)$ *and* $r_i \in R_i$*. If* $\mu < \underline{\mu}$*, then*

$$\sum_{y_i}(\mu\alpha_{i,Y_i}(r_i^C, 1_{(c,d)})[y_i] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i])V_i(\mu(h^*_{-i}|h_i \cdot (r_i, D, y_i)))$$
$$\leq \mu V_i(\hat{\mu}_i).$$

Proof. Let $i \in \{1,2\}$, $h_i \in H_i \setminus H_i^{*0}$, $r_i \in R_i$ and $\mu = \mu(h^*_{-i}|h_i) < \underline{\mu}$. Then for each $y_i \in Y_i$, $\mu(h^*_{-i}|h_i \cdot (r_i, D, y_i)) \leq \hat{\mu}_i$ by Claim A.9.

As in the proof of Claim A.8:

$$\sum_{y_i}(\mu\alpha_{i,Y_i}(r_i^C, 1_{(c,d)})[y_i] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i])\mu(h^*_{-i}|h_i \cdot (r_i, D, y_i)) \leq \mu\hat{\mu}_i.$$

Thus, the value of the problem:

$$\sup_{r_i} \sum_{y_i} (\mu \alpha_{i,Y_i}(r_i^C, 1_{(c,d)})[y_i] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i]) V_i(\mu(h_{-i}^* | h_i \cdot (r_i, D, y_i)))$$

is at most the value of the problem:

$$\sup_{(p_0,p_1)\in[0,1]^2, (\mu_0,\mu_1)\in[0,\hat{\mu}_i]^2} \sum_{j=0}^{1} p_j V_i(\mu_j) \text{ subject to } \sum_{j=0}^{1} p_j \mu_j \leq \mu \hat{\mu}_i \text{ and } \sum_{j=0}^{1} p_j = 1. \quad \text{(A.8)}$$

This is because for any $r_i \in (\Delta Y)^2$:

1. $p_0 = \mu \alpha_{i,Y_i}(r_i^C, 1_{(c,d)})[d] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[d] \in [0,1]$,
2. $p_1 = \mu \alpha_{i,Y_i}(r_i^C, 1_{(c,d)})[c] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[c] \in [0,1]$,
3. $\mu_0 = \mu(h_{-i}^* | h_i \cdot (r_i, D, d)) \in [0, \hat{\mu}_i]$,
4. $\mu_1 = \mu(h_{-i}^* | h_i \cdot (r_i, D, c)) \in [0, \hat{\mu}_i]$,
5. $\sum_{j=0}^{1} p_j \mu_j = \sum_{y_i} (\mu \alpha_{i,Y_i}(r_i^C, 1_{(c,d)})[y_i] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i]) \mu(h_{-i}^* | h_i \cdot (r_i, D, y_i)) \leq \mu \hat{\mu}_i$, and
6. $\sum_{j=0}^{1} p_j = \sum_{y_i} (\mu \alpha_{i,Y_i}(r_i^C, 1_{(c,d)})[y_i] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i]) = 1$.

Thus, p_0, p_1, μ_0, and μ_1 satisfy the constraints of (A.8) and

$$\sum_{j=0}^{1} p_j V_i(\mu_j) = \sum_{y_i} (\mu \alpha_{i,Y_i}(r_i^C, 1_{(c,d)})[y_i] + (1-\mu)\alpha_{i,Y_i}(r_i^D, 1_{(d,c)})[y_i]) V_i(\mu(h_{-i}^* | h_i \cdot (r_i, D, y_i))).$$

Any solution to (A.8) must satisfy the constraint with equality since V_i is strictly increasing. Then, for any p_j and $\mu_j \leq \hat{\mu}_i$ such that $\sum_{j=0}^{1} p_j \mu_j = \mu \hat{\mu}_i$, the convexity of V_i implies that:

$$\begin{aligned}
\sum_{j=0}^{1} p_j V_i(\mu_j) &= \sum_{j=0}^{1} p_j V_i\left(\frac{\mu_j}{\hat{\mu}_i}(\hat{\mu}_i) + \left(1 - \frac{\mu_j}{\hat{\mu}_i}\right)(0)\right) \\
&\leq \sum_{j=0}^{1} p_j \left(\frac{\mu_j}{\hat{\mu}_i} V_i(\hat{\mu}_i) + \left(1 - \frac{\mu_j}{\hat{\mu}_i}\right) V_i(0)\right) \\
&\leq \sum_{j=0}^{1} p_j \frac{\mu_j}{\hat{\mu}_i} V_i(\hat{\mu}_i) \\
&= \mu V_i(\hat{\mu}_i).
\end{aligned}$$

Indeed, the first line is because $\mu_j = \frac{\mu_j}{\hat{\mu}_i}(\hat{\mu}_i) + (1 - \frac{\mu_j}{\hat{\mu}_i})(0)$, the second line is by Jensen's inequality, the third line is because $V_i(0) = 0$, and the last line is by the constraint $\sum_{j=0}^{1} p_j \mu_j = \mu\hat{\mu}_i$. Thus, the value of (A.8) is at most $\mu V_i(\hat{\mu}_i)$. ■

A.1.4 Sequential rationality

Let $i \in \{1, 2\}$. We consider three cases: (i) $h_i \in H_i \setminus H_i^{*0}$ and $\mu(h_{-i}^*|h_i) \geq \mu_i^*$, (ii) $h_i \in H_i \setminus H_i^{*0}$ and $\mu(h_{-i}^*|h_i) < \mu_i^*$, and (iii) $h_i \in H_i^{*0}$.

Case $h_i \in H_i \setminus H_i^{*0}$ **and** $\mu(h_{-i}^*|h_i) \geq \mu_i^*$**:**

Let $\mu = \mu(h_{-i}^*|h_i)$. Then $U_i(h_i) = V_i(\mu) = 2\mu - (1 - \mu)(1 - \delta)$. Consider a one shot deviation to (r_i, s_i). If $s_i = C$, then by Claims A.7 and A.8:

$$U_i^{r_i,C}(h_i) \leq 2\mu(1 - \delta) - (1 - \mu)(1 - \delta) + \delta 2\mu = 2\mu - (1 - \mu)(1 - \delta) = U_i(h_i).$$

If $s_i = D$ and $\mu \geq \underline{\mu}$ (note that $\mu_i^* < \underline{\mu}$ by (A.2)), then by Claims A.7 and A.8:

$$U_i^{r_i,D}(h_i) \leq 3\mu(1 - \delta) + \delta 2\mu\hat{\mu}_i = \mu\left(3(1 - \delta) + 2\hat{\mu}_i\delta\right).$$

Hence,

$$U_i(h_i) - U_i^{r_i,D}(h_i) = -1 + \delta(1 + 2\mu(1 - \hat{\mu}_i)) \geq -1 + \delta(1 + 2\underline{\mu}(1 - \hat{\mu}_i)) > 0$$

by (A.3) since $\delta \geq \delta^*$.

If $s_i = D$ and $\mu < \underline{\mu}$, then by Claims A.7 and A.10:

$$U_i^{r_i,D}(h_i) \leq \mu\big(3(1 - \delta) + \delta V_i(\hat{\mu}_i)\big) \leq 2\mu - (1 - \mu)(1 - \delta) = U_i(h_i)$$

where the second inequality follows because, when $\mu \geq \mu_i^*$, $\mu\left(3(1 - \delta) + \delta V_i(\hat{\mu}_i)\right) \leq 2\mu - (1 - \mu)(1 - \delta)$.

Case $h_i \in H_i \setminus H_i^{*0}$ **and** $\mu(h_{-i}^*|h_i) < \mu_i^*$**:**

Let $\mu = \mu(h_{-i}^*|h_i)$. Then $U_i(h_i) = V_i(\mu) = \mu(3(1 - \delta) + \delta V_i(\hat{\mu}_i))$. Consider a one shot deviation to (r_i, s_i). If $s_i = D$, then by Claims A.7 and A.10 (which applies since $\mu < \mu_i^* < \underline{\mu}$):

$$U_i^{r_i,D}(h_i) \leq \mu(3(1 - \delta) + \delta V_i(\hat{\mu}_i)) = U_i(h_i).$$

If $s_i = C$, then by Claims A.7 and A.8:

$$U_i^{r_i,C}(h_i) \leq 2\mu - (1-\mu)(1-\delta) < \mu(3(1-\delta) + \delta V_i(\hat{\mu}_i)) = U_i(h_i),$$

where the strict inequality follows because, when $\mu < \mu_i^*$, $2\mu - (1-\mu)(1-\delta) < \mu(3(1-\delta) + \delta V_i(\hat{\mu}_i))$.

Case $h_i \in H_i^{*0}$:

We have that $U_i(h_i) = V_i(0) = 0$. Consider a one shot deviation to (r_i, s_i). Since $h_i \cdot (r_i, s_i, y_i) \in H_i^{*0}$ for each y_i, it follows by Claim A.7 that

$$U_i^{r_i,D}(h_i) = 0 = U_i(h_i)$$

and

$$U_i^{r_i,C}(h_i) = -(1-\delta) < 0 = U_i(h_i).$$

A.1.5 Consistency

We show that there exists consistent beliefs satisfying the specification in A.1.2.

Let $\{\sigma^j\}_{j=1}^{\infty}$ be a sequence of totally mixed strategies converging to σ and such that, for each $i \in \{1,2\}$,

1. $\sigma_i^j(1_{(c,d)}, D|h_i) = \frac{1}{j}$ for each $h_i \in H_i \setminus H_i^{*0}$,
2. $\sigma_i^j(r_i, s_i|h_i) = \frac{1}{j^j}$ for each $(r_i, s_i) \notin \{(1_{(c,d)}, D), \sigma(h_i)\}$ and $h_i \in H_i \setminus H_i^{*0}$,
3. $\sigma_i^j(1_{(d,c)}, D|h_i) = \frac{1}{j}$ for each $h_i \in H_i^{*0}$, and
4. $\sigma_i^j(r_i, s_i|h_i) = \frac{1}{j^j}$ for each $(r_i, s_i) \notin \{(1_{(d,c)}, D), \sigma(h_i)\}$ and $h_i \in H_i^{*0}$.

Let $i \in \{1,2\}$, $t \in \mathbb{N}$ and $h_i = (r_i^k, s_i^k, y_i^k)_{k=1}^t \in H_i^t$. Then, for each $h_{-i} = (r_{-i}^k, s_{-i}^k, y_{-i}^k)_{k=1}^t \in H_{-i}^t$ and $j \in \mathbb{N}$,

$$\mu^j(h_{-i}|h_i) = \frac{\prod_{k=1}^t \alpha_i(r_i^{k,s_{-i}^k}, r_{-i}^{k,s_i^k})[y^k]\sigma_{-i}^j(r_{-i}^k, s_{-i}^k|h_{-i}^{k-1})}{\sum_{(\hat{r}_{-i}^k, \hat{s}_{-i}^k, \hat{y}_{-i}^k)_{k=1}^t \in H_{-i}^t} \prod_{k=1}^t \alpha_i(r_i^{k,\hat{s}_{-i}^k}, \hat{r}_{-i}^{k,s_i^k})[(y_i^k, \hat{y}_{-i}^k)]\sigma_{-i}^j(\hat{r}_{-i}^k, \hat{s}_{-i}^k|\hat{h}_{-i}^{k-1})}$$

where $h_{-i}^k = (r_{-i}^n, s_{-i}^n, y_{-i}^n)_{n=1}^k$ and $\hat{h}_{-i}^k = (\hat{r}_{-i}^n, \hat{s}_{-i}^n, \hat{y}_{-i}^n)_{n=1}^k$ for each $k \geq 0$.

Note that the set of histories h_{-i} such that $\prod_{k=1}^{t} \sigma_{-i}(r_{-i}^k, s_{-i}^k | h_{-i}^{k-1}) > 0$ equals $\{h_{-i}^*\} \cup H_{-i}^D$. Define

$$H_i^{D0} = \Big\{ h_i \in H_i : \text{ for all } 1 \le n \le \ell(h_i) \text{ and } (y_{-i}^{n+1}, \dots, y_{-i}^{\ell(h_i)}) \in Y_{-i}^{\ell(h_i)-n},$$
$$\left(\prod_{k=1}^{n-1} \alpha_i(r_i^{k,C}, 1_{(c,s_i^k)})[(y_i^k, c)] \right) \alpha_i(r_i^{n,C}, 1_{(c,s_i^n)})[(y_i^n, d)] \left(\prod_{k=n+1}^{\ell(h_i)} \alpha_i(r_i^{k,D}, 1_{(d,c)})[y^k] \right) = 0 \Big\}.$$

Let $h_i \in H_i$. We consider three cases.

Case $h_i \in H_i \backslash H_i^{*0}$: In this case, it follows that $\mu(h_{-i}^* | h_i) > 0$ since $\sigma_{-i}^j(r_{-i}^*, C | h_{-i}^{*,k-1}) \to 1$ for each $1 \le k \le t$. In addition, if $h_{-i} \in H_{-i}^t \setminus \{h_{-i}^*\}$ is such that $\mu(h_{-i}|h_i) > 0$, then $\prod_{k=1}^{t} \sigma_{-i}(r_{-i}^k, s_{-i}^k | h_{-i}^{k-1}) > 0$. Thus, $h_{-i} \in H_{-i}^D$. In conclusion, $h_{-i}^* \in \operatorname{supp}(\mu(\cdot|h_i)) \subseteq \{h_{-i}^*\} \cup H_{-i}^D$ whenever $h_i \in H_i \setminus H_i^{*0}$.

For later use, we will show by induction on $\ell(h_i)$ that $\lim_j (j^{j-1-\ell(h_i)} \mu^j(h_{-i}|h_i)) = 0$ for each $h_{-i} \in H_{-i} \setminus (H_{-i}^{*0} \cup \{h_{-i}^*\})$. Consider first h_i with $\ell(h_i) = 1$. Then

$$\mu^j(h_{-i}|h_i) = \frac{\alpha_i(r_i^{s_{-i}}, r_{-i}^{s_i})[y] \sigma_{-i}^j(r_{-i}, s_{-i})}{\sum_{(\hat{r}_{-i}, \hat{s}_{-i}, \hat{y}_{-i}) \in H_{-i}^1} \alpha_i(r_i^{\hat{s}_{-i}}, \hat{r}_{-i}^{s_i})[(y_i, \hat{y}_{-i})] \sigma_{-i}^j(\hat{r}_{-i}, \hat{s}_{-i})}.$$

Since $h_i \in H_i \setminus H_i^{*0}$, we have that $\alpha_i(r_i^C, 1_{(c,s_i)})[(y_i, c)] > 0$. In addition,

$$\sum_{(\hat{r}_{-i}, \hat{s}_{-i}, \hat{y}_{-i}) \in H_{-i}^1} \alpha_i(r_i^{\hat{s}_{-i}}, \hat{r}_{-i}^{s_i})[(y_i, \hat{y}_{-i})] \sigma_{-i}^j(\hat{r}_{-i}, \hat{s}_{-i}) \to \alpha_{i,Y_i}(r_i^C, 1_{(c,s_i)})[y_i].$$

Thus, $(r_{-i}^*, C, c) \in \operatorname{supp}(\mu(\cdot|h_i)) \subseteq \{(r_{-i}^*, C, c), (r_{-i}^*, C, d)\}$ and note that $(r_{-i}^*, C, d) \in H_{-i}^{*0}$. We have that $\{(1_{(c,d)}, D, c), (1_{(c,d)}, D, d)\} \subseteq H_{-i}^{*0}$ since $\alpha_{-i}(1_{(c,d)}, 1_{(c,d)})[(y_{-i}, c)] = 0$ by property 1. Hence, for each $h_{-i} \in H_{-i} \setminus (H_{-i}^{*0} \cup \{h_{-i}^*\})$,

$$\lim_j (j^{j-2} \mu^j(h_{-i}|h_i)) = \frac{j^{j-2} \alpha_i(r_i^{s_{-i}}, r_{-i}^{s_i})[y] j^{-j}}{\sum_{(\hat{r}_{-i}, \hat{s}_{-i}, \hat{y}_{-i}) \in H_{-i}^1} \alpha_i(r_i^{\hat{s}_{-i}}, \hat{r}_{-i}^{s_i})[(y_i, \hat{y}_{-i})] \sigma_{-i}^j(\hat{r}_{-i}, \hat{s}_{-i})} \to 0.$$

Let $t > 1$ and assume that we have established that, for each $k = 1, \dots, t-1$ and $h_i \in H_i^k \setminus H_i^{*0}$, $\lim_j (j^{j-1-k} \mu^j(h_{-i}|h_i)) = 0$ for each $h_{-i} \in H_{-i}^k \setminus (H_{-i}^{*0} \cup \{h_{-i}^*\})$.

For each $h_i \in H_i^t$, $h_{-i} \in H_{-i}^t$ and $j \in \mathbb{N}$,

$$\mu^j(h_{-i}|h_i) = \frac{\mu^j(h_{-i}^{t-1}|h_i^{t-1}) \alpha_i(r_i^{t,s_{-i}^t}, r_{-i}^{t,s_i^t})[y^t] \sigma_{-i}^j(r_{-i}^t, s_{-i}^t | h_{-i}^{t-1})}{B_j}$$

where

$$B_j = \sum_{\hat{h}_{-i} \in H_{-i}^{t-1}} \sum_{(\hat{r}_{-i}^t, \hat{s}_{-i}^t, \hat{y}_{-i}^t) \in H_{-i}^1} \mu^j(\hat{h}_{-i}^{t-1} | h_i^{t-1}) \alpha_i(r_i^{t, \hat{s}_{-i}^t}, \hat{r}_{-i}^{t, s_i^t})[(y_i^t, \hat{y}_{-i}^t)] \sigma_{-i}^j(\hat{r}_{-i}^t, \hat{s}_{-i}^t | \hat{h}_{-i}^{t-1}).$$

Let $h_i \in H_i^t \backslash H_i^{*0}$ and $h_{-i} \in H_{-i}^t \backslash (H_{-i}^{*0} \cup \{h_{-i}^*\})$. We have that $\lim_j B_j > 0$ because $h_i \in H_i \setminus H_i^{*0}$. Hence, if $h_{-i}^{t-1} \neq h_{-i}^{*,t-1}$, then $h_{-i}^{t-1} \notin H_{-i}^{*0}$ and $\lim_j (j^{j-1-t} \mu^j(h_{-i}|h_i)) = 0$ since $\lim_j (j^{j-1-(t-1)} \mu^j(h_{-i}^{t-1}|h_i^{t-1})) = 0$.

If, instead, $h_{-i}^{t-1} = h_{-i}^*$, note that $h_{-i}^{*,t-1} \cdot (r_{-i}^*, C, c) = h_{-i}^*$, $h_{-i}^{*,t-1} \cdot (r_{-i}^*, C, d) \in H_{-i}^{*0}$ and that $h_{-i}^{*,t-1} \cdot (1_{(c,d)}, D, y_{-i}) \in H_{-i}^{*0}$ for each $y_{-i} \in Y_{-i}$. Thus, in this case,

$$(r_{-i}^t, s_{-i}^t, y_{-i}^t) \notin \{(r_{-i}^*, C, c), (r_{-i}^*, C, d), (1_{(c,d)}, D, c), (1_{(c,d)}, D, d)\}$$

and the numerator of $(j^{j-1-t} \mu^j(h_{-i}|h_i))$ is

$$j^{j-1-t} \mu^j(h_{-i}^{*,t-1} | h_i^{t-1}) \alpha_i(r_i^{t, s_{-i}^t}, r_{-i}^{t, s_i^t})[y^t] j^{-j}$$

and, hence, $\lim_j (j^{j-1-t} \mu^j(h_{-i}|h_i)) = 0$.

Case $h_i \in H_i^{*0} \backslash H_i^{D0}$: In this case, $\mu(h_{-i}^*|h_i) = 0$ since $h_i \in H_i^{*0}$ and $\text{supp}(\mu(\cdot|h_i)) \subseteq H_{-i}^D \subseteq H_{-i}^{*0}$ exactly as above.

For later use, we will show by induction on $\ell(h_i)$ that $\lim_j (j^{j-1-\ell(h_i)} \mu^j(h_{-i}|h_i)) = 0$ for each $h_{-i} \in H_{-i} \backslash H_{-i}^{*0}$. Consider first h_i with $\ell(h_i) = 1$. Then $\alpha_i(r_i^C, 1_{(c,s_i)})[(y_i, d)] > 0$ and $\alpha_i(r_i^C, 1_{(c,s_i)})[(y_i, c)] = 0$ since, respectively, $h_i \notin H_i^{D0}$ and $h_i \in H_i^{*0}$. In addition, for each j, $\mu^j(h_{-i}^*|h_i) = 0$ and

$$\sum_{(\hat{r}_{-i}, \hat{s}_{-i}, \hat{y}_{-i}) \in H_{-i}^1} \alpha_i(r_i^{\hat{s}_{-i}}, \hat{r}_{-i}^{s_i})[(y_i, \hat{y}_{-i})] \sigma_{-i}^j(\hat{r}_{-i}, \hat{s}_{-i}) \to \alpha_i(r_i^C, 1_{(c,s_i)})[(y_i, d)].$$

Thus, $\text{supp}(\mu(\cdot|h_i)) = \{(r_{-i}^*, C, d)\} \subseteq H_{-i}^{*0}$. We have that $\{(1_{(c,d)}, D, c), (1_{(c,d)}, D, d)\} \subseteq H_{-i}^{*0}$ since $\alpha_{-i}(1_{(c,d)}, 1_{(c,d)})[(y_{-i}, c)] = 0$ by property 1. Hence, for each $h_{-i} \in H_{-i} \backslash H_{-i}^{*0}$,

$$\lim_j (j^{j-2} \mu^j(h_{-i}|h_i)) = \frac{j^{j-2} \alpha_i(r_i^{s_{-i}}, r_{-i}^{s_i})[y] j^{-j}}{\sum_{(\hat{r}_{-i}, \hat{s}_{-i}, \hat{y}_{-i}) \in H_{-i}^1} \alpha_i(r_i^{\hat{s}_{-i}}, \hat{r}_{-i}^{s_i})[(y_i, \hat{y}_{-i})] \sigma_{-i}^j(\hat{r}_{-i}, \hat{s}_{-i})} \to 0.$$

Let $t > 1$ and assume that we have established that, for each $k = 1, \ldots, t-1$ and $h_i \in H_i^k \cap (H_i^{*0} \setminus H_i^{D0})$, $\lim_j (j^{j-1-k} \mu^j(h_{-i}|h_i)) = 0$ for each $h_{-i} \in H_{-i}^k \setminus H_{-i}^{*0}$.

Let $h_i \in H_i^t \cap (H_i^{*0} \setminus H_i^{D0})$. We have that $\lim_j B_j > 0$ because $h_i \in H_i \setminus H_i^{D0}$. Hence, for each $h_{-i} \in H_{-i}^t \setminus H_{-i}^{*0}$, $\lim_j (j^{j-1-t} \mu^j(h_{-i}|h_i)) = 0$ since $\lim_j (j^{j-1-(t-1)} \mu^j(h_{-i}^{t-1}|h_i^{t-1})) = 0$.

Case $h_i \in H_i^{*0} \cap H_i^{D0}$**:** We will show by induction on $\ell(h_i)$ that $\mu(h_{-i}|h_i) = 0$ for each $h_{-i} \in H_{-i} \setminus H_{-i}^{*0}$. Consider first the case where $h_i \in H_i$ has $\ell(h_i) = 1$. Since $h_i \in H_i^{*0} \cap H_i^{D0}$, we have that $\alpha_i(r_i^C, 1_{(c,s_i)})[(y_i, d)] = \alpha_i(r_i^C, 1_{(c,s_i)})[(y_i, c)] = 0$, which implies that $y_i = d$. In addition, for each j, $\mu^j(h_{-i}^*|h_i) = 0$ and, for each $h_{-i} \neq h_{-i}^*$,

$$\mu^j(h_{-i}|h_i) = \frac{\alpha_i(r_i^D, 1_{(d,c)})[(d, y_{-i})]}{\sum_{\hat{y}_{-i}} \alpha_i(r_i^D, 1_{(d,c)})[(d, \hat{y}_{-i})] + j^{-(j-1)} \sum_{(\hat{r}_{-i}, \hat{s}_{-i}, \hat{y}_{-i}) : (\hat{r}_{-i}, \hat{s}_{-i}) \neq (1_{(c,d)}, D)} \alpha_i(r_i^{\hat{s}_{-i}}, \hat{r}_{-i}^{s_i})[(d, \hat{y}_{-i})]}$$

if $(r_{-i}, s_{-i}) = (1_{(c,d)}, D)$ and

$$\mu^j(h_{-i}|h_i) = \frac{\alpha_i(r_i^{s_{-i}}, r_{-i}^{s_i})[(d, y_{-i})] j^{-(j-1)}}{\sum_{\hat{y}_{-i}} \alpha_i(r_i^D, 1_{(d,c)})[(d, \hat{y}_{-i})] + j^{-(j-1)} \sum_{(\hat{r}_{-i}, \hat{s}_{-i}, \hat{y}_{-i}) : (\hat{r}_{-i}, \hat{s}_{-i}) \neq (1_{(c,d)}, D)} \alpha_i(r_i^{\hat{s}_{-i}}, \hat{r}_{-i}^{s_i})[(d, \hat{y}_{-i})]}$$

otherwise. It then follows that $(1_{(c,d)}, D, c) \in \text{supp}(\mu(\cdot|h_i))$ by property 2 and that $\text{supp}(\mu(\cdot|h_i)) \subseteq \{(1_{(c,d)}, D, c), (1_{(c,d)}, D, d)\}$. For each $h_{-i} \in \{(1_{(c,d)}, D, c), (1_{(c,d)}, D, d)\}$, we have that $h_{-i} \in H_{-i}^{*0}$ since $\alpha_{-i}(1_{(c,d)}, 1_{(c,d)})[(y_{-i}, c)] = 0$ by property 1. Hence, if $\ell(h_i) = 1$, then $\text{supp}(\mu(\cdot|h_i)) \subseteq H_{-i}^{*0}$. Furthermore, for each $h_{-i} \notin \text{supp}(\mu(\cdot|h_i))$, $\lim_j (j^{j-2} \mu^j(h_{-i}|h_i)) = 0$; since $H_{-i} \setminus H_{-i}^{*0} \subseteq H_{-i} \setminus \text{supp}(\mu(\cdot|h_i))$, then

$$\lim_j (j^{j-2} \mu^j(h_{-i}|h_i)) = 0 \text{ for each } h_{-i} \in H_{-i} \setminus H_{-i}^{*0}.$$

Let $t > 1$ and assume that we have established that, for each $k = 1, \ldots, t-1$ and $h_i \in H_i^k \cap H_i^{*0} \cap H_i^{D0}$, $\text{supp}(\mu(\cdot|h_i)) \subseteq H_{-i}^{*0}$ and $\lim_j (j^{j-1-k} \mu^j(h_{-i}|h_i)) = 0$ for each $h_{-i} \in H_{-i}^k \setminus H_{-i}^{*0}$.

Let $h_i \in H_i^t \cap H_i^{*0} \cap H_i^{D0}$ and $h_{-i} \in H_{-i} \setminus H_{-i}^{*0}$. Then $h_{-i}^{t-1} \in H_{-i} \setminus H_{-i}^{*0}$ as well. We will show that $\lim_j (j^{j-1-t} \mu^j(h_{-i}|h_i)) = 0$ for each $h_{-i} \in H_{-i} \setminus H_{-i}^{*0}$.

Consider first the case where $h_{-i} = h_{-i}^*$. In this case, $j^{j-1-t} \mu^j(h_{-i}|h_i) = 0$ for each $j \in \mathbb{N}$ since $h_i \in H_i^{*0}$ and the result follows.

Due to the above, we may assume that $h_{-i} \neq h_{-i}^*$. We consider two cases.

Case (i): $h_i^{t-1} \in H_i^{*0} \cap H_i^{D0}$ or $h_i^{t-1} \in H_i^{*0} \cap (H_i \setminus H_i^{D0})$.

Let $\hat{h}_{-i}^{t-1} \in \text{supp}(\mu(\cdot|h_i^{t-1})) \subseteq H_{-i}^{*0}$; since $\sigma_{-i}(1_{(c,d)}, D|\hat{h}_{-i}^{t-1}) = 1$, it follows that $\lim_j B_j > 0$ when $\alpha_i(r_i^{t,D}, 1_{(d,c)})[(y_i^t, \hat{y}_{-i})] > 0$ for some $\hat{y}_{-i} \in Y_{-i}$; in particular,

$\lim_j B_j > 0$ when $y_i^t = d$ by property 2. In this case, $\lim_j(j^{j-1-t}\mu^j(h_{-i}|h_i)) = 0$ since $\lim_j(j^{j-1-(t-1)}\mu^j(h_{-i}^{t-1}|h_i^{t-1})) = 0$.

If $y_i^t = c$ and $\alpha_i(r_i^{t,D}, 1_{(d,c)})[(c, \hat{y}_{-i})] = 0$ for all $\hat{y}_{-i} \in Y_{-i}$, then

$$\lim_j(jB_j) = \lim_j \sum_{\hat{h}_{-i} \in H_{-i}^{t-1} \cap H_{-i}^{*0}} \mu^j(\hat{h}_{-i}^{t-1}|h_i^{t-1}) \times$$
$$\times \left(\frac{1}{j^{j-1}} \sum_{(\hat{r}_{-i}^t, \hat{s}_{-i}^t, \hat{y}_{-i}^t):(\hat{r}_{-i}^t, \hat{s}_{-i}^t) \neq (1_{(d,c)}, D)} \alpha_i(r_i^{t,\hat{s}_{-i}^t}, \hat{r}_{-i}^{t,s_i^t})[(c, \hat{y}_{-i}^t)] + \sum_{\hat{y}_{-i}^t} \alpha_i(r_i^{t,D}, 1_{(c,d)})[(c, \hat{y}_{-i}^t)] \right)$$

which is strictly positive since $\alpha_i(r_i^{t,D}, 1_{(c,d)})[(c,d)] > 0$ by property 2. Since

$$\lim_j \left(j^{j-1-(t-1)} \mu^j(h_{-i}^{t-1}|h_i^{t-1}) \alpha_i(r_i^{t,s_{-i}^t}, r_{-i}^{t,s_i^t})[y^t] \sigma_{-i}^j(r_{-i}^t, s_{-i}^t|h_{-i}^{t-1}) \right) = 0,$$

it follows that $\lim_j(j^{j-1-t}\mu^j(h_{-i}|h_i)) = 0$ for each $h_{-i} \in H_{-i} \setminus H_{-i}^{*0}$.

Case (ii): $h_i^{t-1} \in H_i \setminus H_i^{*0}$.

In this case, we have $\alpha_i(r_i^{t,C}, 1_{(c,s_i^t)})[(y_i^t, c)] = 0$ since $h_i \in H_i^{*0}$ and $\alpha_i(r_i^{t,C}, 1_{(c,s_i^t)})[(y_i^t, d)] = 0$ since $h_i \in H_i^{D0}$. Thus, $y_i^t = d$ and the argument in case (i) can be applied to conclude that $\lim_j B_j > 0$ provided that there is $\hat{h}_{-i}^{t-1} \in \mathrm{supp}(\mu(\cdot|h_i^{t-1})) \cap H_{-i}^{*0}$. Then if $h_{-i}^{t-1} \neq h_{-i}^{*,t-1}$, we have $\lim_j(j^{j-1-(t-1)}\mu^j(h_{-i}^{t-1}|h_i^{t-1})) = 0$ and, hence, $\lim_j(j^{j-1-t}\mu^j(h_{-i}|h_i)) = 0$. If $h_{-i}^{t-1} = h_{-i}^{*,t-1}$, then since $h_{-i} \notin H_{-i}^{*0} \cup \{h_{-i}^{*,t}\}$, the numerator of $\mu^j(h_{-i}|h_i)$ is less than j^{-j}. Thus, $\lim_j(j^{j-1-t}\mu^j(h_{-i}|h_i)) = 0$.

Hence, we are left with the case where $\mathrm{supp}(\mu(\cdot|h_i^{t-1})) = \{h_{-i}^*\}$. In this case,

$$\lim_j(jB_j) = \lim_j \mu^j(h_{-i}^{*,t-1}|h_i^{t-1}) \times$$
$$\times \left(\frac{1}{j^{j-1}} \sum_{(\hat{r}_{-i}^t, \hat{s}_{-i}^t, \hat{y}_{-i}^t):(\hat{r}_{-i}^t, \hat{s}_{-i}^t) \neq (1_{(c,d)}, D)} \alpha_i(r_i^{t,\hat{s}_{-i}^t}, \hat{r}_{-i}^{t,s_i^t})[(c, \hat{y}_{-i}^t)] + \sum_{\hat{y}_{-i}^t} \alpha_i(r_i^{t,D}, 1_{(d,c)})[(d, \hat{y}_{-i}^t)] \right)$$

which is strictly positive since $\alpha_i(r_i^{t,D}, 1_{(d,c)})[(d,c)] > 0$ by property 2. Since

$$\lim_j \left(j^{j-1-(t-1)} \mu^j(h_{-i}^{t-1}|h_i^{t-1}) \alpha_i(r_i^{t,s_{-i}^t}, r_{-i}^{t,s_i^t})[y^t] \sigma_{-i}^j(r_{-i}^t, s_{-i}^t|h_{-i}^{t-1}) \right) = 0,$$

it follows that $\lim_j(j^{j-1-t}\mu^j(h_{-i}|h_i)) = 0$ for each $h_{-i} \in H_{-i} \setminus H_{-i}^{*0}$.

A.2 When the aggregation function is a mixed extension

We show that if α is a mixed extension satisfying (a)–(c), then α is strongly responsive. As noted already, if α satisfies (a) and (b), then α is responsive. Thus, it suffices to establish properties 3 and 4.

Regarding property 3: Note that

$$\alpha_{i,Y_{-i}}(1_y, 1_{(c,d)})[c] = \begin{cases} 0 & \text{if } y = (c,d), \\ 0 & \text{if } y = (d,d), \\ \alpha_i(1_{(c,c)}, 1_{(c,d)})[c,c] & \text{if } y = (c,c), \\ \alpha_i(1_{(d,c)}, 1_{(c,d)})[d,c] & \text{if } y = (d,c) \end{cases}$$

by property (a). Hence, it follows from property (c) that

$$1_{(c,c)} \in \arg\max_y \alpha_{i,Y_{-i}}(1_y, 1_{(c,d)})[c].$$

Thus,

$$\alpha_{i,Y-i}(r, 1_{(c,d)})[c] = \sum_y r(y)\big(\alpha_i(1_y, 1_{(c,d)})[(c,c)] + \alpha_i(1_y, 1_{(c,d)})[(d,c)]\big)$$
$$\leq \sum_y r(y)\big(\alpha_i(1_{(c,c)}, 1_{(c,d)})[(c,c)] + \alpha_i(1_{(c,c)}, 1_{(c,d)})[(d,c)]\big) = \alpha_{i,Y-i}(1_{(c,c)}, 1_{(c,d)})[c].$$

Regarding property 4: Note that $\alpha_i(1_y, 1_{(c,d)})[(c,c)] = 0$ for each $y \neq (c,c)$ by property (a) and that, then, the denominator of $\frac{\alpha_i(r,1_{(c,d)})[(c,c)]}{\alpha_i(r,1_{(c,d)})[(c,c)]+\alpha_i(r,1_{(c,d)})[(c,d)]}$ is $r(c,c)\alpha_i(1_{(c,c)}, 1_{(c,d)})[(c,c)] + \sum_y r(y)\alpha_i(1_y, 1_{(c,d)})[(c,d)]$, which is strictly positive by property (b). Let $r \in \Delta(Y)$ and note that the conclusion is then obvious when $r(c,c) = 0$. If $r(c,c) > 0$, then

$$\begin{aligned} &\frac{\alpha_i(r, 1_{(c,d)})[(c,c)]}{\alpha_i(r, 1_{(c,d)})[(c,c)] + \alpha_i(r, 1_{(c,d)})[(c,d)]} \\ &= \frac{r(c,c)\alpha_i(1_{(c,c)}, 1_{(c,d)})[(c,c)]}{r(c,c)\alpha_i(1_{(c,c)}, 1_{(c,d)})[(c,c)] + \sum_y r(y)\alpha_i(1_y, 1_{(c,d)})[(c,d)]} \\ &\leq \frac{r(c,c)\alpha_i(1_{(c,c)}, 1_{(c,d)})[(c,c)]}{r(c,c)\alpha_i(1_{(c,c)}, 1_{(c,d)})[(c,c)] + r(c,c)\alpha_i(1_{(c,c)}, 1_{(c,d)})[(c,d)]} \\ &= \frac{\alpha_i(1_{(c,c)}, 1_{(c,d)})[(c,c)]}{\alpha_i(1_{(c,c)}, 1_{(c,d)})[(c,c)] + \alpha_i(1_{(c,c)}, 1_{(c,d)})[(c,d)]}. \end{aligned}$$

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