Supplementary Material for "Rosen meets Garicano and Rossi-Hansberg: Stable Matchings in Knowledge Economies"

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February 28, 2025

Contents

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1 Introduction

This paper contains supplementary material to our paper "Rosen meets Garicano and Rossi-Hansberg: Stable Matchings in Knowledge Economies". It contains:

Section 2: Proof of Corollary 1.

Section 3: The stable matchings of the market E^*_{grh} can be obtained from those of an appropriately defined market where the distribution of problems is uniform.

Section 4: A detailed proof of Theorem 5 concerning the stable matchings of the market E_s.

Section 5: Dispensing with an assumption used in the analysis of Rosen markets in Carmona and Laohakunakorn (2024).

Section 6: An example of a Rosen market where all distributions are uniform.

Section 7: A comparison of the earnings function u between $E_{r,\alpha}$ and E_{grh}, and also within E_{grh} for different values of h.

Section 8: The codes used to produce the figures in Section 7.

2 Proof of Corollary 1

Recall that $n(z) = \frac{1}{h(1-F(z))}$, $u(z) = c(z)$ and $c'(z) = f(z)\frac{F(\phi^{-1}(z))-c(z)}{1-F(z)}$ for each $z \in W$. Furthermore, for each $z \in M$, $\phi'(z) = \frac{\theta(z)}{h(1-F(\phi(z)))\theta(\phi(z))}$.

Hence, for each $z \in W$,

$$c'(z) = hf(z)u(\phi^{-1}(z)), \text{ and}$$
$$n'(z) = hf(z)n(z)^2.$$

For each $z \in M$, $u(z) = (F(z) - c(\phi(z)))n(\phi(z))$ and, hence,

$$u'(z) = (f(z) - c'(\phi(z))\phi'(z))n(\phi(z)) + n'(\phi(z))\phi'(z)(F(z) - c(\phi(z))) = f(z)n(\phi(z)).$$

Thus, for each $x \in F(M)$,

$$\frac{du \circ F^{-1}(x)}{dx} = \frac{u'(F^{-1}(x))}{f(F^{-1}(x))} = n(\phi(F^{-1}(x))) \text{ and}$$
$$\frac{d^2u \circ F^{-1}(x)}{dx^2} = \frac{n'(\phi(F^{-1}(x)))\phi'(F^{-1}(x))}{f(F^{-1}(x))} > 0.$$

This shows that $u \circ F^{-1}$ is strictly convex on $F(M)$.

For each $x \in F(W)$,

$$\frac{du \circ F^{-1}(x)}{dx} = \frac{c'(F^{-1}(x))}{f(F^{-1}(x))} = hu(\phi^{-1}((F^{-1}(x)))) \text{ and}$$
$$\frac{d^2u \circ F^{-1}(x)}{dx^2} = \frac{hu'(\phi^{-1}(F^{-1}(x)))}{\phi'(\phi^{-1}(F^{-1}(x)))f(F^{-1}(x))} > 0.$$

This shows that $u \circ F^{-1}$ is strictly convex on $F(W)$.

Note that $u \circ F^{-1}$ is linear in $F(S)$. Hence, to show that $u \circ F^{-1}$ is convex on $F(Z)$, it suffices to show that $(u \circ F^{-1})'_-(F(z_1)) \leq (u \circ F^{-1})'_+(F(z_1))$ and $(u \circ F^{-1})'_-(F(z_2)) \leq (u \circ F^{-1})'_+(F(z_2))$ (apply Lemma 2.1 below twice).

Consider first the case where $S \neq \emptyset$, i.e. $z_1 < z_2$. For each $x \in F(S)$, $u \circ F^{-1}(x) = F(F^{-1}(x)) = x$ and, hence, $(u \circ F^{-1})'_+(F(z_1)) = (u \circ F^{-1})'_-(F(z_2)) = 1$. Using $c(z_1) = F(z_1)$, it follows by what has been shown above that

$$(u \circ F^{-1})'_-(F(z_1)) = hu(\bar{z}) = \frac{F(\bar{z}) - F(z_1)}{1 - F(z_1)} \leq 1 \text{ and}$$
$$(u \circ F^{-1})'_+(F(z_2)) = n(0) = \frac{1}{h} > 1.$$

Finally, consider the case where $S = \emptyset$, i.e. $z_1 = z_2$. Using $c(z_1) \geq F(z_1)$, it follows by what has been shown above that

$$(u \circ F^{-1})'_-(F(z_1)) = hu(\bar{z}) = \frac{F(\bar{z}) - c(z_1)}{1 - F(z_1)} \leq \frac{F(\bar{z}) - F(z_1)}{1 - F(z_1)} \leq 1 \text{ and}$$
$$(u \circ F^{-1})'_+(F(z_1)) = n(0) = \frac{1}{h} > 1.$$

The following lemma on convex functions was used above.

Lemma 2.1 *If $f : [a, b] \to \mathbb{R}$ and $c \in (a, b)$ are such that both $f|_{[a,c]}$ and $f|_{[c,b]}$ are convex and $f'_-(c) \leq f'_+(c)$, then f is convex.*

Proof. We will show that, for each $\alpha, \beta, x \in [a, b]$ such that $\alpha < x < \beta$, $\frac{f(x)-f(\alpha)}{x-\alpha} \leq \frac{f(\beta)-f(\alpha)}{\beta-\alpha}$. This is clear if $\beta \leq c$ or if $c \leq \alpha$; hence, assume that $\alpha < c < \beta$. For concreteness, assume that $x \leq c$ (the case $x > c$ is analogous).

Note first that if $a \leq y_1 < y_2 < y_3 \leq b$ are such that $\frac{f(y_2)-f(y_1)}{y_2-y_1} \leq \frac{f(y_3)-f(y_2)}{y_3-y_2}$, then $\frac{f(y_2)-f(y_1)}{y_2-y_1} \leq \frac{f(y_3)-f(y_1)}{y_3-y_1}$. Indeed, $(f(y_2) - f(y_1))(y_3 - y_2) \leq (f(y_3) - f(y_2))(y_2 - y_1)$ and summing $(f(y_2) - f(y_1))(y_2 - y_1)$ to both sides yields $(f(y_2) - f(y_1))(y_3 - y_1) \leq (f(y_3) - f(y_1))(y_2 - y_1)$, i.e. $\frac{f(y_2)-f(y_1)}{y_2-y_1} \leq \frac{f(y_3)-f(y_1)}{y_3-y_1}$.

Then

$$\frac{f(\beta) - f(c)}{\beta - c} \geq \lim_{y \downarrow c} \frac{f(y) - f(c)}{y - c} = f'_+(c) \geq f'_-(c) = \lim_{y \uparrow c} \frac{f(c) - f(y)}{c - y} \geq \frac{f(c) - f(x)}{c - x}$$

and the above argument implies that

$$\frac{f(c) - f(x)}{c - x} \leq \frac{f(\beta) - f(x)}{\beta - x}$$

by letting $y_1 = x$, $y_2 = c$ and $y_3 = \beta$.

We have that $\frac{f(x)-f(\alpha)}{x-\alpha} \leq \frac{f(c)-f(x)}{c-x}$ by the convexity of $f|_{[a,c]}$ and, hence,

$$\frac{f(x) - f(\alpha)}{x - \alpha} \leq \frac{f(\beta) - f(x)}{\beta - x}.$$

Then the above argument with $y_1 = \alpha$, $y_2 = x$ and $y_3 = \beta$ implies that $\frac{f(x)-f(\alpha)}{x-\alpha} \leq \frac{f(\beta)-f(\alpha)}{\beta-\alpha}$. ■

3 Uniform distribution of problems

We show that assuming that the distribution of problems is uniform is without loss of generality in the sense that stable matchings of E^*_{grh} with an arbitrary distribution of problems F can be obtained from a market with a uniform distribution of problems.

Recall that E^*_{grh} is defined by $(Z, \nu, C, \mathbb{C}, X, (\succ_z)_{z \in Z})$. We define

$$E_u = (Z_u, \nu_u, C, \mathbb{C}_u, X_u, (\succ_{u,x})_{x \in Z_u})$$

as follows:

$$\begin{aligned}
&F(\emptyset) = F^{-1}(\emptyset) = \emptyset, \\
&Z_u = F(Z) = [0, F(\bar{z})], \\
&\nu_u = \nu \circ F^{-1}, \\
&\mathbb{C}_u(x, x') = \mathbb{C}(F^{-1}(x), F^{-1}(x')), \text{ for each } x \in Z_u \text{ and } x' \in Z_{u,\emptyset} \\
&n_u(x) = n(F^{-1}(x)) \text{ for each } x \in Z_{u,\emptyset}, \\
&X_u = \{n_u(x) 1_{(x,c)} : (x, c) \in Z_u \times C\}, \\
&\text{and, for each } (x, x', c) \in Z_u \times Z_u \times C, \\
&U_{u,x}(w, 1_{(x',c)}) = U_{F^{-1}(x)}(w, 1_{(F^{-1}(x'),c)}) = c, \\
&U_{u,x}(s) = U_{F^{-1}(x)}(s) = x, \text{ and} \\
&U_{u,x}(m, n_u(x') 1_{(x',c)}) = U_{F^{-1}(x)}(m, n(F^{-1}(x')) 1_{(F^{-1}(x'),c)}) = \frac{x - c}{h(1 - x')}.
\end{aligned}$$

Define $X_{u,\emptyset} = X_u \cup \{1_{(\emptyset,0)}\}$ and $F_u : Z \times X_\emptyset \to Z_u \times X_{u,\emptyset}$ by setting, for each $(z, n(z') 1_{(z',c)}) \in Z \times X_\emptyset$,

$$F_u(z, n(z') 1_{(z',c)}) = (F(z), n_u(F(z')) 1_{(F(z'),c)}).$$

Theorem 3.1 *Let $\mu \in M(Z \times X_\emptyset)$. Then μ is a stable matching of E^*_{grh} if and only if $\mu \circ F_u^{-1}$ is a stable matching of E_u.*

Proof. (Necessity) Let μ be a stable matching of E^*_{grh}. For each Borel B of Z_u, we have that

$$\begin{aligned}
&\mu \circ F_u^{-1}(B \times X_u) + \mu \circ F_u^{-1}(B \times \{1_{(\emptyset,0)}\}) + \int_{Z_u \times X_u} \delta(B \times C) \mathrm{d}\mu \circ F_u^{-1} \\
&= \mu(F^{-1}(B) \times X) + \mu(F^{-1}(B) \times \{1_{(\emptyset,0)}\}) + \int_{Z \times X} \delta(F^{-1}(B) \times C) \mathrm{d}\mu \\
&= \nu(F^{-1}(B)) = \nu_u(B).
\end{aligned}$$

Let $(x, n_u(x')1_{(x',c)}) \in \mathrm{supp}(\mu \circ F_u^{-1})$. If $n_u(x')1_{(x',c)} \in X_{u,\emptyset} \backslash X_u$, then $n_u(x')1_{(x',c)} = 1_{(\emptyset,0)}$. This and the above argument show that $\mu \circ F_u^{-1}$ is a matching in E_u.

Note that $\mathrm{supp}(\mu \circ F_u^{-1}) = F_u^{-1}(\mathrm{supp}(\mu))$ since F_u is an homeomorphism. Hence, $(F^{-1}(x), n(F^{-1}(x'))1_{(F^{-1}(x'),c)}) \in \mathrm{supp}(\mu)$.

If $n_u(x')1_{(x',c)} \in X_u$, then $n(F^{-1}(x'))1_{(F^{-1}(x'),c)} \in X$ and $\mathrm{supp}(\mu) \subseteq IR(\mu)$ implies that

$$\begin{aligned}
&U_{u,x}(m, n_u(x')1_{(x',c)}) = U_{F^{-1}(x)}(m, n(F^{-1}(x'))1_{(F^{-1}(x'),c)}) \geq U_{F^{-1}(x)}(s) = U_{u,x}(s), \text{ and} \\
&c \geq U_{F^{-1}(x')}(s) = U_{u,x'}(s).
\end{aligned}$$

It then follows that $(x, n_u(x')1_{(x',c)}) \in IR(\mu \circ F_u^{-1})$.

Thus, to complete the proof, it suffices to show that $(x, n_u(x')1_{(x',c)}) \in S_M(\mu \circ F_u^{-1})$. Suppose not and, specifically, that there is $(x^*, c^*) \in T_x^m(\mu \circ F_u^{-1})$ such that $U_{u,x}(m, n_u(x^*)1_{(x^*,c^*)})$ is bigger than $U_{u,x}(m, n_u(x')1_{(x',c)})$ if $n_u(x')1_{(x',c)} \in X_u$ and $U_{u,x}(s)$ if $n_u(x')1_{(x',c)} = 1_{(\emptyset,0)}$.

First note that $(F^{-1}(x^*), c^*) \in T^m_{F^{-1}(x)}(\mu)$. Indeed, if $(\hat{x}, n_u(x^*)1_{(x^*,\hat{c})}) \in \mathrm{supp}(\mu \circ F_u^{-1}) \cap (Z_u \times X_u)$ for some $(\hat{x}, \hat{c}) \in Z_u \times C$ and $c^* > \hat{c}$, then

$$(F^{-1}(\hat{x}), n(F^{-1}(x^*))1_{(F^{-1}(x^*),\hat{c})}) \in \mathrm{supp}(\mu) \cap (Z \times X)$$

and $c^* > \hat{c}$. If $(x^*, 1_{(\emptyset,0)}) \in \mathrm{supp}(\mu \circ F_u^{-1})$ and $c^* > U_{u,x^*}(s)$, then $(F^{-1}(x^*), 1_{(\emptyset,0)}) \in \mathrm{supp}(\mu)$ and $c^* > U_{F^{-1}(x^*)}(s)$. Finally, if $(x^*, n_u(\tilde{x})1_{(\tilde{x},\tilde{c})}) \in \mathrm{supp}(\mu \circ F_u^{-1})$ for some $(\tilde{x}, \tilde{c}) \in Z_u \times C$ and $c^* > U_{u,x^*}(m, n_u(\tilde{x})1_{(\tilde{x},\tilde{c})})$, then $(F^{-1}(x^*), n(F^{-1}(\tilde{x}))1_{(F^{-1}(\tilde{x}),\tilde{c})}) \in \mathrm{supp}(\mu)$ and $c^* > U_{F^{-1}(x^*)}(m, n(F^{-1}(\tilde{x}))1_{(F^{-1}(\tilde{x}),\tilde{c})})$.

Since

$$U_{F^{-1}(x)}(m, n(F^{-1}(x^*))1_{(F^{-1}(x^*),c^*)}) = U_{u,x}(m, n_u(x^*)1_{(x^*,c^*)}),$$
$$U_{F^{-1}(x)}(m, n(F^{-1}(x'))1_{(F^{-1}(x'),c)}) = U_{u,x}(m, n_u(x')1_{(x',c)}) \text{ if } n_u(x')1_{(x',c)} \in X_u \text{ and}$$
$$U_{F^{-1}(x)}(s) = U_{u,x}(s) \text{ if } n_u(x')1_{(x',c)} = 1_{(\emptyset,0)},$$

it then follows that $(F^{-1}(x), n(F^{-1}(x'))1_{(F^{-1}(x'),c)}) \in \text{supp}(\mu)\setminus S_M(\mu)$, a contradiction to the stability of μ.

Then there is $(x^*, c^*) \in T^m_{x'}(\mu \circ F_u^{-1})$ such that $U_{u,x'}(m, n_u(x^*)1_{(x^*,c^*)}) > c$. Then, as above, $(F^{-1}(x^*), c^*) \in T^m_{F^{-1}(x')}(\mu)$ and $U_{F^{-1}(x')}(m, n(F^{-1}(x^*))1_{(F^{-1}(x^*),c^*)}) > c$. But this contradicts the stability of μ. Hence, it follows that $(x, n_u(x')1_{(x',c)}) \in S_M(\mu \circ F_u^{-1})$ and that $\mu \circ F_u^{-1}$ is a stable matching of E_u.

(Sufficiency) Let $\mu \in M(Z \times X_\emptyset)$ be such that $\mu \circ F_u^{-1}$ is a stable matching of E_u. Note that $\mu = (\mu \circ F_u^{-1}) \circ (F_u^{-1})^{-1}$ and let $\hat{\mu} = \mu \circ F_u^{-1}$ and $\hat{F} = F_u^{-1}$. The claim is then that if $\hat{\mu}$ is a stable matching of E_u, then $\hat{\mu} \circ \hat{F}^{-1}$ is a stable matching of E^*_{grh}. The argument is analogous to the one in the necessity part.

For each Borel B of Z, we have that

$$\begin{aligned}
&\hat{\mu} \circ \hat{F}^{-1}(B \times X) + \hat{\mu} \circ \hat{F}^{-1}(B \times \{1_{(\emptyset,0)}\}) + \int_{Z\times X} \delta(B \times C) \mathrm{d}\hat{\mu} \circ \hat{F}^{-1} \\
&= \hat{\mu}(F(B) \times X) + \hat{\mu}(F(B) \times \{1_{(\emptyset,0)}\}) + \int_{Z_u\times X_u} \delta(F(B) \times C) \mathrm{d}\hat{\mu} \\
&= \nu_u(F(B)) = \nu(B).
\end{aligned}$$

Let $(z, n(z')1_{(z',c)}) \in \text{supp}(\hat{\mu}\circ\hat{F}^{-1})$. If $n(z')1_{(z',c)} \in X_\emptyset\setminus X$, then $n(z')1_{(z',c)} = 1_{(\emptyset,0)}$. This and the above argument show that $\mu = \hat{\mu} \circ \hat{F}^{-1}$ is a matching in E^*_{grh}.

Note that $\text{supp}(\hat{\mu} \circ \hat{F}^{-1}) = F_u(\text{supp}(\hat{\mu}))$ since F_u is an homeomorphism. Hence, $(F(z), n_u(F(z'))1_{(F(z'),c)}) \in \text{supp}(\hat{\mu})$.

If $n(z')1_{(z',c)} \in X$, then $n_u(F(z'))1_{(F(z'),c)} \in X_u$ and $\text{supp}(\hat{\mu}) \subseteq IR(\hat{\mu})$ implies that

$$U_z(m, n(z')1_{(z',c)}) = U_{u,F(z)}(m, n_u(F(z'))1_{(F(z'),c)}) \geq U_{u,F(z)}(s) = U_z(s), \text{ and}$$
$$c \geq U_{u,F(z')}(s) = U_{z'}(s).$$

It then follows that $(z, n(z')1_{(z',c)}) \in IR(\mu)$.

Thus, to complete the proof, it suffices to show that $(z, n(z')1_{(z',c)}) \in S_M(\mu)$. Suppose not and, specifically, that there is $(z^*, c^*) \in T_z^m(\mu)$ such that $U_z(m, n(z^*)1_{(z^*,c^*)})$ is bigger than $U_z(m, n(z')1_{(z',c)})$ if $n(z')1_{(z',c)} \in X$ and $U_z(s)$ if $n(z')1_{(z',c)} = 1_{(\emptyset,0)}$.

First note that $(F(z^*), c^*) \in T_{F(z)}^m(\hat{\mu})$. Indeed, if $(\hat{z}, n(z^*)1_{(z^*,\hat{c})}) \in \text{supp}(\hat{\mu} \circ \hat{F}^{-1}) \cap (Z \times X)$ for some $(\hat{z}, \hat{c}) \in Z \times C$ and $c^* > \hat{c}$, then

$$(F(\hat{z}), n_u(F(z^*))1_{(F(z^*),\hat{c})}) \in \text{supp}(\hat{\mu}) \cap (Z_u \times X_u)$$

and $c^* > \hat{c}$. If $(z^*, 1_{(\emptyset,0)}) \in \text{supp}(\hat{\mu} \circ \hat{F}^{-1})$ and $c^* > U_{z^*}(s)$, then $(F(z^*), 1_{(\emptyset,0)}) \in \text{supp}(\hat{\mu})$ and $c^* > U_{F(z^*)}(s)$. Finally, if $(z^*, n(\tilde{z})1_{(\tilde{z},\tilde{c})}) \in \text{supp}(\hat{\mu} \circ \hat{F}^{-1})$ for some $(\tilde{z}, \tilde{c}) \in Z \times C$ and $c^* > U_{z^*}(m, n(\tilde{z})1_{(\tilde{z},\tilde{c})})$, then $(F(z^*), n_u(F(\tilde{z}))1_{(F(\tilde{z}),\tilde{c})}) \in \text{supp}(\hat{\mu})$ and $c^* > U_{F(z^*)}(m, n_u(F(\tilde{z}))1_{(F(\tilde{z}),\tilde{c})})$.

Since

$$U_{u,F(z)}(m, n_u(F(z^*))1_{(F(z^*),c^*)}) = U_z(m, n(z^*)1_{(z^*,c^*)}),$$
$$U_{u,F(z)}(m, n_u(F(z'))1_{(F(z'),c)}) = U_z(m, n(z')1_{(z',c)}) \text{ if } n(z')1_{(z',c)} \in X \text{ and}$$
$$U_{u,F(z)}(s) = U_z(s) \text{ if } n(z')1_{(z',c)} = 1_{(\emptyset,0)},$$

it then follows that $(F(z), n_u(F(z'))1_{(F(z'),c)}) \in \text{supp}(\hat{\mu}) \setminus S_M(\hat{\mu})$, a contradiction to the stability of $\hat{\mu}$.

Then there is $(z^*, c^*) \in T_{z'}^m(\mu)$ such that $U_{z'}(m, n(z^*)1_{(z^*,c^*)}) > c$. Then, as above, $(F(z^*), c^*) \in T_{F(z')}^m(\hat{\mu})$ and $U_{u,F(z')}(m, n_u(F(z^*))1_{(F(z^*),c^*)}) > c$. But this contradicts the stability of $\hat{\mu}$. Hence, it follows that $(z, n(z')1_{(z',c)}) \in S_M(\mu)$ and that μ is stable. ■

4 A detailed proof of Theorem 5

4.1 Existence part

The existence of stable matchings in $E_s = E_{s,0}$ follows by Lemmas 52 and 53.

4.2 Necessity part

Let μ be a stable matching of E_s.

Lemma 4.1 *$\mu(Z \times X) > 0$.*

Proof. It follows from Lemma 51 that $\mathrm{supp}(\mu) \subseteq Z \times X$. This then implies that $\mu(Z \times X) > 0$ since, otherwise, $0 = \mu(Z \times X) = \mu(Z \times X_\emptyset)$ (the latter equality because $\mathrm{supp}(\mu) \subseteq Z \times X$), $\int_{Z\times X} \delta(Z \times X)\mathrm{d}\mu(z,\delta) = 0$ and, hence, $0 = \mu(Z \times X_\emptyset) + \int_{Z\times X} \delta(Z \times X)\mathrm{d}\mu(z,\delta) = \nu(Z) > 0$, a contradiction. ∎

The following results is a simply consequence of the previous lemma and asserts that managers of type less than $\bar{z}$ exist.

Corollary 4.1 $\mathrm{supp}(\mu) \cap ((Z \setminus \{\bar{z}\}) \times X) \neq \emptyset$.

Proof. Suppose not; then $\mathrm{supp}(\mu) \cap (Z \times X) \subseteq \{\bar{z}\} \times X$. Hence,

$$\mu(Z \times X) = \mu(\mathrm{supp}(\mu) \cap (Z \times X)) \leq \mu(\{\bar{z}\} \times X) = \mu_M(\{\bar{z}\}) \leq \nu(\{\bar{z}\}) = 0,$$

a contradiction to Lemma 4.1. ∎

Lemma 4.2 *If $z, \hat{z}, z' \in Z$ and $c \in C$ are such that $(z, n(z')1_{(z',c)}), (\hat{z}, n(z')1_{(z',\hat{c})}) \in \mathrm{supp}(\mu)$, then $c = \hat{c}$.*

Proof. Indeed, if $c > \hat{c}$, then managers of type z can gain by hiring workers of type z' at wage $c - \varepsilon$ for some $\varepsilon > 0$ such that $c - \varepsilon > \hat{c}$, a contradiction to the stability of μ. Thus, $c \leq \hat{c}$ and an analogous argument shows that $c \geq \hat{c}$; hence, $c = \hat{c}$. ∎

Define $c : W \to [0, 1]$ by setting, for each $z \in W$, $c(z) = c$, where $c \in [0, 1]$ is such that $(\hat{z}, n(z)1_{(z,c)}) \in \mathrm{supp}(\mu)$ for some $\hat{z} \in Z$. Lemma 4.2 implies that the function c is well-defined. For convenience, let, for each $z \in Z$ and $z' \in Z$, $U_z(m, z') = U_z(m, n(z')1_{(z',c(z'))})$; then c takes values in $[0, 1]$ since the stability of μ implies that $c(z) \geq 0$ and that $U_{\hat{z}}(m, z) = (F(\hat{z}) - c(z))n(z) \geq 0$ if $(\hat{z}, n(z)1_{(z,c)}) \in \mathrm{supp}(\mu)$.

Lemma 4.3 *c is increasing.*

Proof. Suppose not; then there is $z, z' \in W$ such that $z' > z$ and $c(z') < c(z)$. Note that $n(z') > n(z)$ and let $\hat{z} \in Z$ be such that $(\hat{z}, n(z)1_{(z,c(z))}) \in \mathrm{supp}(\mu)$. Then $U_{\hat{z}}(m, z) = (F(\hat{z}) - c(z))n(z) < (F(\hat{z}) - c(z'))n(z') = U_{\hat{z}}(m, z')$ since $U_{\hat{z}}(m, z) \geq 0$ by the stability of μ. Thus, there is $\varepsilon > 0$ such that $(z', c(z') + \varepsilon) \in T^m_{\hat{z}}(\mu)$ and $U_{\hat{z}}(m, n(z')1_{(z',c(z')+\varepsilon)}) > U_{\hat{z}}(m, z)$, contradicting the stability of μ. ∎

Lemma 4.4 *c is continuous.*

Proof. Suppose not; then there is $z \in W$ such that c is discontinuous at z. Since c is increasing by Lemma 4.3, there are only two possible cases.

Case 1: There exists $\varepsilon > 0$ and a sequence $\{z_k\}_{k=1}^{\infty}$ such that $z_k \to z$ and, for each $k \in \mathbb{N}$, $z_k \in W$, $z_k < z$ and $c(z) > c(z_k) + \varepsilon$. In this case, let $\hat{z} \in Z$ be such that $(\hat{z}, n(z)1_{(z,c(z))}) \in \text{supp}(\mu)$. Then

$$\left(F(\hat{z}) - c(z_k) - \frac{\varepsilon}{2}\right) n(z_k) > \left(F(\hat{z}) - c(z) + \frac{\varepsilon}{2}\right) n(z_k)$$
$$\to \left(F(\hat{z}) - c(z) + \frac{\varepsilon}{2}\right) n(z) > U_{\hat{z}}(m, z).$$

Thus, there is k sufficiently large such that $(z_k, c(z_k) + \frac{\varepsilon}{2}) \in T_{\hat{z}}^m(\mu)$ and $U_{\hat{z}}(m, z) < U_{\hat{z}}(m, n(z_k)1_{(z_k, c(z_k)+\frac{\varepsilon}{2})})$, contradicting the stability of μ.

Case 2: There exists $\varepsilon > 0$ and a sequence $\{z_k\}_{k=1}^{\infty}$ such that $z_k \to z$ and, for each $k \in \mathbb{N}$, $z_k \in W$, $z_k > z$ and $c(z) < c(z_k) - \varepsilon$. In this case, for each $k \in \mathbb{N}$, let $\hat{z}_k \in Z$ be such that $(\hat{z}_k, n(z_k)1_{(z_k, c(z_k))}) \in \text{supp}(\mu)$. Since $(\hat{z}_k, c(z_k)) \in Z \times [0,1]$ for each $k \in \mathbb{N}$ and $Z \times [0,1]$ is compact, we may assume, taking a subsequence if necessary, that $\{(\hat{z}_k, c(z_k))\}_{k=1}^{\infty}$ converges; let $(\hat{z}, c) = \lim_k (\hat{z}_k, c(z_k))$. Then $c(z) + \frac{\varepsilon}{2} \leq c - \frac{\varepsilon}{2}$ and

$$(F(\hat{z}_k) - c(z_k))n(z_k) \to (F(\hat{z}) - c)n(z) < \left(F(\hat{z}) - c + \frac{\varepsilon}{2}\right) n(z)$$
$$\leq \left(F(\hat{z}) - c(z) - \frac{\varepsilon}{2}\right) n(z) = \lim_k \left(F(\hat{z}_k) - c(z) - \frac{\varepsilon}{2}\right) n(z).$$

Thus, there is k sufficiently large such that $(z, c(z) + \frac{\varepsilon}{2}) \in T_{\hat{z}_k}^m(\mu)$ and $U_{\hat{z}_k}(m, z_k) < U_{\hat{z}_k}(m, n(z)1_{(z, c(z)+\frac{\varepsilon}{2})})$, contradicting the stability of μ. ■

Lemma 4.5 *If $z, z', \hat{z} \in Z$ are such that $(z, n(z')1_{(z', c(z'))}), (z, n(\hat{z})1_{(\hat{z}, c(\hat{z}))}) \in \text{supp}(\mu)$, then $U_z(m, z') = U_z(m, \hat{z})$.*

Proof. If $U_z(m, z') > U_z(m, \hat{z})$, then, letting $\varepsilon > 0$ be such that $(F(z) - c(z') - \varepsilon)n(z') > U_z(m, \hat{z})$, it follows that $(z', c(z') + \varepsilon) \in T_z^m(\mu)$ and $U_z(m, n(z')1_{(z', c(z')+\varepsilon)}) > U_z(m, \hat{z})$, a contradiction to the stability of μ. Thus, $U_z(m, z') \leq U_z(m, \hat{z})$ and an analogous argument shows that $U_z(m, z') \geq U_z(m, \hat{z})$; hence, $U_z(m, z') = U_z(m, \hat{z})$. ■

Define $u : M \to \mathbb{R}_+$ by setting, for each $z \in M$, $u(z) = U_z(m, z')$, where $z' \in Z$ is such that $(z, n(z')1_{(z',c(z'))}) \in \text{supp}(\mu)$. Lemma 4.5 implies that the function u is well-defined.

Lemma 4.6 *$u : M \to \mathbb{R}$ is strictly increasing.*

Proof. Suppose not; then there is $z, \hat{z} \in M$ such that $z > \hat{z}$ and $u(z) \leq u(\hat{z})$. Let $z' \in Z$ be such that $(\hat{z}, n(z')1_{(z',c(z'))}) \in \text{supp}(\mu)$. Then $F(\hat{z}) < F(z)$ and

$$u(z) \leq u(\hat{z}) = (F(\hat{z}) - c(z'))n(z') < (F(z) - c(z'))n(z').$$

Thus, there is $\varepsilon > 0$ such that $U_z(m, n(z')1_{(z',c(z')+\varepsilon)}) > u(z)$. Since $(z', c(z') + \varepsilon) \in T_z^m(\mu)$, this contradicts the stability of μ. ■

Lemma 4.7 *$u : M \to \mathbb{R}$ is continuous.*

Proof. Suppose not; then there is $z \in M$ such that u is discontinuous at z. Since u is increasing by Lemma 4.6, there are only two possible cases.

Case 1: There exists $\varepsilon > 0$ and a sequence $\{z_k\}_{k=1}^{\infty}$ such that $z_k \to z$ and, for each $k \in \mathbb{N}$, $z_k \in M$, $z_k < z$ and $u(z) > u(z_k) + \varepsilon$. In this case, let $z' \in Z$ be such that $(z, n(z')1_{(z',c(z'))}) \in \text{supp}(\mu)$. Then $n(z')(F(z_k) - c(z')) \to u(z) > u(z_k) - \varepsilon$. Thus, there is k sufficiently large and $\eta > 0$ such that $n(z')(F(z_k) - c(z') - \eta) > u(z_k)$. Then $(z', c(z') + \eta) \in T_{z_k}^m(\mu)$ and $U_{z_k}(m, n(z')1_{(z',c(z')+\eta)}) > u(z_k)$, contradicting the stability of μ.

Case 2: There exists $\varepsilon > 0$ and a sequence $\{z_k\}_{k=1}^{\infty}$ such that $z_k \to z$ and, for each $k \in \mathbb{N}$, $z_k \in M$, $z_k > z$ and $u(z) < u(z_k) - \varepsilon$. In this case, for each $k \in \mathbb{N}$, let $z'_k \in Z$ be such that $(z_k, n(z'_k)1_{(z'_k,c(z'_k))}) \in \text{supp}(\mu)$. Then, there is k sufficiently large such that $n(z'_k)(F(z) - c(z'_k)) > u(z_k) - \varepsilon > u(z)$ since, using $F(\bar{z}) < 1$,

$$0 \leq u(z_k) - n(z'_k)(F(z) - c(z'_k)) = n(z'_k)(F(z_k) - F(z)) \leq n(\bar{z})(F(z_k) - F(z)) \to 0.$$

Thus, there is $\eta > 0$ such that $n(z'_k)(F(z) - c(z'_k) - \eta) > u(z)$. Then $(z'_k, c(z'_k) + \eta) \in T_z^m(\mu)$ and $U_z(m, n(z'_k)1_{(z'_k,c(z'_k)+\eta)}) > u(z)$, contradicting the stability of μ. ■

Lemma 4.8 *$u(z) = c(z)$ if $z \in M \cap W$.*

Proof. Let $z \in M \cap W$. Suppose that $u(z) > c(z)$ and let $z' \in Z$ be such that $(z, n(z')1_{(z',c(z'))}) \in \text{supp}(\mu)$ and $\varepsilon > 0$ be such that $n(z')(F(z) - c(z') - \varepsilon) > c(z)$. Then $(z', c(z') + \varepsilon) \in T_z^m(\mu)$ and $U_z(m, n(z')1_{(z',c(z')+\varepsilon)}) > c(z)$, contradicting the stability of μ.

If $u(z) < c(z)$, then let $\hat{z} \in Z$ be such that $(\hat{z}, n(z)1_{(z,c(z))}) \in \text{supp}(\mu)$ and $\varepsilon > 0$ be such that $n(z)(F(\hat{z}) - u(z) - \varepsilon) > n(z)(F(\hat{z}) - c(z))$. Then $(z, u(z) + \varepsilon) \in T_{\hat{z}}^m(\mu)$ and $U_{\hat{z}}(m, n(z)1_{(z,u(z)+\varepsilon)}) > U_{\hat{z}}(m, z)$, contradicting the stability of μ. ■

Let $g : \text{supp}(\mu) \to Z \times Z$ be defined by setting, for each $(z, \delta) \in \text{supp}(\mu)$,

$$g(z, \delta) = (z, z')$$

where $z' \in Z$ is such that $\delta = n(z')1_{(z',c(z'))}$. Let $\pi_2(\text{supp}(\mu)) = \{\delta \in X : (z, \delta) \in \text{supp}(\mu)\}$ be the projection of $\text{supp}(\mu)$ onto X and let $g_2 : \pi_2(\text{supp}(\mu)) \to Z$ be defined by setting, for each $\delta \in \pi_2(\text{supp}(\mu))$,

$$g_2(\delta) = z'$$

where $z' \in Z$ is such that $\delta = n(z')1_{(z',c(z'))}$.

Lemma 4.9 *g is a homeomorphism between* $\text{supp}(\mu)$ *and* $g(\text{supp}(\mu))$ *and g_2 is an homeomorphism between* $\pi_2(\text{supp}(\mu))$ *and* $g_2(\pi_2(\text{supp}(\mu)))$.

Proof. Let $\text{id} : Z \to Z$ be the identity. Then $g = (\text{id}, g_2)|_{\text{supp}(\mu)}$. Since id is an homeomorphism, it suffices to show that g_2 is an homeomorphism.

It is clear that $g_2^{-1} : z' \mapsto n(z')1_{(z',c(z'))}$ is 1-1 and continuous, the latter since c is continuous. Let $\delta = n(z)1_{(z,c(z))}$ for some $z \in Z$ and $\{\delta_k\}_{k=1}^{\infty}$ such that, for each $k \in \mathbb{N}$, $\delta_k = n(z_k)1_{(z_k,c(z_k))}$ for some $z_k \in Z$ and $\delta_k \to \delta$. Let $\kappa : Z \times C \to \mathbb{R}$ be defined by setting, for each $(\hat{z}, \hat{c}) \in Z \times C$, $\kappa(\hat{z}, \hat{c}) = |\hat{z} - z|/n(\hat{z})$. Then κ is bounded and continuous, and hence

$$|g_2(\delta_k) - g_2(\delta)| = |z_k - z| = \int \kappa \mathrm{d}\delta_k \to \int \kappa \mathrm{d}\delta = |z - z| = 0.$$

Thus, g_2 is continuous. ■

Lemma 4.10 *M is nonempty, closed and perfect.*

Proof. The nonemptyness of M follows by Corollary 4.1 and the closedness of M follows because X is compact and $\text{supp}(\mu) \subseteq Z \times X$.

Suppose that M has an isolated point z. Then $\text{supp}(\mu) \cap (\{z\} \times X) \neq \emptyset$ and there is $\varepsilon > 0$ such that $B_\varepsilon(z) \cap M = \{z\}$. But this is a contradiction to the definition of $\text{supp}(\mu)$ since

$$\mu(B_\varepsilon(z) \times X) = \mu(\text{supp}(\mu) \cap (B_\varepsilon(z) \times X)) \leq$$
$$\mu((M \cap B_\varepsilon(z)) \times X) = \mu(\{z\} \times X) \leq \nu(\{z\}) = 0,$$

and $\text{supp}(\mu) \setminus (B_\varepsilon(z) \times X)$ is closed and strictly contained in $\text{supp}(\mu)$. Thus, M has no isolated points and is, therefore, perfect. ∎

Lemma 4.11 *W is a nonempty, closed and perfect.*

Proof. It follows by Lemma 4.10 that M is nonempty and, hence, W is nonempty.

The set W is closed since if $z \in Z$ and $\{z_k\}_{k=1}^\infty$ are such that $z_k \to z$ and $z_k \in W$ for each $k \in \mathbb{N}$, then there is, for each $k \in \mathbb{N}$, $\hat{z}_k \in Z$ such that $(\hat{z}_k, n(z_k)1_{(z_k, c(z_k))}) \in \text{supp}(\mu)$. Since Z is compact, we may assume that $\{\hat{z}_k\}_{k=1}^\infty$ converges; let $\hat{z} = \lim_k \hat{z}_k$. Then $(\hat{z}_k, n(z_k)1_{(z_k, c(z_k))}) \to (\hat{z}, n(z)1_{(z, c(z))})$, implying that $(\hat{z}, n(z)1_{(z, c(z))}) \in \text{supp}(\mu)$ and, hence, $z \in W$.

Suppose that W has an isolated point z. Thus, there is $\eta > 0$ such that $B_\eta(z) \cap W = \{z\}$. Then $\text{supp}(\mu) \cap (Z \times \{g_2^{-1}(z)\}) \neq \emptyset$ and there exists $\varepsilon > 0$ such that $\text{supp}(\mu) \cap (Z \times g_2^{-1}(B_\varepsilon(z) \cap W)) = \text{supp}(\mu) \cap (Z \times \{g_2^{-1}(z)\})$. It follows by Lemma 4.9 that $g_2^{-1}(B_\varepsilon(z) \cap W)$ is open in $\pi_2(\text{supp}(\mu))$, hence $\text{supp}(\mu) \cap (Z \times g_2^{-1}(B_\varepsilon(z) \cap W))$ is open in $\text{supp}(\mu)$. Furthermore,

$$0 = \nu(\{z\}) \geq \int_{Z \times X} \delta(\{z\} \times C) \mathrm{d}\mu(z', \delta)$$

implies that $\mu(\{(z', \delta) \in Z \times X : \delta(\{z\} \times C) > 0\}) = 0$. Since $\{(z', \delta) \in Z \times X : \delta(\{z\} \times C) > 0\} = Z \times \{g_2^{-1}(z)\}$, it follows that

$$0 = \mu(Z \times \{g_2^{-1}(z)\}) = \mu(\text{supp}(\mu) \cap (Z \times \{g_2^{-1}(z)\}))$$
$$= \mu(\text{supp}(\mu) \cap (Z \times B_\varepsilon(g_2^{-1}(z) \cap W))).$$

Hence, $\text{supp}(\mu) \setminus (Z \times g_2^{-1}(B_\varepsilon(z) \cap W))$ is closed, strictly contained in $\text{supp}(\mu)$ and such that $\mu(\text{supp}(\mu) \setminus (Z \times g_2^{-1}(B_\varepsilon(z) \cap W))) = \mu(\text{supp}(\mu))$. But this contradicts the definition of $\text{supp}(\mu)$. Thus, W has no isolated points and is, therefore, perfect. ■

Lemma 4.12 $Z = M \cup W$.

Proof. Let $K = M \cup W$ and note that we have that $K \subseteq Z$ by definition.

Conversely, note first that K is closed by Lemmas 4.10 and 4.11. Furthermore, letting $\pi(\text{supp}(\mu))$ be the projection of $\text{supp}(\mu)$ onto Z, we have that $\text{supp}(\mu) \subseteq \pi(\text{supp}(\mu)) \times X = M \times X \subseteq K \times X$ and, hence,

$$\mu(K \times X) \geq \mu(\text{supp}(\mu)) = \mu(\text{supp}(\mu) \cap (Z \times X)) = \mu(Z \times X).$$

Furthermore, for each $(z, \delta) \in \text{supp}(\mu) \cap (Z \times X)$, there is $z' \in Z$ such that $\delta = g_2(z')$ and, hence, $z' \in W$. Thus, $\delta((Z \setminus W) \times C) = 0$, $\delta(W \times C) = \delta(Z \times C)$ and $\delta(K \times C) = \delta(Z \times C)$. Hence,

$$\begin{aligned}
\nu(K) &= \mu(K \times X) + \mu(K \times (X_\emptyset \setminus X)) + \int_{Z \times X} \delta(K \times C)\mathrm{d}\mu(z, \delta) \\
&= \mu(K \times X) + \int_{(Z \times X) \cap \text{supp}(\mu)} \delta(K \times C)\mathrm{d}\mu(z, \delta) \\
&\geq \mu(Z \times X) + \int_{(Z \times X) \cap \text{supp}(\mu)} \delta(Z \times C)\mathrm{d}\mu(z, \delta) = \nu(Z).
\end{aligned}$$

It then follows by the definition of $\text{supp}(\nu)$ that $Z = \text{supp}(\nu) \subseteq K$. ■

Lemma 4.13 *There exists a continuous function $u : Z \to \mathbb{R}$ such that*

1. $u(z) = U_z(m, \delta)$ *for each* $z \in M$ *and* $\delta \in X$ *such that* $(z, \delta) \in \text{supp}(\mu)$,
2. $u(z) = U_z(w, 1_{(\hat{z},c)})$ *for each* $z \in W$ *and* $(\hat{z}, c) \in Z \times C$ *such that* $(\hat{z}, n(z)1_{(z,c)}) \in \text{supp}(\mu)$.

Proof. Define $u : Z \to \mathbb{R}$ by setting, for each $z \in Z$,

$$u(z) = \begin{cases} u(z) & \text{if } z \in M, \\ c(z) & \text{if } z \in W. \end{cases}$$

It follows by Lemma 4.8 that u is well-defined and by Lemmas 4.2 and 4.5 that conditions 1 and 2 in the statement of the lemma hold. We have that u is continuous since M and W are closed (by Lemmas 4.10 and 4.11), $Z = M \cup S$ (by Lemma 4.12) and $u|_M$ and $u|_W$ are continuous. ■

Set $n(\emptyset) = 0$ and let $s : Z \times Z_\emptyset \to \mathbb{R}$ be defined by setting, for each $(z, z') \in Z \times Z_\emptyset$,

$$s(z, z') = F(z)n(z').$$

Define $u(\emptyset) = 0$. Since $\emptyset$ is isolated, $u : Z_\emptyset \to \mathbb{R}$ is continuous.

Lemma 4.14 *$\gamma = \mu \circ g^{-1}$ is a stable assignment.*

Proof. We first show that γ is an assignment. Let B be a Borel subset of Z; then

$$\begin{aligned}
&\gamma(B \times Z_\emptyset) + \int_{Z \times B} n(z')\mathrm{d}\gamma(z, z') = \\
&\gamma(B \times Z_\emptyset) + \int_{Z \times Z} 1_B(z')n(z')\mathrm{d}\gamma(z, z') = \\
&\mu(g^{-1}(B \times Z_\emptyset)) + \int_{Z \times X} 1_B(g(z, \delta))n(g(z, \delta))\mathrm{d}\mu(z, \delta) = \\
&\mu(B \times X_\emptyset) + \int_{Z \times X} \delta(B \times C)\mathrm{d}\mu(z, \delta) = \nu(B).
\end{aligned}$$

where the last equality follows because μ is a matching.

We next show that (γ, u) is stable. Note first that $\text{supp}(\gamma) = g(\text{supp}(\mu))$ by Carmona and Laohakunakorn (2024, Lemma 1) since g is an homeomorphism between two compact spaces by Lemma 19.

Let $(z, z') \in \text{supp}(\gamma) \subseteq Z \times Z$ by Lemma 51. Since $\text{supp}(\gamma) \subseteq g(\text{supp}(\mu))$, $(z, n(z')1_{(z', c(z'))}) \in \text{supp}(\mu)$. Since $z, z' \in Z$, then $z \in M$, $z' \in W$ and

$$u(z) + n(z')u(z') = (F(z) - c(z'))n(z') + c(z')n(z') = s(z, z').$$

Let $(z, z') \in Z \times Z_\emptyset$. If $z' = \emptyset$, then $u(z) + n(z')u(z') = u(z) \geq 0 = s(z, \emptyset)$, where the inequality holds since μ is stable. If $z' \neq \emptyset$, then $s(z, z') = F(z)n(z')$. If $u(z) + n(z')u(z') < s(z, z')$, then, letting $\varepsilon > 0$ be such that $n(z')(F(z) - u(z') - \varepsilon) > u(z)$, it follows that $(z', u(z') + \varepsilon) \in T_z^m(\mu)$ and $U_z(m, n(z')1_{(z', u(z') + \varepsilon)}) > u(z)$, a contradiction to the stability of μ. Thus, $u(z) + n(z')u(z') \geq s(z, z')$. ■

Theorems 7 and 8 imply that:

Lemma 4.15 *γ is surplus maximizing and that* $\mathrm{supp}(\gamma)$ *is s-monotone.*

We have that

$$M = \{z \in Z : (z, z') \in \mathrm{supp}(\gamma) \text{ for some } z' \in Z\} \text{ and}$$
$$W = \{z \in Z : (\hat{z}, z) \in \mathrm{supp}(\gamma) \text{ for some } \hat{z} \in Z\}$$

since $\mathrm{supp}(\gamma) = g(\mathrm{supp}(\mu))$ by Carmona and Laohakunakorn (2024, Lemma 1).

Lemma 4.16 *If $(z, z'), (\hat{z}, \hat{z}') \in Z^2$, $(z, z'), (\hat{z}, \hat{z}') \in \mathrm{supp}(\gamma)$ and $z > \hat{z}$, then $z' \geq \hat{z}'$.*

Proof. Suppose that $z > \hat{z}$ but $\hat{z}' > z'$. Let $\zeta = 1_{(z,z')} + 1_{(\hat{z},\hat{z}')}$ and $\tau = 1_{(z,\hat{z}')} + 1_{(\hat{z},z')}$. Then ζ and τ are finitely-supported, $\mathrm{supp}(\zeta) \subseteq \mathrm{supp}(\gamma)$ and $\tau_Z + \tau_{Z,n} = \zeta_Z + \zeta_{Z,n}$. Since $s(\tau) - s(\zeta) = \big(n(\hat{z}') - n(z')\big)\big(F(z) - F(\hat{z})\big) > 0$, this contradicts Lemma 4.15. ∎

Define $z_1 = \min M$.

Lemma 4.17 *z_1 exists and $z_1 < \bar{z}$.*

Proof. It follows by Lemma 4.10 that z_1 exists and by Corollary 4.1 that $z_1 < \bar{z}$. ∎

Lemma 4.18 *For each $z \in M \setminus \{\bar{z}\}$, there exists $\varepsilon > 0$ such that $(z, z + \varepsilon) \subseteq M \setminus W$.*

Proof. Suppose not; then there exists a sequence $\{z_k\}_{k=1}^{\infty}$ such that, for each $k \in \mathbb{N}$, $z_k > z$, $z_k \in (M \setminus W)^c = M^c \cup W$ and $z_k \to z$; thus, $z_k \in W$ by Lemma 4.12. Let $z' \in Z$ be such that $(z, z') \in \mathrm{supp}(\gamma)$.

For each $k \in \mathbb{N}$, let $\hat{z}_k \in Z$ be such that $(\hat{z}_k, z_k) \in \mathrm{supp}(\gamma)$. Since Z is compact, we may assume that $\{\hat{z}_k\}_{k=1}^{\infty}$ converges; let $\hat{z} = \lim_k \hat{z}_k$. For each $k \in \mathbb{N}$, let $\zeta_k = 1_{(z,z')} + 1_{(\hat{z}_k,z_k)}$ and

$$\tau_k = 1_{(z_k,z')} + \frac{n(z_k) - n(z)}{n(z)(1 + n(z_k))} 1_{(z_k,z)} + \frac{n(z) + 1}{n(z)(1 + n(z_k))} 1_{(\hat{z}_k,z)} + \frac{n(z)n(z_k) - 1}{n(z)(1 + n(z_k))} 1_{(\hat{z}_k,z_k)}.$$

Then ζ_k and τ_k are finitely-supported, $\mathrm{supp}(\zeta_k) \subseteq \mathrm{supp}(\gamma)$, $\tau_{k,Z}+\tau_{k,Z,n} = \zeta_{k,Z}+\zeta_{k,Z,n}$,

$$\begin{aligned}
s(\tau_k) =&F(z_k)n(z') + \frac{n(z_k)-n(z)}{n(z)(1+n(z_k))}F(z_k)n(z) + \frac{n(z)+1}{n(z)(1+n(z_k))}F(\hat{z}_k)n(z) \\
&+ \frac{n(z)n(z_k)-1}{n(z)(1+n(z_k))}F(\hat{z}_k)n(z_k) \text{ and} \\
s(\zeta_k) =&F(z)n(z') + F(\hat{z}_k)n(z_k).
\end{aligned}$$

Furthermore, $n'(z) = hf(z)n(z)^2$. Then,

$$\begin{aligned}
\lim_k \frac{s(\tau_k)-s(\zeta_k)}{z_k - z} &= \lim_k \Bigg(\frac{F(z_k)-F(z)}{z_k - z} n(z') + \frac{n(z_k)-n(z)}{z_k - z}\frac{F(z_k)}{1+n(z_k)} \\
&\quad - \frac{n(z_k)-n(z)}{z_k - z}\frac{n(z)+1}{n(z)(1+n(z_k))}F(\hat{z}_k)\Bigg) \\
&= f(z)\left(n(z') + \frac{hn(z)^2}{1+n(z)}F(z) - hn(z)F(\hat{z})\right).
\end{aligned}$$

Note that $\frac{F(\bar{z})-F(z)}{1-F(z)} \leq F(\bar{z})$ for each $z \in [0,\bar{z}]$ since the function $z \mapsto \frac{F(\bar{z})-F(z)}{1-F(z)}$ is strictly decreasing as its derivative at $z \in [0,\bar{z}]$ equals $\frac{f(z)(F(\bar{z})-1)}{(1-F(z))^2} < 0$. It then follows that

$$\begin{aligned}
&\frac{1}{f(z)} \lim_k \frac{s(\tau_k)-s(\zeta_k)}{z_k - z} = \\
&= \frac{1}{h(1-F(z'))} + \frac{F(z)}{(1-F(z))(1+h(1-F(z)))} - \frac{F(\hat{z})}{1-F(z)} \\
&= \frac{1}{h(1-F(z'))} - \frac{F(\hat{z})-F(z)}{(1-F(z))(1+h(1-F(z)))} - \frac{hF(\hat{z})}{1+h(1-F(z))} \\
&\geq \frac{1}{h} - \left(\frac{F(\bar{z})-F(z)}{1-F(z)} + hF(\bar{z})\right)\frac{1}{1+h(1-F(z))} \\
&> \frac{1}{h} - (F(\bar{z})+hF(\bar{z}))\frac{1}{1+h(1-F(\bar{z}))} \\
&= \frac{1}{h} - \frac{F(\bar{z})(1+h)}{1+h(1-F(\bar{z}))}
\end{aligned}$$

(to see that the strict inequality holds, consider separately the cases $z=\bar{z}$ and $z<\bar{z}$).

Then $\lim_k \frac{s(\tau_k)-s(\zeta_k)}{z_k-z} > 0$ since this is equivalent to

$$
\begin{aligned}
&\frac{1}{h} - \frac{F(\bar{z})(1+h)}{1+h(1-F(\bar{z}))} \geq 0 \Leftrightarrow \\
&1 + h(1-F(\bar{z})) \geq hF(\bar{z}) + h^2 F(\bar{z}) \Leftrightarrow \\
&F(\bar{z})h^2 + (2F(\bar{z})-1)h - 1 \leq 0 \Leftrightarrow \\
&h \leq \frac{1}{2F(\bar{z})} - 1 + \sqrt{1 + \frac{1}{4F(\bar{z})^2}}.
\end{aligned}
$$

Thus, $s(\tau_k) - s(\zeta_k) > 0$ for each k sufficiently large, contradicting Lemma 4.15. ■

Lemma 4.19 *$M = [z_1, \bar{z}]$ and $W = [\underline{z}, z_1]$.*

Proof. Let $\bar{\varepsilon} = \sup\{\varepsilon > 0 : (z_1, z_1 + \varepsilon) \subseteq M \setminus W\}$. Such $\bar{\varepsilon}$ exists because $\{\varepsilon > 0 : (z_1, z_1 + \varepsilon) \subseteq M \setminus W\}$ is nonempty by Lemma 4.18 and is bounded above by $\bar{z} - z_1$. We then have that $(z_1, z_1 + \bar{\varepsilon}) \subseteq M \setminus W$ by the definition of $\bar{\varepsilon}$. Indeed, each $z \in (z_1, z_1 + \bar{\varepsilon})$ belongs to $M \setminus W$ since, letting $\varepsilon > 0$ be such that $z < z_1 + \varepsilon$ and $\varepsilon < \bar{\varepsilon}$, it follows that $z \in (z_1, z_1 + \varepsilon) \subseteq M \setminus W$.

Furthermore, $z_1 + \bar{\varepsilon} \in M \setminus W$. We have that $z_1 + \bar{\varepsilon} \in M$ since M is closed by Lemma 4.10 and every $z < z_1 + \varepsilon$ belongs to M. If $\bar{\varepsilon} = \bar{z} - z_1$, then $z_1 + \bar{\varepsilon} = \bar{z}$ and hence $z_1 + \bar{\varepsilon} \notin W$ since otherwise $\bar{z}$ would be an isolated point of W, a contradiction to Lemma 4.11. Thus, consider $\bar{\varepsilon} < \bar{z} - z_1$ and suppose that $z_1 + \bar{\varepsilon} \in W$. Then, letting $\eta > 0$ be such that $(z_1 + \bar{\varepsilon}, z_1 + \bar{\varepsilon} + \eta) \subseteq M \setminus W$, which exists by Lemma 4.18, it follows that $z_1 + \bar{\varepsilon}$ is an isolated point of W. But this contradicts Lemma 4.11.

It follows that $(z_1, z_1 + \bar{\varepsilon}] \subseteq M \setminus W$. If $\bar{\varepsilon} < \bar{z} - z_1$, then $(z_1 + \bar{\varepsilon}, z_1 + \bar{\varepsilon} + \eta) \subseteq M \setminus W$ for some $\eta > 0$ by Lemma 4.18 and, hence, $(z_1, z_1 + \bar{\varepsilon} + \eta) \subseteq M \setminus W$, contradicting the definition of $\bar{\varepsilon}$. Thus, it follows that $\bar{\varepsilon} = \bar{z} - z_1$ and that $M \setminus W = (z_1, \bar{z}]$. It follows that $W \subseteq [\underline{z}, z_1]$ and, in fact, that $W = [\underline{z}, z_1]$ and $M = [z_1, \bar{z}]$ since $M \cup W = Z$ by Lemma 4.12, M is closed by Lemma 4.10 and W is closed by Lemma 4.11. ■

Lemma 4.20 *$z_1 > 0$.*

Proof. Suppose that $z_1 = 0$. Then, $W = \{0\}$ by Lemma 4.19. Hence, $\text{supp}(\gamma) \subseteq$

$Z \times \{0\}$. Thus,

$$\nu(Z) = \nu(Z \setminus \{0\}) = \gamma((Z \setminus \{0\}) \times Z) + \int_{Z \times (Z \setminus \{0\})} n(z')\mathrm{d}\gamma(z, z')$$
$$= \gamma((Z \setminus \{0\}) \times Z).$$

Furthermore,

$$\gamma((Z \setminus \{0\}) \times Z) \leq \gamma(Z \times Z) \leq \gamma(Z \times Z) + \int_{Z \times Z} n(z')\mathrm{d}\gamma(z, z') = \nu(Z).$$

Hence, $\gamma(Z \times Z) = \nu(Z)$. This then implies that $\int_{Z \times Z} n(z')\mathrm{d}\gamma(z, z') = 0$, which contradicts

$$\int_{Z \times Z} n(z')\mathrm{d}\gamma(z, z') = \int_{\mathrm{supp}(\gamma) \cap (Z \times Z)} n(z')\mathrm{d}\gamma(z, z') = n(0)\gamma(\mathrm{supp}(\gamma) \cap (Z \times Z))$$
$$= n(0)\gamma(Z \times Z) = n(0)\nu(Z) > 0.$$

■

Define $\phi : M \rightrightarrows W$ by setting, for each $z \in M$,

$$\phi(z) = \{z' \in Z : (z, z') \in \mathrm{supp}(\gamma)\}.$$

Then ϕ is nonempty-valued by the definition of M, $\phi(M) = W$ by the definition of W and ϕ has a closed graph since $\mathrm{supp}(\gamma)$ is closed.

Let $Q = \{z \in M : \phi(z) \text{ is not a singleton}\}$.

Lemma 4.21 *Q is countable.*

Proof. For each $z \in Q$, let $r(z) \in \mathbb{Q}$ be such that $\min \phi(z) < r(z) < \max \phi(z)$. This defines a function $r : Q \to \mathbb{Q}$ which, as we now claim, is strictly increasing. Indeed, if $z, \hat{z} \in Q$ are such that $z < \hat{z}$, then $(z, \max \phi(z)) \in \mathrm{supp}(\gamma)$, $(\hat{z}, \min \phi(\hat{z})) \in \mathrm{supp}(\gamma)$ and, hence, $\max \phi(z) \leq \min \phi(\hat{z})$ by Lemma 4.16. Thus, $r(z) < \max \phi(z) \leq \min \phi(\hat{z}) < r(\hat{z})$. Thus, r maps Q in a one-to-one way to a subset of $\mathbb{Q}$, implying that Q is countable. ■

Lemma 4.22 *For each $z \in W$, there exists $z^* \in Z$ such that $\{\hat{z} \in Z : (\hat{z}, z) \in \mathrm{supp}(\gamma)\} = \{z^*\}$.*

Proof. The definition of W implies that $\{\hat{z} \in Z : (\hat{z}, z) \in \text{supp}(\gamma)\}$ is nonempty. Suppose that the conclusion of the lemma fails; then let $z' \in W$ and $z^*, \tilde{z} \in Z$ be such that $z^*, \tilde{z} \in \{\hat{z} \in Z : (\hat{z}, z') \in \text{supp}(\gamma)\}$ and $z^* < \tilde{z}$. Since $z^*, \tilde{z} \in M$ by the definition of M and M is an interval by Lemma 4.19, $[z^*, \tilde{z}] \subseteq M$. Let $z \in (z^*, \tilde{z})$ and $\tilde{z}' \in Z$ be such that $(z, \tilde{z}') \in \text{supp}(\gamma)$. Lemma 4.16 implies that $z' \leq \tilde{z}' \leq z'$, hence $\tilde{z}' = z'$. Thus, $z \in \{\hat{z} \in Z : (\hat{z}, z') \in \text{supp}(\gamma)\}$; since z is arbitrary, it follows that $[z^*, \tilde{z}] \subseteq \{\hat{z} \in Z : (\hat{z}, z') \in \text{supp}(\gamma)\}$.

We have that $(z^*, \tilde{z}) \setminus Q \subseteq M \setminus W$ by Lemma 4.19 and $\phi(z) = \{z'\}$ for each $z \in (z^*, \tilde{z}) \setminus Q$. Thus,

$$\nu([z^*, \tilde{z}] \setminus Q) = \nu((z^*, \tilde{z}) \setminus Q) = \gamma(((z^*, \tilde{z}) \setminus Q) \times Z) = \gamma(((z^*, \tilde{z}) \setminus Q) \times \{z'\}).$$

Since $0 = \nu(\{z'\}) \geq \int_{Z \times \{z'\}} n(\hat{z}) \mathrm{d}\gamma(z, \hat{z})$, it follows that $\int_{Z \times \{z'\}} n(\hat{z}) \mathrm{d}\gamma(z, \hat{z}) = 0$. Thus,

$$\begin{aligned} &0 = \int_{Z \times \{z'\}} n(\hat{z}) \mathrm{d}\gamma(z, \hat{z}) = n(z')\gamma(Z \times \{z'\}) \geq \\ &n(z')\gamma(((z^*, \tilde{z}) \setminus Q) \times \{z'\}) = n(z')\nu([z^*, \tilde{z}] \setminus Q) = n(z')\nu([z^*, \tilde{z}]) > 0, \end{aligned}$$

a contradiction. ■

Lemma 4.23 *If* $(z, z'), (\hat{z}, \hat{z}') \in Z^2$, $(z, z'), (\hat{z}, \hat{z}') \in \text{supp}(\gamma)$ *and* $z > \hat{z}$, *then* $z' > \hat{z}'$.

Proof. We have that $z' \geq \hat{z}'$ by Lemma 4.16 and that $z' \neq \hat{z}'$ by Lemma 4.22. Thus, $z' > \hat{z}'$. ■

Lemma 4.24 ϕ *is a continuous and strictly increasing function,* $\phi(z_1) = 0$ *and* $\phi(\bar{z}) = z_1$.

Proof. We first show that $\phi(z)$ is a singleton for each $z \in M$, i.e. $Q = \emptyset$. Suppose not; then let $z \in M$ and $z^*, \tilde{z} \in \phi(z)$ be such that $z^* < \tilde{z}$. Since W is an interval by Lemma 4.19, $[z^*, \tilde{z}] \subseteq W$. Let $z' \in (z^*, \tilde{z})$ and $\hat{z} \in M$ be such that $z' \in \phi(\hat{z})$. Lemma 4.23 then implies that $z' > \tilde{z}$ if $\hat{z} > z$ and that $z' < z^*$ if $\hat{z} < z$. Thus, $\hat{z} = z$ and $z' \in \phi(z)$; since z' is arbitrary, it follows that $[z^*, \tilde{z}] \subseteq \phi(z)$.

We have that $[z^*, \tilde{z}] \cap \phi(x) = \emptyset$ for each $x \in M \setminus \{z\}$. Indeed, Lemma 4.23 implies that $\min \phi(x) > \tilde{z}$ for each $x > z$ and that $\max \phi(x) < z^*$ for each $x < z$. Since $(z^*, \tilde{z}) \subseteq W \setminus M$, it follows that

$$\nu([z^*, \tilde{z}]) = \gamma([z^*, \tilde{z}] \times Z) + \int_{Z \times [z^*, \tilde{z}]} n(z') \mathrm{d}\gamma(z, z') = 0 + \int_{(Z \setminus \{z\}) \times [z^*, \tilde{z}]} n(z') \mathrm{d}\gamma(z, z') = 0,$$

a contradiction to $\nu([z^*, \tilde{z}]) > 0$. This contradiction shows that $\phi(z)$ is a singleton for each $z \in M$.

It then follows that ϕ is a function. Since the graph of ϕ is closed, it follows that ϕ is continuous. Lemma 4.23 implies that ϕ is strictly increasing.

It follows from $\phi(M) = W$ that ϕ is onto. This then implies that $\phi(z_1) = 0$ and $\phi(\bar{z}) = z_1$ since ϕ is strictly increasing. ■

Lemma 4.25 *If $z \in M$, then $u(z) > 0$.*

Proof. Suppose not; then let $z \in M$ be such that $u(z) = 0$. Since $z \geq z_1 > 0$, it follows that $F(z)n(z) > 0$. Then let $\varepsilon > 0$ be such that $(F(z) - \varepsilon)n(z) > 0$. Then $(z, \varepsilon) \in T_z^m(\mu)$ and $U_z(m, n(z)1_{(z,\varepsilon)}) = (F(z) - \varepsilon)n(z) > 0 = u(z)$, a contradiction to the stability of μ. ■

Lemma 4.26 *c is strictly increasing.*

Proof. Suppose not; then there is $z, z' \in W$ such that $z' > z$ and $c(z') \leq c(z)$. Since $n(z') > n(z)$, it follows that $u(\phi^{-1}(z)) = (F(\phi^{-1}(z)) - c(z))n(z) < (F(\phi^{-1}(z) - c(z'))n(z') = U_{\phi^{-1}(z)}(m, z')$ since $u(\phi^{-1}(z)) > 0$ by Lemma 4.25. Thus, there is $\varepsilon > 0$ such that $(z', c(z') + \varepsilon) \in T^m_{\phi^{-1}(z)}(\mu)$ and $U_{\phi^{-1}(z)}(m, n(z')1_{(z', c(z')+\varepsilon)}) > u(\phi^{-1}(z))$, contradicting the stability of μ. ■

Lemma 4.27 *c is differentiable and, for each $z \in W$, $c'(z) = f(z)\frac{F(\phi^{-1}(z)) - c(z)}{1 - F(z)}$.*

Proof. Let $z \in W = [0, z_1]$ and $\{z_k\}_{k=1}^{\infty}$ be such that, for each $k \in \mathbb{N}$, $z_k \in W$, $z_k \neq z$ and $z_k \to z$. Let $\{\hat{z}_k\}_{k=1}^{\infty}$ be such that $\hat{z}_k = \phi^{-1}(z_k)$ for each $k \in \mathbb{N}$. We have that ϕ^{-1} exists and is continuous by Lemma 4.24 and since M is compact. Thus, $\hat{z}_k \to \phi^{-1}(z)$.

The stability of μ implies that, for each $k \in \mathbb{N}$, $\frac{F(\phi^{-1}(z))-c(z)}{h(1-F(z))} \geq \frac{F(\phi^{-1}(z))-c(z_k)}{h(1-F(z_k))}$. Thus, a simple manipulation of this expression implies that

$$\frac{c(z_k)-c(z)}{z_k-z} \geq \frac{F(z_k)-F(z)}{z_k-z}\frac{F(\phi^{-1}(z))-c(z_k)}{1-F(z_k)};$$

hence, $\liminf_k \frac{c(z_k)-c(z)}{z_k-z} \geq f(z)\frac{F(\phi^{-1}(z))-c(z)}{1-F(z)}$ since c is continuous by Lemma 4.4.

Analogously, the stability of μ implies that, for each $k \in \mathbb{N}$, $\frac{F(\hat{z}_k)-c(z_k)}{h(1-F(z_k))} \geq \frac{F(\hat{z}_k)-c(z)}{h(1-F(z))}$. Thus,

$$\frac{c(z_k)-c(z)}{z_k-z} \leq \frac{F(z_k)-F(z)}{z_k-z}\frac{F(\hat{z}_k)-c(z_k)}{1-F(z_k)};$$

hence, $\limsup_k \frac{c(z_k)-c(z)}{z_k-z} \leq f(z)\frac{F(\phi^{-1}(z))-c(z)}{1-F(z)}$. It thus follows that

$$\lim_k \frac{c(z_k)-c(z)}{z_k-z} = f(z)\frac{F(\phi^{-1}(z))-c(z)}{1-F(z)}.$$

Hence, c is differentiable and, for each $z \in W$, $c'(z) = f(z)\frac{F(\phi^{-1}(z))-c(z)}{1-F(z)}$. ■

Lemma 4.28 $\mu = \nu \circ \sigma^{-1}$.

Proof. Let B be a Borel subset of $Z \times X_\emptyset$. Then

$$\nu \circ \sigma^{-1}(B) = \nu(\{z \in Z : \sigma(z) \in B\}) = \nu(\{z \in [z_1, \bar{z}] : \sigma(z) \in B\}).$$

Furthermore,

$$\begin{aligned}\mu(B) &= \mu(\mathrm{supp}(\mu) \cap B) = \mu(\mathrm{supp}(\mu) \cap B \cap (Z \times X)) \\ &= \mu(\{z \in [z_1, \bar{z}] : \sigma(z) \in B\} \times X).\end{aligned}$$

Let $D = \{z \in (z_1, \bar{z}] : \sigma(z) \in B\}$ and note that

$$\begin{aligned}&\nu(\{z \in [z_1, \bar{z}] : \sigma(z) \in B\}) = \nu(D) = \mu(D \times X) + \int_{Z\times X} \delta(D \times C)\mathrm{d}\mu(z, \delta) \\ &= \mu(D \times X) + 0 = \mu(\{z \in [z_1, \bar{z}] : \sigma(z) \in B\} \times X).\end{aligned}$$

Thus $\nu \circ \sigma^{-1}(B) = \mu(B)$. Since B is arbitrary, $\nu \circ \sigma^{-1} = \mu$. ■

Let $\nu(z') = \nu([0, z'])$ for each $z' \in Z$.

Lemma 4.29 *For each* $z \in [z_1, \bar{z}]$, $\nu(\phi(z)) = \int_{z_1}^{z} \frac{\theta(x)}{h(1-F(\phi(x)))}\mathrm{d}x$.

Proof. Let $z \in [z_1, \bar{z}]$ and let $\tau : Z \times X \to \mathbb{R}$ be defined by setting, for each $(z, \delta) \in Z \times X$, $\tau(z, \delta) = \delta([0, \phi(z)] \times C)$. It follows by Lemma 4.28 that

$$\int_{Z \times X} \delta([0, \phi(z)] \times C) \mathrm{d}\mu(x, \delta) = \int_{[z_1, \bar{z}]} \tau(\sigma(x)) \mathrm{d}\nu(x) = \int_{[z_1, z]} \frac{1}{h(1 - F(\phi(x)))} \mathrm{d}\nu(x).$$

Thus,

$$\nu([0, \phi(z)]) = \int_{[z_1, z]} \frac{1}{h(1 - F(\phi(x)))} \mathrm{d}\nu(x)$$

since $\nu([0, \phi(z)]) = \nu([0, \phi(z)))$ and $[0, \phi(z)) \subseteq W \setminus M$. Since ν has a continuous density θ, it follows that $\nu(\phi(z)) = \nu([0, \phi(z)]) = \int_{z_1}^{z} \frac{\theta(x)}{h(1-F(\phi(x)))} \mathrm{d}x$ for each $z \in [z_1, \bar{z}]$. ■

Lemma 4.30 *ϕ is differentiable and, for each $z \in [z_1, \bar{z}]$,*

$$\phi'(z) = \frac{\theta(z)}{h(1 - F(\phi(z)))\theta(\phi(z))}.$$

Proof. The function $z' \mapsto \nu(z')$ is strictly increasing; let $\lambda : [0, \nu(\bar{z})] \to Z$ be its inverse. It then follows by Lemma 4.29 that, for each $z \in [z_1, \bar{z}]$,

$$\phi(z) = \lambda \left(\int_{z_1}^{z} \frac{\theta(x)}{h(1 - F(\phi(x)))} \mathrm{d}x \right).$$

We have that $z \mapsto \nu(z)$ is differentiable and that its derivative at $z \in Z$ is $\theta(z)$. Then λ is differentiable and $\lambda'(x) = \frac{1}{\theta(\lambda(x))}$ for each $x \in [0, \nu(\bar{z})]$. Let $\zeta : [z_1, \bar{z}] \to \mathbb{R}$ be defined by setting, for each $z \in [z_1, \bar{z}]$, $\zeta(z) = \int_{z_1}^{z} \frac{\theta(x)}{h(1-F(\phi(x)))} \mathrm{d}x$. Then ζ is differentiable with $\zeta'(z) = \frac{\theta(z)}{h(1-F(\phi(z)))}$ for each $z \in [z_1, \bar{z}]$. Since $\phi = \lambda \circ \zeta$, it follows that ϕ is differentiable and that, for each $z \in [z_1, \bar{z}]$, $\phi'(z) = \frac{\theta(z)}{h(1-F(\phi(z)))\theta(\lambda(\zeta(z)))}$. Since $\zeta(z) = \nu(\phi(z))$ by Lemma 4.29, we obtain that $\lambda(\zeta(z)) = \phi(z)$ and, hence, $\phi'(z) = \frac{\theta(z)}{h(1-F(\phi(z)))\theta(\phi(z))}$. ■

Lemma 4.31 $c(z_1) = (F(z_1) - c(0))n(0) > 0$.

Proof. We have that $\phi(\bar{z}) = z_1$, $\phi(z_1) = 0$ and $z_1 \in W \cap M$. It follows from Lemma 4.25 that $U_{z_1}(m, 0) = (F(z_1) - c(0))n(0) > 0$.

Suppose that $c(z_1) > (F(z_1) - c(0))n(0)$ and let $\varepsilon > 0$ be such that $c(z_1) - \varepsilon > (F(z_1) - c(0))n(0)$. Then $(z_1, c(z_1) - \varepsilon) \in T_{\bar{z}}^{m}(\mu)$ (since $z_1 \in M$) and

$$U_{\bar{z}}(m, n(z_1) 1_{(z_1, c(z_1) - \varepsilon)}) > U_{\bar{z}}(m, z_1).$$

But this contradicts the stability of μ since $\bar{z} \in M$. Hence $c(z_1) \leq (F(z_1) - c(0))n(0)$.

Suppose that $(F(z_1) - c(0))n(0) > c(z_1)$ and let $\varepsilon > 0$ be such that $(F(z_1) - c(0) - \varepsilon)n(0) > c(z_1)$. Then $(0, c(0)+\varepsilon) \in T^m_{z_1}(\mu)$ (since $0 \in W$) and $U_{z_1}(m, n(0)1_{(0,c(0)+\varepsilon)}) > U_{z_1}(w, 1_{(\bar{z},c(z_1))})$. But this contradicts the stability of μ since $z_1 \in W$. Hence $(F(z_1) - c(0))n(0) \leq c(z_1)$ and, thus, $(F(z_1) - c(0))n(0) = c(z_1)$. ■

The necessity part of the characterization of the stable matchings of the economy E_s in Theorem 5 then follows by Lemmas 4.17, 4.19, 4.20, 4.24, 4.26–4.28, 4.30 and 4.31.

4.3 Sufficiency part

Lemma 4.32 *If (z_1, ϕ, c) and $(\hat{z}_1, \hat{\phi}, \hat{c})$ satisfy conditions 1–6 in Theorem 5, then $(z_1, \phi, c) = (\hat{z}_1, \hat{\phi}, \hat{c})$.*

Proof. Let (z_1, ϕ, c) and $(\hat{z}_1, \hat{\phi}, \hat{c})$ satisfy conditions 1–6 in Theorem 5.

Recall that $\nu(z) = \nu([0, z]) = \int_0^z \theta(x)\mathrm{d}x$ for each $z \in Z$. Thus, $\nu'(z) = \theta(z)$ for each $z \in Z$.

Claim 1 *For each $z \in [z_1, \bar{z}]$, $\nu(\phi(z)) = \int_{z_1}^z \frac{\theta(x)}{h(1-F(\phi(x)))}\mathrm{d}x$.*

Proof. For each $z \in [z_1, \bar{z}]$, $\nu \circ \phi'(z) = \theta(\phi(z))\phi'(z) = \frac{\theta(z)}{h(1-F(\phi(z)))}$. Since $\nu(\phi(z_1)) = \nu(0) = 0$, it follows that $\nu(\phi(z)) = \int_{z_1}^z \nu \circ \phi'(x)\mathrm{d}x = \int_{z_1}^z \frac{\theta(x)}{h(1-F(\phi(x)))}\mathrm{d}x$ for each $z \in [z_1, \bar{z}]$. ■

Claim 2 *If $z_1 = \hat{z}_1$, then $\phi = \hat{\phi}$ and $c = \hat{c}$.*

Proof. Let $G : [z_1, \bar{z}] \times [0, z_1] \to \mathbb{R}$ be such that $G(z, x) = \frac{\theta(z)}{h(1-F(x))\theta(x)}$ for each $(z, x) \in [z_1, \bar{z}] \times [0, z_1]$. Then the conditions of Lemma 44 hold with ϕ and $\hat{\phi}$ being solutions to the initial value problems and $z_0 = z_1$, the latter since $\phi(z_1) = \hat{\phi}(z_1) = 0$. Then $\phi = \hat{\phi}$.

We next show that $c = \hat{c}$ and start by establishing that $\{z \in [0, z_1] : c(z) = \hat{c}(z)\} \neq \emptyset$. Suppose not; then $\{z \in [0, z_1] : c(z) = \hat{c}(z)\} = \emptyset$. Then $c(0) \neq \hat{c}(0)$ and, since c and $\hat{c}$ are arbitrary, we may assume that $c(0) > \hat{c}(0)$. Then $c(z_1) > \hat{c}(z_1)$ by the intermediate value theorem since c and $\hat{c}$ are continuous. But then

$$c(z_1) > \hat{c}(z_1) = (F(z_1) - \hat{c}(0))n(0) > (F(z_1) - c(0))n(0) = c(z_1),$$

a contradiction. This contradiction then shows that $\{z \in [0, z_1] : c(z) = \hat{c}(z)\} \neq \emptyset$.

Let $G : [0, z_1] \times [0, 1] \to \mathbb{R}$ be such that $G(z, x) = f(z)\frac{F(\phi^{-1}(z))-x}{1-F(z)}$ for each $(z, x) \in [0, z_1] \times [0, 1]$. Then the conditions of Lemma 44 hold with c and $\hat{c}$ being solutions to the initial value problems and $z_0 \in \{z \in [0, z_1] : c(z) = \hat{c}(z)\}$. Hence, $c = \hat{c}$. ■

It follows by Claim 2 that it suffices to show that $z_1 = \hat{z}_1$. Suppose not; then $z_1 \neq \hat{z}_1$ and, since z_1 and $\hat{z}_1$ are arbitrary, we may assume that $z_1 > \hat{z}_1$.

The function $\phi : [z_1, \bar{z}] \to [0, z_1]$ is strictly increasing (condition 3). Hence, let $\varphi : [0, z_1] \to [z_1, \bar{z}]$ be the inverse of ϕ. Then φ is strictly increasing.

We have that $\varphi(0) = z_1 > \hat{z}_1 = \hat{\varphi}(0)$. Assume first that $\{z \in [0, \hat{z}_1] : \varphi(z) = \hat{\varphi}(z)\} \neq \emptyset$. Let $z_0 = \inf\{z \in [0, \hat{z}_1] : \varphi(z) = \hat{\varphi}(z)\}$. Then $\varphi(z_0) = \hat{\varphi}(z_0)$ since φ and $\hat{\varphi}$ are continuous. Thus, $z_0 > 0$.

The definition of z_0, the continuity of both φ and $\hat{\varphi}$ and the intermediate value theorem imply that $\varphi(z) > \hat{\varphi}(z)$ for each $z \in [0, z_0)$. This then implies that $\phi(z) < \hat{\phi}(z)$ for each $z \in [z_1, \varphi(z_0))$. Indeed, $\phi(z_1) = 0 < \hat{\phi}(z_1)$ since otherwise $\hat{\varphi}(0) = z_1$ and, hence, $\hat{z}_1 = \hat{\varphi}(0) = z_1$. Let $z \in (z_1, \varphi(z_0))$ and let x, x' be such that $z = \varphi(x) = \hat{\varphi}(x')$. Since $\varphi(x) > \hat{\varphi}(x)$, it follows that $\hat{\varphi}(x') > \hat{\varphi}(x)$. Then $x' > x$ since $\hat{\varphi}$ is strictly increasing and, thus, $\hat{\phi}(z) = x' > x = \phi(z)$.

It then follows by the above that, for each $z \in [z_1, \varphi(z_0))$,

$$\frac{\theta(z)}{h(1 - F(\phi(z)))} < \frac{\theta(z)}{h(1 - F(\hat{\phi}(z)))}.$$

Thus, by Claim 1,

$$\begin{aligned}
\nu(z_0) = \nu(\phi(\varphi(z_0))) &= \int_{z_1}^{\varphi(z_0)} \frac{\theta(x)}{h(1 - F(\phi(x)))} \mathrm{d}x \\
&< \int_{\hat{z}_1}^{z_1} \frac{\theta(x)}{h(1 - F(\hat{\phi}(x)))} \mathrm{d}x + \int_{z_1}^{\varphi(z_0)} \frac{\theta(x)}{h(1 - F(\hat{\phi}(x)))} \mathrm{d}x \\
&= \int_{\hat{z}_1}^{\hat{\varphi}(z_0)} \frac{\theta(x)}{h(1 - F(\hat{\phi}(x)))} \mathrm{d}x \\
&= \nu(\hat{\phi}(\hat{\varphi}(z_0))) = \nu(z_0),
\end{aligned}$$

a contradiction. This contradiction shows that $\{z \in [0, \hat{z}_1] : \varphi(z) = \hat{\varphi}(z)\} = \emptyset$.

It then follows from $\{z \in [0, \hat{z}_1] : \varphi(z) = \hat{\varphi}(z)\} = \emptyset$, together with the continuity of φ and $\hat{\varphi}$ and $\varphi(0) > \hat{\varphi}(0)$, that $\varphi(z) > \hat{\varphi}(z)$ for each $z \in [0, \hat{z}_1]$ by the intermediate value theorem. Hence, $\varphi(\hat{z}_1) > \hat{\varphi}(\hat{z}_1) = \bar{z}$, a contradiction. This contradiction shows that $z_1 = \hat{z}_1$ and completes the proof of Lemma 4.32. ■

Suppose that there exists (z_1, ϕ, c) such that the conditions 1–6 in the theorem hold and let $\mu = \nu \circ \sigma^{-1}$. Let, by the existence part of Theorem 5, $\hat{\mu}$ be a stable matching and, by the necessity part of Theorem 5, $(\hat{z}_1, \hat{\phi}, \hat{c})$ be such that the conditions 1–6 in the theorem hold and such that $\hat{\mu} = \nu \circ \hat{\sigma}^{-1}$, where $\hat{\sigma} : [\hat{z}_1, \bar{z}] \to Z \times X$ is defined as σ is but with $(\hat{z}_1, \hat{\phi}, \hat{c})$ in place of (z_1, ϕ, c). It then follows by Lemma 4.32 that $(z_1, \phi, c) = (\hat{z}_1, \hat{\phi}, \hat{c})$ and, hence, $\sigma = \hat{\sigma}$. Thus $\mu = \nu \circ \sigma^{-1} = \hat{\mu}$ and μ is a stable matching.

4.4 Uniqueness part

Let μ and $\hat{\mu}$ be stable matchings of E_s and, by Theorem 5, let μ be represented by (z_1, ϕ, c) and $\hat{\mu}$ by $(\hat{z}_1, \hat{\phi}, \hat{c})$; in particular, $\mu = \nu \circ \sigma^{-1}$ and $\hat{\mu} = \nu \circ \hat{\sigma}^{-1}$, where $\hat{\sigma}$ is defined as σ is but with $(\hat{z}_1, \hat{\phi}, \hat{c})$ in place of (z_1, ϕ, c). It then follows by Lemma 4.32 that $(z_1, \phi, c) = (\hat{z}_1, \hat{\phi}, \hat{c})$ and, hence, $\sigma = \hat{\sigma}$. Thus $\mu = \nu \circ \sigma^{-1} = \hat{\mu}$.

5 Allowing $r(0) = 0$ in Rosen markets

The assumption that $r(0) = 0$ is used only in the proof of Corollary 3 in Carmona and Laohakunakorn (2024). Without this assumption, Claim 11 still goes through by noting that it is sufficient to show $\operatorname{supp}(\mu_k) \cap ((Z \setminus \{0\}) \times X_\emptyset) \subseteq Z \times X$ when ν is atomless.

Claim 12 uses $r(0) > 0$ to show that if $(z_{k_j}, \delta_{k_j}) \in \operatorname{supp}(\mu_{k_j})$, $(z'_{k,j}, c_{k_j}) \in \operatorname{supp}(\delta_{k_j})$, and $c_{k_j} \to 0$, then $(z'_{k,j}, c_{k_j} + \varepsilon) \in T^m_{z'_{k,j}}(\mu_{k_j})$ and $z'_{k,j}$ has a profitable deviation to hire $z'_{k,j}$ if $r(z'_{k,j}) > 0$. This argument does not work if $z'_{k,j} = 0$ and $r(0) = 0$. But it is easy to see that, given the specification of $E_{r,\alpha}$, the wage must be the same for all types of workers in any stable matching; hence, if $c_{k_j} \to 0$, then we can pick any worker $z' > 0$ and show that z' has a profitable deviation to hire z'.

Letting c_k be the common wage in μ_k, the above argument also implies that there is $\underline{c} > 0$ such that $c_k \geq c$ for all k sufficiently large. Then $\underline{c}$ is also a lower bound on the manager's payoff, since if there is a manager making less than $\underline{c}$, he could be hired at a lower cost than any existing worker. Thus, in defining the element of $\bar{M}$ that is the (inverse) of the lower bound of the manager's payoff, we can use $1/\underline{c}$ instead of $\frac{2}{g(r(0))\theta(r(0))}$.

6 An example of a Rosen market

We consider the example of $E_{r,\alpha}$ in which $\theta \equiv 1$ and $F(z) = z$ for each $z \in Z$. In this case, for each $w > 0$ and $z \in Z$,

$$n(z, w) = \left(\frac{1-\alpha}{w}\right)^{\frac{1}{\alpha}} z^{\frac{1+\alpha}{\alpha}}, \text{ and}$$
$$wn(z, w) = (1-\alpha)z^{1+\alpha}n(z, w)^{1-\alpha}.$$

This implies that

$$F(z_1)z_1^{\alpha}n(z_1, w)^{1-\alpha} - wn(z_1, w) = \frac{wn(z_1, w)}{1-\alpha} - wn_1(z_1, w) = \frac{\alpha}{1-\alpha}wn(z_1, w).$$

It then follows by condition 4 in Theorem 4 that

$$\frac{\alpha}{1-\alpha}wn(z_1, w) = w \Leftrightarrow n(z_1, w) = \frac{1-\alpha}{\alpha} \Leftrightarrow w = \alpha^{\alpha}(1-\alpha)^{1-\alpha}z_1^{1+\alpha}.$$

Recall from the proof of Theorem 4 that $\nu([0, z_1]) = \int_{[z_1,\bar{z}]} n(z, w)\mathrm{d}\nu(z)$. Since $\nu([0, z_1]) = z_1$ and

$$\int_{[z_1,\bar{z}]} n(z, w)\mathrm{d}\nu(z) = \left(\frac{1-\alpha}{w}\right)^{\frac{1}{\alpha}} \int_{z_1}^{\bar{z}} z^{\frac{1+\alpha}{\alpha}}\mathrm{d}z = \left(\frac{1-\alpha}{w}\right)^{\frac{1}{\alpha}} \frac{\alpha}{1+2\alpha}\left(\bar{z}^{\frac{1+2\alpha}{\alpha}} - z_1^{\frac{1+2\alpha}{\alpha}}\right),$$

it follows that

$$z_1 = \left(\frac{1-\alpha}{w}\right)^{\frac{1}{\alpha}} \frac{\alpha}{1+2\alpha}\left(\bar{z}^{\frac{1+2\alpha}{\alpha}} - z_1^{\frac{1+2\alpha}{\alpha}}\right). \tag{6.1}$$

By substituting $w = \alpha^{\alpha}(1-\alpha)^{1-\alpha}z_1^{1+\alpha}$ in (6.1) and then solving it, we obtain that

$$z_1 = \left(\frac{1-\alpha}{2+\alpha}\right)^{\frac{\alpha}{1+2\alpha}} \bar{z}.$$

The payment u for each individual is

$$u(z) = \begin{cases} w & \text{if } z < z_1, \\ \alpha \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z^{\frac{1+\alpha}{\alpha}} & \text{if } z \geq z_1. \end{cases}$$

The expression for $u(z)$ when $z \geq z_1$ is obtained by noting that, if $z \geq z_1$,

$$u(z) = z^{1+\alpha} n(z,w)^{1-\alpha} - wn(z,w) = \alpha z^{1+\alpha} n(z,w)^{1-\alpha}.$$

It follows that, as $\alpha \to 0$, $z_1 \to \bar{z}$ and $w \to \bar{z}$. Furthermore, by substituting $w = \alpha^\alpha (1-\alpha)^{1-\alpha} z_1^{1+\alpha}$ in $u(\bar{z})$, it follows that

$$u(\bar{z}) = \alpha^\alpha (1-\alpha)^{1-\alpha} \left(\frac{\bar{z}}{z_1}\right)^{\frac{1+\alpha}{\alpha}} z_1^{1+\alpha}.$$

Using $z_1 = \left(\frac{1-\alpha}{2+\alpha}\right)^{\frac{\alpha}{1+2\alpha}} \bar{z}$ to substitute for $\frac{\bar{z}}{z_1}$, it follows that

$$u(\bar{z}) = \alpha^\alpha (1-\alpha)^{1-\alpha} \left(\frac{2+\alpha}{1-\alpha}\right)^{\frac{1+\alpha}{1+2\alpha}} z_1^{1+\alpha}.$$

Hence, $u(\bar{z}) \to 2\bar{z}$ as $\alpha \to 0$.

7 Earnings in the different markets

We plot the earning function for E_{grh}, E_s and $E_{r,\alpha}$ to illustrate their differences in the case where $f = \theta \equiv 1$ and $\bar{z}$ is arbitrary. Then the unique stable matching in E_{grh} is represented by (z_1, z_2, ϕ, c) such that

$$\phi(z) = 1 - \sqrt{1 - \frac{2(z - z_2)}{h}} \text{ for each } z \in [z_2, \bar{z}],$$
$$c(z) = c(0) + (z_2 - c(0))z + \frac{hz^2}{2} \text{ for each } z \in [0, z_1],$$

with

$$z_1 = z_2,$$
$$z_2 = 1 + \frac{1}{h} - \sqrt{1 + \frac{1}{h^2} + \frac{2(1-\bar{z})}{h}},$$
$$c(0) = \frac{z_2\left(1 - h\frac{2+h}{2} z_2\right)}{1 + h(1 - z_2)}$$

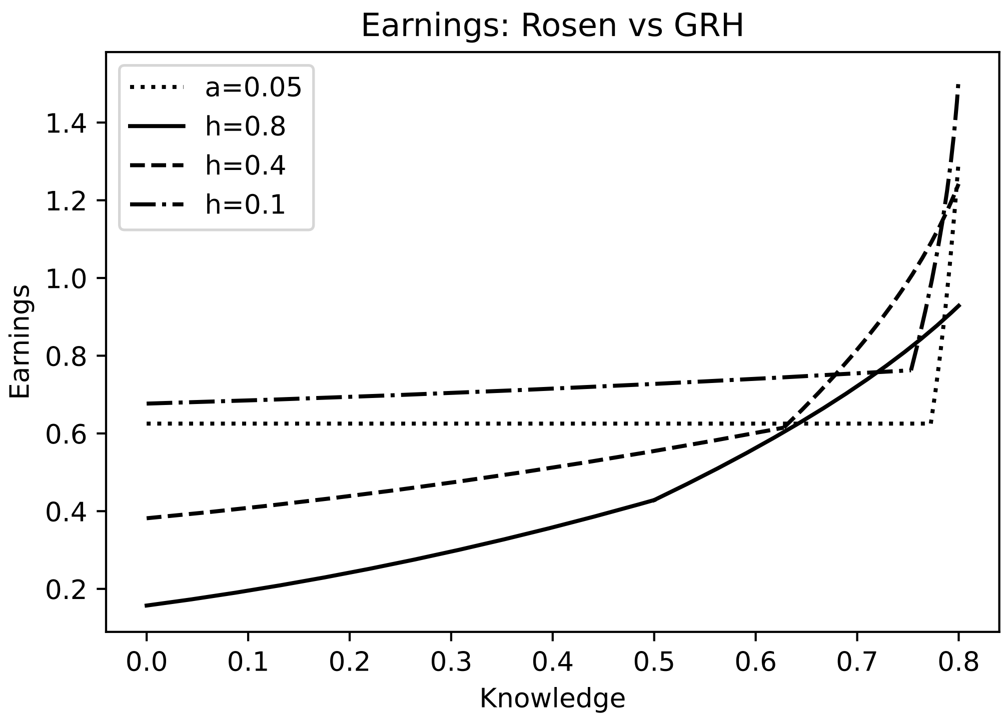

Figure 1: Earning function of $E_{r,\alpha}$ and E_{grh} with $\bar{z} = 0.8$, $\alpha = 0.05$ and $h \in \{0.1, 0.2, 0.8\}$.

if $0 < h \leq \frac{3}{8\bar{z}-4}$ and

$$\begin{aligned}
z_1 &= 1 - \sqrt{1 - \frac{2(\bar{z} - z_2)}{h}}, \\
z_2 &= \frac{2-h}{h} - \sqrt{\frac{3 - (4 - 2(1-\bar{z}))h + h^2}{h^2}}, \\
c(0) &= (1-h)z_2
\end{aligned}$$

if $\frac{3}{8\bar{z}-4} < h < 1$. Furthermore, the unique stable matching in E_s is represented by (z_1, z_2, ϕ, c) in the $z_1 = z_2$ case for each h satisfying (7).

We consider $\bar{z} = 0.8$ and $h \leq 0.8$. This implies that (7) is satisfied and that the stable matching of E_{grh} equals that of E_s. Figure 1 plots the earning function u for $E_{r,\alpha}$ with $\alpha = 0.05$ and that of E_{grh} (and E_s) when $h \in \{0.1, 0.2, 0.8\}$.

We also consider the case $\bar{z} = 1$. Figure 2 plots the stable matching of E_{grh} for $h \in \{0.5, 0.65, 0.8, 0.95\}$.

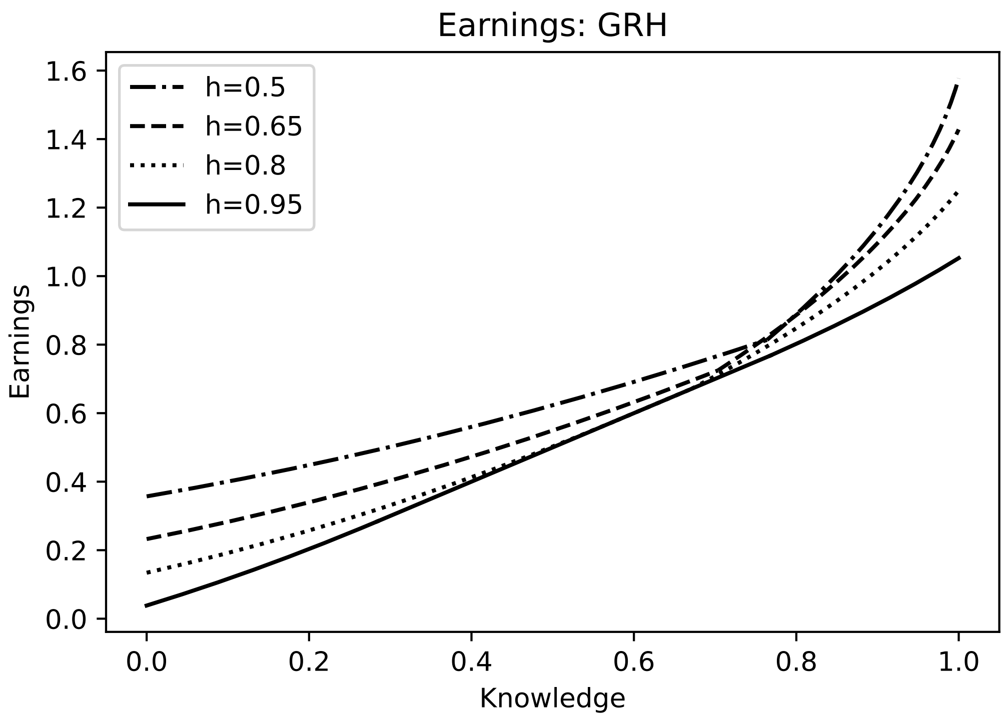

Figure 2: Earning function of E_{grh} with $\bar{z} = 1$ and $h \in \{0.5, 0.65, 0.8, 0.95\}$.

8 Codes

Figures 1 and 2 were produced with `earnings.py` and `earningsgrh.py` respectively. These files use `rosenfunction.py`, `agrhfunction.py` and `grhfunction.py`.

8.1 rosenfunction.py

```
def rosen(a,b):


    z1=b*(((1-a)/(2+a))**(a/(1+2*a)))
    w=(a**a)*((1-a)**(1-a))*(z1**(1+a))


    return [z1,w]


def rosenpay(a,b,z):
```

```
    z1=rosen(a,b)[0]
    w=rosen(a,b)[1]
    if z <= z1:
        return w
    else:
        return a*(((1-a)/w)**((1-a)/a))*(z**((1+a)/a))

def rosenpayw(a,b,z):
    return rosen(a,b)[1]

def rosenpaym(a,b,z):
    w=rosen(a,b)[1]
    return a*(((1-a)/w)**((1-a)/a))*(z**((1+a)/a))

def rosenn(a,b,z):
    z1=rosen(a,b)[0]
    w=rosen(a,b)[1]
    if z <= z1:
        return 0
    else:
        return (((1-a)/w)**(1/a))*(z**((1+a)/a))
```

8.2 agrhfunction.py

```
import numpy as np

def agrh(h,b):
    z1=1+1/h-np.sqrt(1+1/(h**2)+(2*(1-b))/h)
    c0=(z1*(1-h*(2+h)*z1/2))/(1+h*(1-z1))
    return [z1,c0]
```

```
def agrhphi(h,b,z):
    z1=agrh(h,b)[0]
    return 1-np.sqrt(1-2*(z-z1)/h)

def agrhc(h,b,z):
    z1=agrh(h,b)[0]
    c0=agrh(h,b)[1]
    return c0+(z1-c0)*z+(h*(z**2))/2

def agrhpay(h,b,z):
    z1=agrh(h,b)[0]
    if z <= z1:
        return agrhc(h,b,z)
    else:
        x=agrhphi(h,b,z)
        c=agrhc(h,b,x)
        n=1/(h*(1-x))
        return (z-c)*n

def agrhpaym(h,b,z):
    x=agrhphi(h,b,z)
    c=agrhc(h,b,x)
    n=1/(h*(1-x))
    return (z-c)*n

def agrhn(h,b,z):
    z1=agrh(h,b)[0]
    if z <= z1:
        return 0
    else:
```

```
        return 1/(h*(1-agrhphi(h,b,z)))
```

8.3 grhfunction.py

```
import numpy as np


def grh(h,b):
    z2=(2-h)/h-np.sqrt(3-4*h+2*(1-b)*h+h**2)/h
    z1=1-np.sqrt(1-(2*(b-z2))/h)
    c0=(1-h)*z2
    return [z1,z2,c0]


def grhphi(h,b,z):
    z2=grh(h,b)[1]
    return 1-np.sqrt(1-2*(z-z2)/h)


def grhc(h,b,z):
    z2=grh(h,b)[1]
    c0=grh(h,b)[2]
    return c0+(z2-c0)*z+(h*(z**2))/2


def grhpaym(h,b,z):
    x=grhphi(h,b,z)
    c=grhc(h,b,x)
    n=1/(h*(1-x))
    return (z-c)*n


def grhpay(h,b,z):
    z1=grh(h,b)[0]
    z2=grh(h,b)[1]
    if z <= z1:
```

```
        return grhc(h,b,z)
    elif z< z2:
        return z
    else:
        x=grhphi(h,b,z)
        c=grhc(h,b,x)
        n=1/(h*(1-x))
        return (z-c)*n

def grhn(h,b,z):
    z2=grh(h,b)[1]
    if z <= z2:
        return 0
    else:
        return 1/(h*(1-grhphi(h,b,z)))
```

8.4 earnings.py

```
import pylab
import numpy as np
import rosenfunction as r, agrhfunction as s

m=0.0001
a=0.05
b=0.8
h1=0.8
h2=0.4
h3=0.1

zr1=np.arange(0,r.rosen(a,b)[0],m)
zr2=np.arange(r.rosen(a,b)[0],b,m)
```

```
zs1=np.arange(0,s.agrh(h1,b)[0],m)
zs2=np.arange(s.agrh(h1,b)[0],b,m)

zs3=np.arange(0,s.agrh(h2,b)[0],m)
zs4=np.arange(s.agrh(h2,b)[0],b,m)

zs5=np.arange(0,s.agrh(h3,b)[0],m)
zs6=np.arange(s.agrh(h3,b)[0],b,m)

def rw(z):
    return r.rosenpayw(a,b,z)
def rm(z):
    return r.rosenpaym(a,b,z)

def sw1(z):
    return s.agrhc(h1,b,z)
def sm1(z):
    return s.agrhpaym(h1,b,z)

def sw2(z):
    return s.agrhc(h2,b,z)
def sm2(z):
    return s.agrhpaym(h2,b,z)

def sw3(z):
    return s.agrhc(h3,b,z)
def sm3(z):
    return s.agrhpaym(h3,b,z)
```

```
pylab.figure(1)
pylab.plot(zr1,rw(zr1)+zr1-zr1,'k:')
pylab.plot(zr2,rm(zr2),'k:',label='a=0.05')
pylab.plot(zs1,sw1(zs1),'k-')
pylab.plot(zs2,sm1(zs2),'k-',label='h=0.8')
pylab.plot(zs3,sw2(zs3),'k--')
pylab.plot(zs4,sm2(zs4),'k--',label='h=0.4')
pylab.plot(zs5,sw3(zs5),'k-.')
pylab.plot(zs6,sm3(zs6),'k-.',label='h=0.1')
pylab.title('Earnings: Rosen vs GRH')
pylab.xlabel('Knowledge')
pylab.ylabel('Earnings')
pylab.legend()
pylab.savefig('earnings.pdf')
pylab.show()
```

8.5 earningsgrh.py

```
import pylab
import numpy as np
import agrhfunction as s, grhfunction as g

m=0.0001
b=1
h0=0.95
h1=0.8
h2=0.65
h3=0.5

xa0=np.arange(0,g.grh(h0,b)[0],m)
xb0=np.arange(g.grh(h0,b)[0],g.grh(h0,b)[1],m)
```

```
xc0=np.arange(g.grh(h0,b)[1],b,m)

xa1=np.arange(0,g.grh(h1,b)[0],m)
xb1=np.arange(g.grh(h1,b)[0],g.grh(h1,b)[1],m)
xc1=np.arange(g.grh(h1,b)[1],b,m)

xa2=np.arange(0,s.agrh(h2,b)[0],m)
xb2=np.arange(s.agrh(h2,b)[0],b,m)

xa3=np.arange(0,s.agrh(h3,b)[0],m)
xb3=np.arange(s.agrh(h3,b)[0],b,m)

def w0(z):
    return g.grhc(h0,b,z)
def m0(z):
    return g.grhpaym(h0,b,z)

def w1(z):
    return g.grhc(h1,b,z)
def m1(z):
    return g.grhpaym(h1,b,z)

def w2(z):
    return s.agrhc(h2,b,z)
def m2(z):
    return s.agrhpaym(h2,b,z)

def w3(z):
    return s.agrhc(h3,b,z)
def m3(z):
```

```
    return s.agrhpaym(h3,b,z)

pylab.figure(1)


pylab.plot(xa3,w3(xa3),'k-.')
pylab.plot(xb3,m3(xb3),'k-.',label='h=0.5')
pylab.plot(xa2,w2(xa2),'k--')
pylab.plot(xb2,m2(xb2),'k--',label='h=0.65')
pylab.plot(xa1,w1(xa1),'k:')
pylab.plot(xb1,xb1,'k:')
pylab.plot(xc1,m1(xc1),'k:',label='h=0.8')
pylab.plot(xa0,w0(xa0),'k-')
pylab.plot(xb0,xb0,'k-')
pylab.plot(xc0,m0(xc0),'k-',label='h=0.95')
pylab.title('Earnings: GRH')
pylab.xlabel('Knowledge')
pylab.ylabel('Earnings')
pylab.legend()
pylab.savefig('earnings4h.pdf')
pylab.show()
```

References

Carmona, G., and K. Laohakunakorn (2024): "Stable Matching in Large Markets with Occupational Choice," *Theoretical Economics*, 19, 1261–1304.