Rosen meets Garicano and Rossi-Hansberg: Stable Matchings in Knowledge Economies*

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Abstract

We use the framework of large many-to-one matching markets with occupational choice introduced in Carmona and Laohakunakorn (2024) to formally compare the knowledge-based theories of Rosen (1982) and Garicano and Rossi-Hansberg (2004). We show that these theories differ only in three elements: the factor share of labor in the goods' production function, the payment to self-employed individuals and the number of workers each manager can hire. These differences imply starkly different properties of the stable matchings of the two theories. We decompose these differences by characterizing the stable matchings of a sequence of markets that allows us to move from Rosen's (1982) market to Garicano and Rossi-Hansberg's (2004) market by changing only one element at each step. This shows that the difference in the number of workers each manager can hire accounts for the qualitative difference in the properties of the stable matchings of the two theories.

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1 Introduction

Understanding several important economic issues, such as the wage and firm size distributions and their evolution, requires the study of the internal organization of firms in a market setting. This has been convincingly argued in Lucas (1978), Rosen (1982), Garicano and Rossi-Hansberg (2004) and Garicano and Rossi-Hansberg (2006), among others, which provide different ways of incorporating organization in competitive models.[1]

The relationship between these models can be better understood by placing them in a unifying framework. In Carmona and Laohakunakorn (2024), we introduce a framework of large many-to-one matching markets with occupational choice and show that the above models are particular cases of our general framework. We use this framework in this paper to show that Rosen's (1982) and Garicano and Rossi-Hansberg's (2004) knowledge-based theories are remarkably similar, that they differ essentially in one dimension – the limits they impose on firm size – and this is, therefore, the reason why their conclusions differ.

Rosen (1982) and Garicano and Rossi-Hansberg (2004) have a similar motivation and purpose, namely to study the matching and occupational choice of individuals in a competitive market in which organization shapes the relationship between workers and managers. We focus on these two models because each is representative of a class of models that have been widely used in economics.[2] Their settings appear to differ in several details but, by placing them in the unifying framework of Carmona and Laohakunakorn (2024), we highlight the essential elements of each model which are similar in many respects. Nevertheless, the remaining differences matter and

[1] See Garicano and Rossi-Hansberg (2015) for a survey and endorsement of this literature.

[2] Rosen (1982) features a neoclassical production function which has been widely used in macroeconomics to explain the firm size distribution, whereas variants of Garicano and Rossi-Hansberg (2004) have been used to successfully explain the distributional effects of improvements in information and communication technology (Garicano and Rossi-Hansberg (2006)), offshoring (Antràs, Garicano, and Rossi-Hansberg (2006)) and optimal contracting arrangements in knowledge-intensive industries (Fuchs, Garicano, and Rayo (2015)).

this is clearly seen from the differences in their conclusions: in the specific setting considered in this paper, Rosen (1982) implies a constant wage for the workers and an indeterminate matching between managers and workers; in contrast, in Garicano and Rossi-Hansberg (2004), the wage is strictly increasing and matching is positive assortative. These properties make the latter a better fit to real-world data and a better theory to explain e.g. the shadow that superstar managers cast on workers discussed in Garicano and Rossi-Hansberg (2015).

Our contribution is a clear understanding of the economic mechanism that drives the above differences and, thus, of which economic factors account for the empirical plausibility of Garicano and Rossi-Hansberg's (2004) model and its successors. In the stripped-down description that is the starting point of our exercise, the latter is a setting where the production function of goods is such that total factor productivity is increasing in the manager's skill, as in Rosen (1982), and the factor share of labor is one. We extend the conclusions of Garicano and Rossi-Hansberg (2004) to the case, allowed in Rosen (1982), where the factor share of labor is close to but less than one; hence, the two settings are similar as far as the production function of goods is concerned. The main difference, i.e. the one which is responsible for the distinct conclusions, is that managers are unrestricted in the number of workers they can hire in Rosen (1982), whereas they can only hire a bounded number which is increasing in the workers' skill in Garicano and Rossi-Hansberg (2004). These limits to firm size are due to the time cost of communication between different members of a firm and thus, more broadly, due to what Becker and Murphy (1992) term coordination costs.[3] Our results demonstrate that coordination costs not only shape the pattern of specialization (as Becker and Murphy (1992) emphasized) but also, through their effect on the limits to firm size, the matching of individuals skills and wages in a more realistic way as compared to when they are absent.

[3]As they put it: "A variable of great importance is the cost of combining specialized workers. Modern work on principal-agent conflicts, free-riding, and the difficulties of communication implies that the cost of coordinating a group of complementary specialized workers grows as the number of specialists increases."

Highlighting the importance of having a limit on firm size that is increasing in the workers' skill is important for further development and testing of knowledge-based theories. Considering coordination costs broadly as determinants to such limit can therefore widen the applicability of the main insights in Garicano and Rossi-Hansberg (2004) by freeing them from the specific mechanism they emphasized and by allowing for more flexible functional forms for the resulting bound.

Similar considerations apply more generally. As we show in this paper, the knowledge-based theories of Rosen (1982) and Garicano and Rossi-Hansberg (2004) differ in three aspects: The number of workers a manager can hire (which we argue is the main one accounting for their distinct conclusions), the factor share of labor in the production function of goods and the payment of self-employed individuals. Thus, we obtain a unifying theory by considering general functional forms for all these elements which may be useful to calibrate the resulting model and to use it to match real-world data.[4]

Our main results are obtained by comparing the stable matchings of several markets. The sequence of markets we consider allows us to move from Rosen's (1982) setting to that of Garicano and Rossi-Hansberg (2004) by changing only one element at each step. Thus, we also consider a market that differs from the former only in the number of workers a manager can hire and a market that differs from the latter only in the self-employed payment.[5] These two additional markets then differ from one another only by the factor share of labor in the production function and, as we show, have qualitatively similar outcomes to Garicano and Rossi-Hansberg (2004). Thus, this decomposition reveals that the difference in the number of workers each manager can hire accounts for the qualitative difference in the properties of the stable matchings of the knowledge-based theories of Rosen (1982) and Garicano and Rossi-Hansberg (2004).

To reach the above conclusion, we establish several technical results, stated at a broad level of generality, that provide a toolkit for analyzing stable matchings in

[4]See Gabaix and Landier (2008) for the calibration of a related model.

[5]The second of these two markets is then the one in Antràs, Garicano, and Rossi-Hansberg (2006).

general knowledge economies. Stable matchings are formalized as measures over skill levels (describing the skill or knowledge of a manager or self-employed individual) and workforces (describing the number of workers hired, their skill and wage). When the pairs of skill levels and workforces lie on a compact metric space, then stable matchings are measures on a compact metric space and, thus, lie on a space which is convenient for establishing existence results. We established such an existence result in Carmona and Laohakunakorn (2024); here we show the existence of stable matchings in the knowledge economies we consider by building on it to deal with the failure of some compactness assumptions.

To characterize stable matchings, we use the fact that wages are transfers from managers to workers. This means that our setting is one of matching with transferable utility but, unlike in e.g. Chiappori (2017), there is no a priori given sets of individuals to match since each individual can choose his own occupation. Nevertheless, we show that every stable matching defines a stable assignment, in an analogous way to Chiappori (2017, Section 2.1) but which, in our setting, is defined by a measure over pairs of skill levels (z, z'), indicating roughly how many managers with skill z are matched with workers of skill z', and a continuous function of skill levels, indicating the payment of each individual with a given skill. This then allows us to establish that the assignment solves an optimal transport problem which differs from the one in Chiappori (2017, Theorem 1, p. 45) in its constraint; due to occupational choice, the marginal distributions of the assignment are not fixed as in the optimal transport problem of Chiappori (2017, Theorem 1, p. 45) but rather the requirement that an assignment needs to satisfy in our setting is that the sum of the measures of managers, self-employed and workers must equal the given measure describing the distributions of skills in the population. We then build on a result by Beiglböck and Griessler (2019) to establish a condition that the support of any solution to our optimal transport problem must satisfy and which greatly facilitates the characterization of such solutions.

We use the above tools to characterize the stable matchings in the settings of Garicano and Rossi-Hansberg (2004) and Antràs, Garicano, and Rossi-Hansberg (2006).

We show that stable matchings in these two settings, which are defined as measures over skill levels and workforces, have a simpler description. Indeed, they are represented by two skill levels, z_1 and z_2 with $z_1 \leq z_2$, and two functions c and ϕ such that (i) workers are those with skills no greater than z_1, (ii) managers are those with skill no lower than z_2, (iii) self-employed (if any) are those with skill between z_1 and z_2, (iv) ϕ is a (strictly increasing and differentiable) assignment function with $\phi(z)$ being the skill of workers hired by each manager of skill z and (v) c is a (strictly increasing and differentiable) wage function with $c(z')$ being the wage of workers with skill z'. These properties are the ones used by Garicano and Rossi-Hansberg (2004) and Antràs, Garicano, and Rossi-Hansberg (2006) to define competitive equilibrium in their settings and, therefore, these results provide a sense in which they are without loss of generality, i.e. their equilibrium notion corresponds exactly to stable matchings and this provides a justification for using (i)–(v) as the defining properties of the solution concept in their settings. Our approach and tools thus dispense with the need to make such simplifying assumptions from the outset since these assumptions can be derived from the definition of stable matching with rigorous and transparent arguments.[6]

There is also a conceptual advantage of our approach, namely that a single parsimonious solution concept is applied broadly to a variety of models. Settings featuring matching of a large number of individuals and occupational choice, such as those

[6] To illustrate this point, consider for example the argument in Antràs, Garicano, and Rossi-Hansberg (2006) to show that ϕ is increasing. It relies on the implicit function theorem and on its derivative being strictly positive, which then requires its domain to be an interval, i.e. it requires assuming that the set of skill levels of managers is an interval. As another example, consider the argument in Fuchs, Garicano, and Rayo (2015) to show that workers have lower skills than managers. It considers an optimal transport problem and consists in showing that if an assignment γ does not have this property, then there is another one γ' which increases the objective function. But γ' is not defined and, in fact, it would be difficult to define; moreover, in the argument there is a transport of equal mass of workers to managers and vice-versa, which would be appropriate in the optimal transport in Chiappori (2017, Theorem 1, p. 45) but not in the one describing the allocation of people in their setting since production requires more workers than managers. Both these difficulties can be dealt with by our result on the support of the solutions to our optimal transport problem.

in Garicano and Rossi-Hansberg (2004) and Antràs, Garicano, and Rossi-Hansberg (2006), are technically challenging and the standard approach is to define an intuitive notion of competitive equilibrium that is tailor-made to each such setting.[7] This makes the comparison of different models difficult since some differences in their conclusions can come from unintended differences in the solution concept. This problem can be avoided by showing how the setting fits into classical general equilibrium theory (e.g. into the framework of Debreu's (1959) Theory of Value or some more general general equilibrium framework allowing for e.g. an infinite-dimensional commodity space) and then apply the classical notion of competitive equilibrium. This exercise has been carried out in settings unrelated to the knowledge economies we consider by e.g. Prescott and Townsend (1984) and Cole and Prescott (1997) but is not a straightforward one. A simpler way, as we advocated in Carmona and Laohakunakorn (2024) and carried out in this paper, is to represent knowledge economies in the general framework of large many-to-one matching markets with occupational choice and then systematically use stable matching as a solution concept. We view this approach and the development of the above tools as important steps to use knowledge economies, along the lines of those in Rosen (1982) and Garicano and Rossi-Hansberg (2004), to address important economic problems, such as the evolution of the wage and firm size distributions and others that require the study of the internal organization of firms in a market setting.

The paper is organized as follows. In Section 2, we describe Garicano and Rossi-Hansberg's (2004) setting, represent it as a large many-to-one matching market with occupational choice and recall the definition of stable matchings. In Section 3 we develop our tools and use them to establish existence and to characterize the stable matchings in the setting of Garicano and Rossi-Hansberg (2004).[8] We also obtain results for the analysis of the differential equations that are part of this characteriza-

[7]Ide and Talamàs (2025) provide a recent example of this approach. See also Amaral and Rivera-Padilla (2024) and Ritto (2024) among others.

[8]As a by-product of this analysis, we show that there is no change to the stable matchings when workers are allowed to be more knowledgeable than managers and when production is allowed, in such case, to depend only on the workers' knowledge.

tion and use them to show that there is a unique stable matching. The comparison between the stable matchings in the settings of Rosen (1982) and Garicano and Rossi-Hansberg (2004) is done in Section 4. Some concluding remarks are in Section 5 and the proofs of our results are in the (online) Appendix and in the supplementary material to this paper.[9]

2 Model

In this section we describe the knowledge economy of Garicano and Rossi-Hansberg (2004), represent it as a large many-to-one matching market with occupational choice and recall the definition of stable matchings for such markets.

2.1 Knowledge economies

The setting we consider is that of Garicano and Rossi-Hansberg (2004) apart from some slight change of notation. There is a large number of individuals characterized by their knowledge, with $Z = [0, \bar{z}]$ denoting the set of knowledge levels and where $\bar{z} \in \mathbb{R}_{++}$. The knowledge distribution is denoted by ν and is such that it has a continuously differentiable and strictly positive density θ. Then ν is atomless and $\mathrm{supp}(\nu) = Z$. Individuals can be workers, managers or self-employed.

A firm consists of one manager and several workers of the same type. Production happens when a worker solves a problem with which he is faced. Problems are drawn according to a cumulative distribution function F on $\mathbb{R}_+$ with a continuous and decreasing density f. We allow for the case where $F(\bar{z}) < 1$ as well as $F(\bar{z}) = 1$. Each worker is allowed to ask the manager for the solution of the problem he has drawn if he cannot solve it himself. Knowledge is cumulative: If someone has knowledge $z \in Z$, then he can solve all problems in $[0, z]$. Thus, a worker with knowledge z asks for help with probability $1 - F(z)$. Asking for help incurs a communication cost: The manager incurs a cost of $0 < h < 1$ units of time to attempt solving the problem regardless of whether or not he succeeds. Individuals have one unit of time, which

[9] The latter is available at https://sites.google.com/site/gmbbcarmona/home/papers.

will be entirely spent working in the case of workers and on helping workers in the case of managers. Workers draw one problem per unit of time spent in production. Thus, a firm with a manager with knowledge z and workers with knowledge z' can have a measure n of workers provided that

$$nh(1 - F(z')) = 1,$$

i.e. such that the manager exhausts his time helping the workers. Problems in $[0, \max\{z, z'\}]$ get to be solved, either by the workers or by the manager, and output is 1 if the problem is solved and 0 otherwise. Expected production is then $F(\max\{z, z'\})n$ and the managers' rent is $(F(\max\{z, z'\}) - c)n$, where c is the wage paid to the workers.

A self-employed individual with knowledge z solves the problems that he can and, thus, produces $F(z)$. Let $U_z(s)$ denote the production of a self-employed individual and set $U_z(s) = F(z)$ for each $z \in Z$.[10]

2.2 Matching with occupational choice

We represent this setting in the general framework of markets with occupational choice of Carmona and Laohakunakorn (2024) and use its notion of a stable matching as our solution concept. A matching market with occupational choice (a market, henceforth) is $E = (Z, \nu, C, \mathbb{C}, X, (\succ_z)_{z \in Z})$, where Z is the set of individual types, ν is the type distribution, C is the set of contracts, $\mathbb{C}$ is the contract correspondence, X is the set of possible matches of managers and $\succ_z$ describes the preferences of an individual of type z, for each $z \in Z$.

The representation of Garicano and Rossi-Hansberg's (2004) setting described above as a market is as follows. First, let Z and ν be as in Section 2.1. Second, we let contracts be wages and set $C = \mathbb{R}_+$. As already mentioned, an individual can choose to be a manager, a worker or be self-employed, thus, the set of occupations is $A = \{w, s, m\}$. A dummy type $\emptyset \notin Z$ is used to represent self-employed individuals,

[10]This seemingly redundant notation will be helpful in Section 4 since there we will consider the case where $U_z(s) = 0$ for each $z \in Z$.

and we let $Z_\emptyset = Z \cup \{\emptyset\}$, with the assumption that $\emptyset$ is an isolated point in $Z_\emptyset$. The contract correspondence $\mathbb{C}$ maps $Z \times Z_\emptyset$ into C with $\mathbb{C}(z, z')$ describing the set of contracts that are feasible for a manager of type z and a worker of type z'.[11] We specify that $\mathbb{C}(z, z') = C$ and $\mathbb{C}(z, \emptyset) = \{0\}$ for each $z, z' \in Z$, i.e. any nonnegative wage is feasible for a manager and a worker and we set a wage of zero for a self-employed individual (as they receive a rent, not a wage).

We incorporate the time constraint of managers in the set X of feasible matches for managers and let

$$X = \{n1_{(z,c)} : (z, c) \in Z \times C \text{ and } n \in \mathbb{R}_+ \text{ such that } nh(1 - F(z)) = 1\}$$

since managers can hire several workers all of the same type such that his time constraint is satisfied.[12] Let

$$X_\emptyset = X \cup \{1_{(\emptyset,c)} : c \in C\}$$

be the set of possible matches of managers and self-employed individuals and, for each $z \in [0, \bar{z})$, let

$$n(z) = \frac{1}{h(1 - F(z))};$$

if $F(\bar{z}) < 1$, then let also $n(\bar{z}) = \frac{1}{h(1-F(\bar{z}))}$.

Preferences are defined by specifying payoff functions as follows. A worker's preferences are over managers' types and contracts, thus, over the set $X_w = \{1_{(z,c)} : (z, c) \in Z \times C\}$ and are represented by a utility function $(z, 1_{(z',c)}) \mapsto U_z(w, 1_{(z',c)})$.

[11] When $z' = \emptyset$ the manager is, in fact, self-employed and $\mathbb{C}(z, \emptyset)$ describes the feasible contracts for a self-employed individual of type z.

[12] A manager's workforce and their wages are described as in Carmona and Laohakunakorn (2024) by measures over $Z \times C$, i.e. as measures over knowledge levels and wages. The measure that assigns measure one to $(z, c) \in Z \times C$ is denoted by $1_{(z,c)}$ and, hence, $n1_{(z,c)}$ assigns measure n to (z, c). Such measures are elements of $M(Z \times C)$. In general, given a metric space T, $M(T)$ denotes the set of finite, Borel measures on T endowed with the weak (narrow) topology (see Varadarajan (1958) for details). If T' is another metric space and $\delta \in M(T \times T')$, δ_T denotes the marginal of δ on T and $\delta_{T'}$ denotes the marginal of δ on T'. If $q : T \to T'$ is Borel-measurable and $\pi \in M(T)$, then $\pi \circ q^{-1} \in M(T')$ is defined by setting, for each Borel subset B of T', $\pi \circ q^{-1}(B) = \pi(q^{-1}(B))$.

We specify that

$$U_z(w, 1_{(z',c)}) = c \text{ for each } 1_{(z',c)} \in X_w$$

so that each worker's preferences are represented by his wage.

A self-employed's preferences are, in general, over contracts, thus, over the set $X_s = \{1_{(\emptyset,c)} : c \in C\}$ and are represented by a utility function $(z, 1_{(\emptyset,c)}) \mapsto U_z(s, 1_{(\emptyset,c)})$. We specify that

$$U_z(s, 1_{(\emptyset,c)}) = U_z(s) \text{ for each } 1_{(\emptyset,c)} \in X_s$$

so that each self-employed's preferences are represented by his expected output.

Finally, a manager's preferences are over measures describing whom to hire and the contracts offered, thus, over the set X and are represented by a utility function $(z, n1_{(z',c)}) \mapsto U_z(m, n1_{(z',c)})$. We specify that

$$U_z(m, n1_{(z',c)}) = (F(\max\{z, z'\}) - c)n \text{ for each } n1_{(z',c)} \in X$$

so that each manager's preferences are represented by his rent.

Overall, then, for each $z \in Z$, an individual of type z's preferences are represented by the utility function U_z which depends both on his occupational choice and whom he matches with.

Let E_{grh} denote the market just defined. Let E^*_{grh} be exactly as E_{grh} but for $U_z(m, n1_{(z',c)}) = (F(z) - c)n$ for each $z \in Z$ and $n1_{(z',c)} \in X$. The formalization in Garicano and Rossi-Hansberg (2004) is actually E^*_{grh}, which is simpler than that of E_{grh}, but the latter has the advantage of corresponding more closely to the description in Garicano (2000). As we will show, both markets have the same stable matchings and are, therefore, equivalent as far as stable matchings are concerned.

2.3 Stable matchings

A matching in E_{grh} is a measure μ on $Z \times X_\emptyset$ that describes the occupational choices of individuals and the way they are matched. Thus, matches are of the form (z, δ) and the occupational choices are described by the place in the match each individual occupies: if $\delta \in X$, then the first coordinate refers to a manager and the second to workers (as part of a firm) and, when $\delta \in X_\emptyset \setminus X$, the first coordinate refers to a self-employed individual and the second describes the individual's contract. Then μ roughly specifies how many matches described by (z, δ) there are.

Formally, a *matching* is a Borel measure μ on $Z \times X_\emptyset$ such that

1. $\delta = 1_{(\emptyset,0)}$ for each $(z, \delta) \in \operatorname{supp}(\mu) \cap (Z \times (X_\emptyset \setminus X))$, and

2. $\nu_M + \nu_S + \nu_W = \nu$

where, for each Borel subset B of Z, $\nu_M(B) = \mu(B \times X)$, $\nu_S(B) = \mu(B \times (X_\emptyset \setminus X))$ and $\nu_W(B) = \int_{Z \times X} \delta(B \times C) \mathrm{d}\mu(z, \delta)$. Condition 1 requires that the contract is feasible according to the contract correspondence and condition 2 requires that everyone in the market is accounted for.[13]

Given a matching μ and $z \in Z$, individuals of type z can target certain types and contracts (z^*, c) in the sense that someone of type z^* is better off with someone of type z at contract c than in his current match. The set of such (z^*, c) also depends on the occupational choice a of z in the prospective match, and is denoted by $T_z^a(\mu)$.

The targets for the prospective self-employed are the contracts that are feasible when someone is unmatched: For each $z \in Z$, let $T_z^s(\mu) = \{(\emptyset, 0)\}$.

The targets of prospective managers are as follows. For each $z \in Z$, let $T_z^m(\mu)$ be the set of $(z^*, c) \in Z \times C$ such that there exists

(a) $(z', c', \delta') \in Z \times C \times X$ such that $(z', \delta') \in \operatorname{supp}(\mu)$, $(z^*, c') \in \operatorname{supp}(\delta')$ and $c > c'$, or

(b) $\delta' \in X_\emptyset \setminus X$ such that $(z^*, \delta') \in \operatorname{supp}(\mu)$ and $c > U_{z^*}(s)$, or

(c) $\delta' \in X$ such that $(z^*, \delta') \in \operatorname{supp}(\mu)$ and $c > U_{z^*}(m, \delta')$.

Anyone of type z can be a manager if he finds workers, here of type z^*, who prefer to work for him than to be in their current occupation. Each of these workers can be

[13] Indeed, $\nu_a(B)$ is the measure of people with occupation a and with a type in B, hence $\nu_M(B) + \nu_S(B) + \nu_W(B) = \nu(B)$ for each Borel subset B of Z means that the measure of people with a type in B and some occupation equals to the measure of people with a type in B, i.e. everyone has an occupation.

someone who was already a worker in μ as described in condition (a), or self-employed as described by condition (b), or a manager as described by condition (c).[14]

Stability of a matching μ requires that, for each matched pair (z, δ), neither the manager or self-employed individual of type z, nor the workers that are part of the workforce δ (when $\delta \in X$) can gain by being managers and hiring a workforce from their targets $T_z^m(\mu)$. Furthermore, when $\delta \in X$, neither the manager nor the workers can gain by becoming self-employed.[15]

The above requirements are formalized as follows. Let $S_M(\mu)$ be the set of $(z, \delta) \in Z \times X_\emptyset$ such that, if $\delta \in X$, then

(i) there does not exist $\delta' \in X$ such that $\text{supp}(\delta') \subseteq T_z^m(\mu)$ and $U_z(m, \delta') > U_z(m, \delta)$,[16]

(ii) for each $(z', c) \in \text{supp}(\delta)$, there does not exist $\delta' \in X$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu)$ and $U_{z'}(m, \delta') > c$,

and, if $\delta \in X_\emptyset \setminus X$, then

(iii) there does not exist $\delta' \in X$ such that $\text{supp}(\delta') \subseteq T_z^m(\mu)$ and $U_z(m, \delta') > U_z(s)$.

The set $S_M(\mu)$ describes matches (z, δ) such that no one in it would like to change their occupation-match pair by becoming a manager or changing his match while remaining a manager.

Let $IR(\mu)$ be the set of $(z, \delta) \in Z \times X_\emptyset$ such that, if $\delta \in X$, then

(i) $U_z(m, \delta) \geq U_z(s)$ and

[14] It follows by Theorem 1 in Carmona and Laohakunakorn (2024) that the targets $T_z^w(\mu)$ of prospective workers are not needed for the definition of a stable matching.

[15] Individuals are also allowed to change their occupation to become workers and to change the manager who employs them when they are already workers in μ. It follows again by Theorem 1 in Carmona and Laohakunakorn (2024) that these changes can be ignored in the definition of a stable matching.

[16] This condition is equivalent to the one in Carmona and Laohakunakorn (2024) which has $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ instead of $\text{supp}(\delta') \subseteq T_z^m(\mu)$. Indeed, if $\delta, \delta' \in X$ are such that $\text{supp}(\delta') \subseteq \text{supp}(\delta)$, then $\delta = \delta' = n(z')1_{(z',c')}$ for some $(z', c') \in Z \times C$ and $U_z(m, \delta') > U_z(m, \delta)$ cannot hold.

(ii) $c \geq U_{z'}(s)$ for each $(z', c) \in \text{supp}(\delta)$.

The set $IR(\mu)$ describes matches (z, δ) such that no one in it would like to change their occupation to become self-employed. A matching μ is *stable* if $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$.

3 Existence, characterization and uniqueness

In this section we describe the general properties of stable matchings in Garicano and Rossi-Hansberg's (2004) framework. To do so, we develop several tools that are useful for the analysis of stable matchings of knowledge economies more generally and, in fact, we will use them to analyse the setting of Antràs, Garicano, and Rossi-Hansberg (2006) in Section 4.

3.1 Existence

Our first main result establishes the existence of a stable matching in E_{grh} and E^*_{grh}. It builds on the existence result in Carmona and Laohakunakorn (2024) but requires additional arguments because some of the assumptions of that result are not satisfied in the setting of the current paper.

The main difficulty regarding the existence of stable matchings concerns the set X of feasible matches for managers since the equality $nh(1 - F(z)) = 1$ cannot hold when $z = \bar{z}$ and $F(\bar{z}) = 1$. Nevertheless, any sequence of markets truncated by prohibiting individuals of type close to $\bar{z}$ to be workers satisfies the conditions of the existence result in Carmona and Laohakunakorn (2024) and this allows us to show that stable matchings exist. Intuitively, individuals of type close to $\bar{z}$ will not be workers in any stable matching but rather managers, hence the truncation is eventually innocuous. This also addresses other potential issues, namely, the number of workers hired by a manager are bounded once worker types are bounded away from $\bar{z}$. Furthermore, wages are also bounded, in fact, by one since otherwise managers would have a negative payoff.

Theorem 1 *A stable matching exists in E_{grh} and in E^*_{grh}.*

3.2 Characterization

Our second main result fully characterizes stable matchings. For each matching μ of a market, let

$$M = \{z \in Z : (z, \delta) \in \text{supp}(\mu) \text{ for some } \delta \in X\}, \tag{1}$$

$$S = \{z \in Z : (z, \delta) \in \text{supp}(\mu) \text{ for some } \delta \in X_\emptyset \setminus X\} \text{ and} \tag{2}$$

$$W = \{z \in Z : z \in \text{supp}(\delta_Z) \text{ for some } (\hat{z}, \delta) \in \text{supp}(\mu)\} \tag{3}$$

denote the set of manager, self-employed and workers types, respectively. Theorem 2 shows that E_{grh} and E^*_{grh} have the same set of stable matchings. It also establishes that each stable matching is characterized by four elements (z_1, z_2, ϕ, c), where $z_1 \in Z$ is the highest worker type, $z_2 \in Z$ is the lowest manager type, $\phi : M \to W$ assigns manager types to worker types (i.e. describes what is the type of workers that each type of managers hires) and $c : W \to [0, 1]$ is the wage function, and describes their properties.

Theorem 2 *The following conditions are equivalent:*

(a) μ *is a stable matching of* E_{grh}.

(b) μ *is a stable matching of* E^*_{grh}.

(c) There exists (z_1, z_2, ϕ, c) *such that*

1. $0 < z_1 \leq z_2 < \bar{z}$,
2. $W = [0, z_1]$, $M = [z_2, \bar{z}]$,
3. $S \neq \emptyset$ *if and only if* $z_1 < z_2$, *in which case* $S = [z_1, z_2]$,
4. $\phi : M \to W$ *is strictly increasing, differentiable,* $\phi(z_2) = 0$, $\phi(\bar{z}) = z_1$ *and, for each* $z \in M$,

$$\phi'(z) = \frac{\theta(z)}{h(1 - F(\phi(z)))\theta(\phi(z))},$$

5. *$c : W \to [0,1]$ is strictly increasing, differentiable and, for each $z \in W$,*

$$c'(z) = f(z)\frac{F(\phi^{-1}(z)) - c(z)}{1 - F(z)},$$

6. *$\mu = \nu \circ \sigma^{-1}$ where $\sigma : [z_1, \bar{z}] \to Z \times X_\emptyset$ is defined by setting, for each $z \in [z_1, \bar{z}]$,*

$$\sigma(z) = \begin{cases} (z, n(\phi(z))1_{(\phi(z), c(\phi(z)))}) & \textit{if } z \in [z_2, \bar{z}], \\ (z, 1_{(\emptyset, 0)}) & \textit{if } z \in [z_1, z_2), \end{cases}$$

7. *$c(z_1) = F(z_1)$ and $F(z_2) = (F(z_2) - c(0))n(0)$ if $S \neq \emptyset$, and*

8. *$c(z_2) = (F(z_2) - c(0))n(0) \geq F(z_2)$ if $S = \emptyset$.*

We say that μ is *represented by* (z_1, z_2, ϕ, c) when condition (c) of Theorem 2 holds.

Theorem 2 shows that there is a strictly positive measure of workers and managers, the lowest types (those in $[0, z_1]$) are workers and the highest types (those in $[z_2, \bar{z}]$) are managers. There may or may not be self-employed types; if there are, then they are in the middle of worker and manager types (i.e. are those in $[z_1, z_2]$).

The assignment function ϕ is strictly increasing and assigns the lowest manager type (z_2) to the lowest worker type (0) and the highest manager type ($\bar{z}$) to the highest worker type (z_1). Thus, the matching is positive assortative. The assignment function is also differentiable and satisfies an initial value problem that is equivalent to feasibility, i.e. to the requirement that, for each Borel subset B of types, the measure of types in B equals the measure of types in B that are either managers or self-employed or workers.

The wage function is strictly increasing, thus more knowledgable workers are paid more. It is also differentiable and satisfies an ordinary differentiable equation consisting of the first order condition of a maximization problem that managers of type $\phi^{-1}(z)$ solve, namely the problem

$$\max_{z' \in W} \frac{F(\phi^{-1}(z)) - c(z')}{h(1 - F(z'))}$$

of finding their preferred type of worker of which z is the solution.

Each stable matching μ is then fully described by (z_1, z_2, ϕ, c), which means that μ is the inverse measure of the type distribution ν with respect to σ and that σ is fully described by (z_1, z_2, ϕ, c).

The definitions of M, S and W imply that each of these sets are closed and their union is Z. Thus, they intersect. More precisely, when $S = \emptyset$, $z_1 = z_2$ belongs to $M \cap W$ and, therefore, type z_2 is indifferent between being a worker and a manager, which he weakly prefers to being self-employed: $c(z_2) = (F(z_2) - c(0))n(0) \geq F(z_2)$. When $S \neq \emptyset$, then z_1 belongs to $W \cap S$ and z_2 belongs to $S \cap M$. Thus, $c(z_1) = F(z_1)$ and $F(z_2) = (F(z_2) - c(0))n(0)$.

The proof of Theorem 2 is quite involved as it has to deal with several difficulties. The following outline describes its key steps and highlights the generality of the argument, thus providing a toolkit to analyze stable matchings in general knowledge economies. Section 3.2.5 concerns the sufficiency part of Theorem 2, whereas Sections 3.2.1–3.2.4 are for its necessity part. Due to the equivalence of the stable matchings of E_{grh} and E^*_{grh}, we focus on E^*_{grh} which is the simpler of the two and, thus, let μ be a fixed stable matching of E^*_{grh} for the latter sections.

3.2.1 High types are managers

The possibility that $F(\bar{z}) = 1$ causes some technical difficulties, namely, it implies that the set X of feasible matches for managers is not closed. This complicates some arguments, as it was already the case regarding existence of stable matchings, and, as in there, it is convenient to show that all types close to $\bar{z}$ cannot be workers. Lemma 1 shows a stronger conclusion, namely that all types close to $\bar{z}$ must be managers.

Lemma 1 *There exists $\xi > 0$ such that $[\bar{z} - \xi, \bar{z}] \cap (W \cup S) = \emptyset$.*

The conclusion of Lemma 1 is a particular case of parts 1 and 2 of condition (c) in Theorem 2 and, accordingly, it will be strengthened. Lemma 1 is a preliminary result which is only needed when $F(\bar{z}) = 1$ to make the set of feasible matches that are actually used by managers be compact.

3.2.2 Equal treatment

In a match $(z, n(z')1_{(z',c)})$ between a manager and workers, the wage c is a transfer from the former to the latter. This then implies an equal treatment property, namely, individuals of the same type must be equally well off. The reason is that, e.g., a type z manager with a lower rent than another type z manager matched with $n(z')1_{(z',c)}$ can hire type z' workers at a slightly higher wage $c + \varepsilon$, with $\varepsilon > 0$, to obtain a rent virtually equal to that of the latter and, thus, higher than his own. But this is a contradiction to the stability of the matching.

Focusing on managers, the above argument shows that there exists a function $u : M \to \mathbb{R}$ such that $u(z)$ is the rent of a manager of type z for each $z \in M$. An analogous argument shows that this function is continuous since, e.g., a type z manager with a rent lower and bounded away from the rent of some close by manager $\tilde{z}$ could attract the workers of the latter by paying slightly more and obtain a rent virtually equal to $u(\tilde{z})$ which is higher than $u(z)$. This would again contradict the stability of the matching.

Lemma 2 *There exists a continuous function $u : Z \to \mathbb{R}$ such that*

1. $u(z) = U_z(m, \delta)$ *for each* $z \in M$ *and* $\delta \in X$ *such that* $(z, \delta) \in \operatorname{supp}(\mu)$,
2. $u(z) = U_z(s)$ *for each* $z \in S$,
3. $u(z) = U_z(w, 1_{(\hat{z},c)})$ *for each* $z \in W$ *and* $(\hat{z}, c) \in Z \times C$ *such that* $(\hat{z}, n(z)1_{(z,c)}) \in \operatorname{supp}(\mu)$.

3.2.3 Stable assignments

The equal treatment property is essentially a consequence of the observation that the wage is a transfer from managers to workers. This also brings the model into the realm of matching with transferable utility as we next show.

Let $Y = [0, \bar{z} - \xi]$, $Y_\emptyset = Y \cup \{\emptyset\}$, $F(\emptyset) = 0$ and $n(\emptyset) = 1$. Define the surplus function $s : Z \times Y_\emptyset \to \mathbb{R}$ by setting, for each $(z, z') \in Z \times Y_\emptyset$,

$$s(z, z') = F(z)n(z').$$

Note that s is just total production, is continuous and satisfies $s(z, \emptyset) = F(z)$ for each $z \in Z$. Stability in the transferable utility setting is typically defined using a measure on the set of types to match, here $Z \times Y_\emptyset$, the surplus function s and a function like u in Lemma 2. The latter is required to be defined on $Z_\emptyset$, hence we set $u(\emptyset) = 0$.

An *assignment* is $\gamma \in M(Z \times Y_\emptyset)$ such that, for each Borel subset B of Z,

$$\gamma(B \times Y_\emptyset) + \int_{Z \times (B \cap Y)} n(z') \mathrm{d}\gamma(z, z') = \nu(B). \tag{4}$$

Note that $\gamma(B \times Y_\emptyset)$ is the measure of those who are managers or self-employed and have a type in B and that $\int_{Z \times (B \cap Y)} n(z') \mathrm{d}\gamma(z, z')$ is the measure of those who are workers and have a type in B; thus (4) is equivalent to the feasibility requirement (condition 2) in the definition of a matching.

Let γ be an assignment and $v \in C(Z_\emptyset)$.[17] Then (γ, v) is *stable* if $v(z)+n(z')v(z') = s(z, z')$ for each $(z, z') \in \text{supp}(\gamma)$ and $v(z) + n(z')v(z') \geq s(z, z')$ for each $(z, z') \in Z \times Y_\emptyset$. Moreover, γ is *stable* if there exists $v \in C(Z_\emptyset)$ such that (γ, v) is stable.

The following lemma shows that any stable matching yields a stable assignment via a simple transformation and the function u. The transformation uses the fact that a match is either between a manager and workers, hence of the form $(z, n(z')1_{(z',c(z'))})$, or between a self-employed individual and the null type, hence of the form $(z, 1_{(\emptyset,0)})$; in both cases, it is fully described by a pair $(z, z') \in Z \times Y_\emptyset$.

Formally, let $c : W \to \mathbb{R}$ be the restriction of u to W, $X^* = \{n(z)1_{(z,c(z))} : z \in Y\}$ and $X^*_\emptyset = X^* \cup \{1_{(\emptyset,0)}\}$. It follows by Lemma 1 and by the definition of a matching that $\text{supp}(\mu) \subseteq Z \times X^*_\emptyset$. Let, for convenience, $c(\emptyset) = 0$ and recall that $n(\emptyset) = 1$; thus, $1_{(\emptyset,0)} = n(\emptyset)1_{(\emptyset,c(\emptyset))}$ and, for each $\delta \in X^*_\emptyset$, there is $z' \in Y_\emptyset$ such that $\delta = n(z')1_{(z',c(z'))}$. Let $g : Z \times X^*_\emptyset \to Z \times Y_\emptyset$ be defined by setting, for each $(z, \delta) \in Z \times X^*_\emptyset$,

$$g(z, \delta) = (z, z')$$

where $z' \in Y_\emptyset$ is such that $\delta = n(z')1_{(z',c(z'))}$. Then g is a homeomorphism between $\text{supp}(\mu)$ and $g(\text{supp}(\mu)) \subseteq Z \times Y_\emptyset$ (see Lemma 19 in the appendix). More importantly, $\mu \circ g^{-1}$ is a measure on $Z \times Y_\emptyset$ and it turns out that $(\mu \circ g^{-1}, u)$ is stable.

[17] Given a metric space T, $C(T)$ denotes the space of bounded and continuous real-valued functions on T.

Lemma 3 *$\gamma = \mu \circ g^{-1}$ is a stable assignment.*

3.2.4 Surplus maximization

The conclusion that stable matchings transform into stable assignments allows us to use the techniques that are standard in the analysis of matching problems with transferable utility, namely those of optimal transport.

For each $\gamma \in M(Z \times Y_\emptyset)$, recall from Footnote 12 that $\gamma_Z \in M(Z)$ is the marginal of γ on Z, and that $\gamma_{Y_\emptyset} \in M(Y_\emptyset)$ is the marginal of γ on $Y_\emptyset$. Let $\gamma_{Y,n} \in M(Z)$ be defined by setting, for each Borel subset B of Z, $\gamma_{Y,n}(B) = \int_{B\cap Y} n(z')\mathrm{d}\gamma_{Y_\emptyset}(z')$. Define $\Gamma = \{\gamma \in M(Z \times Y_\emptyset) : \gamma_Z + \gamma_{Y,n} = \nu\}$; the set Γ is simply the set of assignments since the equation $\gamma_Z + \gamma_{Y,n} = \nu$ is just a succinct way of writing the feasibility condition (4).

An assignment γ is *surplus maximizing* if it solves

$$\max_{\tau\in\Gamma} \int_{Z\times Y_\emptyset} s\mathrm{d}\tau.^{18} \tag{5}$$

Lemma 4 shows that any stable assignment solves the optimal transport problem (5).

Lemma 4 *If γ is a stable assignment, then γ is surplus maximizing.*

One way of deriving the properties of stable matchings is by directly applying the definition of stability. Lemma 4 gives us a alternative way to characterize who matches with whom. This alternative approach is useful when solving (5) is easier than analysing stable matchings directly and the following lemma is helpful in this regard by providing a necessary condition for its solutions.

A set $C \subseteq Z \times Y_\emptyset$ is *s-monotone* if $\int_{Z\times Y_\emptyset} s\mathrm{d}\zeta \geq \int_{Z\times Y_\emptyset} s\mathrm{d}\tau$ for each finite measure ζ concentrated on finitely many points of C and for each finitely-supported measure τ on $Z \times Y_\emptyset$ such that $\tau_Z + \tau_{Y,n} = \zeta_Z + \zeta_{Y,n}$. In words, no finitely-supported measure on C can be improved, in the sense of yielding a higher surplus, by transporting

[18] In the appendix we often use $s(\tau)$ to denote $\int_{Z\times Y_\emptyset} s\mathrm{d}\tau$.

mass in a feasible way. Lemma 5 shows that the support of any solution to (5) is s-monotone.[19]

Lemma 5 *If* γ *is surplus maximizing, then* $\text{supp}(\gamma)$ *is s-monotone.*

3.2.5 Uniqueness of the conditions in Theorem 2

The previous lemmas are useful tools to establish necessary conditions that stable matchings need to satisfy. The conditions 1–8 in Theorem 2 are parsimonious in the sense that there is a considerable distance between them and the conditions that express individual rationality and that no one can increase his payoff by changing either his occupation to become a manager or his match while remaining a manager.

Lemma 6 forms the core of the argument showing that conditions 1–8 in Theorem 2 are sufficient for stable matchings. It shows that there can only be one (z_1, z_2, ϕ, c) satisfying these conditions. This then implies that if μ is represented by (z_1, z_2, ϕ, c) and $\hat{\mu}$ is a stable matching whose existence is guaranteed by Theorem 1, then μ must equal $\hat{\mu}$ since the latter is also represented by (z_1, z_2, ϕ, c); hence, μ is a stable matching.

Lemma 6 *If* (z_1, z_2, ϕ, c) *and* $(\hat{z}_1, \hat{z}_2, \hat{\phi}, \hat{c})$ *satisfy conditions 1–8 in Theorem 2, then* $(z_1, z_2, \phi, c) = (\hat{z}_1, \hat{z}_2, \hat{\phi}, \hat{c})$.

The proof of Lemma 6 relies considerably on properties of solutions to initial value problems.[20]

[19] Lemma 5 is analogous to Beiglböck and Griessler (2019, Theorem 1.4) and its proof follows that of the latter. Beiglböck and Griessler (2019) consider the weaker notion of finitely minimal/c-monotone sets which adds the requirement that $\tau(Z \times Y_\emptyset) = \zeta(Z \times Y_\emptyset)$; their Theorem 4.1 also requires γ and all elements of Γ to be probability measures. Lemma 5 dispenses with both requirements.

[20] Specifically, it relies on the following result for which we found no reference and which may be widely useful: Let $G : [a, b] \times [\hat{a}, \hat{b}] \to \mathbb{R}$ be continuous and such that $(z, x) \mapsto \frac{\partial G(z,x)}{\partial x}$ is continuous, where $a < b$ and $\hat{a} < \hat{b}$. Then consider two solutions to the ordinary differential equation $x' = G(z, x)$ that are allowed to differ at the end point a; namely, let $g : [a, b] \to [\hat{a}, \hat{b}]$ be a solution to the initial value problem $x' = G(z, x)$ and $x(a) = g(a)$, and $\hat{g} : [a, b] \to [\hat{a}, \hat{b}]$ be a solution to the initial value problem $x' = G(z, x)$ and $x(a) = \hat{g}(a)$. If one of them, g say, is strictly increasing and the two coincide at some point $z_0 \in [a, b]$, i.e. $g(z_0) = \hat{g}(z_0)$, then must coincide everywhere, i.e. $g = \hat{g}$.

3.3 Uniqueness

Our third main result shows that there is a unique stable matching.

Theorem 3 *There is a unique stable matching in* E_{grh} *and in* E^{*}_{grh}.

Theorem 3 follows easily from Theorem 2 and Lemma 6 since, given two stable matchings, they are represented by the same (z_1, z_2, ϕ, c) and, hence, are equal. It is possible to provide a shorter proof of the theorem using measure theoretical methods (e.g. via Ahmad, Kim, and McCann (2011, Lemma 3.1)) instead of Lemma 6; using the latter has the advantage of showing easily that conditions 1–8 in Theorem 2 are sufficient for stable matchings.

3.4 An example

We illustrate our main results using the example in Proposition 1 of Garicano and Rossi-Hansberg (2004). In this example, $\bar{z} = 1$ and $f = \theta \equiv 1$ (in particular, we then have that $F(z) = z$ for each $z \in Z$). Then the unique stable matching is represented by (z_1, z_2, ϕ, c) such that

$$\phi(z) = 1 - \sqrt{1 - \frac{2(z - z_2)}{h}} \text{ for each } z \in [z_2, 1],$$
$$c(z) = c(0) + (z_2 - c(0))z + \frac{hz^2}{2} \text{ for each } z \in [0, z_1],$$

with

$$z_1 = z_2,$$
$$z_2 = 1 + \frac{1}{h} - \sqrt{1 + \frac{1}{h^2}},$$
$$c(0) = \frac{z_2\left(1 - h\frac{2+h}{2}z_2\right)}{1 + h(1 - z_2)}$$

if $0 < h \leq 3/4$ and

$$z_1 = 1 - \sqrt{1 - \frac{2(1 - z_2)}{h}},$$
$$z_2 = \frac{2 - h}{h} - \sqrt{\frac{3 - 4h + h^2}{h^2}},$$
$$c(0) = (1 - h)z_2$$

if $3/4 < h < 1$.

It follows by Theorems 1 and 3 that there is a unique stable matching. By Theorem 2, it is represented by (z_1, z_2, ϕ, c). Hence, it remains to show that the above (z_1, z_2, ϕ, c) satisfies the conditions that Theorem 2 imposes on them, namely conditions 1, 4, 5 plus 7 if $z_1 < z_2$ and 8 if $z_1 = z_2$. This is relatively straightforward and the details are in Appendix A.7.

3.5 Convexity of earnings

The convexity of individual payments — or income or earnings — means, as Rosen (1981) noted, that "small differences in talent become magnified in larger earnings differences" and serves as an explanation "for differential skew between the distributions of income and talent." Earnings in our setting are described by the function $u : Z \to \mathbb{R}$ in Lemma 2. However, the elements in its domain Z have no economic meaning; instead, for each $z \in Z$, it is the fraction of problems $F(z)$ that an individual with knowledge z can solve that has economic meaning. This suggests that $F(z)$ should be used as a measure of talent and, thus, that it is $u \circ F^{-1}$ rather than u that can be expected to be convex. Corollary 1 shows that $u \circ F^{-1}$ is indeed convex.

Corollary 1 *The function $u \circ F^{-1}$ is convex on $F(Z)$ and strictly convex both on $F(W)$ and on $F(M)$.*

Antràs, Garicano, and Rossi-Hansberg (2006) have established the conclusion of Corollary 1 when F is the cumulative distribution function of the uniform distribution, in which case $u \circ F^{-1} = u$. We establish it simply by examining the second derivative of $u \circ F^{-1}$ but note that focusing on the uniform distribution case suffices. Indeed, if E_u is obtained from E^*_{grh} by setting the set of types to be $F(Z)$ with distribution $\nu \circ F^{-1}$ and the distribution of problems to be uniform, then the stable matchings of E_u are in a one-to-one relationship with those of E^*_{grh}.[21]

[21] This result provides a justification for the claim in footnote 4 of Fuchs, Garicano, and Rayo (2015). See the supplementary material to this paper for details and also for a proof of Corollary 1.

4 Rosen meets Garicano and Rossi-Hansberg

Our formal comparison between Rosen (1982) and Garicano and Rossi-Hansberg (2004) will reveal that their differences fall in three categories. The first concerns the factor share of workers: This equals $1 - \alpha \in (0, 1)$ under a Cobb-Douglas specification of the production function in Rosen (1982) and equals 1 in Garicano and Rossi-Hansberg (2004). A second difference concerns the payoff of self-employed individuals: This is zero in Rosen (1982) and $F(z)$ for a self-employed individual of type z in Garicano and Rossi-Hansberg (2004). The third difference comes from the number of workers each manager can hire: This number is $n(z) = 1/h(1 - F(z))$ for workers of type z in Garicano and Rossi-Hansberg (2004) and unbounded in Rosen (1982).

These differences have a significant impact on the conclusions derived from these two settings. Indeed, matching is (strictly) positive assortative and wages are strictly increasing in the worker's knowledge in Garicano and Rossi-Hansberg (2004) whereas the matching is indeterminate and wages are constant in Rosen (1982).

We will show that these differences in conclusions obtained from the settings of Rosen (1982) and Garicano and Rossi-Hansberg (2004) are due to the differences in the number of workers each manager can hire. We define, for each $\alpha \in (0, 1)$, a market $E_{r,\alpha}$ which belongs to Rosen's (1982) setting and differs from E^*_{grh} in the above three elements. We then decompose the differences between $E_{r,\alpha}$ and E^*_{grh} as follows. First, we consider a market $E_{s,\alpha}$ which differs from $E_{r,\alpha}$ only in the number of workers each manager can hire to see the effects of changes in the latter. Then we consider a market E_s which differs from $E_{s,\alpha}$ only because the factor share of workers is 1 in the former. This also means that E_s differs from E^*_{grh} only due to the self-employed's payoff.[22] Our results can then be described as showing that

$$E_{r,\alpha} \neq E_{s,\alpha} \to E_s \simeq E^*_{\text{grh}} = E_{\text{grh}}. \tag{6}$$

In this section we will elaborate on (6), which is meant as a simply mnemonic device.

[22]To put it differently, E_s is the market considered in Antràs, Garicano, and Rossi-Hansberg (2006).

Its meaning is that the predictions of $E_{r,\alpha}$ are considerably different from those of $E_{s,\alpha}$ when α is close to zero; in this case, the predictions of $E_{s,\alpha}$ are close to those of E_s, which in turn are similar to those of E^*_{grh} and of E_{grh} (the predictions of E^*_{grh} and E_{grh} are the same by Theorem 2). Hence, differences in either the factor share of workers or the self-employment payoffs cannot account for the differences between the predictions of $E_{r,\alpha}$ and E_{grh} and, in fact, these are due to the difference in the number of workers a manager can hire. As we have argued in the introduction, this difference follow from the presence of coordination costs in Garicano and Rossi-Hansberg (2004) and their absence in Rosen (1982).

The reason why the number of workers a manager can hire plays such an important role is as follows. The specification of the production function that unifies the two settings is that output is $F(z)z^{\alpha}n^{1-\alpha}$ when a manager with knowledge z hires n workers, where $\alpha \in (0,1)$ in Rosen (1982) and $\alpha = 0$ in Garicano and Rossi-Hansberg (2004). Thus, when $\alpha \in (0,1)$, the marginal productivity of labor and, hence, its demand when each manager can hire an unbounded number of workers does not depend on the worker's knowledge. Workers are therefore undifferentiated from the managers' viewpoint and a single wage suffices, its role being only to determine the relative attractiveness of the two occupations to make the total number of workers be equal to the demand by managers. Furthermore, the matching is indeterminate since each manager is indifferent between all types of workers. In contrast, when each manager can only hire $n(z')$ workers with knowledge z' and α is equal (or very close to) zero, then the constraint binds. If wages were constant, then each manager would only demand the most knowledgeable worker since $z' \mapsto n(z')$ is strictly increasing; this would require a measure zero of workers to fulfill the entire demand for workers which is impossible. Thus, wages must be strictly increasing. Furthermore, better managers benefit more than worse managers from hiring more workers, hence this makes matching positive assortative. In fact, these two properties go hand in hand: the best manager hires the best worker and the wage is such that the best worker is optimal for the best manager and not optimal for all the remaining managers. Changing the payoff of self-employed individuals may change the set of self-employed individuals and its measure but it won't change the properties of the allocation of workers to managers; these will also not change by changing α from zero to close to zero since what matters is whether or not the constraint on the number of workers a manager can hire is binding.

We represent all the above knowledge economies as markets which, recall, are described by a list $(Z, \nu, C, \mathbb{C}, X, (\succ_z)_{z\in Z})$. The following elements are common to all the markets we consider in this section and are (roughly) as in Section 2.1: $Z = [0, \bar{z}]$, with $F(\bar{z}) < 1$ to avoid technical issues, ν is the type distribution and has a continuously differentiable and strictly positive density θ, $C = \mathbb{R}_+$ is the set of possible wages, $\mathbb{C}(z, z') = C$ and $\mathbb{C}(z, \emptyset) = \{0\}$, preferences $\succ_z$ are represented by utility functions U_z and workers' utility equals their wage: $U_z(w, 1_{(z',c)}) = c$ for each $z, z' \in Z$ and $c \in C$. In this section we assume that

$$h \leq \frac{1}{2F(\bar{z})} - 1 + \sqrt{1 + \frac{1}{4F(\bar{z})^2}}. \tag{7}$$

For example, if the most knowledgeable worker can solve 80% of the problems, i.e. $F(\bar{z}) = 0.8$, then (7) is satisfied if $h \leq 0.8$. Furthermore, (7) is satisfied independently of $F(\bar{z})$ if $h \leq (\sqrt{5} - 1)/2$, hence if $h \leq 0.61$.

We define $E_{r,\alpha}$ for each $\alpha \in (0, 1)$ by setting

$$\begin{aligned} &X = \{n1_{(z,c)} : (z, c) \in Z \times C \text{ and } n \in \mathbb{R}_+\}, \\ &U_z(s) = 0 \text{ for each } z \in Z, \text{ and} \\ &U_z(m, n1_{(z',c)}) = F(z) z^\alpha n^{1-\alpha} - cn \text{ for each } (z, z', c, n) \in Z^2 \times C \times \mathbb{R}_+. \end{aligned}$$

The market $E_{r,\alpha}$ is a particular case of Rosen's (1982) setting when represented as a Rosen market, i.e. as a large many-to-one matching market with occupational choice, as shown in Carmona and Laohakunakorn (2024, Section 6.2).[23]

[23] We set $\mathbb{C}(z, \emptyset) = C$ for each $z \in Z$ in Carmona and Laohakunakorn (2024, Section 6.2), which differs from Section 2.2 where $\mathbb{C}(z, \emptyset) = \{0\}$. This difference has no importance since $\text{supp}(\mu) \subseteq Z \times X$ in any stable matching μ of a Rosen market (see Carmona and Laohakunakorn (2024, Claim 11). The only assumption of Carmona and Laohakunakorn (2024) which is not satisfied here is $r(0) > 0$, but as we show in the supplementary material to this paper, the assumption that $r(0) > 0$ can be dropped when the production function takes the form specified by $E_{r,\alpha}$.

In Rosen (1982), the number of workers that a manager can hire is unbounded and the self-employed payoff is zero. Thus, the differences in X and $U_z(s)$ between $E_{r,\alpha}$ and E^*_{grh} are unavoidable. The market $E_{r,\alpha}$ is useful to compare the settings of Rosen (1982) and Garicano and Rossi-Hansberg (2004) because it reduces further differences to a single parameter, which is the factor share of workers $1-\alpha$. Indeed, when $\alpha = 0$,

$$F(z)z^{\alpha}n^{1-\alpha} - cn = (F(z) - c)n$$

for each $(z, c, n) \in Z \times C \times \mathbb{R}_+$ and, hence, $E_{\alpha,r}$ differs from E^*_{grh} only due to the differences in α, X and $U_z(s)$.

These differences have a significant impact on the stable matchings of these markets as can be seen by comparing Theorem 2 to the following result that builds on Theorem 3 in Carmona and Laohakunakorn (2024) to characterize the stable matchings of $E_{r,\alpha}$.

Theorem 4 *A matching μ is stable in $E_{r,\alpha}$ if and only if there exists $\gamma \in M(Z^2)$, $z_1 \in (0, \bar{z})$ and $w > 0$ such that*

1. $\mu = \gamma \circ \tilde{g}^{-1}$*, where, for each* $(z, z') \in Z^2$*,*

$$\tilde{g}(z, z') = (z, n(z, w)1_{(z',w)}), \text{ and}$$
$$n(z, w) = \left(\frac{(1-\alpha)F(z)}{w}\right)^{\frac{1}{\alpha}} z,$$

2. $\gamma(B \times Z) + \int_{Z \times B} n(z, w) \mathrm{d}\gamma(z, z') = \nu(B)$ *for each Borel* $B \subseteq Z$*,*

3. $W = [0, z_1]$*,* $M = [z_1, \bar{z}]$*, and*

4. $F(z_1)z_1^{\alpha}n(z_1, w)^{1-\alpha} - wn(z_1, w) = w$*.*

5. *Furthermore,* z_1 *and* w *are unique.*

Theorem 4 shows that, in any stable matching of $E_{r,\alpha}$, workers, who are those with a knowledge level in $[0, z_1]$, receive a wage $w > 0$ and that every manager with knowledge $z \in [z_1, \bar{z}]$ hires the same number $n(z, w)$ of workers. The actual matching between workers and managers is indeterminate and can be done in any way such that the feasibility condition (i.e. 2 in its statement) holds. The marginal type is z_1 and is, therefore, indifferent between being a worker or a manager.[24]

Thus, in contrast to the case of E^*_{grh}, the stable matchings of $E_{r,\alpha}$ are not unique, the matching of managers and workers is indeterminate and, thus, need not be positive assortative, and wages are constant. We show in what follows that the differences in α and $U_z(s)$ are unimportant to explain these differences in the properties of stable matchings of $E_{r,\alpha}$ and E^*_{grh} (and E_{grh}) and, thus, we trace these differences to differences in X.

We decompose the differences in $E_{r,\alpha}$ and E^*_{grh} by introducing markets $E_{s,\alpha}$, $\alpha \in (0,1)$, and E_s as follows. Define E_s by setting

$$
\begin{aligned}
&X = \{n(z)1_{(z,c)} : (z,c) \in Z \times C\},\\
&U_z(s) = 0 \text{ for each } z \in Z, \text{ and}\\
&U_z(m, n1_{(z',c)}) = (F(z) - c)n \text{ for each } (z, z', c, n) \in Z^2 \times C \times \mathbb{R}_+.
\end{aligned}
$$

Thus E_s differs from E^*_{grh} only by the self-employed's payoff $U_z(s)$.[25] This change implies that, unlike in E^*_{grh}, there are no self-employed individuals in the stable matchings of E_s regardless of the parameters f, θ, h and $\bar{z}$ but otherwise the properties of the stable matchings are the same. The latter is a consequence of the following characterization of the stable matchings of E_s.

Theorem 5 *The market E_s has a unique stable matching and μ is a stable matching of E_s if and only if there exists (z_1, ϕ, c) such that*

1. $0 < z_1 < \bar{z}$,

2. $W = [0, z_1]$, $M = [z_1, \bar{z}]$,

[24] See the supplementary material to this paper for the case where both $\theta = f \equiv 1$, in which closed forms for z_1 and w can be obtained.

[25] The use of the subscript s in E_s emphasizes this difference. Note that the definition of stable matchings in Section 2.3 was provided for a general function $U_z(s)$ and, hence, applies to E_s too.

3. $\phi : M \to W$ *is strictly increasing, differentiable,* $\phi(z_1) = 0$, $\phi(\bar{z}) = z_1$ *and, for each* $z \in M$,

$$\phi'(z) = \frac{\theta(z)}{h(1 - F(\phi(z)))\theta(\phi(z))},$$

4. $c : W \to [0, 1]$ *is strictly increasing, differentiable and, for each* $z \in W$,

$$c'(z) = f(z)\frac{F(\phi^{-1}(z)) - c(z)}{1 - F(z)},$$

5. $\mu = \nu \circ \sigma^{-1}$ *where* $\sigma : [z_1, \bar{z}] \to Z \times X$ *is defined by setting, for each* $z \in [z_1, \bar{z}]$,

$$\sigma(z) = \left(z, n(\phi(z))1_{(\phi(z), c(\phi(z)))}\right), \text{ and}$$

6. $c(z_1) = (F(z_1) - c(0))n(0) > 0.$

Theorems 2 and 5 imply that the stable matchings of E^*_{grh} and E_s have the same properties. In fact, when $(f, \theta, h, \bar{z})$ are such that the stable matching of E^*_{grh} has no self-employed individuals, then E^*_{grh} and E_s have the same stable matching.[26] When $(f, \theta, h, \bar{z})$ are such that the stable matching of E^*_{grh} has self-employed individuals, then the stable matchings of E^*_{grh} and E_s are not the same but have the same properties: occupations are ordered in the same way, managers and workers are matched in a strictly increasing i.e. positive assortative way, wages are strictly increasing and, although with different initial conditions, the wage and assignment functions c and ϕ satisfy the same differential equations. In the sense of this paragraph, the difference in the self-employed payoffs between E^*_{grh} and $E_{r,\alpha}$ — which is the only difference between E^*_{grh} and E_s — cannot explain the differences in their stable matchings.

We next argue that the difference in α cannot explain the differences in the stable matchings of E^*_{grh} and $E_{r,\alpha}$. To this end, define $E_{s,\alpha}$ by setting

$X = \{n(z)1_{(z,c)} : (z, c) \in Z \times C\}$,

$U_z(s) = 0$ for each $z \in Z$, and

$U_z(m, n1_{(z',c)}) = F(z)z^\alpha n^{1-\alpha} - cn$ for each $(z, z', c, n) \in Z^2 \times C \times \mathbb{R}_+$.

[26] Indeed, if μ is the stable matching of E^*_{grh} and $S = \emptyset$, then μ is a stable matching of E_s since the incentive to become self-employed is not greater in E_s than in E^*_{grh}. The conclusion then follows because E_s has a unique stable matching.

Thus $E_{s,\alpha}$ differs from E_s only on α.[27] Theorem 6 below shows that the stable matchings of $E_{s,\alpha}$ are close to the stable matching of E_s when α is close to zero. Let $E_{s,0} = E_s$ and, for each $\alpha \in [0, 1)$, let $\Phi(\alpha)$ be the set of stable matchings of $E_{s,\alpha}$.

Theorem 6 *The correspondence Φ has nonempty values for each $\alpha \in [0, 1)$ and is continuous at $\alpha = 0$.*

In the sense of Theorem 6, the difference in the production function i.e. in α between E^*_{grh} and $E_{r,\alpha}$ — which is the only difference between $E_{s,\alpha}$ and E_s — cannot explain the differences in their stable matchings. Since the difference in the self-employed payoffs between E^*_{grh} and $E_{r,\alpha}$ also cannot explain the differences in their stable matchings, these must be caused by the differences in the number of workers a manager can hire i.e. by the differences in X. This can also be seen by comparing $E_{r,\alpha}$ and $E_{s,\alpha}$ since they differ only in X.

5 Concluding remarks

In this paper we used the framework of large matching markets with occupational choice in Carmona and Laohakunakorn (2024) to compare Rosen (1982) and Garicano and Rossi-Hansberg (2004) in a detailed way. They differ in three ways — factor share of labor, self-employed payoff and number of workers managers can hire — with the latter accounting for their distinct conclusions. The difference in the number of workers managers can hire can be understood in light of Becker and Murphy (1992) as differences in coordination costs, specifically as differences in communication costs between members of a firm, since coordination costs lead to bounds on the number of workers managers can hire, are present in Garicano and Rossi-Hansberg (2004) and absent in Rosen (1982).

Beyond a formal understanding of the economic mechanisms in those knowledge-based theories and the main driver of their distinct conclusions, the comparison readily

[27] Note that the definition of stable matchings in Section 2.3 was provided for a general function $U_z(m, \delta)$ and, hence, applies to $E_{s,\alpha}$ too.

suggests a unified and flexible setting. Namely, one in which, in addition to the common elements of the two settings, goods are produced according to a production function $\rho(r(z))\psi(r(z), nq(z'))$, where z is the manager's skill, z' is the workers' skills and n is the number of workers, each manager can hire at most $n(z, z')$ workers and self-employed individuals with skill z receive $U_z(s)$.

In light of the above formalization, it is natural to ask: Under what conditions on the functions ρ, r, q, ψ, n and $z \mapsto U_z(s)$ do we obtain the conclusions in Garicano and Rossi-Hansberg (2004)? Are there choices for them different from the ones in Garicano and Rossi-Hansberg (2004) that are useful to match real-world data?

We leave the above questions for further research. Answering them will likely require advances in the techniques we used in this paper. Indeed, while some results — such as Theorems 7 and 8 in the Appendix — are stated in general terms, others are, at least apparently, specific to the settings of Rosen (1982) and Garicano and Rossi-Hansberg (2004). Hence, their extension and unification will be useful to analyse more general models such as the one above.

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A Online Appendix

A.1 Proof of Theorems 1–3

In the section we describe how we will establish Theorems 1–3. An important simplification arises from the following lemma showing that the set of stable matchings of E_{grh} is contained in the set of stable matchings of E^*_{grh}.

Lemma 7 *If μ is a stable matching of E_{grh}, then μ is a stable matching of E^*_{grh}.*

Lemma 7 then implies that the conclusion of Theorem 1 follows once we show that E_{grh} has a stable matching. Furthermore, to establish Theorems 2 and 3, it suffices to show that E^*_{grh} has a unique stable matching and that conditions (b) and (c) in Theorem 2 are equivalent.

In the remainder of this section, we establish Lemma 7. We start by a lemma which will also be used in the proof of Theorem 1 and, since it uses a limit argument, is applied also to the following sequence of markets. For each $k \in \mathbb{N}$, let $C_k = [0,1]$, $\mathbb{C}_k(z,z') = C_k$ and $\mathbb{C}_k(z,\emptyset) = \{0\}$ for each $z, z' \in Z$, and $X_k = \{n1_{(z,c)} : (z,c) \in [0, \bar{z} - 1/k] \times C_k$ and $n \in \mathbb{R}_+$ such that $nh(1 - F(z)) = 1\}$ and E_k be equal to E_{grh} except for this change to X_k.

Lemma 8 below shows that if $(z, \delta) \in Z \times X$ is in the support of a stable matching of either E_{grh}, E^*_{grh} or E_k for some k, then (by the definition of X) $\delta = n(z')1_{(z',c)}$ for some $(z', c) \in Z \times C$ and, more importantly, $F(z') \le c \le F(z)$. This just shows that the manager is at least as knowledgable as the workers he employs (i.e. $z' \le z$) and that the wage is in between $F(z')$ and $F(z)$ by individual rationality. Furthermore, this shows that $c \in [0, F(\bar{z})] \subseteq [0,1]$.

Lemma 8 *Let E be E_{grh}, E^*_{grh} or E_k for some $k \in \mathbb{N}$. If μ is a stable matching of E and $(z, n(z')1_{(z',c)}) \in \text{supp}(\mu) \cap (Z \times X)$, then $F(z') \le c \le F(z)$.*

Proof. We write $U_z(m, n1_{(z',c)}) = (\max\{F(z), F(z')\} - c)n$ and $U^*_z(m, n1_{(z',c)}) = (F(z) - c)n$ for each $z, z' \in Z$, $c \in C$ and $n \in \mathbb{R}_+$.

Individual rationality implies that $U_{z'}(w, 1_{(z,c)}) = c \geq F(z') = U_{z'}(s)$. Individual rationality then implies that $c \leq F(z)$. Indeed, if $c > F(z)$, then $U^*_z(m, n1_{(z',c)}) = (F(z) - c)n(z') < 0 \leq U_z(s)$ and, if $F(z) > 0$,

$$U_z(m, n(z')1_{(z',c)}) = (\max\{F(z), F(z')\} - c)n(z') \leq 0 < U_z(s).$$

Thus, the conclusion of the lemma holds when $E = E^*_{\text{grh}}$ and also when $E \in \{E_{\text{grh}}, E_k\}$ if $F(z) > 0$.

Consider then the remaining case of $F(z) = 0$ and $E \in \{E_{\text{grh}}, E_k\}$. In this case we have that $c \leq F(z')$ by individual rationality of type z since $U_z(m, n(z')1_{(z',c)}) = (F(z') - c)n(z')$. Thus, $c = F(z')$ and $U_z(m, \delta) = 0$. If $F(z') > 0$, then type z' can hire workers of type z at wage $\varepsilon > 0$ such that $\varepsilon < F(z')(n(z) - 1)/n(z)$ to obtain $U_{z'}(m, 1_{(z,\varepsilon)}) = (F(z') - \varepsilon)n(z) > F(z') = c = U_{z'}(w, 1_{(z,c)})$, a contradiction to the stability of μ. Thus, it follows that $F(z') = 0$ and, hence, that $c = F(z') = 0 = F(z)$. ■

It follows from Lemma 8 that, if μ is a stable matching of E and $(z, n(z')1_{(z',c)}) \in \text{supp}(\mu) \cap (Z \times X)$, then $z' \leq z$ and that $U_z(m, n(z')1_{(z',c)}) = (F(z) - c)n(z')$.

We turn to the proof of Lemma 7

Proof of Lemma 7. Let μ be a stable matching of E_{grh}. To distinguish between E_{grh} and E^*_{grh}, we will write $T^m_z(\mu; E)$, $S_M(\mu; E)$ and $IR(\mu; E)$ where $E \in \{E_{\text{grh}}, E^*_{\text{grh}}\}$. Furthermore, let, for each $z \in Z$ and $n1_{(z',c)} \in X$, $U_z(m, n1_{(z',c)}) = (F(\max\{z, z'\}) - c)n$ and $U^*_z(m, n1_{(z',c)}) = (F(z) - c)n$. Note that there is no need to make $T^s_z(\mu)$ depend on $E \in \{E_{\text{grh}}, E^*_{\text{grh}}\}$ since $T^s_z(\mu) = \{(\emptyset, 0)\}$ in either case.

It follows by Lemma 8 that

$$U_z(m, \delta) = U^*_z(m, \delta) \text{ for each } (z, \delta) \in \text{supp}(\mu) \cap (Z \times X). \tag{8}$$

This then implies that $T^m_z(\mu; E_{\text{grh}}) = T^m_z(\mu; E^*_{\text{grh}})$ for each $z \in Z$ since condition (c) in the definition of these sets becomes the same (and condition (a) and (b) are the same too regardless of (8)).

It also follows that $\text{supp}(\mu) \cap IR(\mu; E_{\text{grh}}) = \text{supp}(\mu) \cap IR(\mu; E^*_{\text{grh}})$ since (8) implies that condition (i) in the definition of these sets becomes the same (and condition (ii) is the same too regardless of (8)).

We next show that $\text{supp}(\mu) \cap S_M(\mu; E_{\text{grh}}) \subseteq \text{supp}(\mu) \cap S_M(\mu; E^*_{\text{grh}})$. Indeed, if $(z, \delta) \in \text{supp}(\mu) \cap S_M(\mu; E_{\text{grh}})$ and δ' is as in condition (i), (ii) or (iii), then $U_z(m, \delta) = U^*_z(m, \delta)$ by (8) and $U_z(m, \delta') \geq U^*_z(m, \delta')$ by definition. Thus, $(z, \delta) \in \text{supp}(\mu) \cap S_M(\mu; E^*_{\text{grh}})$.

The stability of μ in E_{grh} then implies that

$$\text{supp}(\mu) = \text{supp}(\mu) \cap S_M(\mu; E_{\text{grh}}) \cap IR(\mu; E_{\text{grh}}) \subseteq \text{supp}(\mu) \cap S_M(\mu; E^*_{\text{grh}}) \cap IR(\mu; E^*_{\text{grh}}).$$

Hence, μ is a stable matching of E^*_{grh}. ■

A.2 Proof of Theorem 1

It is enough to show that E_{grh} has a stable matching to establish Theorem 1. A further simplification is obtained by the following lemma which shows that we may assume that wages are bounded by one.

Write $E_{\text{grh}} = (Z, \nu, C, \mathbb{C}, X, (\succ_z)_{z \in Z})$ and let $\hat{E} = (Z, \nu, \hat{C}, \hat{\mathbb{C}}, \hat{X}, (\succ_z)_{z \in Z})$ where $\hat{C} = [0, 1]$, $\hat{\mathbb{C}}(z, z') = \hat{C}$ and $\hat{\mathbb{C}}(z, \emptyset) = \{0\}$ for each $z, z' \in Z \times Z$, and $\hat{X} = \{n(z) 1_{(z,c)} : (z, c) \in Z \times \hat{C}$ and $n \in \mathbb{R}_+$ such that $nh(1 - F(z)) = 1\}$, i.e. $\hat{E}$ is equal to E_{grh} except for these changes to C, $\mathbb{C}$ and X.

Lemma 9 *If μ is a stable matching of $\hat{E}$, then μ is a stable matching of E_{grh}.*

Proof. We write $IR(\mu; E)$ and $S_M(\mu; E)$ for each $E \in \{E_{\text{grh}}, \hat{E}\}$. Let μ be a stable matching of $\hat{E}$. Then $\text{supp}(\mu) \subseteq S_M(\mu; \hat{E}) \cap IR(\mu; \hat{E})$. Since $IR(\mu; \hat{E}) = IR(\mu; E_{\text{grh}})$, it follows that $\text{supp}(\mu) \subseteq IR(\mu; E_{\text{grh}})$.

Thus, if μ is not a stable matching of E_{grh}, then there is $(z, z', c) \in Z \times Z_\emptyset \times C$ such that $(z, n(z') 1_{(z',c)}) \in \text{supp}(\mu) \setminus S_M(\mu; E_{\text{grh}})$ (with $n(\emptyset) = 0$) and, hence, $(\hat{z}, \tilde{z}, \tilde{c}) \in Z^2 \times C$ such that $\hat{z} \in \{z, z'\} \setminus \{\emptyset\}$, $\tilde{c} > 1$ and $U_{\hat{z}}(m, n(\tilde{z}) 1_{(\tilde{z},\tilde{c})}) > U_{\hat{z}}(a, \hat{\delta})$ with $a = m$ and $\hat{\delta} = n(z') 1_{(z',c)}$ if $\hat{z} = z$ and $a = w$ and $\hat{\delta} = 1_{(z,c)}$ if $\hat{z} = z'$ and $z' \neq \emptyset$. But $\tilde{c} > 1$ implies that $U_{\hat{z}}(m, n(\tilde{z}) 1_{(\tilde{z},\tilde{c})}) = (\max\{F(\hat{z}), F(\tilde{z})\} - \tilde{c}) n(\tilde{z}) < 0$. Hence, $U_{\hat{z}}(a, \hat{\delta}) < 0$ and, thus, either $U_z(m, n(z') 1_{(z',c)}) < 0$ or $z' \neq \emptyset$ and $U_{z'}(w, 1_{(z,c)}) < 0$, contradicting $(z, n(z') 1_{(z',c)}) \in IR(\mu; E_{\text{grh}})$. This contradiction then shows that μ is a stable matching of E_{grh}. ■

It follows by Lemma 9 that it suffices to shows that $\hat{E}$ has a stable matching.

For each $k \in \mathbb{N}$, let $X_k = \{n(z)1_{(z,c)} : (z,c) \in [0, \bar{z}-1/k] \times \hat{C}\}$ and E_k be equal to $\hat{E}$ except for this change to X_k. It follows by Theorem 2 in Carmona and Laohakunakorn (2024) that there exists a stable matching μ_k of E_k. Indeed, E_k is rational since the preferences of each type $z \in Z$ are represented by utility functions. It is also continuous since $(z, a, \delta) \mapsto U_z(a, \delta)$ is continuous, $\hat{\mathbb{C}}$ is continuous with nonempty and compact values and X_k is closed; the latter follows because X_k is the image of the continuous function $(z,c) \mapsto \frac{1}{h(1-F(z))} 1_{(z,c)}$ whose domain is compact.[28] We have that E_k is bounded since $\delta(Z \times \hat{C}) \leq \frac{1}{h(1-F(\bar{z}-1/k))} < \infty$ for each $\delta \in X$. Finally, E_k is rich since it satisfies conditions (α) and (β) in Carmona and Laohakunakorn (2024): (β) is straightforward since each $\delta \in X$ has finite support and (α) follows from the continuity of $(z,c) \mapsto \frac{1}{h(1-F(z))} 1_{(z,c)}$.

It follows from Lemma 8 that, if $(z, n(z')1_{(z',c)}) \in \text{supp}(\mu_k) \cap (Z \times X_k)$, then $z' \leq z$ and that $U_z(m, n(z')1_{(z',c)}) = (F(z) - c)n(z')$.

We will show that $\{\mu_k\}_{k=1}^\infty$ has a convergent subsequence and its limit point μ is a stable matching of $\hat{E}$. The key lemma in the convergence argument is that there is $M \in \mathbb{N}$ such that $\delta((\bar{z} - 1/M, \bar{z}] \times \hat{C}) = 0$ for each $(z, \delta) \in \text{supp}(\mu_k) \cap (Z \times X_k)$ and k sufficiently large, i.e. no one with type in $(\bar{z} - 1/M, \bar{z}]$ is a worker. This then bounds the number of workers that a manager hires and allow us to conclude that $\{\mu_k\}_{k=1}^\infty$ has indeed a convergent subsequence.

Lemma 10 *There exist $K, M \in \mathbb{N}$ such that, for each $k \geq K$ and $(z, \delta) \in \text{supp}(\mu_k) \cap (Z \times X_k)$, $\delta((\bar{z} - 1/M, \bar{z}] \times \hat{C}) = 0$.*

Proof. Suppose that the conclusion of the lemma fails. Then, for each $j \in \mathbb{N}$, there exists $k_j \geq j$ and $(z_{k_j}, \delta_{k_j}) \in \text{supp}(\mu_{k_j}) \cap Z \times X_k$ such that $\delta_{k_j}((\bar{z}-1/j, \bar{z}] \times \hat{C}) > 0$. Thus, for each $j \in \mathbb{N}$, there is $(z'_{k_j}, c_{k_j}) \in (\bar{z}-1/j, \bar{z}] \times \hat{C}$ such that $\delta_{k_j} = n(z'_{k_j}) 1_{(z'_{k_j}, c_{k_j})}$. It follows by Lemma 8 that $F(z'_{k_j}) \leq c_{k_j} \leq F(z_{k_j})$ for each j.

[28] To see the continuity of $(z,c) \mapsto \frac{1}{h(1-F(z))} 1_{(z,c)}$, let $f : [0, \bar{z} - 1/k] \times \hat{C} \to \mathbb{R}$ be bounded and continuous and let $(z,c) \in [0, \bar{z}-1/k] \times \hat{C}$ and $\{(z_j, c_j)\}_{j=1}^\infty \subseteq [0, \bar{z}-1/k] \times \hat{C}$ be such $(z_j, c_j) \to (z, c)$. Then $\int_{[0,\bar{z}-1/k]\times\hat{C}} f \mathrm{d} \frac{1}{h(1-F(z_j))} 1_{(z_j,c_j)} = \frac{f(z_j,c_j)}{h(1-F(z_j))} \to \frac{f(z,c)}{h(1-F(z))} = \int_{[0,\bar{z}-1/k]\times\hat{C}} f \mathrm{d} \frac{1}{h(1-F(z))} 1_{(z,c)}$.

Claim 1 *There exists* $\varepsilon > 0$ *and* $J \in \mathbb{N}$*, such that* $\text{supp}(\mu_{k_j}) \cap ([0, \varepsilon] \times X_{k_j,\emptyset}) = \emptyset$ *for each* $j \geq J$*.*

Proof. Let $\eta > 0$ be such that $(F(\bar{z}) - \eta)/h > F(\bar{z})$. Then let $0 < \varepsilon < \bar{z}$ be such that $\frac{F(z)}{h(1-F(z))} < \eta$ for all $z \in [0, \varepsilon]$.

If the claim fails, then, taking a subsequence if necessary, we may assume that $\text{supp}(\mu_{k_j}) \cap ([0, \varepsilon] \times X_{k_j,\emptyset}) \neq \emptyset$ and $z'_{k_j} > \varepsilon$ for each j. Consider first the case where $\text{supp}(\mu_{k_j}) \cap ([0, \varepsilon] \times (X_{k_j,\emptyset} \setminus X_{k_j})) \neq \emptyset$ for infinitely many js. In this case, taking a subsequence if necessary, we may assume that there exists, for each j, $\tilde{z}_{k_j} \in [0, \varepsilon]$ such that $(\tilde{z}_{k_j}, 1_{(\emptyset,0)}) \in \text{supp}(\mu_{k_j})$. Since $U_{\tilde{z}_{k_j}}(s, 1_{(\emptyset,0)}) = F(\tilde{z}_{k_j})$, then type z'_{k_j} can hire workers of type $\tilde{z}_{k_j}$ at wage $\eta > \frac{F(\tilde{z}_{k_j})}{h(1-F(\tilde{z}_{k_j}))} > F(\tilde{z}_{k_j})$ so that

$$
\begin{aligned}
U_{z'_{k_j}}(m, n(\tilde{z}_{k_j})1_{(\tilde{z}_{k_j},\eta)}) = \frac{F(z'_{k_j}) - \eta}{h(1 - F(\tilde{z}_{k_j}))} &\geq \frac{F(z'_{k_j}) - \eta}{h} \to \frac{F(\bar{z}) - \eta}{h} \\
&> F(\bar{z}) \geq c_{k_j} = U_{z'_{k_j}}(w, 1_{(z_{k_j},c_{k_j})}).
\end{aligned}
$$

This contradicts the stability of μ_{k_j} whenever j is sufficiently large.

Consider next the remaining case where $\text{supp}(\mu_{k_j}) \cap ([0, \varepsilon] \times X_{k_j}) \neq \emptyset$ for all j sufficiently large. In this case, there exists, for each j sufficiently large, $\tilde{z}_{k_j} \in [0, \varepsilon]$ and $\tilde{\delta}_{k_j} = n(\hat{z}_{k_j})1_{(\hat{z}_{k_j},\tilde{c}_{k_j})}$ such that $(\tilde{z}_{k_j}, \tilde{\delta}_{k_j}) \in \text{supp}(\mu_{k_j})$. Since

$$
U_{\tilde{z}_{k_j}}(m, \tilde{\delta}_{k_j}) = \frac{F(\tilde{z}_{k_j}) - \tilde{c}_{k_j}}{h(1 - F(\hat{z}_{k_j}))} \leq \frac{F(\tilde{z}_{k_j})}{h(1 - F(\tilde{z}_{k_j}))} < \eta
$$

due to $F(\hat{z}_{k_j}) \leq \tilde{c}_{k_j} \leq F(\tilde{z}_{k_j})$ which follows by Lemma 8, then type z'_{k_j} can hire workers of type $\tilde{z}_{k_j}$ at wage η as in the previous paragraph, thus contradicting the stability of μ_{k_j} whenever j is sufficiently large. ■

Claim 2 *For each* $j \in \mathbb{N}$*, there is* $(z, c) \in Z \times \hat{C}$ *such that* $(z, \frac{1}{h}1_{(0,c)}) \in \text{supp}(\mu_{k_j})$*.*

Proof. Let $j \in \mathbb{N}$ be fixed and suppose that, for each $(z, \delta) \in \text{supp}(\mu_{k_j})$, $0 \notin \text{supp}(\delta_Z)$. Then, due to the compactness of $\hat{C}$, Z and X_{k_j}, and, thus, of $\text{supp}(\mu_{k_j})$, there is $0 < \eta < \varepsilon$ such that $0 \notin B_\eta(\text{supp}(\delta_Z))$ for each $(z, \delta) \in \text{supp}(\mu_{k_j})$. This, together with Claim 1, implies that

$$
\nu(B_\eta(0)) = \int_{Z \times X_{k_j}} \delta(B_\eta(0) \times \hat{C}) \text{d}\mu_{k_j}(z, \delta) = 0.
$$

But this is a contradiction since $0 \in \text{supp}(\nu)$ and, hence, $\nu(B_\eta(0)) > 0$. ■

For each $j \in \mathbb{N}$, let $\underline{z}_{k_j} = \min\{z \in Z : (z, \delta) \in \text{supp}(\mu_{k_j})$ for some $\delta \in X_{k_j}\}$; since Z and X_{k_j} are compact, $\underline{z}_{k_j}$ exists.

Claim 3 *For each $j \in \mathbb{N}$, $(\underline{z}_{k_j}, \frac{1}{h}1_{(0,c)}) \in \text{supp}(\mu_{k_j})$ for some $c \in \hat{C}$.*

Proof. Let $j \in \mathbb{N}$ be fixed. It follows by Claim 2 that there is $(z, c) \in Z \times \hat{C}$ such that $(z, \frac{1}{h}1_{(0,c)}) \in \text{supp}(\mu_{k_j})$. It follows by the definition of $\underline{z}_{k_j}$ that $z \geq \underline{z}_{k_j}$.

Suppose, in order to reach a contradiction, that $(\underline{z}_{k_j}, \frac{1}{h}1_{(0,\hat{c})}) \notin \text{supp}(\mu_{k_j})$ for each $\hat{c} \in \hat{C}$. Then, in particular, $z > \underline{z}_{k_j}$ (by considering $\hat{c} = c$) and, letting $\delta = n(z')1_{(z',c')} \in X_{k_j}$ be such that $(\underline{z}_{k_j}, \delta) \in \text{supp}(\mu_{k_j})$, $z' > 0$ (by considering $\hat{c} = c'$). It follows that $U_{\underline{z}_{k_j}}(m, \delta) \geq U_{\underline{z}_{k_j}}(m, \frac{1}{h}1_{(0,c+\eta)})$ for each $\eta > 0$ and, hence, $U_{\underline{z}_{k_j}}(m, \delta) - U_{\underline{z}_{k_j}}(m, \frac{1}{h}1_{(0,c)}) \geq 0$. For each $\tilde{z} \geq \underline{z}_{k_j}$, we have that

$$U_{\tilde{z}}(m, \delta) - U_{\tilde{z}}(m, \frac{1}{h}1_{(0,c)}) = \frac{1}{h}\left(F(\tilde{z})\left(\frac{1}{1 - F(z')} - 1\right) - \frac{c'}{1 - F(z')} + c\right)$$

is strictly increasing in $\tilde{z}$ since $z' > 0$ and, hence, $F(z') > 0$. Thus, for each $\tilde{z} > \underline{z}_{k_j}$, there is $\eta > 0$ such that $U_{\tilde{z}}(m, n(z')1_{(z',c'+\eta)}) > U_{\tilde{z}}(m, \frac{1}{h}1_{(0,c)})$, which implies that $(\tilde{z}, \frac{1}{h}1_{(0,c)}) \notin \text{supp}(\mu_{k_j})$. But this is a contradiction to $(z, \frac{1}{h}1_{(0,c)}) \in \text{supp}(\mu)$ and $z > \underline{z}_{k_j}$. Thus, $(\underline{z}_{k_j}, \frac{1}{h}1_{(0,c)}) \in \text{supp}(\mu_{k_j})$. ■

We next claim that the sequence $\{\underline{z}_{k_j}\}_{j=1}^\infty$ is bounded away from $\bar{z}$.

Claim 4 *There is $\xi > 0$ and $J \in \mathbb{N}$ such that $\underline{z}_{k_j} \leq \bar{z} - \xi$ for each $j \geq J$.*

Proof. Suppose not; then, taking a subsequence if necessary, we may assume that $\underline{z}_{k_j} \to \bar{z}$ and that $\{\underline{z}_{k_j}\}_{j=1}^\infty$ is increasing. Let $\varepsilon > 0$ be as in Claim 1. Then, for each j, it follows by the definition of $\underline{z}_{k_j}$ and by Claim 1 that

$$\nu([0, \varepsilon]) = \int_{[\underline{z}_{k_j}, \bar{z}] \times X_{k_j}} \delta([0, \varepsilon] \times \hat{C}) \mathrm{d}\mu_{k_j}(z, \delta) \leq \frac{\mu_{k_j}([\underline{z}_{k_j}, \bar{z}] \times X_{k_j})}{h(1 - F(\varepsilon))}$$

$$\leq \frac{\nu([\underline{z}_{k_j}, \bar{z}])}{h(1 - F(\varepsilon))} \to \frac{\nu(\{\bar{z}\})}{h(1 - F(\varepsilon))} = 0.$$

But this contradicts $0 \in \text{supp}(\nu)$. ■

For each $j \in \mathbb{N}$, let, by Claim 3, $c'_{k_j} \in \hat{C}$ be such that $(\underline{z}_{k_j}, \frac{1}{h}1_{(0,c'_{k_j})}) \in \text{supp}(\mu_{k_j})$. Individual rationality implies that $(F(\underline{z}_{k_j}) - c'_{k_j})/h \geq F(\underline{z}_{k_j})$ and, hence,

$$c'_{k_j} \leq (1-h)F(\underline{z}_{k_j}) \leq (1-h)F(\bar{z}-\xi) < (1-h)F(\bar{z}-\xi'),$$

where $0 < \xi' < \xi$ and $\xi > 0$ is as in Claim 4. Thus,

$$\begin{aligned} U_{z'_{k_j}}\left(m, \frac{1}{h}1_{(0,(1-h)F(\bar{z}-\xi'))}\right) &= \frac{F(z'_{k_j}) - (1-h)F(\bar{z}-\xi')}{h} \\ &\to \frac{F(\bar{z}) - (1-h)F(\bar{z}-\xi')}{h} > F(\bar{z}) \geq c_{k_j}. \end{aligned}$$

But this contradicts the stability of μ_{k_j} whenever j is sufficiently large. This conclusion the proof of the lemma. ■

It follows from Lemma 10 that $\mu_k \in M(Z \times X_{M,\emptyset})$ for each $k \geq K$. Since $M(Z \times X_{M,\emptyset})$ is compact, we may assume, taking a subsequence if necessary, that $\{\mu_k\}_{k=1}^{\infty}$ converges; let $\mu = \lim_k \mu_k$.

We establish that μ is a stable matching of $\hat{E}$ in the following claims.

Claim 5 *μ is a matching of $\hat{E}$.*

Proof. We first consider condition 2 of the definition of a matching. Let $\pi : Z \times \hat{X}_\emptyset \to Z$ be the projection of $Z \times \hat{X}_\emptyset$ onto Z and note that, for each Borel subset B of Z, $\nu_M(B) + \nu_S(B) = \mu(B \times \hat{X}_\emptyset) = \mu \circ \pi^{-1}(B)$ and $\nu_{M,k}(B) + \nu_{S,k}(B) = \mu_k(B \times \hat{X}_\emptyset) = \mu_k \circ \pi^{-1}(B)$ for each $k \in \mathbb{N}$. Since π is continuous, $\mu_k \circ \pi^{-1} \to \mu \circ \pi^{-1}$. Hence, $\nu_M + \nu_S = \mu \circ \pi^{-1} = \lim_k \mu_k \circ \pi^{-1} = \lim_k(\nu_{M,k} + \nu_{S,k})$.

Also, for each Borel subset B of Z, $\nu_W(B) = \int_{Z\times\hat{X}} \delta(B\times\hat{C})\text{d}\mu(z,\delta)$ and $\nu_{W,k}(B) = \int_{Z\times\hat{X}} \delta(B\times\hat{C})\text{d}\mu_k(z,\delta)$ for each $k \in \mathbb{N}$. We show that $\nu_{W,k} \to \nu_W$. For each Borel subset of Z, let $g_B : \hat{X} \to \mathbb{R}$ be defined by setting, for each $\delta \in \hat{X}$, $g_B(\delta) = \delta(B \times \hat{C})$. We have that g_B is bounded (by $n(\bar{z} - 1/M)$) for each Borel subset B of Z. Since g_Z is continuous, it follows that

$$\lim_k \nu_{W,k}(Z) = \lim_k \int_{Z\times\hat{X}} g_Z \text{d}\mu_k(z,\delta) = \int_{Z\times\hat{X}} g_Z \text{d}\mu(z,\delta) = \nu_W(Z);$$

furthermore, if B is closed, then g_B is upper semi-continuous and, by (a suitable adaptation of) Aliprantis and Border (2006, Theorem 15.5),

$$\limsup_k \nu_{W,k}(B) = \limsup_k \int_{Z\times\hat{X}} g_B \text{d}\mu_k \leq \int_{Z\times\hat{X}} g_B \text{d}\mu = \nu_W(B).$$

This shows that $\nu_{W,k} \to \nu_W$. Condition 2 of the definition of a matching then holds since

$$\nu_M + \nu_S + \nu_W = \lim_k (\nu_{M,k} + \nu_{S,k}) + \lim_k \nu_{W,k} = \lim_k (\nu_{M,k} + \nu_{S,k} + \nu_{W,k}) = \nu.$$

Condition 1 holds because, for each $(z, \delta) \in \text{supp}(\mu) \cap (Z \times (\hat{X}_\emptyset \setminus \hat{X}))$, there exists a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding $\{(z_{k_j}, \delta_{k_j})\}_{j=1}^\infty$ such that $(z_{k_j}, \delta_{k_j}) \to (z, \delta)$ and, for each $j \in \mathbb{N}$, $(z_{k_j}, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$. Hence, $\delta_{k_j} = 1_{(\emptyset,0)}$ for each $j \in \mathbb{N}$ and, hence, $\delta = 1_{(\emptyset,0)}$. ■

We claim that μ is stable. For convenience, let $Y = [0, \bar{z} - 1/M]$ and $n(\emptyset) = 1$. Let $(z, \delta) \in \text{supp}(\mu)$, $\delta = n(z')1_{(z',c)}$ for some $(z', c) \in Y_\emptyset \times [0, 1]$. Let, by Lemma 12 in Carmona and Podczeck (2009), $\{(z_{k_j}, z'_{k_j}, c_{k_j})\}_{j=1}^\infty$ be such that $(z_{k_j}, z'_{k_j}, c_{k_j}) \to (z, z', c)$ and $(z_{k_j}, n(z'_{k_j})1_{(z'_{k_j}, c_{k_j})}) \in \text{supp}(\mu_{k_j})$ for each $j \in \mathbb{N}$. In particular, note that $z'_{k_j} \in Y_\emptyset$ for each j. Note that, in case $z' \in Z$, Lemma 8 implies that $F(z'_{k_j}) \leq c_{k_j} \leq F(z_{k_j})$, hence $\max\{F(z), F(z')\} = F(z)$. Thus, for any $(z, n(z')1_{(z',c)}) \in \text{supp}(\mu) \cap (Z \times X_M)$, $\max\{F(z), F(z')\} = F(z)$.

Claim 6 *$(z, \delta) \in IR(\mu)$.*

Proof. Suppose not; then $(z, \delta) \in Z \times X_M$ and either (i) $F(z) > (F(z) - c)n(z')$ or (ii) $F(z') > c$. It then follows that (i) $F(z_{k_j}) > (F(z_{k_j}) - c_{k_j})n(z'_{k_j})$ or (ii) $F(z'_{k_j}) > c_{k_j}$ for all j sufficiently large, a contradiction to the stability of μ_{k_j}. Thus, $(z, \delta) \in IR(\mu)$. ■

Claim 7 *$(z, \delta) \in S_M(\mu)$.*

Proof. Suppose not; then there is $(\hat{z}, \hat{c}) \in Z \times \hat{C}$ and $z^* \in Z$ such that $(\hat{z}, \hat{c}) \in T^m_{z^*}(\mu)$, $n(\hat{z})1_{(\hat{z},\hat{c})} \in \hat{X}$ and (i) $z^* = z$ and $(\max\{F(z), F(\hat{z})\} - \hat{c})n(\hat{z}) > (F(z) - c)n(z')$, or (ii) $z^* = z'$ and $(\max\{F(z'), F(\hat{z})\} - \hat{c})n(\hat{z}) > c$, or (iii) $z^* = z$ and $(\max\{F(z), F(\hat{z})\} - \hat{c})n(\hat{z}) > F(z)$.

It follows from $(\hat{z}, \hat{c}) \in T^m_{z^*}(\mu)$ that we may assume, taking a subsequence if necessary, that there exists $\{\hat{z}_{k_j}\}_{j=1}^\infty$ and $J \in \mathbb{N}$ such that $\hat{z}_{k_j} \to \hat{z}$ and $(\hat{z}_{k_j}, \hat{c}) \in T^m_{z^*_{k_j}}(\mu_{k_j})$

for each $j \geq J$, where, for each $j \geq J$, $z^*_{k_j} = z_{k_j}$ if $z^* = z$ and $z^*_{k_j} = z'_{k_j}$ if $z^* = z'$. Indeed, suppose first that there exists $(\tilde{z}, \tilde{c}) \in Z \times \hat{C}$ such that $(\tilde{z}, n(\hat{z})1_{(\hat{z},\tilde{c})}) \in \text{supp}(\mu)$ and $\hat{c} > \tilde{c}$. Then, taking a subsequence if necessary, there exists $\{(\tilde{z}_{k_j}, \tilde{c}_{k_j}, \hat{z}_{k_j})\}_{j=1}^\infty$ such that $(\tilde{z}_{k_j}, \tilde{c}_{k_j}, \hat{z}_{k_j}) \to (\tilde{z}, \tilde{c}, \hat{z})$ and $(\tilde{z}_{k_j}, n(\hat{z}_{k_j})1_{(\hat{z}_{k_j},\tilde{c}_{k_j})}) \in \text{supp}(\mu_{k_j})$. Thus, $\hat{c} > \tilde{c}_{k_j}$ for each j sufficiently large and, thus, $(\hat{z}_{k_j}, \hat{c}) \in T^m_{z^*_{k_j}}(\mu_{k_j})$.

Suppose next that $(\hat{z}, 1_{(\emptyset,0)}) \in \text{supp}(\mu)$ and $\hat{c} > F(\hat{z})$. Then, taking a subsequence if necessary, there exists $\{\hat{z}_{k_j}\}_{j=1}^\infty$ such that $\hat{z}_{k_j} \to \hat{z}$ and $(\hat{z}_{k_j}, 1_{(\emptyset,0)}) \in \text{supp}(\mu_{k_j})$. Thus, $\hat{c} > F(\hat{z}_{k_j})$ for each j sufficiently large and, thus, $(\hat{z}_{k_j}, \hat{c}) \in T^m_{z^*_{k_j}}(\mu_{k_j})$.

Finally, suppose that there exists $(\tilde{z}, \tilde{c}) \in Z \times \hat{C}$ such that $(\hat{z}, n(\tilde{z})1_{(\tilde{z},\tilde{c})}) \in \text{supp}(\mu)$ and $\hat{c} > (F(\hat{z}) - \tilde{c})n(\tilde{z})$. Then, taking a subsequence if necessary, there exists $\{(\tilde{z}_{k_j}, \tilde{c}_{k_j}, \hat{z}_{k_j})\}_{j=1}^\infty$ such that $(\tilde{z}_{k_j}, \tilde{c}_{k_j}, \hat{z}_{k_j}) \to (\tilde{z}, \tilde{c}, \hat{z})$ and $(\hat{z}_{k_j}, n(\tilde{z}_{k_j})1_{(\tilde{z}_{k_j},\tilde{c}_{k_j})}) \in \text{supp}(\mu_{k_j})$. We have that $n(\tilde{z}_{k_j}) \to n(\tilde{z})$: this is clear if $F(\bar{z}) < 1$ and, in the case where $F(\bar{z}) = 1$, this follows because $\tilde{z} < \bar{z}$ since $n(\tilde{z})1_{(\tilde{z},\tilde{c})} \in \hat{X}$. Thus, $\hat{c} > (F(\hat{z}_{k_j}) - \tilde{c}_{k_j})n(\tilde{z}_{k_j})$ for each j sufficiently large and, hence, $(\hat{z}_{k_j}, \hat{c}) \in T^m_{z^*_{k_j}}(\mu_{k_j})$.

We have that $n(z'_{k_j}) \to n(z')$ since $z' = \emptyset$ or $z' \leq \bar{z} - 1/M$. Since $n(\hat{z})1_{(\hat{z},\hat{c})} \in \hat{X}$, we have that $\hat{z} < \bar{z}$ whenever $F(\bar{z}) = 1$ and, hence, $n(\hat{z}_{k_j}) \to n(\hat{z})$. It then follows from $(\hat{z}_{k_j}, \hat{c}) \in T^m_{z^*_{k_j}}(\mu_{k_j})$ for each $j \geq J$ together with (a) $(\max\{F(z_{k_j}), F(\hat{z}_{k_j})\} - \hat{c})n(\hat{z}_{k_j}) > (F(z_{k_j}) - c_{k_j})n(z'_{k_j})$ for each j sufficiently large in case (i), and (b) $(\max\{F(z'_{k_j}), F(\hat{z}_{k_j})\} - \hat{c})n(\hat{z}_{k_j}) > c_{k_j}$ for each j sufficiently large in case (ii), and $(\max\{F(z_{k_j}), F(\hat{z}_{k_j})\} - \hat{c})n(\hat{z}_{k_j}) > F(z_{k_j})$ for each j sufficiently large in case (iii) that $(z_{k_j}, n(z'_{k_j})1_{(z'_{k_j},c_{k_j})}) \in \text{supp}(\mu_{k_j}) \setminus S_M(\mu_{k_j})$ for each j sufficiently large. But this contradicts the stability of μ_{k_j}. Thus, it follows that $(z, \delta) \in S_M(\mu)$. ■

It follows from Claims 6 and 7 that $(z, \delta) \in S_M(\mu) \cap IR(\mu)$. Since $(z, \delta) \in \text{supp}(\mu)$ is arbitrary, it follows that that μ is stable.

A.3 Proof of Theorems 2 and 3

We establish Theorems 2 and 3 as follows. We start by showing that conditions (b) and (c) in Theorem 2 are equivalent; we show in Section A.4 that (b) implies (c) and, in Section A.5 that (c) implies (b). Then we prove that E^*_{grh} has a unique stable matching μ in Section A.6. It then follows by Lemma 7 that μ is the unique stable matching of E_{grh}, thus establishing Theorem 3, the equivalence between conditions (a) and (b) in Theorem 2 and, thus, completing the proof of Theorem 2.

A.4 Proof of Theorem 2: Necessity

A.4.1 Proof of Lemma 1

The proof of Lemma 1 uses two additional lemmas. The first of these shows that there are managers; indeed, if not, then everyone would be self-employed, type 0 individuals would get $F(0) = 0$ and, hence, any type $\bar{z}$ could become a manager and hire type 0 workers to obtain virtually $F(\bar{z})/h$, thus more than $F(\bar{z})$ which is what he gets by being self-employed.

Lemma 11 $\mu(Z \times X) > 0$.

Proof. Suppose not; then, for each Borel subset B of Z, $\nu_M(B) = \nu_W(B) = 0$ and, hence, $\nu_S(B) = \nu_M(B) + \nu_S(B) + \nu_W(B)$. Thus, $\nu_S = \nu$. In particular, $\mu(Z \times (X_\emptyset \setminus X)) = \nu(Z)$ and, thus, $\text{supp}(\mu) \subseteq Z \times \{1_{(\emptyset,0)}\}$ since $\{z\} \times \text{supp}(\delta) \subseteq \text{graph}(\mathbb{C})$ for each $(z, \delta) \in \text{supp}(\mu)$. Conversely, if there is $z \in Z$ such that $(z, 1_{(\emptyset,0)}) \notin \text{supp}(\mu)$, then $\mu(O \times \{1_{(\emptyset,0)}\}) = 0$ for some open neighborhood O of z. Hence,

$$\nu(O) = \mu((O \times (X_\emptyset \setminus X)) \cap \text{supp}(\mu)) \leq \mu(O \times \{1_{(\emptyset,0)}\}) = 0,$$

a contradiction to $\text{supp}(\nu) = Z$. Thus, $\text{supp}(\mu) = Z \times \{1_{(\emptyset,0)}\}$.

We have that $U_0(s) = 0$ and, hence, $(0, \varepsilon) \in T^m_{\bar{z}}(\mu)$ for each $\varepsilon > 0$. Thus, type $\bar{z}$ can get $U_{\bar{z}}(m, \frac{1}{h} 1_{(0,\varepsilon)}) = \frac{F(\bar{z})-\varepsilon}{h} > F(\bar{z}) = U_{\bar{z}}(s)$ by hiring type 0 workers at wage $0 < \varepsilon < (1-h)F(\bar{z})$. But this contradicts the stability of μ. ■

The following results is a simply consequence of the previous lemma and asserts that managers of type less than $\bar{z}$ exist.

Corollary 2 $\text{supp}(\mu) \cap ((Z \setminus \{\bar{z}\}) \times X) \neq \emptyset$.

Proof. Suppose not; then $\text{supp}(\mu) \cap (Z \times X) \subseteq \{\bar{z}\} \times X$. Hence,

$$\mu(Z \times X) = \mu(\text{supp}(\mu) \cap (Z \times X)) \leq \mu(\{\bar{z}\} \times X) = \mu_M(\{\bar{z}\}) \leq \nu(\{\bar{z}\}) = 0,$$

a contradiction to Lemma 11. ■

We turn to the proof of Lemma 1. The idea is to take a manager of type $z < \bar{z}$, whose rent must be at least $F(z)$. The difference between the rent and the self-employed payoff is strictly increasing in z, hence type $\bar{z}$ and those close by must make strictly more that $F(\bar{z})$. No worker or self-employed can make as much, hence all those types must be managers.

Proof of Lemma 1. Let, by Corollary 2, $z, z' \in Z$ and $c \in C$ be such that $(z, n(z')1_{(z',c)}) \in \text{supp}(\mu)$ and $z < \bar{z}$. Then, $(F(z) - c)n(z') = U_z(m, n(z')1_{(z',c)}) \geq U_z(s) = F(z)$. Since $\hat{z} \mapsto (F(\hat{z}) - c)n(z') - F(\hat{z})$ is strictly increasing,

$$U_{\bar{z}}(m, n(z')1_{(z',c)}) > U_{\bar{z}}(s) = F(\bar{z})$$

and, hence, there is $\xi > 0$ such that $U_{\hat{z}}(m, n(z')1_{(z',c)}) > F(\bar{z})$ for each $\hat{z} \in [\bar{z} - \xi, \bar{z}]$.

It then follows that $[\bar{z} - \xi, \bar{z}] \cap S = \emptyset$. Indeed, if $z \in [\bar{z} - \xi, \bar{z}] \cap S$, then $(z', c + \varepsilon) \in T_z^m(\mu)$ for each $\varepsilon > 0$ and, hence, $F(z) = U_z(s) \geq U_z(m, n(z')1_{(z',c+\varepsilon)})$. Thus, letting $\varepsilon \to 0$, $F(\bar{z}) \geq F(z) \geq U_z(m, n(z')1_{(z',c)}) > F(\bar{z})$, a contradiction.

We also have that $[\bar{z} - \xi, \bar{z}] \cap W = \emptyset$. This is clear since $U_{\hat{z}}(m, n(z')1_{(z',c)}) > F(\bar{z})$ for each $\hat{z} \in [\bar{z} - \xi, \bar{z}]$ and because Lemma 8 implies that $\tilde{c} \leq F(\bar{z})$ whenever $(z, n(\tilde{z})1_{(\tilde{z},\tilde{c})}) \in \text{supp}(\mu)$. ■

A.4.2 Proof of Lemma 2

We prove Lemma 2 in a series of lemmas, the first of which establishes the equal treatment property for workers.

Lemma 12 *If $z, \hat{z}, z' \in Z$ and $c \in C$ are such that $(z, n(z')1_{(z',c)}), (\hat{z}, n(z')1_{(z',\hat{c})}) \in \text{supp}(\mu)$, then $c = \hat{c}$.*

Proof. Indeed, if $c > \hat{c}$, then managers of type z can gain by hiring workers of type z' at wage $c - \varepsilon$ for some $\varepsilon > 0$ such that $c - \varepsilon > \hat{c}$, a contradiction to the stability of μ. Thus, $c \leq \hat{c}$ and an analogous argument shows that $c \geq \hat{c}$; hence, $c = \hat{c}$. ■

Define $c : W \to [0, 1]$ by setting, for each $z \in W$, $c(z) = c$, where $c \in [0, 1]$ is such that $(\hat{z}, n(z)1_{(z,c)}) \in \text{supp}(\mu)$ for some $\hat{z} \in Z$. Lemma 12 implies that the function

c is well-defined and Lemma 8 implies that c takes values in $[0, 1]$. For convenience, let, for each $z \in Z$ and $z' \in Z \setminus \{\bar{z}\}$, $U_z(m, z') = U_z(m, n(z')1_{(z', c(z'))})$.

Managers prefer workers with higher z since they can hire more of them, hence, the wage function c is strictly increasing.

Lemma 13 *c is strictly increasing.*

Proof. Suppose not; then there is $z, z' \in W$ such that $z' > z$ and $c(z') \leq c(z)$. Let $\hat{z} \in Z$ be such that $(\hat{z}, n(z)1_{(z,c(z))}) \in \text{supp}(\mu)$. Then $n(z') > n(z)$ and $\hat{z} \geq z$. If $\hat{z} > 0$, then $U_{\hat{z}}(m, z) = (F(\hat{z}) - c(z))n(z) < (F(\hat{z}) - c(z'))n(z') = U_{\hat{z}}(m, z')$ since $U_{\hat{z}}(m, z) \geq F(\hat{z}) > 0$. Thus, there is $\varepsilon > 0$ such that $(z', c(z') + \varepsilon) \in T^m_{\hat{z}}(\mu)$ and $U_{\hat{z}}(m, n(z')1_{(z',c(z')+\varepsilon)}) > U_{\hat{z}}(m, z)$, contradicting the stability of μ.

If $\hat{z} = 0$, then $z = 0$ (since $\hat{z} \geq z$) and $c(z') \leq c(z) = 0$ (since $F(z) \leq c \leq F(\hat{z})$ by Lemma 8). Since $F(z') > 0$, this contradicts the stability of μ. ■

The function c is continuous since, e.g., a type $\hat{z}$ manager hiring workers of type z with a wage higher and bounded away from the wage of some close by worker type z' can attract the latter by paying slightly more and in this way obtain a rent higher than when hiring the former.

Lemma 14 *c is continuous.*

Proof. Suppose not; then there is $z \in W$ such that c is discontinuous at z. Since c is increasing by Lemma 13, there are only two possible cases.

Case 1: There exists $\varepsilon > 0$ and a sequence $\{z_k\}_{k=1}^{\infty}$ such that $z_k \to z$ and, for each $k \in \mathbb{N}$, $z_k \in W$, $z_k < z$ and $c(z) > c(z_k) + \varepsilon$. In this case, let $\hat{z} \in Z$ be such that $(\hat{z}, n(z)1_{(z,c(z))}) \in \text{supp}(\mu)$. Then

$$\frac{F(\hat{z}) - c(z_k) - \frac{\varepsilon}{2}}{h(1 - F(z_k))} > \frac{F(\hat{z}) - c(z) + \frac{\varepsilon}{2}}{h(1 - F(z_k))} \to \frac{F(\hat{z}) - c(z) + \frac{\varepsilon}{2}}{h(1 - F(z))} > U_{\hat{z}}(m, z).$$

Thus, there is k sufficiently large such that $(z_k, c(z_k) + \frac{\varepsilon}{2}) \in T^m_{\hat{z}}(\mu)$ and $U_{\hat{z}}(m, z) < U_{\hat{z}}(m, n(z_k)1_{(z_k,c(z_k)+\frac{\varepsilon}{2})})$, contradicting the stability of μ.

Case 2: There exists $\varepsilon > 0$ and a sequence $\{z_k\}_{k=1}^{\infty}$ such that $z_k \to z$ and, for each $k \in \mathbb{N}$, $z_k \in W$, $z_k > z$ and $c(z) < c(z_k) - \varepsilon$. In this case, for each $k \in \mathbb{N}$, let $\hat{z}_k \in Z$ be such that $(\hat{z}_k, n(z_k)1_{(z_k,c(z_k))}) \in \text{supp}(\mu)$. Since $(\hat{z}_k, c(z_k)) \in Z \times [0,1]$ for each $k \in \mathbb{N}$ and $Z \times [0,1]$ is compact, we may assume, taking a subsequence if necessary, that $\{(\hat{z}_k, c(z_k))\}_{k=1}^{\infty}$ converges; let $(\hat{z}, c) = \lim_k(\hat{z}_k, c(z_k))$. Then $c(z) + \frac{\varepsilon}{2} \leq c - \frac{\varepsilon}{2}$ and

$$\frac{F(\hat{z}_k) - c(z_k)}{h(1 - F(z_k))} \to \frac{F(\hat{z}) - c}{h(1 - F(z))} < \frac{F(\hat{z}) - c + \frac{\varepsilon}{2}}{h(1 - F(z))}$$
$$\leq \frac{F(\hat{z}) - c(z) - \frac{\varepsilon}{2}}{h(1 - F(z))} = \lim_k \frac{F(\hat{z}_k) - c(z) - \frac{\varepsilon}{2}}{h(1 - F(z))}.$$

Thus, there is k sufficiently large such that $(z, c(z) + \frac{\varepsilon}{2}) \in T^m_{\hat{z}_k}(\mu)$ and $U_{\hat{z}_k}(m, z_k) < U_{\hat{z}_k}(m, n(z)1_{(z,c(z)+\frac{\varepsilon}{2})})$, contradicting the stability of μ. ■

The following is the equal treatment property for managers.

Lemma 15 *If $z, z', \hat{z} \in Z$ are such that $(z, n(z')1_{(z',c(z'))}), (z, n(\hat{z})1_{(\hat{z},c(\hat{z}))}) \in \text{supp}(\mu)$, then $U_z(m, z') = U_z(m, \hat{z})$.*

Proof. If $U_z(m, z') > U_z(m, \hat{z})$, then, letting $\varepsilon > 0$ be such that $\frac{F(z)-c(z')-\varepsilon}{h(1-F(z'))} > U_z(m, \hat{z})$, it follows that $(z', c(z')+\varepsilon) \in T^m_z(\mu)$ and $U_z(m, n(z')1_{(z',c(z')+\varepsilon)}) > U_z(m, \hat{z})$, a contradiction to the stability of μ. Thus, $U_z(m, z') \leq U_z(m, \hat{z})$ and an analogous argument shows that $U_z(m, z') \geq U_z(m, \hat{z})$; hence, $U_z(m, z') = U_z(m, \hat{z})$. ■

Define $u : M \to \mathbb{R}_+$ by setting, for each $z \in M$, $u(z) = U_z(m, z')$, where $z' \in Z$ is such that $(z, n(z')1_{(z',c(z'))}) \in \text{supp}(\mu)$. Lemma 15 implies that the function u is well-defined.

Managers with higher z are more productive, hence, the rent function $u : M \to \mathbb{R}$ is strictly increasing.

Lemma 16 *$u : M \to \mathbb{R}$ is strictly increasing.*

Proof. Suppose not; then there is $z, \hat{z} \in M$ such that $z > \hat{z}$ and $u(z) \leq u(\hat{z})$. Let $z' \in Z$ be such that $(\hat{z}, n(z')1_{(z',c(z'))}) \in \text{supp}(\mu)$. Then $F(\hat{z}) < F(z)$ and

$$u(z) \leq u(\hat{z}) = \frac{F(\hat{z}) - c(z')}{h(1 - F(z'))} < \frac{F(z) - c(z')}{h(1 - F(z'))}.$$

Thus, there is $\varepsilon > 0$ such that $(z', c(z') + \varepsilon) \in T^m_z(\mu)$ and $U_z(m, n(z')1_{(z',c(z')+\varepsilon)}) > u(z)$, contradicting the stability of μ. ■

The function $u : M \to \mathbb{R}$ is continuous since, as already noticed, a type z manager with rent lower and bounded away from the rent of some close by manager $\tilde{z}$ can attract the workers of the latter by paying slightly more and obtain a rent virtually equal to $u(\tilde{z})$ which is higher than $u(z)$.

Lemma 17 *$u : M \to \mathbb{R}$ is continuous.*

Proof. Suppose not; then there is $z \in M$ such that u is discontinuous at z. Since u is increasing by Lemma 16, there are only two possible cases.

Case 1: There exists $\varepsilon > 0$ and a sequence $\{z_k\}_{k=1}^{\infty}$ such that $z_k \to z$ and, for each $k \in \mathbb{N}$, $z_k \in M$, $z_k < z$ and $u(z) > u(z_k) + \varepsilon$. In this case, let $z' \in Z$ be such that $(z, n(z')1_{(z',c(z'))}) \in \text{supp}(\mu)$. Then $n(z')(F(z_k) - c(z')) \to u(z) > u(z_k) - \varepsilon$. Thus, there is k sufficiently large and $\eta > 0$ such that $n(z')(F(z_k) - c(z') - \eta) > u(z_k)$. Then $(z', c(z') + \eta) \in T^m_{z_k}(\mu)$ and $U_{z_k}(m, n(z')1_{(z',c(z')+\eta)}) > u(z_k)$, contradicting the stability of μ.

Case 2: There exists $\varepsilon > 0$ and a sequence $\{z_k\}_{k=1}^{\infty}$ such that $z_k \to z$ and, for each $k \in \mathbb{N}$, $z_k \in M$, $z_k > z$ and $u(z) < u(z_k) - \varepsilon$. In this case, for each $k \in \mathbb{N}$, let $z'_k \in Z$ be such that $(z_k, n(z'_k)1_{(z'_k,c(z'_k))}) \in \text{supp}(\mu)$. Then, there is k sufficiently large such that $n(z'_k)(F(z) - c(z'_k)) > u(z_k) - \varepsilon > u(z)$ since, using Lemma 1,

$$0 \le u(z_k) - n(z'_k)(F(z) - c(z'_k)) = n(z'_k)(F(z_k) - F(z)) \le n(\bar{z} - \xi)(F(z_k) - F(z)) \to 0.$$

Thus, there is $\eta > 0$ such that $n(z'_k)(F(z) - c(z'_k) - \eta) > u(z)$. Then $(z'_k, c(z'_k) + \eta) \in T^m_z(\mu)$ and $U_z(m, n(z'_k)1_{(z'_k,c(z'_k)+\eta)}) > u(z)$, contradicting the stability of μ. ■

Self-employed individuals of type z receive a payoff of $F(z)$. The follows lemma shows that, for some type z, there are type z individuals with different occupations, then the payoffs from such occupations are the same at z.

Lemma 18 *The following holds: (i) $u(z) = c(z)$ if $z \in M \cap W$, (ii) $u(z) = F(z)$ if $z \in M \cap S$ and (iii) $c(z) = F(z)$ if $z \in W \cap S$.*

Proof. Let $z \in M \cap W$. Suppose that $u(z) > c(z)$ and let $z' \in Z$ be such that $(z, n(z')1_{(z',c(z'))}) \in \text{supp}(\mu)$ and $\varepsilon > 0$ be such that $n(z')(F(z) - c(z') - \varepsilon) >$

$c(z)$. Then $(z', c(z') + \varepsilon) \in T_z^m(\mu)$ and $U_z(m, n(z')1_{(z', c(z'))}) > c(z)$, contradicting the stability of μ.

If $u(z) < c(z)$, then let $\hat{z} \in Z$ be such that $(\hat{z}, n(z)1_{(z,c(z))}) \in \text{supp}(\mu)$ and $\varepsilon > 0$ be such that $n(z)(F(\hat{z}) - u(z) - \varepsilon) > n(z)(F(\hat{z}) - c(z))$. Then $(z, u(z) + \varepsilon) \in T_{\hat{z}}^m(\mu)$ and $U_{\hat{z}}(m, n(z)1_{(z,u(z)+\varepsilon)}) > U_{\hat{z}}(m, z)$, contradicting the stability of μ. Thus, (i) follows.

Let $z \in M \cap S$. The stability of μ implies that $u(z) \geq F(z)$. If $u(z) > F(z)$, then let $z' \in Z$ be such that $(z, n(z')1_{(z', c(z'))}) \in \text{supp}(\mu)$ and $\varepsilon > 0$ be such that $n(z')(F(z) - c(z') - \varepsilon) > F(z)$. It then follows that $(z', c(z') + \varepsilon) \in T_z^m(\mu)$ and $U_z(m, n(z')1_{(z', c(z')+\varepsilon)}) > F(z)$, contradicting the stability of μ. Thus, (ii) follows.

Let $z \in W \cap S$. The stability of μ implies that $c(z) \geq F(z)$. If $c(z) > F(z)$, then $\hat{z} \in Z$ be such that $(\hat{z}, n(z)1_{(z,c(z))}) \in \text{supp}(\mu)$ and $\varepsilon > 0$ be such that $n(z)(F(\hat{z}) - F(z) - \varepsilon) > n(z)(F(\hat{z}) - c(z))$. Then $(z, F(z) + \varepsilon) \in T_{\hat{z}}^m(\mu)$ and $U_{\hat{z}}(m, n(z)1_{(z,F(z)+\varepsilon)}) > U_{\hat{z}}(m, z)$, contradicting the stability of μ. Thus, (iii) follows. ■

The next step is to combine $c : W \to [0, 1]$, $u : M \to \mathbb{R}$ and $z \mapsto F(z)$. Before that, we establish some results on the sets M, S and W that are needed for the argument.

Recall that $Y = [0, \bar{z} - \xi]$, $Y_\emptyset = Y \cup \{\emptyset\}$, $X^* = \{n(z)1_{(z,c(z))} : z \in Y\}$ and $X_\emptyset^* = X^* \cup \{1_{(\emptyset,0)}\}$. It follows by Lemma 1 and by the definition of a matching that $\text{supp}(\mu) \subseteq Z \times X_\emptyset^*$. Recall that $g : \text{supp}(\mu) \to Z \times Y_\emptyset$ is defined by setting, for each $(z, \delta) \in \text{supp}(\mu)$, $g(z, \delta) = (z, z')$ where $z' \in Y_\emptyset$ is such that $\delta = n(z')1_{(z', c(z'))}$. Let $\pi_2(\text{supp}(\mu)) = \{\delta \in X^* : (z, \delta) \in \text{supp}(\mu)\}$ be the projection of $\text{supp}(\mu)$ onto X^* and let $g_2 : \pi_2(\text{supp}(\mu)) \to Y_\emptyset$ be defined by setting, for each $\delta \in \pi_2(\text{supp}(\mu))$,

$$g_2(\delta) = z'$$

where $z' \in Y_\emptyset$ is such that $\delta = n(z')1_{(z', c(z'))}$.

The following lemma shows that a match $(z, n(z')1_{(z', c(z'))})$ can be represented in an equivalent way simply by the pair (z, z').

Lemma 19 *g is a homeomorphism between* $\text{supp}(\mu)$ *and* $g(\text{supp}(\mu))$ *and g_2 is an homeomorphism between* $\pi_2(\text{supp}(\mu))$ *and* $g_2(\pi_2(\text{supp}(\mu)))$.

Proof. Let id : $Z \to Z$ be the identity. Then $g = (\text{id}, g_2)|_{\text{supp}(\mu)}$. Since id is an homeomorphism, it suffices to show that g_2 is an homeomorphism.

It is clear that $g_2^{-1} : z' \mapsto n(z')1_{(z',c(z'))}$ is 1-1 and continuous, the latter since c is continuous and $\emptyset$ is an isolated point of $Y_\emptyset$. If $1_{(\emptyset,0)} \in \pi_2(\text{supp}(\mu))$, then g_2 is continuous at $1_{(\emptyset,0)}$ since this is an isolated point of $\pi_2(\text{supp}(\mu))$. Thus, consider $\delta = n(z)1_{(z,c(z))}$ for some $z \in Y$ and $\{\delta_k\}_{k=1}^{\infty}$ such that, for each $k \in \mathbb{N}$, $\delta_k = n(z_k)1_{(z_k,c(z_k))}$ for some $z_k \in Y$ and $\delta_k \to \delta$. Let $\kappa : Z \times C \to \mathbb{R}$ be defined by setting, for each $(\hat{z}, \hat{c}) \in Z \times C$, $\kappa(\hat{z}, \hat{c}) = |\hat{z} - z|/n(\hat{z})$. Then κ is bounded and continuous, and hence

$$|g_2(\delta_k) - g_2(\delta)| = |z_k - z| = \int \kappa \mathrm{d}\delta_k \to \int \kappa \mathrm{d}\delta = |z - z| = 0.$$

Thus, g_2 is continuous. ■

The following lemmas assert that M, W and S are closed, which follows mostly from their definitions, and perfect. Recall that a set is perfect if it has no isolated points and this holds for M, S and W roughly because if some of these sets were to have an isolated point, then there would be an open subset of the support of μ with zero measure. But this is impossible by the definition of support.

Lemma 20 *M is nonempty, closed and perfect.*

Proof. The nonemptyness of M follows by Corollary 2 and the closedness of M follows because X^* is compact and $\text{supp}(\mu) \cap (Z \times X) \subseteq Z \times X^*$.

Suppose that M has an isolated point z. Then $\text{supp}(\mu) \cap (\{z\} \times X) \neq \emptyset$ and there is $\varepsilon > 0$ such that $B_\varepsilon(z) \cap M = \{z\}$. But this is a contradiction to the definition of $\text{supp}(\mu)$ since

$$\mu(B_\varepsilon(z) \times X) = \mu(\text{supp}(\mu) \cap (B_\varepsilon(z) \times X)) \leq$$
$$\mu((M \cap B_\varepsilon(z)) \times X) = \mu(\{z\} \times X) \leq \nu(\{z\}) = 0,$$

and $\text{supp}(\mu) \setminus (B_\varepsilon(z) \times X)$ is closed and strictly contained in $\text{supp}(\mu)$. Thus, M has no isolated points and is, therefore, perfect. ■

Lemma 21 *W is a nonempty, closed and perfect subset of $[0, \bar{z} - \xi]$.*

Proof. It follows from Lemma 1 that $W \subseteq [0, \bar{z} - \xi]$. Furthermore, W is nonempty since M is nonempty.

The set W is closed since if $z \in Z$ and $\{z_k\}_{k=1}^{\infty}$ are such that $z_k \to z$ and $z_k \in W \subseteq [0, \bar{z} - \xi]$ for each $k \in \mathbb{N}$, then there is, for each $k \in \mathbb{N}$, $\hat{z}_k \in Z$ such that $(\hat{z}_k, n(z_k)1_{(z_k, c(z_k))}) \in \text{supp}(\mu)$. Since Z is compact, we may assume that $\{\hat{z}_k\}_{k=1}^{\infty}$ converges; let $\hat{z} = \lim_k \hat{z}_k$. Then $(\hat{z}_k, n(z_k)1_{(z_k, c(z_k))}) \to (\hat{z}, n(z)1_{(z, c(z))})$, implying that $(\hat{z}, n(z)1_{(z,c(z))}) \in \text{supp}(\mu)$ and, hence, $z \in W$.

Suppose that W has an isolated point z. Thus, there is $\eta > 0$ such that $B_\eta(z) \cap W = \{z\}$. Then $\text{supp}(\mu) \cap (Z \times \{g_2^{-1}(z)\}) \neq \emptyset$ and there exists $\varepsilon > 0$ such that $\text{supp}(\mu) \cap (Z \times g_2^{-1}(B_\varepsilon(z) \cap W)) = \text{supp}(\mu) \cap (Z \times \{g_2^{-1}(z)\})$. It follows by Lemma 19 that $g_2^{-1}(B_\varepsilon(z) \cap W)$ is open in $\pi_2(\text{supp}(\mu))$, hence $\text{supp}(\mu) \cap (Z \times g_2^{-1}(B_\varepsilon(z) \cap W))$ is open in $\text{supp}(\mu)$. Furthermore,

$$0 = \nu(\{z\}) \geq \int_{Z \times X} \delta(\{z\} \times C) \mathrm{d}\mu(z', \delta)$$

implies that $\mu(\{(z', \delta) \in Z \times X : \delta(\{z\} \times C) > 0\}) = 0$. Since $\{(z', \delta) \in Z \times X : \delta(\{z\} \times C) > 0\} = Z \times \{g_2^{-1}(z)\}$, it follows that

$$\begin{aligned} 0 &= \mu(Z \times \{g_2^{-1}(z)\}) = \mu(\text{supp}(\mu) \cap (Z \times \{g_2^{-1}(z)\})) \\ &= \mu(\text{supp}(\mu) \cap (Z \times B_\varepsilon(g_2^{-1}(z) \cap W))). \end{aligned}$$

Hence, $\text{supp}(\mu) \setminus (Z \times g_2^{-1}(B_\varepsilon(z) \cap W))$ is closed, strictly contained in $\text{supp}(\mu)$ and such that $\mu(\text{supp}(\mu) \setminus (Z \times g_2^{-1}(B_\varepsilon(z) \cap W))) = \mu(\text{supp}(\mu))$. But this contradicts the definition of $\text{supp}(\mu)$. Thus, W has no isolated points and is, therefore, perfect. ■

Lemma 22 *S is closed and perfect.*

Proof. The closedness of S follows because $\text{supp}(\mu)$ is closed and $\text{supp}(\mu) \cap (Z \times (X_\emptyset \setminus X)) \subseteq Z \times \{1_{(\emptyset, 0)}\}$.

Suppose that S has an isolated point z. Then $\text{supp}(\mu) \cap (\{z\} \times (X_\emptyset \setminus X)) \neq \emptyset$ and there is $\varepsilon > 0$ such that $B_\varepsilon(z) \cap S = \{z\}$. But this is a contradiction to the definition of $\text{supp}(\mu)$ since

$$\begin{aligned} &\mu(B_\varepsilon(z) \times (X_\emptyset \setminus X)) = \mu(\text{supp}(\mu) \cap (B_\varepsilon(z) \times (X_\emptyset \setminus X))) \leq \\ &\mu((S \cap B_\varepsilon(z)) \times (X_\emptyset \setminus X)) = \mu(\{z\} \times (X_\emptyset \setminus X)) \leq \nu(\{z\}) = 0, \end{aligned}$$

and $\mathrm{supp}(\mu) \setminus (B_\varepsilon(z) \times (X_\emptyset \setminus X))$ is closed and strictly contained in $\mathrm{supp}(\mu)$. Thus, S has no isolated points and is, therefore, perfect. ■

The following lemma shows that M, S and W cover Z, i.e. any type has an occupation. The idea is that $\mu((M \cup S \cup W) \times X_\emptyset) = \mu(Z \times X_\emptyset)$ and $\delta((M \cup S \cup W) \times C) = \delta(Z \times C)$ mostly by the definition of M, S and W, and then feasibility implies that $\nu(M \cup S \cup W) = \nu(Z)$. Thus, $Z = \mathrm{supp}(\nu) = M \cup S \cup W$.

Lemma 23 $Z = M \cup S \cup W$.

Proof. Let $K = M \cup S \cup W$ and note that we have that $K \subseteq Z$ by definition.

Conversely, note first that K is closed by Lemmas 20, 21 and 22. Furthermore, letting $\pi(\mathrm{supp}(\mu))$ be the projection of $\mathrm{supp}(\mu)$ in Z, we have that $\mathrm{supp}(\mu) \subseteq \pi(\mathrm{supp}(\mu)) \times X_\emptyset = (M \cup S) \times X_\emptyset \subseteq K \times X_\emptyset$ and, hence,

$$\mu(K \times X_\emptyset) \geq \mu(\mathrm{supp}(\mu)) = \mu(\mathrm{supp}(\mu) \cap (Z \times X_\emptyset)) = \mu(Z \times X_\emptyset).$$

Furthermore, for each $(z, \delta) \in \mathrm{supp}(\mu) \cap (Z \times X)$, there is $z' \in Z$ such that $\delta = g_2(z')$ and, hence, $z' \in W$. Thus, $\delta((Z \setminus W) \times C) = 0$, $\delta(W \times C) = \delta(Z \times C)$ and $\delta(K \times C) = \delta(Z \times C)$. Hence,

$$\begin{aligned}
\nu(K) &= \mu(K \times X) + \mu(K \times (X_\emptyset \setminus X)) + \int_{Z \times X} \delta(K \times C) \mathrm{d}\mu(z, \delta) \\
&= \mu(K \times X_\emptyset) + \int_{(Z \times X) \cap \mathrm{supp}(\mu)} \delta(K \times C) \mathrm{d}\mu(z, \delta) \\
&\geq \mu(Z \times X_\emptyset) + \int_{(Z \times X) \cap \mathrm{supp}(\mu)} \delta(Z \times C) \mathrm{d}\mu(z, \delta) = \nu(Z).
\end{aligned}$$

It then follows by the definition of $\mathrm{supp}(\nu)$ that $Z = \mathrm{supp}(\nu) \subseteq K$. ■

To complete the proof of Lemma 2, define $u : Z \to \mathbb{R}$ by setting, for each $z \in Z$,

$$u(z) = \begin{cases} u(z) & \text{if } z \in M, \\ c(z) & \text{if } z \in W, \\ F(z) & \text{if } z \in S. \end{cases}$$

It follows by Lemma 18 that u is well-defined. It follows by $U_z(s, 1_{(\emptyset,c)}) = F(z)$ for each $(z, c) \in Z \times C$ and by Lemmas 12 and 15 that conditions 1–3 in the statement of the lemma hold. We have that u is continuous since M, W and S are closed (by Lemmas 20, 21, 22), $Z = M \cup W \cup S$ (by Lemma 23) and $u|_M$, $u|_W$ and $u|_S$ are continuous.

A.4.3 Proof of Lemma 3

Recall that u is extended to $Z_\emptyset$ by setting $u(\emptyset) = 0$. Since $\emptyset$ is isolated, $u : Z_\emptyset \to \mathbb{R}$ is continuous.

We first show that γ is an assignment. Let B be a Borel subset of Z; then

$$\begin{aligned}
&\gamma(B \times Y_\emptyset) + \int_{Z\times(B\cap Y)} n(z')\mathrm{d}\gamma(z,z') = \\
&\gamma(B \times Y_\emptyset) + \int_{Z\times Y} 1_B(z')n(z')\mathrm{d}\gamma(z,z') = \\
&\mu(g^{-1}(B \times Y_\emptyset)) + \int_{Z\times X^*} 1_B(g(z,\delta))n(g(z,\delta))\mathrm{d}\mu(z,\delta) = \\
&\mu(B \times X^*_\emptyset) + \int_{Z\times X^*} \delta(B \times C)\mathrm{d}\mu(z,\delta) = \\
&\mu(B \times X_\emptyset) + \int_{Z\times X} \delta(B \times C)\mathrm{d}\mu(z,\delta) = \nu(B),
\end{aligned}$$

where the penultimate equality follows because $\mathrm{supp}(\mu) \subseteq Z \times X^*_\emptyset$ and the last equality follows because μ is a matching.

We next show that (γ, u) is stable. Note first that $\mathrm{supp}(\gamma) = g(\mathrm{supp}(\mu))$ by Carmona and Laohakunakorn (2024, Lemma 1) since g is an homeomorphism between two compact spaces by Lemma 19.

Let $(z, z') \in \mathrm{supp}(\gamma)$. Since $\mathrm{supp}(\gamma) \subseteq g(\mathrm{supp}(\mu))$, $(z, n(z')1_{(z',c(z'))}) \in \mathrm{supp}(\mu)$. If $z' \in Z$, then $z \in M$ and $z' \in W$. Thus,

$$u(z) + n(z')u(z') = (F(z) - c(z'))n(z') + c(z')n(z') = s(z, z').$$

If $z' = \emptyset$, then $z \in S$ and $u(z) + n(z')u(z') = F(z) = s(z, \emptyset)$.

Let $(z, z') \in Z \times Y_\emptyset$. If $z' = \emptyset$, then $u(z) + n(z')u(z') = u(z) \geq F(z) = s(z, \emptyset)$, where the inequality holds since μ is stable. If $z' \neq \emptyset$, then $s(z, z') = F(z)n(z')$. If $u(z) + n(z')u(z') < s(z, z')$, then $(F(z) - u(z'))n(z') > u(z)$. Letting $\varepsilon > 0$ be such that $(F(z) - u(z') - \varepsilon)n(z') > u(z)$, it follows that $(z', u(z') + \varepsilon) \in T^m_z(\mu)$ and

$U_z(m, n(z')1_{(z',u(z')+\varepsilon)}) > u(z)$, a contradiction to the stability of μ. Thus, $u(z) + n(z')u(z') \geq s(z, z')$.

A.4.4 Proof of Lemma 4

We state a general result which implies Lemma 4. Note that the definitions of a stable and of a surplus maximizing assignment extend without change to a general case where Z is any Polish space, Y is any closed subset of Z and s is an arbitrary bounded and continuous real-valued function on $Z \times Y_\emptyset$.

In the case of Lemma 4, Z is a compact metric case, Y is closed and s is continuous, hence the lemma follows from the following result.

Theorem 7 *If Z is a Polish space, Y a closed subset of Z, $s : Z \times Y_\emptyset \to \mathbb{R}$ is bounded and continuous and $\gamma \in M(Z \times Y_\emptyset)$ is a stable assignment, then γ is surplus maximizing.*

Proof. Let $U = \{u \in C(Z_\emptyset) : u(z)+n(z')u(z') \geq s(z, z') \text{ for each } (z, z') \in Z\times Y_\emptyset\}$ and $\gamma \in \Gamma$ be stable. Then there is $u \in U$ such that $u(z) + n(z')u(z') = s(z, z')$ for each $(z, z') \in \text{supp}(\gamma)$ and, hence,

$$\begin{aligned}\int_{Z\times Y_\emptyset} s(z, z')\mathrm{d}\gamma(z, z') &= \int_{Z\times Y_\emptyset} u(z)\mathrm{d}\gamma(z, z') + \int_{Z\times Y_\emptyset} n(z')u(z')\mathrm{d}\gamma(z, z') \\ &= \int_Z u\mathrm{d}\gamma_Z + \int_Y u\mathrm{d}\gamma_{Y,n} = \int_Z u\mathrm{d}\nu.\end{aligned} \tag{9}$$

Suppose that there is $\gamma' \in \Gamma$ such that $\int_{Z\times Y_\emptyset} s\mathrm{d}\gamma' > \int_{Z\times Y_\emptyset} s\mathrm{d}\gamma$. Since $u \in U$, we have that $u(z) + n(z')u(z') \geq s(z, z')$ for each $(z, z) \in Z \times Y_\emptyset$ and it follows as in (9) that $\int_{Z\times Y_\emptyset} s\mathrm{d}\gamma' \leq \int_Z u\mathrm{d}\nu$. Thus,

$$\int_{Z\times Y_\emptyset} s\mathrm{d}\gamma' \leq \int_Z u\mathrm{d}\nu = \int_{Z\times Y_\emptyset} s\mathrm{d}\gamma < \int_{Z\times Y_e s} s\mathrm{d}\gamma',$$

a contradiction. Hence, for each $\gamma' \in \Gamma$, $\int_{Z\times Y_\emptyset} s\mathrm{d}\gamma' \leq \int_{Z\times Y_\emptyset} s\mathrm{d}\gamma$. ■

A.4.5 Proof of Lemma 5

We state a general result which implies Lemma 5. Note that the definitions of a surplus maximizing assignment and of a s-monotone set extend without change to a general case where Z is any Polish space, Y is any closed subset of Z and s is an arbitrary bounded and continuous real-valued function on $Z \times Y_\emptyset$.

In the case of Lemma 5, Z is a compact metric case, Y is closed and s is continuous, hence the lemma follows from the following result.

Theorem 8 *If Z is a Polish space, Y a closed subset of Z, $s : Z \times Y_\emptyset \to \mathbb{R}$ is bounded and continuous and $\gamma \in M(Z \times Y_\emptyset)$ is surplus maximizing, then* $\mathrm{supp}(\gamma)$ *is s-monotone.*

Proof. We start by establishing two preliminary claims. The first is a straightforward variation of Lemma 4.1 in Beiglböck and Griessler (2019). For each $l \in \mathbb{N}$, $1 \leq i \leq l$ and Polish space D, $\pi_i : D^l \to D$ is the projection of D^l onto the ith coordinate.

Claim 8 *If (D, m) is a Polish measure space, $0 < m(D) < \infty$, $l \in \mathbb{N}$ and K is an analytic subset of D^l, then one of the following conditions holds:*

(i) There exist m-null sets $K_1, \ldots, K_l \subseteq E$ such that $K \subseteq \cup_{i=1}^{l} \pi_i^{-1}(K_i)$.

(ii) There is a measure η on E^l such that $\eta(K) > 0$ and $\eta \circ \pi_i^{-1} \leq m$ for each $i = 1, \ldots, l$.

Proof. Apply Beiglböck and Griessler (2019, Lemma 4.1) to (D, m') where $m' = m/m(D)$. If condition (i) of the lemma holds, then there exist m'-null sets, hence m-null sets, $K_1, \ldots, K_l \subseteq E$ such that $K \subseteq \cup_{i=1}^{l} \pi_i^{-1}(M_i)$. If condition (ii) of the lemma holds, there is a measure η' on E^l such that $\eta'(M) > 0$ and $\eta' \circ \pi_i^{-1} \leq m'$ for each $i = 1, \ldots, l$. Letting $\eta = m(D)\eta'$, it follows that condition (ii) of this claim holds. ■

For convenience, let $D = Z \times Y_\emptyset$ and $s(\tau) = \int_{Z \times Y_\emptyset} s \mathrm{d}\tau$ for each $\tau \in M(Z \times Y_\emptyset)$. For each $g \in C(Z)$, let $g(\emptyset) = 0$ and $f_g : D \to \mathbb{R}$ be defined by setting, for each $(z, z') \in D$, $f_g(z, z') = g(z) + n(z')g(z')$. Then let $\mathcal{F} = \{f_g : g \in C(Z)\}$ and note that $\mathcal{F} \subseteq C(D)$, i.e. each element of $\mathcal{F}$ is continuous and bounded.

Claim 9 *For each $\zeta, \tau \in M(D)$, $\tau_1 + \tau_{2,n} = \zeta_1 + \zeta_{2,n}$ if and only if $\int_D f \mathrm{d}\tau = \int_D f \mathrm{d}\zeta$ for each $f \in \mathcal{F}$.*

Proof. Let $\zeta, \tau \in M(D)$. If $\tau_1 + \tau_{2,n} = \zeta_1 + \zeta_{2,n}$ then, for each $f \in \mathcal{F}$, it follows that, letting $g \in C(Z)$ be such that $f = f_g$,

$$\int_D f \mathrm{d}\tau = \int_Z g \mathrm{d}\tau_1 + \int_Z g \mathrm{d}\tau_{2,n} = \int_Z g \mathrm{d}\zeta_1 + \int_Z g \mathrm{d}\zeta_{2,n} = \int_D f \mathrm{d}\zeta.$$

Conversely, if $\int_D f \mathrm{d}\tau = \int_D f \mathrm{d}\zeta$ for each $f \in \mathcal{F}$, then $\int_Z g \mathrm{d}(\tau_1 + \tau_{2,n}) = \int_Z g \mathrm{d}(\zeta_1 + \zeta_{2,n})$ for each $g \in C(Z)$; hence, by e.g. Parthasarathy (1967, Theorem 5.9, p. 39), $\tau_1 + \tau_{2,n} = \zeta_1 + \zeta_{2,n}$. ■

Let γ be a surplus maximizing assignment. The following claim is the core of the argument.

Claim 10 *For each $l \in \mathbb{N}$, there is a subset S_l of D such that $\gamma(S_l^c) = 0$ and the following holds:*

$$\text{for each } \zeta \in M(D) \text{ with } \operatorname{supp}(\zeta) \subseteq S_l,\ |\operatorname{supp}(\zeta)| \leq l \text{ and } \zeta(D) \leq 1, \tag{10}$$

$$\text{and each } \tau \in M(D) \text{ with } |\operatorname{supp}(\tau)| \leq l,\ \tau(D) \leq l \text{ and } \tau_1 + \tau_{2,n} = \zeta_1 + \zeta_{2,n}, \tag{11}$$

$$s(\zeta) \geq s(\tau). \tag{12}$$

Proof. Let $l \in \mathbb{N}$ and define

$$\begin{aligned} K = \{&(d_1, \ldots, d_l) \in D^l : \text{ there exist } \zeta, \tau \in M(D) \text{ such that} \\ &\zeta(D) \leq 1, \operatorname{supp}(\zeta) \subseteq \{d_1, \ldots, d_l\}, \\ &|\operatorname{supp}(\tau)| \leq l, \tau(D) \leq l, \tau_1 + \tau_{2,n} = \zeta_1 + \zeta_{2,n} \text{ and } s(\tau) > s(\zeta)\}. \end{aligned}$$

Note that K is the projection of the set

$$\begin{aligned} \hat{K} = \{&(d_1, \ldots, d_l, \zeta_1, \ldots, \zeta_l, d'_1, \ldots, d'_l, \tau_1, \ldots, \tau_l) \in D^l \times \mathbb{R}^l_+ \times D^l \times \mathbb{R}^l_+ : \\ &\sum_{i=1}^l \zeta_i \leq 1, \sum_{i=1}^l \tau_i \leq l, \sum_{i=1}^l \zeta_i f(d_i) = \sum_{i=1}^l \tau_i f(d'_i) \text{ for each } f \in \mathcal{F} \text{ and} \\ &\sum_{i=1}^l \zeta_i s(d_i) < \sum_{i=1}^l \tau_i f(d'_i)\}. \end{aligned}$$

onto the first l coordinates by Claim 9. Since the set $\hat{K}$ is Borel, it follows that K is analytic.

We apply Claim 8 to the space (D, γ) and the set K: if (i) holds, then define $S_l = \cap_{i=1}^l K_i^c$. Then $\gamma(S_l^c) = \gamma(\cup_{i=1}^l K_i) \le \sum_{i=1}^l \gamma(M_i) = 0$. Furthermore, let ζ, τ be such that (10) and (11) hold and assume, to get a contradiction, that (12) fails. Then letting $d_1, \ldots, d_l$ be such that $\{d_1, \ldots, d_l\} = \text{supp}(\zeta)$, $d'_1, \ldots, d'_l$ be such that $\{d'_1, \ldots, d'_l\} = \text{supp}(\tau)$ and, for each $i = 1, \ldots, l$, $\zeta_i = \zeta(d_i)$ and $\tau_i = \tau(d'_i)$, it follows that $(d_1, \ldots, d_l, \zeta_1, \ldots, \zeta_l, d'_1, \ldots, d'_l, \tau_1, \ldots, \tau_l) \in \hat{K}$. Thus, $(d_1, \ldots, d_l) \in K$ and there is $1 \le i \le l$ such that $(d_1, \ldots, d_l) \in \pi_i^{-1}(M_i)$, i.e. $d_i \in M_i$. But this is a contradiction since $d_i \in \text{supp}(\zeta) \subseteq S_l \subseteq M_i^c$. This contradiction shows that (12) holds and this completes the proof of the claim when condition (i) in Claim 8 holds.

It remains to show that condition (ii) in Claim 8 cannot hold. Suppose that there is a measure η on E^l as in condition (ii) in Claim 8. We may assume that η is concentrated on K and satisfies $\eta \circ \pi_i^{-1} \le \gamma/l$ for each $i = 1, \ldots, l$; indeed, if not, use η' defined by setting $\eta'(B) = \eta(B \cap K)/l$ for each Borel $B \subseteq D^l$.

We apply the Jankow–von Neumann selection theorem (e.g. Bogachev (2007, Theorem 6.9.2, p. 35) to the set $\hat{K}$ to define a mapping $\phi : K \to \mathbb{R}_+^l \times D^l \times \mathbb{R}_+^l$ such that, letting

$$\phi(d) = (\zeta_1(d), \ldots, \zeta_l(d), d'_1(d), \ldots, d'_l(d), \tau_1(d), \ldots, \tau_l(d)),$$

$(d, \phi(d)) \in \hat{K}$ for each $d \in K$, and ϕ is measurable with respect to the σ-field generated by the analytic subsets of D^l. Setting

$$\zeta_d = \sum_{i=1}^l \zeta_i(d) 1_{d_i} \text{ and } \tau_d = \sum_{i=1}^l \tau_i(d) 1_{d'_i(d)},$$

we thus obtain kernels $d \mapsto \zeta_d$ and $d \mapsto \tau_d$ from D^l with the σ-field generated by its analytic subsets to D with its Borel sets. We use these kernels to define measures ω and ω' on the Borel subsets of D by setting, for each Borel subset $B \subseteq D$,

$$\omega(B) = \int \zeta_d(B) \mathrm{d}\eta(d) \text{ and } \omega'(B) = \int \tau_d(B) \mathrm{d}\eta(d).$$

We have that $\omega \le \gamma$ since, for each Borel $B \subseteq D$, letting $r(e) = 1_B(e)$ for each $e \in D$

and noting that $\zeta_i(d) \leq 1$ for each $d \in K$ and $1 \leq i \leq l$,

$$\begin{aligned}\omega(B) &= \int \sum_{i=1}^{l} \zeta_i(d) 1_{d_i}(B) \mathrm{d}\eta(d) = \sum_{i=1}^{l} \int \zeta_i(d) 1_{d_i}(B) \mathrm{d}\eta(d) \\ &\leq \sum_{i=1}^{l} \int 1_{d_i}(B) \mathrm{d}\eta(d) = \sum_{i=1}^{l} \int r \circ \pi_i(d) \mathrm{d}\eta(d) = \sum_{i=1}^{l} \int r(e) \mathrm{d}\eta \circ \pi_i^{-1}(e) \\ &= \sum_{i=1}^{l} \int 1_B \mathrm{d}\eta \circ \pi_i^{-1} = \sum_{i=1}^{l} \eta \circ \pi_i^{-1}(B) \leq \sum_{i=1}^{l} \frac{\gamma(B)}{l} = \gamma(B).\end{aligned}$$

Moreover, for each $f \in \mathcal{F}$, we have that $\int f \mathrm{d}\tau_d = \int f \mathrm{d}\tau_d$ for each $d \in K$ and, hence,

$$\int f \mathrm{d}\omega' = \int \left(\int f \mathrm{d}\tau_d \right) \mathrm{d}\eta(d) = \int \left(\int f \mathrm{d}\zeta_d \right) \mathrm{d}\eta(d) = \int f \mathrm{d}\omega,$$

where the first and last equality are justified since f is bounded, $\zeta_d(D) \leq 1$ and $\tau_d(D) \leq l$ for each $d \in K$. Similarly, since $\int s \mathrm{d}\tau_d = s(\tau_d) > s(\zeta_d) = \int s \mathrm{d}\zeta_d$ for each $d \in K$ and s is continuous and bounded, we obtain that

$$s(\omega') = \int s \mathrm{d}\omega' = \int \left(\int s \mathrm{d}\tau_d \right) \mathrm{d}\eta(d) > \int \left(\int s \mathrm{d}\zeta_d \right) \mathrm{d}\eta(d) = \int s \mathrm{d}\omega = s(\omega).$$

In conclusion, $\gamma' = \gamma - \omega + \omega'$ belongs to Γ because $\omega \leq \gamma$, $\int f \mathrm{d}\omega' = \int f \mathrm{d}\omega$ for each $f \in \mathcal{F}$ and Claim 9, and is such that $s(\gamma') = s(\gamma) - s(\omega) + s(\omega') > s(\gamma)$. But this is a contradiction since γ is a surplus maximizing assignment. ■

Let $S = \cap_{l=1}^{\infty} S_l$. Then $\gamma(S^c) = 0$ and S is s-monotone. Indeed, $\gamma(S^c) \leq \sum_{l=1}^{\infty} \gamma(S_l^c) = 0$ and let $\zeta \in M(D)$ be supported on finitely many points of S and $\tau \in M(D)$ be finitely-supported and such that $\tau_1 + \tau_{2,n} = \zeta_1 + \zeta_{2,n}$. Let $\zeta' = \zeta / \max\{1, \zeta(D)\}$, $\tau' = \tau / \max\{1, \zeta(D)\}$ and $l = \max\{|\mathrm{supp}(\zeta)|, |\mathrm{supp}(\tau)|, \lceil \tau'(D) \rceil\}$; then $|\mathrm{supp}(\zeta')| \leq l$, $|\mathrm{supp}(\tau')| \leq l$, $\mathrm{supp}(\zeta') \subseteq S \subseteq S_l$, $\tau'(D) \leq l$ and $\tau_1' + \tau_{2,n}' = \zeta_1' + \zeta_{2,n}'$. It then follows by the properties of S_l, i.e. (10)–(12) that $s(\zeta') \geq s(\tau')$ and, hence, $s(\zeta) \geq s(\tau)$. Thus, S is s-monotone.

Note that any subset of an s-monotone set is s-monotone and that the closure of an s-monotone set is also s-monotone since s is continuous; thus, any subset of $\bar{S}$ is s-monotone. Since $\mathrm{supp}(\gamma) \subseteq \bar{S}$, it follows that $\mathrm{supp}(\gamma)$ is s-monotone. ■

A.4.6 Completing the proof of the necessity part of Theorem 2

We have that

$$M = \{z \in Z : (z, z') \in \operatorname{supp}(\gamma) \text{ for some } z' \in Y\},$$
$$S = \{z \in Z : (z, \emptyset) \in \operatorname{supp}(\gamma)\} \text{ and}$$
$$W = \{z \in Z : (\hat{z}, z) \in \operatorname{supp}(\gamma) \text{ for some } \hat{z} \in Z\}$$

since $\operatorname{supp}(\gamma) = g(\operatorname{supp}(\mu))$ by Carmona and Laohakunakorn (2024, Lemma 1).

The following lemma shows that matching is positive assortative in the sense the better managers have at least as good workers than worse managers.

Lemma 24 *If $(z, z'), (\hat{z}, \hat{z}') \in Z^2$, $(z, z'), (\hat{z}, \hat{z}') \in \operatorname{supp}(\gamma)$ and $z > \hat{z}$, then $z' \geq \hat{z}'$.*

Proof. Suppose that $z > \hat{z}$ but $\hat{z}' > z'$. Let $\zeta = 1_{(z,z')} + 1_{(\hat{z},\hat{z}')}$ and $\tau = 1_{(z,\hat{z}')} + 1_{(\hat{z},z')}$. Then ζ and τ are finitely-supported, $\operatorname{supp}(\zeta) \subseteq \operatorname{supp}(\gamma)$ and $\tau_Z + \tau_{Y,n} = \zeta_Z + \zeta_{Y,n}$. Since $s(\tau) - s(\zeta) = \big(n(\hat{z}') - n(z')\big)\big(F(z) - F(\hat{z})\big) > 0$, this contradicts Lemma 5. ■

Define $z_2 = \min M$.

Lemma 25 *z_2 exists and $z_2 < \bar{z}$.*

Proof. It follows by Lemma 20 that z_2 exists and by Corollary 2 that $z_2 < \bar{z}$. ■

The following lemma carries out the computations needed in Lemma 27 below, which is a key part of the argument showing that M, S and W are intervals. It consider a match (z, z') where the manager is of type z and the workers of type z' and a sequence of matches $(\hat{z}_k, z_k)$ such that $z_k \to z$.

Lemma 26 *Let $z' \in Z_\emptyset$, $z, \hat{z} \in Z$, $\{z_k\}_{k=1}^\infty \subseteq Z$ and $\{\hat{z}_k\}_{k=1}^\infty \subseteq Z$ be such that $z' \leq z < z_k \leq \hat{z}_k$ for each $k \in \mathbb{N}$, $z_k \to z$ and $\hat{z}_k \to \hat{z}$. If, for each $k \in \mathbb{N}$, $\zeta_k = 1_{(z,z')} + 1_{(\hat{z}_k,z_k)}$ and*

$$\tau_k = 1_{(z_k,z')} + \frac{1}{n(z)} 1_{(\hat{z}_k,z)} + \left(1 - \frac{1}{n(z)}\right) 1_{(\hat{z}_k,z_k)} + \left(\frac{n(z_k)}{n(z)} - 1\right) 1_{(z_k,\emptyset)},$$

then

$$\lim_k \frac{s(\tau_k) - s(\zeta_k)}{z_k - z} = f(z)\left(n(z') - \frac{F(\hat{z}) - F(z)}{1 - F(z)}\right).$$

Proof. We have that

$$s(\zeta) = F(z)n(z') + F(\hat{z}_k)n(z_k), \text{ and}$$
$$s(\tau) = F(z_k)n(z') + F(\hat{z}_k) + \left(1 - \frac{1}{n(z)}\right) F(\hat{z}_k)n(z_k) + \left(\frac{n(z_k)}{n(z)} - 1\right) F(z_k).$$

Furthermore, $n'(z) = hf(z)n(z)^2$ and, hence,

$$\frac{n'(z)}{n(z)} = hf(z)n(z) = \frac{f(z)}{1 - F(z)}.$$

Then,

$$\begin{aligned}\frac{s(\tau_k) - s(\zeta_k)}{z_k - z} &= \frac{F(z_k) - F(z)}{z_k - z} n(z') - \frac{F(\hat{z}_k)}{n(z)} \frac{n(z_k) - n(z)}{z_k - z} + \frac{n(z_k) - n(z)}{z_k - z} \frac{F(z_k)}{n(z)} \\ &\to f(z)n(z') - (F(\hat{z}) - F(z)) \frac{n'(z)}{n(z)} \\ &= f(z)\left(n(z') - \frac{F(\hat{z}) - F(z)}{1 - F(z)}\right).\end{aligned}$$

■

Lemma 27 shows that if there are managers of type z, then all individuals slightly more knowledgeable than z must be managers.

Lemma 27 *For each $z \in M \backslash \{\bar{z}\}$, there exists $\varepsilon > 0$ such that $(z, z+\varepsilon) \subseteq M \backslash (S \cup W)$.*

Proof. Suppose not; then there exists a sequence $\{z_k\}_{k=1}^{\infty}$ such that, for each $k \in \mathbb{N}$, $z_k > z$, $z_k \in (M \setminus (S \cup W))^c = M^c \cup (S \cup W)$ and $z_k \to z$; thus, $z_k \in S \cup W$ by Lemma 23. Let $z' \in Z$ be such that $(z, z') \in \text{supp}(\gamma)$.

Suppose that $z_k \in S$ for some $k \in \mathbb{N}$ and let $\zeta = 1_{(z,z')} + 1_{(z_k,\emptyset)}$ and $\tau = 1_{(z_k,z')} + 1_{(z,\emptyset)}$. Then ζ and τ are finitely-supported, $\text{supp}(\zeta) \subseteq \text{supp}(\gamma)$ and $\tau_Z + \tau_{Y,n} = \zeta_Z + \zeta_{Y,n}$. Since $s(\tau) - s(\zeta) = \big(n(z') - 1\big)\big(F(z_k) - F(z)\big) > 0$, this contradicts Lemma 5.

Thus, $z_k \notin S$ for each $k \in \mathbb{N}$ and, hence, $z_k \in W$ for each $k \in \mathbb{N}$ by Lemma 23. For each $k \in \mathbb{N}$, let $\hat{z}_k \in Z$ be such that $(\hat{z}_k, z_k) \in \text{supp}(\gamma)$. Since Z is compact, we may assume that $\{\hat{z}_k\}_{k=1}^{\infty}$ converges; let $\hat{z} = \lim_k \hat{z}_k$. For each $k \in \mathbb{N}$, let $\zeta_k = 1_{(z,z')} + 1_{(\hat{z}_k,z_k)}$ and

$$\tau_k = 1_{(z_k,z')} + \frac{1}{n(z)} 1_{(\hat{z}_k,z)} + \left(1 - \frac{1}{n(z)}\right) 1_{(\hat{z}_k,z_k)} + \left(\frac{n(z_k)}{n(z)} - 1\right) 1_{(z_k,\emptyset)}.$$

Then ζ_k and τ_k are finitely-supported, $\text{supp}(\zeta_k) \subseteq \text{supp}(\gamma)$, $\tau_{k,Z} + \tau_{k,Y,n} = \zeta_{k,Z} + \zeta_{k,Y,n}$, and, by Lemma 26,

$$\lim_k \frac{s(\tau_k) - s(\zeta_k)}{z_k - z} = f(z) \left(\frac{1}{h(1 - F(z'))} - \frac{F(\hat{z}) - F(z)}{1 - F(z)} \right) \geq f(z) \left(\frac{1}{h} - 1 \right) > 0.$$

Thus, $s(\tau_k) - s(\zeta_k) > 0$ for each k sufficiently large, contradicting Lemma 5. ■

The following lemma shows that M and $S \cup W$ are intervals and that, except for z_2, any workers and self-employed are less knowledgeable than managers. It proof simply extends globally the local conclusion of Lemma 27.

Lemma 28 *$M = [z_2, \bar{z}]$ and $S \cup W = [0, z_2]$.*

Proof. Let $\bar{\varepsilon} = \sup\{\varepsilon > 0 : (z_2, z_2 + \varepsilon) \subseteq M \setminus (S \cup W)\}$. Such $\bar{\varepsilon}$ exists because $\{\varepsilon > 0 : (z_2, z_2 + \varepsilon) \subseteq M \setminus (S \cup W)\}$ is nonempty by Lemma 27 and is bounded above by $\bar{z} - z_2$. We then have that $(z_2, z_2 + \bar{\varepsilon}) \subseteq M \setminus (S \cup W)$ by the definition of $\bar{\varepsilon}$. Indeed, each $z \in (z_2, z_2 + \bar{\varepsilon})$ belongs to $M \setminus (S \cup W)$ since, letting $\varepsilon > 0$ be such that $z < z_2 + \varepsilon$ and $\varepsilon < \bar{\varepsilon}$, it follows that $z \in (z_2, z_2 + \varepsilon) \subseteq M \setminus (S \cup W)$.

Furthermore, $z_2 + \bar{\varepsilon} \in M \setminus (S \cup W)$. We have that $z_2 + \bar{\varepsilon} \in M$ since M is closed by Lemma 20 and every $z < z_2 + \varepsilon$ belongs to M; then the claim clearly holds by Lemma 1 if $\bar{\varepsilon} = \bar{z} - z_2$. Thus, consider $\bar{\varepsilon} < \bar{z} - z_2$ and suppose that $z_2 + \bar{\varepsilon} \in S \cup W$. Then, letting $\eta > 0$ be such that $(z_2 + \bar{\varepsilon}, z_2 + \bar{\varepsilon} + \eta) \subseteq M \setminus (S \cup W)$, which exists by Lemma 27, it follows that $z_2 + \bar{\varepsilon}$ is an isolated point of S or W. But this contradicts Lemmas 21 and 22.

It follows that $(z_2, z_2 + \bar{\varepsilon}] \subseteq M \setminus (S \cup W)$. If $\bar{\varepsilon} < \bar{z} - z_2$, then $(z_2 + \bar{\varepsilon}, z_2 + \bar{\varepsilon} + \eta) \subseteq M \setminus (S \cup W)$ for some $\eta > 0$ by Lemma 27 and, hence, $(z_2, z_2 + \bar{\varepsilon} + \eta) \subseteq M \setminus (S \cup W)$, contradicting the definition of $\bar{\varepsilon}$. Thus, it follows that $\bar{\varepsilon} = \bar{z} - z_2$ and that $M \setminus (S \cup W) = (z_2, \bar{z}]$. It follows that $S \cup W \subseteq [0, z_2]$ and, in fact, that $S \cup W = [0, z_2]$ and $M = [z_2, \bar{z}]$ since $M \cup S \cup W = Z$ by Lemma 23, M is closed by Lemma 20 and $S \cup W$ is closed by Lemmas 21 and 22. ■

Let $z_1 = \min S$ when S is nonempty and $z_1 = z_2$ otherwise.

Lemma 29 *$z_1 \leq z_2$ and $S \neq \emptyset$ if and only if $z_1 < z_2$.*

Proof. It follows by Lemma 28 that $z_1 \leq z_2$.

The definition of z_1 implies that $S \neq \emptyset$ if $z_1 < z_2$. For the converse, suppose that $S \neq \emptyset$ and $z_1 = z_2$. Then $S = \{z_2\}$ and $(z_2, \emptyset) \in \text{supp}(\gamma)$ by the definition of S. Since $\gamma(\{(z_2, \emptyset)\}) \leq \nu(\{z_2\}) = 0$, it follows that $\text{supp}(\gamma) \cap (Z \times Y)$ is closed and a strict subset of $\text{supp}(\gamma)$ with $\gamma(\text{supp}(\gamma) \cap (Z \times Y)) = \gamma(\text{supp}(\gamma))$. This contradicts the definition of $\text{supp}(\gamma)$ and shows that $z_1 < z_2$. ■

Lemma 27 shows that if all individuals of type z are self-employed, then all individuals slightly more knowledgeable than z must be self-employed.

Lemma 30 *If $S \neq \emptyset$, then, for each $z \in S \setminus \{z_2\}$, there exists $0 < \varepsilon < z_2 - z$ such that $(z, z+\varepsilon) \subseteq S \setminus W$.*

Proof. Consider first the case where $z \in W$ and $(\bar{z}, z) \in \text{supp}(\gamma)$. In this case, let $0 < \varepsilon < z_2 - z$. If there is $z' \in (z, z+\varepsilon) \cap S^c$, then there is an open neighborhood V of z' such that $\gamma(V \times \{\emptyset\}) = 0$. We may assume that $V \subseteq (z, z_2)$. Then $V \cap M = \emptyset$ by Lemma 28 and, hence, $\gamma(V \times Y) = 0$. Furthermore, for each $\tilde{z} > z$, if $(\hat{z}, \tilde{z}) \in \text{supp}(\mu)$, then $\hat{z} = \bar{z}$ by Lemma 24. Thus,

$$\begin{aligned} 0 < \nu(V) &= \gamma(V \times Y) + \mu(V \times \{\emptyset\}) + \int_{Z \times V} n(z') \mathrm{d}\gamma(z, z') \\ &= \int_{(Z \setminus \{\bar{z}\}) \times V} n(z') \mathrm{d}\gamma(z, z') = 0, \end{aligned}$$

a contradiction. Thus, $(z, z+\varepsilon) \subseteq S$.

Furthermore, $(z, z+\varepsilon) \subseteq W^c$. Indeed, if $(z, z+\varepsilon) \cap W \neq \emptyset$, then $\text{supp}(\gamma) \cap (Z \times (z, z+\varepsilon)) \neq \emptyset$. Lemma 24 implies that $\text{supp}(\gamma) \cap (Z \times (z, z+\varepsilon)) \subseteq \{\bar{z}\} \times X$ and, hence,

$$\gamma(Z \times (z, z+\varepsilon)) = \gamma(\text{supp}(\gamma) \cap (Z \times (z, z+\varepsilon))) \leq \gamma(\{\bar{z}\} \times Y) = 0.$$

Since $Z \times (z, z+\varepsilon)$ is open, it follows that $\text{supp}(\gamma) \setminus (Z \times (z, z+\varepsilon))$ is closed, strictly contained in $\text{supp}(\gamma)$ and $\gamma(\text{supp}(\gamma)) = \gamma(\text{supp}(\gamma) \setminus (Z \times (z, z+\varepsilon)))$. But this contradicts the definition of $\text{supp}(\gamma)$. Thus, $(z, z+\varepsilon) \subseteq W^c$ and this, together what has been shown above, implies that $(z, z+\varepsilon) \subseteq S \setminus W$.

Hence, we may consider the case where $z \notin W$ or $(\bar{z}, z) \notin \text{supp}(\gamma)$. Suppose that there is no $0 < \varepsilon < z_2 - z$ such that $(z, z+\varepsilon) \subseteq S \setminus W$. Then there exists a sequence $\{z_k\}_{k=1}^{\infty}$ such that, for each $k \in \mathbb{N}$, $z < z_k < z_2$, $z_k \in S^c \cup W$ and $z_k \to z$. Thus, $z_k \in W$ for each $k \in \mathbb{N}$ since $M = [z_2, \bar{z}]$ and $Z = M \cup S \cup W$ by Lemmas 23 and 28. Hence, $z \in W$ since W is closed by Lemma 21 and it follows that $(\bar{z}, z) \notin \text{supp}(\gamma)$.

Let $z' = \emptyset$ and, for each $k \in \mathbb{N}$, let $\hat{z}_k \in Z$ be such that $(\hat{z}_k, z_k) \in \text{supp}(\gamma)$. Since Z is compact, we may assume that $\{\hat{z}_k\}_{k=1}^{\infty}$ converges; let $\hat{z} = \lim_k \hat{z}_k$. Then $\hat{z} < \bar{z}$ since $(\hat{z}, z) \in \text{supp}(\gamma)$ and $(\bar{z}, z) \notin \text{supp}(\gamma)$.

For each $k \in \mathbb{N}$, let $\zeta_k = 1_{(z,z')} + 1_{(\hat{z}_k, z_k)}$ and

$$\tau_k = 1_{(z_k,z')} + \frac{1}{n(z)} 1_{(\hat{z}_k,z)} + \left(1 - \frac{1}{n(z)}\right) 1_{(\hat{z}_k,z_k)} + \left(\frac{n(z_k)}{n(z)} - 1\right) 1_{(z_k,\emptyset)}.$$

Then ζ_k and τ_k are finitely-supported, $\text{supp}(\zeta_k) \subseteq \text{supp}(\gamma)$, $\tau_{k,Z} + \tau_{k,Y,n} = \zeta_{k,Z} + \zeta_{k,Y,n}$, and, by Lemma 26,

$$\lim_k \frac{s(\tau_k) - s(\zeta_k)}{z_k - z} = f(z)\left(1 - \frac{F(\hat{z}) - F(z)}{1 - F(z)}\right) > f(z)(1-1) = 0.$$

Thus, $s(\tau_k) - s(\zeta_k) > 0$ for each k sufficiently large, contradicting Lemma 5. This contradiction shows that there exists $0 < \varepsilon < z_2 - z$ such that $(z, z+\varepsilon) \subseteq S \setminus W$. ■

The local conclusion of Lemma 30 extends globally and this shows that, when S is nonempty and with the exception of z_1, workers are less knowledgeable than managers.

Lemma 31 *$W = [0, z_1]$ and, if $S \neq \emptyset$, then $S = [z_1, z_2]$.*

Proof. We can assume that S is nonempty since otherwise the conclusion follows from Lemma 28. Then $z_1 < z_2$ by Lemma 29.

Let $\bar{\varepsilon} = \sup\{\varepsilon > 0 : (z_1, z_1 + \varepsilon) \subseteq S \setminus W\}$. Such $\bar{\varepsilon}$ exists because $\{\varepsilon > 0 : (z_1, z_1+\varepsilon) \subseteq S \setminus W\}$ is nonempty by Lemma 30 and is bounded above by $z_2 - z_1$. We then have that $(z_1, z_1+\bar{\varepsilon}) \subseteq S \setminus W$ by the definition of $\bar{\varepsilon}$. Indeed, each $z \in (z_1, z_1+\bar{\varepsilon})$ belongs to $S \setminus W$ since, letting $\varepsilon > 0$ be such that $z < z_1 + \varepsilon$ and $\varepsilon < \bar{\varepsilon}$, it follows that $z \in (z_1, z_1 + \varepsilon) \subseteq S \setminus W$.

We next claim that $\bar{\varepsilon} = z_2 - z_1$. Suppose not; then $\bar{\varepsilon} < z_2 - z_1$. We have that $z_1 + \bar{\varepsilon} \in S$ since S is closed. If $z_1 + \bar{\varepsilon} \in W$, then, letting $\eta > 0$ be such that

$(z_1+\bar{\varepsilon}, z_1+\bar{\varepsilon}+\eta) \subseteq S \setminus W$, it follows that $z_1+\bar{\varepsilon}$ is an isolated point of W. But this contradicts Lemma 21. Thus, $(z_1, z_1+\bar{\varepsilon}] \subseteq S \setminus W$. But then $(z_1+\bar{\varepsilon}, z_1+\bar{\varepsilon}+\eta) \subseteq S \setminus W$ for some $\eta > 0$ and, hence, $(z_1, z_1+\bar{\varepsilon}+\eta) \subseteq S \setminus W$, contradicting the definition of $\bar{\varepsilon}$. Thus, it follows that $\bar{\varepsilon} = z_2 - z_1$.

It follows from $\bar{\varepsilon} = z_2 - z_1$ that $(z_1, z_2) \subseteq S \setminus W$. We have that $z_1, z_2 \in S$ since S is closed, hence $S = [z_1, z_2]$. Then $[0, z_1) \subseteq W$ by Lemma 28 and that $z_1 \in W$ since W is closed. Since $(z_1, z_2) \subseteq W^c$ and W is perfect, it follows that $z_2 \notin W$. Thus, $W = [0, z_1]$ and $S = [z_1, z_2]$. ■

The following lemma asserts that $z_1 > 0$; this happens because there is a positive measure of managers (since $z_2 < \bar{z}$) and, hence, there must be a positive measure of workers.

Lemma 32 $z_1 > 0$.

Proof. Suppose that $z_1 = 0$. Then $W = \{0\}$ and $\mathrm{supp}(\gamma) \cap (Z \times Y) \subseteq Z \times \{0\}$. Thus,

$$\begin{aligned}\nu(Z) &= \nu(Z \setminus \{0\}) = \gamma((Z \setminus \{0\}) \times Y_\emptyset) + \int_{Z \times (Y \setminus \{0\})} n(z')\mathrm{d}\gamma(z, z') \\ &= \gamma((Z \setminus \{0\}) \times Y_\emptyset).\end{aligned}$$

Furthermore,

$$\gamma((Z \setminus \{0\}) \times Y_\emptyset) \leq \gamma(Z \times Y_\emptyset) \leq \gamma(Z \times Y_\emptyset) + \int_{Z \times Y} n(z')\mathrm{d}\gamma(z, z') = \nu(Z).$$

Hence, $\gamma(Z \times Y_\emptyset) = \nu(Z)$. This then implies that $\int_{Z \times Y} n(z')\mathrm{d}\gamma(z, z') = 0$. But this contradicts

$$\begin{aligned}\int_{Z \times Y} n(z')\mathrm{d}\gamma(z, z') &= \int_{\mathrm{supp}(\gamma) \cap (Z \times Y)} n(z')\mathrm{d}\gamma(z, z') = n(0)\gamma(\mathrm{supp}(\gamma) \cap (Z \times Y)) \\ &= \frac{\gamma(Z \times Y)}{h} = \frac{\mu(Z \times X)}{h} > 0,\end{aligned}$$

where the last inequality follows by Lemma 11. ■

Define $\phi : M \rightrightarrows W$ by setting, for each $z \in M$,

$$\phi(z) = \{z' \in Z : (z, z') \in \mathrm{supp}(\gamma)\}.$$

Then ϕ is nonempty-valued by the definition of M, $\phi(M) = W$ by the definition of W and ϕ has a closed graph since $\text{supp}(\gamma)$ is closed.

Let $Q = \{z \in M : \phi(z) \text{ is not a singleton}\}$. The correspondence ϕ is increasing due to positive assortativeness and, hence, the set Q of (lower hemi) discontinuities of ϕ is countable.

Lemma 33 *Q is countable.*

Proof. For each $z \in Q$, let $r(z) \in \mathbb{Q}$ be such that $\min \phi(z) < r(z) < \max \phi(z)$. This defines a function $r : Q \to \mathbb{Q}$ which, as we now claim, is strictly increasing. Indeed, if $z, \hat{z} \in Q$ are such that $z < \hat{z}$, then $(z, \max \phi(z)) \in \text{supp}(\gamma)$, $(\hat{z}, \min \phi(\hat{z})) \in \text{supp}(\gamma)$ and, hence, $\max \phi(z) \leq \min \phi(\hat{z})$ by Lemma 24. Thus, $r(z) < \max \phi(z) \leq \min \phi(\hat{z}) < r(\hat{z})$. Thus, r maps Q in a one-to-one way to a subset of $\mathbb{Q}$, implying that Q is countable. ■

We next show that each worker type is matched with a unique manager type. The reason is roughly that if worker type z' were matched with manager types z^* and $\tilde{z}$, then positive assortativeness implies that type z' is matched with all types in $[z^*, \tilde{z}]$; furthermore, those types in $[z^*, \tilde{z}] \setminus Q$ are only matched with z', which means that a zero measure of workers (those with type z') is matched with a positive measure of managers (those with type in $[z^*, \tilde{z}] \setminus Q$).

Lemma 34 *For each $z \in W$, there exists $z^* \in Z$ such that $\{\hat{z} \in Z : (\hat{z}, z) \in \text{supp}(\gamma)\} = \{z^*\}$.*

Proof. The definition of W implies that $\{\hat{z} \in Z : (\hat{z}, z) \in \text{supp}(\gamma)\}$ is nonempty. Suppose that the conclusion of the lemma fails; then let $z' \in W$ and $z^*, \tilde{z} \in Z$ be such that $z^*, \tilde{z} \in \{\hat{z} \in Z : (\hat{z}, z') \in \text{supp}(\gamma)\}$ and $z^* < \tilde{z}$. Since $z^*, \tilde{z} \in M$ by the definition of M and M is an interval by Lemma 28, $[z^*, \tilde{z}] \subseteq M$. Let $z \in (z^*, \tilde{z})$ and $\tilde{z}' \in Z$ be such that $(z, \tilde{z}') \in \text{supp}(\gamma)$. Lemma 24 implies that $z' \leq \tilde{z}' \leq z'$, hence $\tilde{z}' = z'$. Thus, $z \in \{\hat{z} \in Z : (\hat{z}, z') \in \text{supp}(\gamma)\}$; since z is arbitrary, it follows that $[z^*, \tilde{z}] \subseteq \{\hat{z} \in Z : (\hat{z}, z') \in \text{supp}(\gamma)\}$.

We have that $(z^*, \tilde{z}) \setminus Q \subseteq M \setminus (W \cup S)$ by Lemma 28 and $\phi(z) = \{z'\}$ for each $z \in (z^*, \tilde{z}) \setminus Q$. Thus,

$$\nu([z^*, \tilde{z}] \setminus Q) = \nu((z^*, \tilde{z}) \setminus Q) = \gamma(((z^*, \tilde{z}) \setminus Q) \times Y_\emptyset) = \gamma(((z^*, \tilde{z}) \setminus Q) \times \{z'\}).$$

Since $0 = \nu(\{z'\}) \geq \int_{Z \times \{z'\}} n(\hat{z}) \mathrm{d}\gamma(z, \hat{z})$, it follows that $\int_{Z \times \{z'\}} n(\hat{z}) \mathrm{d}\gamma(z, \hat{z}) = 0$. Thus,

$$\begin{aligned} &0 = \int_{Z \times \{z'\}} n(\hat{z}) \mathrm{d}\gamma(z, \hat{z}) = n(z')\gamma(Z \times \{z'\}) \geq \\ &n(z')\gamma(((z^*, \tilde{z}) \setminus Q) \times \{z'\}) = n(z')\nu([z^*, \tilde{z}] \setminus Q) = n(z')\nu([z^*, \tilde{z}]) > 0, \end{aligned}$$

a contradiction. ■

It then follows that matching is strictly positive assortative in the sense the better managers have better workers.

Lemma 35 *If $(z, z'), (\hat{z}, \hat{z}') \in Z^2$, $(z, z'), (\hat{z}, \hat{z}') \in \text{supp}(\gamma)$ and $z > \hat{z}$, then $z' > \hat{z}'$.*

Proof. We have that $z' \geq \hat{z}'$ by Lemma 24 and that $z' \neq \hat{z}'$ by Lemma 34. Thus, $z' > \hat{z}'$. ■

Strict positive assortativeness then implies that ϕ is a function, i.e. $Q = \emptyset$. This happens because if manager type z were matched with worker types z^* and $\tilde{z}$, then strict positive assortativeness implies that type z is matched with all types in $[z^*, \tilde{z}]$ and that these types are not matched with any other manager type. But then a zero measure of managers (those with type z) is matched with a positive measure of workers (those with type in $[z^*, \tilde{z}]$).

Lemma 36 *ϕ is a continuous and strictly increasing function. Furthermore, $\phi(z_2) = 0$ and $\phi(\bar{z}) = z_1$.*

Proof. We first show that $\phi(z)$ is a singleton for each $z \in M$, i.e. $Q = \emptyset$. Suppose not; then let $z \in M$ and $z^*, \tilde{z} \in \phi(z)$ be such that $z^* < \tilde{z}$. Since W is an interval by Lemma 31, $[z^*, \tilde{z}] \subseteq W$. Let $z' \in (z^*, \tilde{z})$ and $\hat{z} \in M$ be such that $z' \in \phi(\hat{z})$. Lemma 35 then implies that $z' > \tilde{z}$ if $\hat{z} > z$ and that $z' < z^*$ if $\hat{z} < z$. Thus, $\hat{z} = z$ and $z' \in \phi(z)$; since z' is arbitrary, it follows that $[z^*, \tilde{z}] \subseteq \phi(z)$.

We have that $[z^*, \tilde{z}] \cap \phi(x) = \emptyset$ for each $x \in M \setminus \{z\}$. Indeed, Lemma 35 implies that $\min \phi(x) > \tilde{z}$ for each $x > z$ and that $\max \phi(x) < z^*$ for each $x < z$. Since $(z^*, \tilde{z}) \subseteq W \setminus (M \cup S)$, it follows that

$$\nu([z^*, \tilde{z}]) = \gamma([z^*, \tilde{z}] \times Y_\emptyset) + \int_{Z \times [z^*, \tilde{z}]} n(z') \mathrm{d}\gamma(z, z') = 0 + \int_{(Z \setminus \{z\}) \times [z^*, \tilde{z}]} n(z') \mathrm{d}\gamma(z, z') = 0,$$

a contradiction to $\nu([z^*, \tilde{z}]) > 0$. This contradiction shows that $\phi(z)$ is a singleton for each $z \in M$.

It then follows that ϕ is a function. Since the graph of ϕ is closed, it follows that ϕ is continuous. Lemma 35 implies that ϕ is strictly increasing.

It follows from $\phi(M) = W$ that ϕ is onto. This then implies that $\phi(z_2) = 0$ and $\phi(\bar{z}) = z_1$ since ϕ is strictly increasing. ■

The properties of ϕ above are then used to show that the wage function c is differentiable.

Lemma 37 *c is differentiable and, for each $z \in W$, $c'(z) = f(z)\frac{F(\phi^{-1}(z)) - c(z)}{1 - F(z)}$.*

Proof. Let $z \in W = [0, z_1]$ and $\{z_k\}_{k=1}^\infty$ be such that, for each $k \in \mathbb{N}$, $z_k \in W$, $z_k \neq z$ and $z_k \to z$. Let $\{\hat{z}_k\}_{k=1}^\infty$ be such that $\hat{z}_k = \phi^{-1}(z_k)$ for each $k \in \mathbb{N}$. We have that ϕ^{-1} exists and is continuous by Lemma 36 and since M is compact. Thus, $\hat{z}_k \to \phi^{-1}(z)$.

The stability of μ implies that, for each $k \in \mathbb{N}$, $\frac{F(\phi^{-1}(z)) - c(z)}{h(1 - F(z))} \geq \frac{F(\phi^{-1}(z)) - c(z_k)}{h(1 - F(z_k))}$. Thus, a simple manipulation of this expression implies that

$$\frac{c(z_k) - c(z)}{z_k - z} \geq \frac{F(z_k) - F(z)}{z_k - z} \frac{F(\phi^{-1}(z)) - c(z_k)}{1 - F(z_k)};$$

hence, $\liminf_k \frac{c(z_k) - c(z)}{z_k - z} \geq f(z)\frac{F(\phi^{-1}(z)) - c(z)}{1 - F(z)}$ since c is continuous by Lemma 14.

Analogously, the stability of μ implies that, for each $k \in \mathbb{N}$, $\frac{F(\hat{z}_k) - c(z_k)}{h(1 - F(z_k))} \geq \frac{F(\hat{z}_k) - c(z)}{h(1 - F(z))}$. Thus,

$$\frac{c(z_k) - c(z)}{z_k - z} \leq \frac{F(z_k) - F(z)}{z_k - z} \frac{F(\hat{z}_k) - c(z_k)}{1 - F(z_k)};$$

hence, $\limsup_k \frac{c(z_k) - c(z)}{z_k - z} \leq f(z)\frac{F(\phi^{-1}(z)) - c(z)}{1 - F(z)}$. It thus follows that

$$\lim_k \frac{c(z_k) - c(z)}{z_k - z} = f(z)\frac{F(\phi^{-1}(z)) - c(z)}{1 - F(z)}.$$

Hence, c is differentiable and, for each $z \in W$, $c'(z) = f(z)\frac{F(\phi^{-1}(z))-c(z)}{1-F(z)}$. ■

We have shown so far that types in $[z_1, z_2]$ are self-employed and, thus, matched with $1_{(\emptyset,0)}$ and types in $[z_2, \bar{z}]$ are managers and matched with workers of type $\phi(z)$, thus, with $n(\phi(z))1_{(\phi(z),c(\phi(z)))}$. Hence, the matching μ is fully described by the distribution ν of types and the function σ.

Lemma 38 $\mu = \nu \circ \sigma^{-1}$.

Proof. Let B be a Borel subset of $Z \times X_\emptyset$. Then

$$\begin{aligned}\nu \circ \sigma^{-1}(B) &= \nu(\{z \in Z : \sigma(z) \in B\}) \\ &= \nu(\{z \in [z_1, z_2) : \sigma(z) \in B\}) + \nu(\{z \in [z_2, \bar{z}] : \sigma(z) \in B\}).\end{aligned}$$

Furthermore,

$$\begin{aligned}\mu(B) &= \mu(\text{supp}(\mu) \cap B) \\ &= \mu(\text{supp}(\mu) \cap B \cap (Z \times \{1_{(\emptyset,0)}\})) + \mu(\text{supp}(\mu) \cap B \cap (Z \times X)) \\ &= \mu(\{z \in [z_1, z_2] : \sigma(z) \in B\} \times \{1_{(\emptyset,0)}\}) + \mu(\{z \in [z_2, \bar{z}] : \sigma(z) \in B\} \times X).\end{aligned}$$

Let $\hat{D} = \{z \in (z_1, z_2) : \sigma(z) \in B\}$ and note that

$$\begin{aligned}&\nu(\{z \in [z_1, z_2) : \sigma(z) \in B\}) = \nu(\hat{D}) \\ &= \mu(\hat{D} \times X) + \mu(\{z \in (z_1, z_2) : \sigma(z) \in B\} \times (X_\emptyset \setminus X)) + \int_{Z \times X} \delta(\hat{D} \times C) \mathrm{d}\mu(z, \delta) \\ &= 0 + \mu(\{z \in (z_1, z_2) : \sigma(z) \in B\} \times \{1_{(\emptyset,0)}\}) + 0 \\ &= \mu(\{z \in [z_1, z_2] : \sigma(z) \in B\} \times \{1_{(\emptyset,0)}\}).\end{aligned}$$

Let $D = \{z \in (z_2, \bar{z}] : \sigma(z) \in B\}$ and note that

$$\begin{aligned}&\nu(\{z \in [z_2, \bar{z}] : \sigma(z) \in B\}) = \nu(D) \\ &= \mu(D \times X) + \mu(D \times (X_\emptyset \setminus X)) + \int_{Z \times X} \delta(D \times C) \mathrm{d}\mu(z, \delta) \\ &= \mu(D \times X) + 0 + 0 = \mu(\{z \in [z_2, \bar{z}] : \sigma(z) \in B\} \times X).\end{aligned}$$

Thus $\nu \circ \sigma^{-1}(B) = \mu(B)$. Since B is arbitrary, $\nu \circ \sigma^{-1} = \mu$. ■

Let $\nu(z') = \nu([0, z'])$ for each $z' \in Z$. For each $z \geq z_2$, individuals of knowledge up to $\phi(z)$ are workers and are matched with managers of knowledge in $[z_2, z]$, hence their measure $\nu(\phi(z))$ equals the measure of workers hired by managers of knowledge between z_2 and z, which is $\int_{[z_2,z]} \frac{1}{h(1-F(\phi(x)))} \mathrm{d}\nu(x) = \int_{z_2}^{z} \frac{\theta(x)}{h(1-F(\phi(x)))} \mathrm{d}x$.

Lemma 39 *For each $z \in [z_2, \bar{z}]$, $\nu(\phi(z)) = \int_{z_2}^{z} \frac{\theta(x)}{h(1-F(\phi(x)))} \mathrm{d}x$.*

Proof. Let $z \in [z_2, \bar{z}]$ and let $\tau : Z \times X \to \mathbb{R}$ be defined by setting, for each $(z, \delta) \in Z \times X$, $\tau(z, \delta) = \delta([0, \phi(z)] \times C)$. It follows by Lemma 38 that

$$\int_{Z \times X} \delta([0, \phi(z)] \times C) \mathrm{d}\mu(x, \delta) = \int_{[z_2, \bar{z}]} \tau(\sigma(x)) \mathrm{d}\nu(x) = \int_{[z_2, z]} \frac{1}{h(1 - F(\phi(x)))} \mathrm{d}\nu(x).$$

Thus,

$$\nu([0, \phi(z)]) = \int_{[z_2, z]} \frac{1}{h(1 - F(\phi(x)))} \mathrm{d}\nu(x)$$

since $\nu([0, \phi(z)]) = \nu([0, \phi(z)))$ and $[0, \phi(z)) \subseteq W \setminus (S \cup M)$. Since ν has a continuous density θ, it follows that $\nu(\phi(z)) = \nu([0, \phi(z)]) = \int_{z_2}^{z} \frac{\theta(x)}{h(1-F(\phi(x)))} \mathrm{d}x$ for each $z \in [z_2, \bar{z}]$.
■

The feasibility of the matching μ is fully captured by the equality $\nu(\phi(z)) = \int_{z_2}^{z} \frac{\theta(x)}{h(1-F(\phi(x)))} \mathrm{d}x$ for each $z \in [z_2, \bar{z}]$ as stated in the previous lemma. It can be equivalently stated in terms of the derivative of ϕ as the following lemma shows.

Lemma 40 *ϕ is differentiable and, for each $z \in [z_2, \bar{z}]$,*

$$\phi'(z) = \frac{\theta(z)}{h(1 - F(\phi(z)))\theta(\phi(z))}.$$

Proof. The function $z' \mapsto \nu(z')$ is strictly increasing; let $\lambda : [0, \nu(\bar{z})] \to Z$ be its inverse. It then follows by Lemma 39 that, for each $z \in [z_2, \bar{z}]$,

$$\phi(z) = \lambda \left(\int_{z_2}^{z} \frac{\theta(x)}{h(1 - F(\phi(x)))} \mathrm{d}x \right).$$

We have that $z \mapsto \nu(z)$ is differentiable and that its derivative at $z \in Z$ is $\theta(z)$. Then λ is differentiable and $\lambda'(x) = \frac{1}{\theta(\lambda(x))}$ for each $x \in [0, \nu(\bar{z})]$. Let $\zeta : [z_2, \bar{z}] \to \mathbb{R}$ be defined by setting, for each $z \in [z_2, \bar{z}]$, $\zeta(z) = \int_{z_2}^{z} \frac{\theta(x)}{h(1-F(\phi(x)))} \mathrm{d}x$. Then ζ is differentiable with $\zeta'(z) = \frac{\theta(z)}{h(1-F(\phi(z)))}$ for each $z \in [z_2, \bar{z}]$. Since $\phi = \lambda \circ \zeta$, it follows that ϕ is differentiable and that, for each $z \in [z_2, \bar{z}]$, $\phi'(z) = \frac{\theta(z)}{h(1-F(\phi(z)))\theta(\lambda(\zeta(z)))}$. Since $\zeta(z) = \nu(\phi(z))$ by Lemma 39, we obtain that $\lambda(\zeta(z)) = \phi(z)$ and, hence, $\phi'(z) = \frac{\theta(z)}{h(1-F(\phi(z)))\theta(\phi(z))}$. ■

The following two results show that individuals who belong to two of the sets M, S and W must be indifferent between the corresponding occupations. Lemma 41 considers the case where there are self-employed individuals, in which case $z_1 < z_2$, $z_1 \in W \cap S$ and $z_2 \in S \cap M$. Consequently, those with knowledge z_1 are indifferent between being a worker or self-employed and those with knowledge z_2 are indifferent between being a manager or self-employed.

Lemma 41 *If $S \neq \emptyset$, then $c(z_1) = F(z_1)$ and $c(0) = (1-h)F(z_2)$.*

Proof. It follows from $(\emptyset, 0) \in T^s_{z_1}(\mu)$, $\phi(\bar{z}) = z_1$ and $z_1 \in W$ that $c(z_1) = U_{z_1}(w, 1_{(\bar{z}, c(z_1))}) \geq U_{z_1}(s, 1_{(\emptyset,0)}) = F(z_1)$. Similarly, $(\emptyset, 0) \in T^s_{z_2}(\mu)$, $z_2 \in M$ and $\phi(z_2) = 0$ that $\frac{F(z_2)-c(0)}{h} = U_{z_2}(m, 0) \geq U_{z_2}(s, 1_{(\emptyset,0)}) = F(z_2)$.

Suppose that $c(z_1) > F(z_1)$ and let $\varepsilon > 0$ be such that $c(z_1) - \varepsilon > F(z_1)$. Then $(z_1, c(z_1) - \varepsilon) \in T^m_{\bar{z}}(\mu)$ (since $z_1 \in S$) and $U_{\bar{z}}(m, n(z_1)1_{(z_1, c(z_1)-\varepsilon)}) = \frac{F(\bar{z})-c(z_1)+\varepsilon}{h(1-F(z_1))} > \frac{F(\bar{z})-c(z_1)}{h(1-F(z_1))} = U_{\bar{z}}(m, z_1)$. But this contradicts the stability of μ since $\bar{z} \in M$. Hence $c(z_1) \leq F(z_1)$ and, thus, $c(z_1) = F(z_1)$.

Suppose that $\frac{F(z_2)-c(0)}{h} > F(z_2)$ and let $\varepsilon > 0$ be such that $\frac{F(z_2)-c(0)-\varepsilon}{h} > F(z_2)$. Then $(0, c(0) + \varepsilon) \in T^m_{z_2}(\mu)$ (since $0 \in W$) and $U_{z_2}(m, n(0)1_{(0,c(0)+\varepsilon)}) > U_{z_2}(s, 1_{(\emptyset,0)})$. But this contradicts the stability of μ since $z_2 \in S$. Hence $\frac{F(z_2)-c(0)}{h} \leq F(z_2)$ and, thus, $c(0) = (1-h)F(z_2)$. ■

The following lemma considers the case where there are no self-employed individuals, in which case $z_1 = z_2$ and $z_2 \in W \cap M$. Consequently, those with knowledge z_2 are indifferent between being a worker or a manager.

Lemma 42 *If $S = \emptyset$, then $c(z_2) = \frac{F(z_2)-c(0)}{h} \geq F(z_2)$.*

Proof. We have that $\phi(\bar{z}) = z_2$ and $z_2 \in W \cap M$ since $S = \emptyset$. It follows from $(\emptyset, 0) \in T^s_{z_2}(\mu)$, $z_2 \in M$ and $\phi(z_2) = 0$ that $\frac{F(z_2)-c(0)}{h} = U_{z_2}(m, 0) \geq U_{z_2}(s, 1_{(\emptyset,0)}) = F(z_2)$.

Suppose that $c(z_2) > \frac{F(z_2)-c(0)}{h}$ and let $\varepsilon > 0$ be such that $c(z_2) - \varepsilon > \frac{F(z_2)-c(0)}{h}$. Then $(z_2, c(z_2) - \varepsilon) \in T^m_{\bar{z}}(\mu)$ (since $z_2 \in M$) and $U_{\bar{z}}(m, n(z_2)1_{(z_2,c(z_2)-\varepsilon)}) > U_{\bar{z}}(m, z_2)$. But this contradicts the stability of μ since $\bar{z} \in M$. Hence $c(z_2) \leq \frac{F(z_2)-c(0)}{h}$.

Suppose that $\frac{F(z_2)-c(0)}{h} > c(z_2)$ and let $\varepsilon > 0$ be such that $\frac{F(z_2)-c(0)-\varepsilon}{h} > c(z_2)$. Then $(0, c(0)+\varepsilon) \in T^m_{z_2}(\mu)$ (since $0 \in W$) and $U_{z_2}(m, n(0)1_{(0,c(0)+\varepsilon)}) > U_{z_2}(w, 1_{(\bar{z},c(z_2))})$. But this contradicts the stability of μ since $z_2 \in W$. Hence $\frac{F(z_2)-c(0)}{h} \leq c(z_2)$ and, thus, $\frac{F(z_2)-c(0)}{h} = c(z_2)$. ■

The necessity part of Theorem 2 then follows by Lemmas 13, 25, 28, 29, 31, 32, 37 and 36–42.

A.5 Proof of Theorem 2: Sufficiency

A.5.1 Proof of Lemma 6

Let (z_1, z_2, ϕ, c) and $(\hat{z}_1, \hat{z}_2, \hat{\phi}, \hat{c})$ satisfy the conditions 1–8 in Theorem 2.

Recall that $\nu(z) = \nu([0, z]) = \int_0^z \theta(x)\mathrm{d}x$ for each $z \in Z$. Thus, $\nu'(z) = \theta(z)$ for each $z \in Z$. The following lemma applies the fundamental theorem of calculus to the function $\nu \circ \phi$.

Lemma 43 *For each $z \in [z_2, \bar{z}]$, $\nu(\phi(z)) = \int_{z_2}^z \frac{\theta(x)}{h(1-F(\phi(x)))}\mathrm{d}x$.*

Proof. For each $z \in [z_2, \bar{z}]$, $\nu \circ \phi'(z) = \theta(\phi(z))\phi'(z) = \frac{\theta(z)}{h(1-F(\phi(z)))}$. Since $\nu(\phi(z_2)) = \nu(0) = 0$, it follows that $\nu(\phi(z)) = \int_{z_2}^z \nu \circ \phi'(x)\mathrm{d}x = \int_{z_2}^z \frac{\theta(x)}{h(1-F(\phi(x)))}\mathrm{d}x$ for each $z \in [z_2, \bar{z}]$. ■

The following is the key technical lemma in the proof of Lemma 6. It considers two solutions of initial value problems that differ (at most) on the initial conditions, and shows that if they coincide at some point and one of them is strictly increasing, then they must coincide everywhere.

Lemma 44 *Let $a, b, \hat{a}, \hat{b} \in \mathbb{R}$ be such that $a < b$ and $\hat{a} < \hat{b}$, $G : [a, b] \times [\hat{a}, \hat{b}] \to \mathbb{R}$ be continuous and such that $(z, x) \mapsto \frac{\partial G(z,x)}{\partial x}$ is continuous, $g : [a, b] \to [\hat{a}, \hat{b}]$ be a solution to the initial value problem $x' = G(z, x)$ and $x(a) = g(a)$ and $\hat{g} : [a, b] \to [\hat{a}, \hat{b}]$ be* *a solution to the initial value problem $x' = G(z,x)$ and $x(a) = \hat{g}(a)$. If g is strictly increasing and there exists $z_0 \in [a,b]$ such that $g(z_0) = \hat{g}(z_0)$, then $g = \hat{g}$.*

Proof. Let g be strictly increasing and $z_0 \in [a,b]$ be such that $g(z_0) = \hat{g}(z_0)$. We first show that $g(z) = \hat{g}(z)$ for each $z \geq z_0$. If $z_0 = b$, then this conclusion holds; hence, we may assume that $z_0 < b$.

Suppose that $\{z \in [z_0,b] : g(z) \neq \hat{g}(z)\} \neq \emptyset$ and let $z^* = \inf\{z \in [z_0,b] : g(z) \neq \hat{g}(z)\}$. Then $g(z^*) = \hat{g}(z^*)$: this is clear if $z^* = z_0$ and, if $z^* > z_0$, then $z^* \in \{z \in [a,b] : g(z) = \hat{g}(z)\}$ since this set is closed (due to the continuity of g and $\hat{g}$, which follows from the fact that they are solutions to an initial value problem) and $g(z) = \hat{g}(z)$ for each $z_0 \leq z < z^*$. Furthermore, $z^* < b$ since otherwise the definition of z^* implies that $\{z \in [z_0,b] : g(z) \neq \hat{g}(z)\} = \emptyset$.

We have that $g(z^*) < g(b) \leq \hat{b}$ since g is strictly increasing. Let $\eta > 0$ be such that $g(z^*) + \eta \leq \hat{b}$ and let, by the continuity of g and $\hat{g}$, $\varepsilon > 0$ be such that $\max\{g(z), \hat{g}(z)\} \leq g(z^*) + \eta$ for each $z \in [z^*, z^* + \varepsilon]$. Thus, both g and $\hat{g}$ are solutions of the initial value problem $x' = \tilde{G}(z,x)$ and $x(z^*) = g(z^*) = \hat{g}(z^*)$, where $\tilde{G}$ is the restriction of G to $[z^*, z^* + \varepsilon] \times [g(z^*), g(z^*) + \eta]$. The definition of z^* implies that $g(z) \neq \hat{g}(z)$ for some $z \in (z^*, z^* + \varepsilon)$. But this contradicts the Picard-Lindelöf Theorem.[29] This contradiction shows that $\{z \in [z_0,b] : g(z) \neq \hat{g}(z)\} = \emptyset$, i.e. $g(z) = \hat{g}(z)$ for each $z \geq z_0$.

We next show that $g(z) = \hat{g}(z)$ for each $z \leq z_0$. If $z_0 = a$, then this conclusion holds; hence, we may assume that $z_0 > a$.

[29] See Tesch (2012, Theorem 2.2, p. 38) for a statement of this result. Since this statement differs slightly from the version we are using, here is a sketch of the proof of this result, based on DePree and Swartz (1989, Example 3, p. 285): Let $\Gamma = [z^*, z^* + \varepsilon] \times [g(z^*), g(z^*) + \eta]$ for convenience and $M, L > 0$ be such that $|G(z,x)| \leq M$ and $|G(z,x) - G(z,x')| \leq L|x - x'|$ for each $(z,x), (z,x') \in \Gamma$; since Γ is compact, M exists because G is continuous and L exists because $\frac{\partial G}{\partial x}$ is continuous. Let $\delta > 0$ be such that $[z^*, z^* + \delta] \times [g(z^*), g(z^*) + M\delta] \subseteq \Gamma$ and $\delta L < 1$, i.e. $\delta < \min\{\varepsilon, \eta/M, 1/L\}$. Let $\mathcal{C}$ be the space of continuous functions on $[z^*, z^* + \delta]$ whose range is contained in $[g(z^*), g(z^*) + M\delta]$; then $\mathcal{C}$, endowed with the sup norm, is a complete metric space. Finally, define $\Lambda : \mathcal{C} \to \mathcal{C}$ by setting, for each $\lambda \in \mathcal{C}$ and $z \in [z^*, z^* + \delta]$, $\Lambda(\lambda)(z) = g(z^*) + \int_{z^*}^{z} G(y, \lambda(y))\mathrm{d}y$. Then, indeed $\Lambda(\lambda) \in \mathcal{C}$ and Λ is a contraction. Thus, Λ has a unique fixed point.

Suppose that $\{z \in [a, z_0] : g(z) \neq \hat{g}(z)\} \neq \emptyset$ and let $z^* = \sup\{z \in [a, z_0] : g(z) \neq \hat{g}(z)\}$. Then $g(z^*) = \hat{g}(z^*)$: this is clear if $z^* = z_0$ and, if $z^* < z_0$, then $z^* \in \{z \in [a, b] : g(z) = \hat{g}(z)\}$ since this set is closed and $g(z) = \hat{g}(z)$ for each $z^* < z \leq z_0$. Furthermore, $z^* > a$ since otherwise the definition of z^* implies that $\{z \in [a, z_0] : g(z) \neq \hat{g}(z)\} = \emptyset$.

We have that $g(z^*) > g(a) \geq \hat{a}$ since g is strictly increasing. Let $\eta > 0$ be such that $g(z^*) - \eta \geq \hat{a}$ and let, by the continuity of g and $\hat{g}$, $\varepsilon > 0$ be such that $\min\{g(z), \hat{g}(z)\} \geq g(z^*) - \eta$ for each $z \in [z^* - \varepsilon, z^*]$. Thus, both g and $\hat{g}$ are solutions of the initial value problem $x' = \tilde{G}(z, x)$ and $x(z^*) = g(z^*) = \hat{g}(z^*)$, where $\tilde{G}$ is the restriction of G to $[z^*, z^* - \varepsilon] \times [g(z^*) - \eta, g(z^*)]$. The definition of z^* implies that $g(z) \neq \hat{g}(z)$ for some $z \in (z^* - \varepsilon, z^*)$. But this contradicts the Picard-Lindelöf Theorem. This contradiction shows that $\{z \in [a, z_0] : g(z) \neq \hat{g}(z)\} = \emptyset$, i.e. $g(z) = \hat{g}(z)$ for each $z \leq z_0$. ■

The following lemma applies Lemma 44 to conclude that it suffices to show that $z_2 = \hat{z}_2$.

Lemma 45 *If $z_2 = \hat{z}_2$, then $\phi = \hat{\phi}$, $z_1 = \hat{z}_1$ and $c = \hat{c}$.*

Proof. We divide the proof of this lemma in four parts.

Part 1: If $z_2 = \hat{z}_2$, then $\phi = \hat{\phi}$ and $z_1 = \hat{z}_1$.

Let $\tilde{z}_1 = \max\{z_1, \hat{z}_1\}$ and $G : [z_2, \bar{z}] \times [0, \tilde{z}_1] \to \mathbb{R}$ be such that $G(z, x) = \frac{\theta(z)}{h(1-F(x))\theta(x)}$ for each $(z, x) \in [z_2, \bar{z}] \times [0, \tilde{z}_1]$. Then the conditions of Lemma 44 hold with ϕ and $\hat{\phi}$ being solutions to the initial value problems and $z_0 = z_2$, the latter since $\phi(z_2) = \hat{\phi}(z_2) = 0$. Then $\phi = \hat{\phi}$ and $z_1 = \phi(\bar{z}) = \hat{\phi}(\bar{z}) = \hat{z}_1$.

Part 2: If $z_2 = \hat{z}_2$ and $z_1 < z_2$, then $c = \hat{c}$.

By part 1, $\phi = \hat{\phi}$ and $z_1 = \hat{z}_1$. Let $G : [0, z_1] \times [0, 1] \to \mathbb{R}$ be such that $G(z, x) = f(z)\frac{F(\phi^{-1}(z))-x}{1-F(z)}$ for each $(z, x) \in [0, z_1] \times [0, 1]$. Then the conditions of Lemma 44 hold with c and $\hat{c}$ being solutions to the initial value problems and $z_0 = z_1$, the latter since $c(z_1) = \hat{c}(z_1) = F(z_1)$. Hence, $c = \hat{c}$.

Part 3: If $z_2 = \hat{z}_2$ and $z_1 = z_2$, then $c = \hat{c}$.

The argument for part 2 applies provided that $\{z \in [0, z_2] : c(z) = \hat{c}(z)\} \neq \emptyset$, which we establish in what follows. Suppose that $\{z \in [0, z_2] : c(z) = \hat{c}(z)\} = \emptyset$. Then $c(0) \neq \hat{c}(0)$ and, since c and $\hat{c}$ are arbitrary, we may assume that $c(0) > \hat{c}(0)$. Then $c(z_2) > \hat{c}(z_2)$ by the intermediate value theorem since c and $\hat{c}$ are continuous. But then

$$c(z_2) > \hat{c}(z_2) = \frac{F(z_2) - \hat{c}(0)}{h} > \frac{F(z_2) - c(0)}{h} = c(z_2),$$

a contradiction. This contradiction then shows that $\{z \in [0, z_2] : c(z) = \hat{c}(z)\} \neq \emptyset$.

Part 4: If $z_2 = \hat{z}_2$, then $\phi = \hat{\phi}$, $z_1 = \hat{z}_1$ and $c = \hat{c}$.

Part 4 follows by parts 1, 2 and 3. ■

The function $\phi : [z_2, \bar{z}] \to [0, z_1]$ is strictly increasing (condition 4). Hence, let $\varphi : [0, z_1] \to [z_2, \bar{z}]$ be the inverse of ϕ. Then φ is strictly increasing and differentiable and, for each $z \in [0, z_1]$,

$$\varphi'(z) = \frac{h(1 - F(z))\theta(z)}{\theta(\varphi(z))}.$$

As noted, Lemma 45 implies that it remains to establish that $z_2 = \hat{z}_2$. Lemma 46 derives some consequences of the assumption that $z_2 > \hat{z}_2$. Namely it shows that if there are less managers in $[z_2, \bar{z}]$ than in $[\hat{z}_2, \bar{z}]$ (i.e. $[z_2, \bar{z}] \subset [\hat{z}_2, \bar{z}]$), then it must be less workers in $[0, z_1]$ than in $[0, \hat{z}_1]$ (i.e. $[0, z_1] \subset [0, \hat{z}_1]$) and that every worker in $[0, z_1]$ is assigned to a more knowledgeable manager by φ as compared to $\hat{\varphi}$.

Lemma 46 *If $z_2 > \hat{z}_2$, then $z_1 < \hat{z}_1$ and $\varphi(z) > \hat{\varphi}(z)$ for each $z \in [0, z_1]$.*

Proof. We have that $\varphi(0) = z_2 > \hat{z}_2 = \hat{\varphi}(0)$. Let $\tilde{z}_1 = \min\{z_1, \hat{z}_1\}$ and assume that $\{z \in [0, \tilde{z}_1] : \varphi(z) = \hat{\varphi}(z)\} \neq \emptyset$. Let $z_0 = \inf\{z \in [0, \tilde{z}_1] : \varphi(z) = \hat{\varphi}(z)\}$. Then $\varphi(z_0) = \hat{\varphi}(z_0)$ since φ and $\hat{\varphi}$ are continuous. Thus, $z_0 > 0$.

The definition of z_0, the continuity of both φ and $\hat{\varphi}$ and the intermediate value theorem imply that $\varphi(z) > \hat{\varphi}(z)$ for each $z \in [0, z_0)$. This then implies that $\phi(z) < \hat{\phi}(z)$ for each $z \in [z_2, \varphi(z_0))$. Indeed, $\phi(z_2) = 0 < \hat{\phi}(z_2)$ since otherwise $\hat{\varphi}(0) = z_2$ and, hence, $\hat{z}_2 = \hat{\varphi}(0) = z_2$. Let $z \in (z_2, \varphi(z_0))$ and let x, x' be such that $z = \varphi(x) = \hat{\varphi}(x')$. Since $\varphi(x) > \hat{\varphi}(x)$, it follows that $\hat{\varphi}(x') > \hat{\varphi}(x)$. Then $x' > x$ since $\hat{\varphi}$ is strictly increasing and, thus, $\hat{\phi}(z) = x' > x = \phi(z)$.

It then follows by the above that, for each $z \in [z_2, \varphi(z_0))$,

$$\frac{\theta(z)}{h(1-F(\phi(z)))} < \frac{\theta(z)}{h(1-F(\hat{\phi}(z)))}.$$

Thus, by Lemma 43,

$$\begin{aligned}
\nu(z_0) = \nu(\phi(\varphi(z_0))) &= \int_{z_2}^{\varphi(z_0)} \frac{\theta(x)}{h(1-F(\phi(x)))}\mathrm{d}x \\
&< \int_{\hat{z}_2}^{z_2} \frac{\theta(x)}{h(1-F(\hat{\phi}(x)))}\mathrm{d}x + \int_{z_2}^{\varphi(z_0)} \frac{\theta(x)}{h(1-F(\hat{\phi}(x)))}\mathrm{d}x \\
&= \int_{\hat{z}_2}^{\hat{\varphi}(z_0)} \frac{\theta(x)}{h(1-F(\hat{\phi}(x)))}\mathrm{d}x \\
&= \nu(\hat{\phi}(\hat{\varphi}(z_0))) = \nu(z_0),
\end{aligned}$$

a contradiction. This contradiction shows that $\{z \in [0, \tilde{z}_1] : \varphi(z) = \hat{\varphi}(z)\} = \emptyset$.

It then follows from $\{z \in [0, \tilde{z}_1] : \varphi(z) = \hat{\varphi}(z)\} = \emptyset$, together with the continuity of φ and $\hat{\varphi}$ and $\varphi(0) > \hat{\varphi}(0)$, that $\varphi(z) > \hat{\varphi}(z)$ for each $z \in [0, \tilde{z}_1]$ by the intermediate value theorem. Hence, $\varphi(\tilde{z}_1) > \hat{\varphi}(\tilde{z}_1)$. If $\tilde{z}_1 = \hat{z}_1$, then $\varphi(\hat{z}_1) > \hat{\varphi}(\hat{z}_1) = \bar{z}$, a contradiction. Thus, $\tilde{z}_1 < \hat{z}_1$. It then follows that $z_1 = \tilde{z}_1 < \hat{z}_1$ and $\varphi(z) > \hat{\varphi}(z)$ for each $z \in [0, z_1]$. ■

To conclude the argument, we consider three cases: (i) $z_1 = z_2$ and $\hat{z}_1 = \hat{z}_2$, (ii) $z_1 < z_2$ and $\hat{z}_1 < \hat{z}_2$, and (iii) $z_1 < z_2$ and $\hat{z}_1 = \hat{z}_2$. The case $z_1 = z_2$ and $\hat{z}_1 < \hat{z}_2$ is covered by case (iii) since (z_1, z_2) and $(\hat{z}_1, \hat{z}_2)$ are arbitrary, case (i) is considered in part 1 of 47, case (ii) in Lemma 49 and case (iii) in Lemma 50.

Lemma 47 *If $z_1 = z_2$ and $\hat{z}_1 = \hat{z}_2$, then $z_2 = \hat{z}_2$.*

Proof. Suppose not and assume that $z_2 > \hat{z}_2$. Then Lemma 46 implies that $z_2 = z_1 < \hat{z}_1 = \hat{z}_2 < z_2$, a contradiction. ■

The following is a technical lemma that will be used in cases (ii) and (iii).

Lemma 48 *If $z_1 < z_2$ and $z_2 > \hat{z}_2$, then $c(z_1) > \hat{c}(z_1)$.*

Proof. It follows from $z_2 > \hat{z}_2$ that $z_1 < \hat{z}_1$ by Lemma 46. Since $z_1 < z_2$,

$$c(0) = (1-h)F(z_2) > (1-h)F(\hat{z}_2) \geq \hat{c}(0).$$

We claim that $\{z \in [0, z_1] : c(z) = \hat{c}(z)\} = \emptyset$. Suppose not and let $z_0 = \inf\{z \in [0, z_1] : c(z) = \hat{c}(z)\}$. Then $z_0 > 0$ since $c(0) > \hat{c}(0)$, $c(z_0) = \hat{c}(z_0)$ by the continuity of c and $\hat{c}$, and $c(z) \neq \hat{c}(z)$ for each $z \in [0, z_0)$ by the definition of z_0. Then $c(z) > \hat{c}(z)$ for each $z \in [0, z_0)$ by the intermediate value theorem since $c(0) > \hat{c}(0)$.

Lemma 46 implies that $\varphi(z_0) > \hat{\varphi}(z_0)$. Thus,

$$c'(z_0) = f(z_0)\frac{F(\varphi(z_0)) - c(z_0)}{1 - F(z_0)} > f(z_0)\frac{F(\hat{\varphi}(z_0)) - c(z_0)}{1 - F(z_0)} = \hat{c}'(z_0)$$

and, hence, there is $0 < \varepsilon < z_0$ such that $c'(z) > \hat{c}'(z)$ for each $z \in (z_0 - \varepsilon, z_0)$. Thus, by the mean value theorem, there is $z \in (z_0 - \varepsilon, z_0)$ such that

$$0 = c(z_0) - \hat{c}(z_0) = c(z_0 - \varepsilon) - \hat{c}(z_0 - \varepsilon) + (c'(z) - \hat{c}'(z))\varepsilon > 0,$$

a contradiction. This contradiction shows that $\{z \in [0, z_1] : c(z) = \hat{c}(z)\} = \emptyset$.

It then follows that $c(z) > \hat{c}(z)$ for each $z \in [0, z_1]$ by the intermediate value theorem since $c(0) > \hat{c}(0)$. Thus, $c(z_1) > \hat{c}(z_1)$. ■

Lemma 49 considers case (ii) above, i.e. shows that $z_2 = \hat{z}_2$ whenever $z_1 < z_2$ and $\hat{z}_1 < \hat{z}_2$. The idea is that $z_2 > \hat{z}_2$ would imply that $F(z_1) = c(z_1) > \hat{c}(z_1)$; this will then imply that $F(\hat{z}_1) > \hat{c}(\hat{z}_1)$ since $z_2 > \hat{z}_2$ implies $\hat{z}_1 > z_1$ and $z \mapsto \hat{c}(z) - F(z)$ is strictly decreasing, thus violating condition 7 in Theorem 2.

Lemma 49 *If $z_1 < z_2$ and $\hat{z}_1 < \hat{z}_2$, then $z_2 = \hat{z}_2$.*

Proof. Suppose that $z_2 > \hat{z}_2$. Then $z_1 < \hat{z}_1$ by Lemma 46 and $c(z_1) > \hat{c}(z_1)$ by Lemma 48.

Define $\lambda : [0, z_1] \to \mathbb{R}$ by setting, for each $z \in [0, z_1]$, $\lambda(z) = c(z) - F(z)$. Likewise, define $\hat{\lambda} : [0, \hat{z}_1] \to \mathbb{R}$ by setting, for each $z \in [0, \hat{z}_1]$, $\hat{\lambda}(z) = \hat{c}(z) - F(z)$. Then, for each $z \in (0, \hat{z}_1)$,

$$\begin{aligned}\hat{\lambda}'(z) = \hat{c}'(z) - f(z) = f(z)\left(\frac{F(\hat{\varphi}(z)) - \hat{c}(z)}{1 - F(z)} - 1\right) \\ = \frac{f(z)}{1 - F(z)}\left(F(\hat{\varphi}(z)) - 1 - (\hat{c}(z) - F(z))\right) < 0\end{aligned}$$

since $z < \hat{z}_1$ implies that $\hat{c}(z) \geq F(z)$ and that $\hat{\varphi}(z) < \bar{z}$ and, hence, $F(\hat{\varphi}(z)) < 1$. Thus,

$$\hat{\lambda}(\hat{z}_1) < \hat{\lambda}(z_1) = \hat{c}(z_1) - F(z_1) < c(z_1) - F(z_1) = 0.$$

But this contradicts $\hat{\lambda}(\hat{z}_1) = 0$. ∎

Lemma 50 considers the final case (iii).

Lemma 50 *If* $z_1 < z_2$ *and* $\hat{z}_1 = \hat{z}_2$*, then* $z_2 = \hat{z}_2$*.*

Proof. Suppose that $z_2 > \hat{z}_2$. Then $z_1 < \hat{z}_1$ by Lemma 46 and $c(z_1) > \hat{c}(z_1)$ by Lemma 48.

Note that we have that

$$\frac{F(z_2) - c(0)}{h} - F(z_2) = c(z_1) - F(z_1), \text{ and} \tag{13}$$

$$\frac{F(\hat{z}_2) - \hat{c}(0)}{h} - F(\hat{z}_2) = \hat{c}(\hat{z}_1) - F(\hat{z}_1). \tag{14}$$

Letting λ and $\hat{\lambda}$ be as in the proof of Lemma 49, it follows from that proof that $\hat{\lambda}(z_1) > \hat{\lambda}(\hat{z}_1)$ and, since $c(z_1) > \hat{c}(z_1)$, that $\lambda(z_1) > \hat{\lambda}(z_1)$. This two properties together with $\hat{c}(0) \leq (1-h)F(\hat{z}_2)$ and (13) imply that

$$\frac{F(\hat{z}_2) - \hat{c}(0)}{h} - F(\hat{z}_2) \geq 0 = \frac{F(z_2) - c(0)}{h} - F(z_2) = c(z_1) - F(z_1)$$
$$= \lambda(z_1) > \hat{\lambda}(z_1) > \hat{\lambda}(\hat{z}_1) = \hat{c}(\hat{z}_1) - F(\hat{z}_1),$$

a contradiction to (14). This contradiction shows that $z_2 = \hat{z}_2$. ∎

Since (z_1, z_2) and $(\hat{z}_1, \hat{z}_2)$ are arbitrary in Lemma 50, this lemma also shows that if $z_1 = z_2$ and $\hat{z}_1 < \hat{z}_2$, then $z_2 = \hat{z}_2$. It then follows from Lemmas 47–50 that $z_2 = \hat{z}_2$ and by Lemma 45 that $z_1 = \hat{z}_1$, $\phi = \hat{\phi}$ and $c = \hat{c}$. Hence, Lemma 6 follows.

A.5.2 Completing the proof of the sufficiency part of Theorem 2

Suppose that there exists (z_1, z_2, ϕ, c) such that the conditions 1–8 in the theorem hold and let $\mu = \nu \circ \sigma^{-1}$. Let, by Theorem 1, $\hat{\mu}$ be a stable matching and, by the necessity part of Theorem 2 just established, $(\hat{z}_1, \hat{z}_2, \hat{\phi}, \hat{c})$ be such that the conditions 1–8 in the theorem hold and such that $\hat{\mu} = \nu \circ \hat{\sigma}^{-1}$, where $\hat{\sigma} : [\hat{z}_1, \bar{z}] \to Z \times X_\emptyset$ is defined as σ is but with $(\hat{z}_1, \hat{z}_2, \hat{\phi}, \hat{c})$ in place of (z_1, z_2, ϕ, c). It then follows by Lemma 6 that $(z_1, z_2, \phi, c) = (\hat{z}_1, \hat{z}_2, \hat{\phi}, \hat{c})$ and, hence, $\sigma = \hat{\sigma}$. Thus $\mu = \nu \circ \sigma^{-1} = \hat{\mu}$ and μ is a stable matching.

A.6 Proof of Theorem 3

Let μ and $\hat{\mu}$ be stable matchings and, by Theorem 2, let μ be represented by (z_1, z_2, ϕ, c) and $\hat{\mu}$ by $(\hat{z}_1, \hat{z}_2, \hat{\phi}, \hat{c})$; in particular, $\mu = \nu \circ \sigma^{-1}$ and $\hat{\mu} = \nu \circ \hat{\sigma}^{-1}$, where $\hat{\sigma}$ is defined as σ is but with $(\hat{z}_1, \hat{z}_2, \hat{\phi}, \hat{c})$ in place of (z_1, z_2, ϕ, c). It then follows by Lemma 6 that $(z_1, z_2, \phi, c) = (\hat{z}_1, \hat{z}_2, \hat{\phi}, \hat{c})$ and, hence, $\sigma = \hat{\sigma}$. Thus $\mu = \nu \circ \sigma^{-1} = \hat{\mu}$.

A.7 Details for Section 3.4

We show that conditions 1, 4, 5 plus 7 if $z_1 < z_2$ and 8 if $z_1 = z_2$ in Theorem 2 hold.

Consider condition 4 first. Note that ϕ is well defined since, for each $z \in [z_2, 1]$, $1 - \frac{2(z-z_2)}{h} \geq 1 - \frac{2(1-z_2)}{h} > 0$. The latter inequality is equivalent to $1 - z_2 < \frac{h}{2}$. In the case $0 < h \leq 3/4$, we have that $1 - z_2 = \frac{1}{h}\left(-1 + \sqrt{1+h^2}\right)$ and, thus,

$$\begin{aligned}1 - z_2 < \frac{h}{2} &\Leftrightarrow \frac{1}{h}\left(-1 + \sqrt{1+h^2}\right) < \frac{h}{2} \Leftrightarrow \sqrt{1+h^2} < 1 + \frac{h^2}{2} \\ &\Leftrightarrow 1 + h^2 < 1 + h^2 + \frac{h^4}{4},\end{aligned}$$

which holds; in the case $3/4 < h < 1$, $1 - z_2 = 1 - \frac{2-h}{h} + \sqrt{\frac{3-4h+h^2}{h^2}}$ and, thus,

$$\begin{aligned}&1 - z_2 < \frac{h}{2} \Leftrightarrow 2h(1-z_1) < h^2 \Leftrightarrow h^2 - 4h + 4 - 2\sqrt{h^2 - 4h + 3} > 0 \\ &\Leftrightarrow (2-h)^2 - 2\sqrt{(2-h)^2 - 1} > 0 \Leftrightarrow 1 + \left((2-h)^2 - 1\right) - 2\sqrt{(2-h)^2 - 1} > 0,\end{aligned}$$

which holds since $1 + x - 2\sqrt{x} > 0 \Leftrightarrow (1+x)^2 > 4x \Leftrightarrow x^2 - 2x + 1 > 0 \Leftrightarrow (x-1)^2 > 0 \Leftrightarrow x \neq 1$ and $(2-h)^2 - 1 = 1$ only if $h = 2 - \sqrt{2} < 3/4$.

Furthermore, for each $z \in [z_2, 1]$,

$$\phi'(z) = \frac{1}{h\sqrt{1 - \frac{2(z-z_2)}{h}}} > 0$$

showing that ϕ is strictly increasing and

$$\frac{\theta(z)}{h(1 - F(\phi(z)))\theta(\phi(z))} = \frac{1}{h(1-\phi(z))} = \frac{1}{h\sqrt{1 - \frac{2(z-z_2)}{h}}} = \phi'(z).$$

Finally, $\phi(z_2) = 0$ and we have that $\phi(1) = z_1$ holds. This is clear in the case where $h \in (3/4, 1)$ and also holds in the case where $h \in (0, 3/4]$ since then, due to $z_1 = z_2$,

$\phi(1) = z_1$ is equivalent to

$$1 - \sqrt{1 - \frac{2(1-z_2)}{h}} = z_2 \Leftrightarrow (1-z_2)^2 = 1 - \frac{2(1-z_2)}{h} \Leftrightarrow 0 = 1 - (1+h)z_2 + \frac{h}{2}z_2^2,$$

which holds.

Consider next condition 5. We have that c is strictly increasing since $z_2 > c(0)$. This is clear when $h \in (3/4, 1)$ since then $c(0) = (1-h)z_2$; in the case of $h \in (0, 3/4]$,

$$z_2 > c(0) \Leftrightarrow 1 + h(1-z_2) > 1 - h\frac{2+h}{2}z_2 \Leftrightarrow 1 - z_2 > -\frac{2+h}{2}z_2,$$

which holds. Let $\varphi : [0, z_1] \to [z_2, 1]$ be the inverse of ϕ. Then $\varphi(x) = z_2 + hx - \frac{hx^2}{2}$ for each $x \in [0, z_1]$ since, for each $z \in [z_2, 1]$, letting $\alpha = 1 - \frac{2(z-z_2)}{h}$,

$$\varphi(\phi(z)) = z_2 + h(1-\sqrt{\alpha}) - \frac{h}{2}\left(1 - 2\sqrt{\alpha} + 1 - \frac{2(z-z_2)}{h}\right) = z.$$

It then follows that, for each $z \in [0, z_1]$,

$$\begin{aligned} &c'(z) = z_2 - c(0) + hz, \\ &f(z)\frac{F(\phi^{-1}(z)) - c(z)}{1 - F(z)} = \frac{(1-z)(z_2 - c(0) + hz)}{1-z} = c'(z). \end{aligned}$$

Condition 1 is satisfied when $h \in (0, 3/4]$ since then $z_2 \in (0, 1)$. Indeed,

$$\begin{aligned} &z_2 = \frac{1}{h}(1 + h - \sqrt{1+h^2}), \text{ thus} \\ &z_2 > \frac{1}{h}(1 + h - \sqrt{1 + 2h + h^2}) = \frac{1}{h}(1 + h - \sqrt{(1+h)^2}) = 0, \text{ and} \\ &z_2 < \frac{1}{h}(1 + h - \sqrt{1}) = \frac{1}{h}(1 + h - 1) = 1. \end{aligned}$$

In the case where $h \in (3/4, 1)$, we have that $z_1 = \phi(1) > \phi(z_2) = 0$. We next show that $z_1 < z_2 < 1$ for each $h \in (3/4, 1)$. Let $z_2(h) = \frac{2-h}{h} - \sqrt{\frac{3-4h+h^2}{h^2}}$, $z_1(h, z_2) = 1 - \sqrt{1 - \frac{2(1-z_2)}{h}}$ and $z_1(h) = z_1(h, z_2(h))$ for each $h \in [3/4, 1)$. Then $z_2(3/4) = z_1(3/4) = 2/3$, $\lim_{h \to 1} z_2(h) = 1$ and $h \mapsto z_2(h)$ is strictly increasing (as we next show), from which it follows that $h \mapsto z_1(h)$ is strictly decreasing. Thus, it follows that $z_1 < z_2 < 1$ for each $h \in (3/4, 1)$.

To see that $h \mapsto z_2(h)$ is strictly increasing, note that

$$z_2(h) = \frac{2}{h} - 1 - \sqrt{1 - \frac{4}{h} + \frac{3}{h^2}} \text{ and}$$
$$z_2'(h) = \frac{2}{h^2}\left(-1 + \frac{1}{\sqrt{1-\frac{4}{h}+\frac{3}{h^2}}}\left(\frac{3}{2h} - 1\right)\right).$$

Thus,

$$z_2'(h) > 0 \Leftrightarrow \frac{1}{\sqrt{1-\frac{4}{h}+\frac{3}{h^2}}}\frac{3-2h}{2h} > 1 \Leftrightarrow \left(\frac{3-2h}{2h}\right)^2 > 1 - \frac{4}{h} + \frac{3}{h^2}$$
$$\Leftrightarrow \frac{9-12h+4h^2}{4h^2} > \frac{h^2-4h+3}{h^2} \Leftrightarrow h > \frac{3}{4}.$$

Consider next condition 7 when $h \in (3/4, 1)$. Then $c(0) = (1-h)z_2 = (1-h)F(z_2)$ and, as we next show, $c(z_1) = z_1$, which then implies that $c(z_1) = F(z_1)$. We have that $c(z_1) = (1-h)z_2 + hz_1z_2 + \frac{hz_1^2}{2}$. Thus, $c(z_1) = z_1$ is equivalent to

$$0 = (1-h)z_2 - (1-hz_2)z_1 + \frac{hz_1^2}{2}$$

which, letting $\alpha = 1 - \frac{2(1-z_2)}{h}$ and using $z_1 = 1 - \sqrt{1 - \frac{2(1-z_2)}{h}} = 1 - \sqrt{\alpha}$, is equivalent to

$$0 = (1-h)z_2 - (1-hz_2)(1-\sqrt{\alpha}) + \frac{h}{2}(1-\sqrt{\alpha})^2.$$

Simplifying this equation yields

$$0 = 2(z_2 - 1) + h + \sqrt{\alpha}(1 - hz_2 - h) \Leftrightarrow 0 = 1 - \frac{2(1-z_2)}{h} + \sqrt{\alpha}\left(\frac{1}{h} - z_2 - 1\right)$$
$$\Leftrightarrow 0 = \alpha + \sqrt{\alpha}\left(\frac{1}{h} - z_2 - 1\right).$$

In turn, this equation is equivalent to

$$\sqrt{\alpha} = 1 + z_2 - \frac{1}{h} \Leftrightarrow \alpha = \left(1 + z_2 - \frac{1}{h}\right)^2.$$

Simplifying this equation yields

$$0 = \frac{1}{2h^2} + \frac{h-2}{h}z_2 + \frac{z_2^2}{2},$$

which holds.

Finally consider condition 8 when $h \in (0, 3/4]$. We have that $\frac{F(z_2)-c(0)}{h} = \frac{z_2-c(0)}{h}$ and $c(z_2) = c(0) + (z_2 - c(0))z_2 + \frac{hz_2^2}{2}$. Hence, $c(z_2) = \frac{F(z_2)-c(0)}{h}$ holds since $c(0) = \frac{z_2\left(1-h\frac{2+h}{2}z_2\right)}{1+h(1-z_2)}$. Thus, it remains to show that $\frac{F(z_2)-c(0)}{h} \geq F(z_2)$, which is equivalent to $c(0) \leq (1-h)z_2$. We have that

$$\begin{aligned} c(0) \leq (1-h)z_2 &\Leftrightarrow \frac{1 - h\left(1+\frac{h}{2}\right)z_2}{1+h(1-z_2)} \leq 1-h \\ &\Leftrightarrow 1 - hz_2 - \frac{h^2}{2}z_2 \leq 1 + h - hz_2 - h - h^2 + h^2 z_2 \\ &\Leftrightarrow h^2 \leq \frac{3}{2}h^2 z_2 \\ &\Leftrightarrow z_2 \geq \frac{2}{3}. \end{aligned}$$

Letting $z_2(h) = 1 + \frac{1}{h} - \sqrt{1+\frac{1}{h^2}}$, we have that

$$\frac{dz_2(h)}{dh} = -\frac{1}{h^2} + \frac{h}{\sqrt{1+\frac{1}{h^2}}} = \frac{h^3 - \sqrt{1+\frac{1}{h^2}}}{h^2\sqrt{1+\frac{1}{h^2}}} < 0$$

since $h^3 < 1 < \sqrt{1+\frac{1}{h^2}}$; thus $h \mapsto z_2(h)$ is strictly decreasing. Furthermore,

$$z_2(3/4) = \frac{1+\frac{3}{4}-\sqrt{1+\frac{9}{16}}}{\frac{3}{4}} = \frac{4}{3}\left(\frac{7}{4} - \sqrt{\frac{25}{16}}\right) = \frac{4}{3}\frac{2}{4} = \frac{2}{3}.$$

Hence, it follows that $z_2 \geq \frac{2}{3}$ for each $h \leq \frac{3}{4}$ and, thus, that $c(0) \leq (1-h)z_2$ holds.

A.8 Proof of Theorem 4

Theorem 4 follows in part from Theorem 3 in Carmona and Laohakunakorn (2024). The latter result applies to a general production function $g(r(z))\psi(r(z), nq(z'))$ and thus set, for each $z \in Z$, $r \in \mathbb{R}_+$ and $(x, y) \in \mathbb{R}_+^2$, $r(z) = z$, $q(z) = 1$, $g(r) = F(r)$ and $\psi(x, y) = x^\alpha y^{1-\alpha}$ to obtain $F(z)z^\alpha n^{1-\alpha}$ as in $E_{r,\alpha}$. In this case and with a constant wage $w > 0$, the optimal number $n(z, w)$ of workers for a manager of ability z, and the manager's rent $R(z, w) = F(z)z^\alpha n(z, w)^{1-\alpha} - wn(z, w)$ equal

$$\begin{aligned} n(z, w) &= \left(\frac{(1-\alpha)F(z)}{w}\right)^{\frac{1}{\alpha}} z, \\ R(z, w) &= \alpha F(z)^{\frac{1}{\alpha}} z \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}}. \end{aligned}$$

It follows by Carmona and Laohakunakorn (2024, Theorem 3) that μ is a stable matching of $E_{r,\alpha}$ if and only if there exists $\gamma \in M(Z^2)$ and $w > 0$ such that

$$\gamma(B \times Z) + \int_{Z\times B} n(z,w)\mathrm{d}\gamma(z,z') = \nu(B) \text{ for each Borel } B \subseteq Z, \tag{15}$$

$$\operatorname{supp}(\gamma) \subseteq \{z \in Z : R(z,w) \geq w\} \times \{z \in Z : w \geq R(z,w)\}, \text{ and} \tag{16}$$

$$\mu = \gamma \circ \tilde{g}^{-1}, \tag{17}$$

where $\tilde{g} : Z^2 \to Z \times X$ is defined by setting, for each $(z,z') \in Z^2$,

$$\tilde{g}(z,z') = (z, n(z,w)1_{(z',w)}).$$

Thus, it remains to show that (16) is equivalent to conditions 3 and 4 in the statement of Theorem 4, and that w and z_1 are unique.

Note that $\tilde{g}$ is continuous. Furthermore the restriction of $\tilde{g}$ to $\operatorname{supp}(\gamma)$ is 1-1 both under (16) and under condition 3: in both cases, $(z,z') \in \operatorname{supp}(\gamma)$ implies $z > 0$ (since $R(0,w) = 0$ and $z_1 > 0$) and $n(z,w) > 0$ if $z > 0$. Thus, $\tilde{g}$ is a homeomorphism between $\operatorname{supp}(\gamma)$ and $\tilde{g}(\operatorname{supp}(\gamma))$. Then $M = \{z \in Z : (z,z') \in \operatorname{supp}(\gamma) \text{ for some } z' \in Z\}$ and $W = \{z \in Z : (\hat{z},z) \in \operatorname{supp}(\gamma) \text{ for some } \hat{z} \in Z\}$ since $\operatorname{supp}(\mu) = \tilde{g}(\operatorname{supp}(\gamma))$ by Carmona and Laohakunakorn (2024, Lemma 1). We have that $\operatorname{supp}(\gamma)$ is compact since it is a closed subset of Z^2 and, hence, M and W are compact. Then $M \cup W = Z$ since

$$\begin{aligned}\nu(M\cup W) =& \gamma((M\cup W)\times Z) + \int_{Z\times(M\cup W)} n(z,w)\mathrm{d}\gamma(z,z') \\ &= \gamma(Z\times Z) + \int_{Z\times Z} n(z,w)\mathrm{d}\gamma(z,z') = \nu(Z)\end{aligned}$$

and, hence, $Z = \operatorname{supp}(\mu) \subseteq M \cup W \subseteq Z$.

It follows by (16) that $M \subseteq \{z \in Z : R(z,w) \geq w\}$ and $W \subseteq \{z \in Z : w \geq R(z,w)\}$. Let $\lambda : Z \to \mathbb{R}$ be defined by setting, for each $z \in Z$, $\lambda(z) = R(z,w) - w$. Then λ is continuous, strictly increasing and $\lambda(0) < 0$. Furthermore, $\lambda(\bar{z}) > 0$ since otherwise $\{z \in Z : R(z,w) \geq w\} \subseteq \{\bar{z}\}$ and hence

$$\begin{aligned}\gamma(Z\times Z) &= \gamma(\{z \in Z : R(z,w) \geq w\} \times Z) \\ &\leq \gamma(\{\bar{z}\}\times Z) + \int_{Z\times\{\bar{z}\}} n(z,w)\mathrm{d}\gamma(z,z') = \nu(\{\bar{z}\}) = 0.\end{aligned}$$

But this is a contradiction since $\gamma(Z \times Z) = 0$ implies that $\int_{Z\times Z} n(z,w)\mathrm{d}\gamma(z,z') = 0$ and, hence, $0 = \gamma(Z \times Z) + \int_{Z\times Z} n(z,w)\mathrm{d}\gamma(z,z') = \nu(Z) > 0$. It then follows that there is a unique $z_1 \in (0, \bar{z})$ such that $\lambda(z_1) = 0$. Thus, $M \subseteq \{z \in Z : R(z,w) \geq w\} = [z_1, \bar{z}]$ and $W \subseteq \{z \in Z : w \geq R(z,w)\} = [z_1, \bar{z}]$, which, together with $M \cup W = Z$, implies that $W = [0, z_1]$ and $M = [z_1, \bar{z}]$. The definition of z_1 implies that $R(z_1, w) = w$ and, hence $F(z_1)z_1^\alpha n(z_1,w)^{1-\alpha} - wn(z_1,w) = w$, since $R(z_1,w) = F(z_1)z_1^\alpha n(z_1,w)^{1-\alpha} - wn(z_1,w)$.

Conversely, suppose that conditions 3 and 4 in the statement of the Theorem hold. Thus, $\lambda(z_1) = 0$ and, since λ is strictly increasing, $\{z \in Z : R(z,w) \geq w\} = [z_1, \bar{z}] = M$ and $\{z \in Z : R(z,w) \leq w\} = [0, z_1] = W$. Hence, for each $(z,z') \in \mathrm{supp}(\gamma)$, it follows that $(z,z') \in M \times W$ and, thus, $R(z,w) \geq w$ and $R(z',w) \leq w$, i.e. (16) holds.

We conclude the proof of Theorem 4 by establishing the uniqueness of z_1 and w. Let (γ, z_1, w) and $(\hat{\gamma}, \hat{z}_1, \hat{w})$ be such that conditions 1–4 hold. Let λ be as above and define $\hat{\lambda}(z) = R(z, \hat{w}) - \hat{w}$. Note that $w = \hat{w}$ implies that $z_1 = \hat{z}_1$ since then $\lambda = \hat{\lambda}$.

Suppose that $w > \hat{w}$. Then $R(z,w) < R(z,\hat{w})$ for each $z > 0$ and, hence, $\lambda(z) < \hat{\lambda}(z)$ for each $z \in Z$. Thus, $z_1 > \hat{z}_1$ since z_1 (resp. $\hat{z}_1$) is the unique $z \in (0, \bar{z})$ such that $\lambda(z_1) = 0$ (resp. $\hat{\lambda}(\hat{z}_1) = 0$).

Let γ_1 denote the marginal of γ on the first coordinate. For each Borel $B \subseteq [z_1, \bar{z}]$, condition 2 implies that $\gamma_1(B) = \gamma(B \times Z) = \nu(B)$ since $\int_{Z\times B} n(z,w)\mathrm{d}\gamma(z,z') = 0$ due to $W = [0, z_1]$. Thus, γ_1 is the restriction of ν to $M = [z_1, \bar{z}]$. Analogously, $\hat{\gamma}_1$ is the restriction of ν to $[\hat{z}_1, \bar{z}]$.

It then follows that

$$\int_{Z\times Z} n(z,w)\mathrm{d}\gamma(z,z') = \int_{Z} n(z,w)\mathrm{d}\gamma_1(z) = \int_{[z_1,\bar{z}]} n(z,w)\mathrm{d}\nu(z).$$

Analogously, $\int_{Z\times Z} n(z,\hat{w})\mathrm{d}\hat{\gamma}(z,z') = \int_{[\hat{z}_1,\bar{z}]} n(z,\hat{w})\mathrm{d}\nu(z)$.

Since $W = [0, z_1]$, condition 2 implies that

$$\int_{Z\times Z} n(z,w)\mathrm{d}\gamma(z,z') = \gamma([0,z_1] \times Z) + \int_{Z\times[0,z_1]} n(z,w)\mathrm{d}\gamma(z,z') = \nu([0,z_1])$$

and, analogously, $\int_{Z\times Z} n(z,\hat{w})\mathrm{d}\hat{\gamma}(z,z') = \nu([0,\hat{z}_1])$. Thus, $z_1 > \hat{z}_1$ implies that

$\nu([0,z_1]) > \nu([0,\hat{z}_1])$ and, hence,

$$\int_{Z\times Z} n(z,w)\mathrm{d}\gamma(z,z') > \int_{Z\times Z} n(z,\hat{w})\mathrm{d}\hat{\gamma}(z,z'). \tag{18}$$

Since $n(z,w) < n(z,\hat{w})$ for each $z > 0$, it follows that

$$\begin{aligned}\int_{Z\times Z} n(z,w)\mathrm{d}\gamma(z,z') = \int_{[z_1,\bar{z}]} n(z,w)\mathrm{d}\nu(z) &< \int_{[z_1,\bar{z}]} n(z,\hat{w})\mathrm{d}\nu(z)\\ < \int_{[\hat{z}_1,\bar{z}]} n(z,\hat{w})\mathrm{d}\nu(z) &= \int_{Z\times Z} n(z,\hat{w})\mathrm{d}\hat{\gamma}(z,z'),\end{aligned}$$

a contradiction to (18). This contradiction shows that $w = \hat{w}$ and, hence, $z_1 = \hat{z}_1$.

A.9 Proof of Theorem 5

The proof of Theorem 5 is analogous to the proofs of Theorems 1–3 and is available in the supplementary material to this paper.

A.10 Proof of Theorem 6

For each $\alpha \in [0,1)$, write $E_{s,\alpha} = (Z,\nu,C,\mathbb{C},X,(\succ_z)_{z\in Z})$ and let $\hat{E}_{s,\alpha} = (Z,\nu,\hat{C},\hat{\mathbb{C}},\hat{X},(\succ_z)_{z\in Z})$ where $\hat{C} = [0,\max\{1,\bar{z}h\}]$, $\hat{\mathbb{C}}(z,z') = \hat{C}$ and $\hat{\mathbb{C}}(z,\emptyset) = \{0\}$ for each $z,z' \in Z\times Z$, and $\hat{X} = \{n(z)1_{(z,c)} : (z,c) \in Z\times\hat{C}\}$, i.e. $\hat{E}_{s,\alpha}$ is equal to $E_{s,\alpha}$ except for these changes to C, $\mathbb{C}$ and X.

Lemma 51 *If* $\alpha \in [0,1)$ *and* μ *is a stable matching of* $\hat{E}_{s,\alpha}$*, then* $\mathrm{supp}(\mu) \subseteq Z\times\hat{X}$.

Proof. Note that it is enough to show that $\mathrm{supp}(\mu)\cap((Z\setminus\{0\})\times\hat{X}_\emptyset) \subseteq Z\times\hat{X}$. Indeed, this implies that

$$\begin{aligned}\mathrm{supp}(\mu) = &\big(\mathrm{supp}(\mu)\cap((Z\setminus\{0\})\times\hat{X}_\emptyset)\big)\cup\big(\mathrm{supp}(\mu)\cap(\{0\}\times\hat{X})\big)\cup\\ &\cup\big(\mathrm{supp}(\mu)\cap(\{0\}\times(\hat{X}_\emptyset\setminus\hat{X}))\big)\\ &= \big(\mathrm{supp}(\mu)\cap(Z\times\hat{X})\big)\cup\big(\mathrm{supp}(\mu)\cap(\{0\}\times(\hat{X}_\emptyset\setminus\hat{X}))\big).\end{aligned}$$

Since $\mathrm{supp}(\mu)\cap(Z\times\hat{X})$ is closed and $\mu(\{0\}\times(\hat{X}_\emptyset\setminus\hat{X})) \le \nu(\{0\}) = 0$, it follows that $\mathrm{supp}(\mu) = \mathrm{supp}(\mu)\cap(Z\times\hat{X})$, i.e. $\mathrm{supp}(\mu)\subseteq Z\times\hat{X}$.

Hence, it remains to show that $\text{supp}(\mu) \cap ((Z \setminus \{0\}) \times \hat{X}_\emptyset) \subseteq Z \times \hat{X}$. Suppose not; then let $(z, \delta) \in \text{supp}(\mu) \cap ((Z \setminus \{0\}) \times (\hat{X}_\emptyset \setminus \hat{X}))$. Let $\varepsilon > 0$ be such that $(zh(1 - F(z)))^\alpha F(z) - \varepsilon > 0$, which exists since $z > 0$ and $F(\bar{z}) < 1$. Then $(z, \varepsilon) \in T_z^m(\mu)$ since $(z, \delta) \in \text{supp}(\mu)$ and $U_z(w, 1_{(z,\varepsilon)}) = \varepsilon > 0 = U_z(s, \delta)$. Thus, letting $\delta' = n(z)1_{(z,\varepsilon)}$, it follows that $\text{supp}(\delta') \subseteq T_z^m(\mu)$ and $U_z(m, \delta') = \big((zh(1 - F(z)))^\alpha F(z) - \varepsilon\big)n(z) > 0 = U_z(s, \delta)$. Hence, $(z, \delta) \notin S(\mu)$, a contradiction to the stability of μ. ■

Lemma 52 *For each $\alpha \in [0, 1)$, $\hat{E}_{s,\alpha}$ has a stable matching.*

Proof. We argue that $\hat{E}_{s,\alpha}$ satisfies the conditions of Carmona and Laohakunakorn (2024, Theorem 2), which follows the proof of Theorem 1. Indeed, $\hat{E}_{s,\alpha}$ is rational since the preferences of each type $z \in Z$ are represented by utility functions, is bounded since $\delta(Z \times C) \leq \frac{1}{h(1-F(\bar{z}))} < \infty$ for each $\delta \in \hat{X}$ (since $F(\bar{z}) < 1$) and is rich by the same argument as in the proof of Theorem 1; this argument also shows that $\hat{X}$ is closed. It then follows that $\hat{E}_{s,\alpha}$ is continuous since $(z, a, \delta) \mapsto U_z(a, \delta)$ is continuous and $\mathbb{C}$ is continuous with nonempty and compact values.

It then follows that $\hat{E}_{s,\alpha}$ has a stable matching μ. ■

Lemma 53 *For each $\alpha \in [0, 1)$, μ is a stable matching of $\hat{E}_{s,\alpha}$ if and only if μ is a stable matching of $E_{s,\alpha}$.*

Proof. Let μ be a stable matching of $\hat{E}_{s,\alpha}$. Note that, for each $(z, \delta) \in \text{supp}(\mu)$, $\delta \in \hat{X}$ and thus $\delta = n(z')1_{(z',c)}$ for some $(z', c) \in Z \times Z$ by Lemma 51. Furthermore, $U_z(m, n(z')1_{(z',c)}) \geq 0$ and $U_{z'}(w, 1_{(z,c)}) \geq 0$ since μ is individually rational in $\hat{E}_{s,\alpha}$.

We claim that μ is a stable matching of $E_{s,\alpha}$. Indeed, if μ is not a stable matching of $E_{s,\alpha}$, then there is $(z, z', c) \in Z^2 \times C$ such that $(z, n(z')1_{(z',c)}) \in \text{supp}(\mu)$ and $(\hat{z}, \tilde{z}, \tilde{c}) \in Z^2 \times C$ such that $\hat{z} \in \{z, z'\}$, $\tilde{c} > \max\{1, \bar{z}h\} \geq (\bar{z}h)^\alpha$ and $U_{\hat{z}}(m, n(\tilde{z})1_{(\tilde{z},\tilde{c})}) > U_{\hat{z}}(a, \hat{\delta})$ with $a = m$ and $\hat{\delta} = n(z')1_{(z',c)}$ if $\hat{z} = z$ and $a = w$ and $\hat{\delta} = 1_{(z,c)}$ if $\hat{z} = z'$. But $\tilde{c} > (\bar{z}h)^\alpha$ implies that $0 > U_{\hat{z}}(m, n(\tilde{z})1_{(\tilde{z},\tilde{c})})$ since $U_{\hat{z}}(m, n(\tilde{z})1_{(\tilde{z},\tilde{c})}) \geq 0$ is equivalent to $\tilde{c} \leq (\hat{z}h(1 - F(\tilde{z})))^\alpha F(\hat{z})$ and, thus, implies that $\tilde{c} \leq (\bar{z}h)^\alpha$; hence, $0 > U_{\hat{z}}(a, \hat{\delta})$. But this contradicts $U_z(m, n(z')1_{(z',c)}) \geq 0$ and $U_{z'}(w, 1_{(z,c)}) \geq 0$. Thus, it follows that μ is a stable matching of $E_{s,\alpha}$.

Let μ be a stable matching of $E_{s,\alpha}$. Then for each $(z,\delta) \in \text{supp}(\mu)$ and $(z',c) \in \text{supp}(\delta)$, individual rationality implies that $c \in [0, \max\{1, \bar{z}h\}]$; hence μ is a stable matching of $\hat{E}_{s,\alpha}$. ■

Lemmas 52 and 53 imply that Φ is nonempty valued.

To prove continuity, we first show that Φ is upper hemi-continuous at $\alpha = 0$. By Lemma 53, $\Phi(\alpha)$ equals the set of stable matching of $\hat{E}_{s,\alpha}$. Let $\{\alpha_k\}_{k=1}^{\infty}$ be such that $\alpha_k \in (0,1)$ for each $k \in \mathbb{N}$ and $\alpha_k \to 0$ and let $\{\mu_k\}_{k=1}^{\infty}$ be such that $\mu_k \in \Phi(\alpha_k)$ for each $k \in \mathbb{N}$. Since μ_k is a stable matching of $\hat{E}_{s,\alpha_k}$, $\mu_k \in M(Z \times \hat{X})$ for each $k \in \mathbb{N}$ by Lemma 51. Since $\hat{X}$ is compact due to $1/h(1-F(z)) \leq 1/h(1-F(\bar{z})) < \infty$ for each $z \in Z$, we may assume that $\{\mu_k\}_{k=1}^{\infty}$ converges. Let $\mu = \lim_k \mu_k$.

We claim that $\mu \in \Phi(0)$. This claim follows by Lemma 5 (and its proof) in Carmona and Laohakunakorn (2024). To see this, let $E_k = \hat{E}_{s,\alpha_k}$ and note that $Z_k = Z$, $\nu_k = \nu$, $C_k = \hat{C}$, $\mathbb{C}_k(z,z') = \hat{C}$ and $\mathbb{C}_k(z,\emptyset) = \{0\}$ for $(z,z') \in Z$ and $X_k = \hat{X}$. Conditions (a) and (b) of part 4 of Lemma 5 in Carmona and Laohakunakorn (2024) are satisfied since $\{\gamma \in X_k : \{z_k\} \times \text{supp}(\gamma) \subseteq \text{graph}(\mathbb{C}_k)\} = \hat{X}$ for each $k \in \mathbb{N}$ and $z_k \in Z_k = Z$ since $\text{graph}(\mathbb{C}_k) = Z \times Z_{\emptyset} \times C$. The assumption of Lemma 5 in Carmona and Laohakunakorn (2024) that $U_z^k(m,\delta) = U_z(m,\delta)$ for each $(z,\delta) \in Z \times \hat{X}$ does not hold but the proofs of parts 3 and 4 go through provided that $\sup_{(z,z',c)\in Z^2\times C} |U_z^k(m, n(z')1_{(z',c)}) - U_z(m, n(z')1_{(z',c)})| \to 0$. This condition holds since

$$\begin{aligned}
&\left|U_z^k(m, n(z')1_{(z',c)}) - U_z(m, n(z')1_{(z',c)})\right| = |(zh(1-F(z')))^{\alpha_k} - 1| \frac{F(z)}{h(1-F(z'))} \\
&\leq \max\{1 - (\underline{z}h(1-F(\bar{z})))^{\alpha_k}, (\bar{z}h(1-F(\underline{z})))^{\alpha_k} - 1\} \frac{F(\bar{z})}{h(1-F(\bar{z}))} \to 0.
\end{aligned}$$

Thus, it follows that $\mu \in \Phi(0)$ and, hence, Φ is upper hemi-continuous at $\alpha = 0$.

Finally, since $\Phi(0)$ is a singleton by Theorem 5, it follows that Φ is lower hemi-continuous at $\alpha = 0$. Thus, Φ is continuous at $\alpha = 0$.