Knowledge Economies*

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Abstract

We develop a general framework for knowledge economies which feature occupational choice between workers and managers and many-to-one matching of workers to managers into firms whose production depends on members' knowledge. The framework is flexible and nests several existing models as special cases. We establish existence of a stable outcome and show that, in our setting, stability is equivalent to competitive equilibrium and surplus maximization. We provide sufficient conditions under which stable outcomes admit a sharp characterization featuring positive assortativeness, occupation stratification, and strictly increasing earnings, and we show that these conditions hold in a range of knowledge economies. Quantitatively, the model in Garicano and Rossi-Hansberg (2004) accounts for U.S. wage polarization over 1988-2008, but a more flexible model is needed to account for both wage polarization and the U.S. establishment-size distribution. The flexible economy also replicates top income shares and their evolution remarkably well.

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1 Introduction

Understanding important economic issues, such as the wage and firm size distributions and their evolution, requires studying the internal organization of firms in a market setting. This point has been convincingly made in Lucas (1978), Rosen (1982), Garicano and Rossi-Hansberg (2004) and Garicano and Rossi-Hansberg (2006), which propose different ways of incorporating organization into competitive models. A common feature of these models is that production depends explicitly on the knowledge (or talent or ability) of the agents involved; we therefore refer to them as *knowledge economies*. Knowledge economies have been used to shed light on central issues—ranging from offshoring to the impact of artificial intelligence—yet, as noted by Rosen (2002) and Garicano and Rossi-Hansberg (2015), their use has been limited, in part because of the absence of a unified framework.

This paper develops a general framework for knowledge economies that is flexible enough to encompass existing models as special cases. It also allows for new knowledge economies by combining elements from different settings and by introducing novel ones. The knowledge-economy model is tractable: under considerable generality, we show that it admits a solution and provide a method for computing it. The paper therefore broadens the applicability of knowledge economies and lowers the barrier to using them.

The knowledge economies we consider are defined succinctly by (i) a set of knowledge levels, (ii) a distribution over these levels, (iii) a production function determining the output generated by managers and workers of different knowledge levels, (iv) bounds on the firm size, and (v) the payoffs for self-employed individuals. Our solution concept generalizes the standard definition of stable matching in transferable utility settings (see, e.g. Chiappori (2017)) to allow for many-to-one matching and occupational choice. A stable outcome consists of an allocation that specifies, for each knowledge level, how many individuals become managers of each possible workforce (described by the knowledge and number of workers), and a continuous earnings function that assigns a value to each knowledge level. We show that stable outcomes exist under mild assumptions. We also show that the allocation solves an optimal transport problem and the earnings function solves its dual. This result underpins our computational method. Finally, we show that the notion of competitive equilibrium applied to knowledge economies is equivalent to our notion of stability.

Our general framework nests various existing models as special cases and can therefore be used to study a wide range of questions, such as offshoring, sectoral variation in inequality, and artificial intelligence, as Antràs, Garicano, and Rossi-Hansberg (2006), Gola (2021), and Ide and Talamàs (2025) have already demonstrated. It also enables the exploration of variations of these models to address these as well as new questions. Moreover, the framework has the potential to accommodate new elements, including alternative functional forms for firm-size bounds and production technologies recently proposed by Adenbaum (2023), Freund (2023) and Carmona and Laohakunakorn (2024a).

We use the flexibility and tractability of our framework to study worker-manager matching and the wage distribution, both theoretically and empirically. In a model based on the organizational theory of Garicano (2000), Garicano and Rossi-Hansberg (2004) show that equilibrium has empirically relevant properties: earnings are strictly increasing in knowledge, matching is strictly positive assortative (i.e. more knowledgeable workers are hired by more knowledgeable managers), and the equilibrium features occupational stratification (i.e. workers are the least knowledgeable individuals, followed by the self-employed and then by the managers). We provide sufficient conditions for these properties to hold in a general knowledge economy and show that they hold in several cases, such as one-dimensional versions of Mak and Siow (2025), several variations of Garicano and Rossi-Hansberg (2004) and a modification of Rosen (1982) with bounded span of control and production linear in span of control.

One application of these results is to a knowledge economy in which neither production nor span of control depends on worker knowledge. With bounded span of control, the solution shares key properties with Garicano and Rossi-Hansberg (2004). This contrasts with the unbounded case of Rosen (1982), in which workers earn a constant wage and worker–manager matching is indeterminate. Why does Garicano and Rossi-Hansberg's (2004) knowledge economy deliver implications so different from Rosen's (1982)? Placing both economies in our unified framework shows that they differ only in three elements: the labor share in the production function, self-employment payoffs, and the number of workers each manager can hire. We decompose these differences by considering two intermediate knowledge economies that move from one model to the other by changing one element at a time. This decomposition exercise reveals that the qualitative gap between the two economies is driven by bounded versus unbounded span of control.[1]

The theoretical analysis above implies that, with bounded span of control, the knowledge-economy model can explain real-world worker-manager matching and the wage distribution. We thus evaluate whether it can quantitatively match trends in U.S. earnings inequality. We focus on the wage polarization documented by Acemoglu and Autor (2011): between 1988 and 2008, earnings percentiles below and above the median rise relative to the median, producing a U-shaped pattern, in stark contrast to the broadly increasing profile in 1974–1988 (see Acemoglu and Autor (2011, Figure 9)). We show that the knowledge economy of Garicano and Rossi-Hansberg (2004) accounts well for the wage polarization when the most knowledgeable managers become relatively more productive than the least knowledgeable ones and communication costs decrease between 1988 and 2008. This result echoes the emphasis in Caicedo, Lucas, and Rossi-Hansberg (2019) on technology becoming more complex relative to available knowledge but less so for more knowledgeable managers, i.e. in a skill-biased manner, and aligns with Garicano and Rossi-Hansberg (2015) in highlighting the role of declining communication costs in wage polarization.

However, the Garicano and Rossi-Hansberg (2004) economy cannot match the U.S. firm-size distribution, as it generates too little dispersion in establishment size. We therefore introduce a more flexible economy that can jointly match wage polarization and the establishment-size distribution through changes in production and span-of-control technologies. As in Acemoglu and Autor (2011), we target only the 5th to 95th percentiles of the wage distribution; nevertheless, the flexible economy also matches the untargeted top 1% income share and its change remarkably well. This suggests that matching the establishment-size distribution can also help account for top income shares, i.e. that large inequality requires highly knowledgeable individuals to leverage their knowledge through large firm sizes. Wage polarization is explained by three technological changes that are often highlighted in the

[1] In Garicano and Rossi-Hansberg (2004), limits to firm size arise from the time cost of communication between different members of a firm and, more broadly, from what Becker and Murphy (1992) term coordination costs. Our results demonstrate that coordination costs shape not only the pattern of specialization (as Becker and Murphy (1992) emphasized) but also, through their effect on the limits to firm size, the matching of individuals skills and wages in a more realistic way as compared to when they are absent.

literature: a skill-biased technological change that increases the relative productivity of the best managers, lower communication costs (an ICT channel) and a decline in the sensitivity of span of control to worker knowledge (consistent with task routinization). Relative to Garicano and Rossi-Hansberg (2004), span of control in the flexible economy is more sensitive to worker knowledge, which is key to jointly matching wage polarization and the establishment-size distribution. To our knowledge, this is the first quantitative attempt to use a knowledge-economy model to explain the establishment-size distribution, an application envisioned by Lucas (1978). More broadly, the result shows that adding a new element to an existing model—an extension that our general framework accommodates easily—makes knowledge-economy models more informative about why organization theory matters economically beyond wages.

Related Literature

Our paper builds on Carmona and Laohakunakorn (2024b), which embeds several canonical knowledge economies in a unifying framework of large matching markets. We advance this agenda by deriving results that provide a toolkit for analyzing general knowledge economies within this framework. The key step is to exploit transferable utility: when a worker matches with a manager, the wage is a transfer from the manager to the worker. This allows us to show that stable outcomes, defined for transferable utility settings analogously to Chiappori (2017, Section 2.1) but allowing many-to-one matching and occupational choice, are equivalent to the stable matchings of Carmona and Laohakunakorn (2024b). We also show that stable outcomes solve an optimal transport problem and its dual, which underpins our computational method.[2] We then build on Beiglböck and Griessler (2019) to establish a condition that the support of any solution to our optimal transport problem must satisfy, which greatly facilitates the characterization of such solutions.

Our approach also has a conceptual advantage: it applies a single solution concept,

[2] Our optimal transport problem differs from the one in Chiappori (2017, Theorem 1, p. 45) in its constraint; with occupational choice, marginal distributions of the assignment are not fixed as in the optimal transport problem of Chiappori (2017); instead, the requirement that an assignment needs to satisfy in our setting is that the sum of the measures of managers, self-employed and workers must equal the given measure describing the distributions of skills in the population. Related results appear in Gretsky, Ostroy, and Zame (1992), Gretsky, Ostroy, and Zame (1999), Villani (2009), Chiappori, McCann, and Nesheim (2010) and McCann and Trokhimtchouk (2010), but none of them allow for the simultaneous presence of many-to-one matching and occupational choice.

defined broadly for a general framework featuring many-to-one matching of a large number of individuals and occupational choice, across knowledge economies. Several previous work, such as Garicano and Rossi-Hansberg (2004) and Antràs, Garicano, and Rossi-Hansberg (2006), dealt with the technical challenges by defining an intuitive notion of competitive equilibrium tailored to the specific setting.[3] This complicates comparisons across models, because differences in conclusions may come from differences in the solution concept. We address this issue by defining a general notion of competitive equilibrium for knowledge economies in a general equilibrium setting that combines the classical framework of Debreu's (1959) Theory of Value, Hildenbrand (1974) (to allows for a continuum of individuals), and Jones (1984) (to allow for differentiated labor inputs by knowledge).[4] We then show that competitive equilibria are fully described by stable outcomes thus rendering both notions—and stable matchings—equivalent solution concepts for knowledge economies.

In the specific cases of Garicano and Rossi-Hansberg (2004) and Antràs, Garicano, and Rossi-Hansberg (2006), competitive equilibrium is defined by two knowledge cutoffs, z_1 and z_2 with $z_1 \leq z_2$, and two functions c and ϕ such that: (i) workers have knowledge at most z_1, (ii) managers have knowledge at least z_2, (iii) self-employed agents (if any) are those with knowledge between z_1 and z_2, (iv) ϕ is a (strictly increasing and differentiable) assignment function with $\phi(z)$ being the knowledge of workers hired by each manager of knowledge z, and (v) c is a (strictly increasing and differentiable) wage function with $c(z')$ being the wage of workers with knowledge z'. We provide sufficient conditions under which stable matchings (hence, stable outcomes and competitive equilibria) of knowledge economies admit this characterization. Thus, this result shows that the equilibrium notion used in Garicano and Rossi-Hansberg (2004) and Antràs, Garicano, and Rossi-Hansberg (2006) corresponds exactly to stable matchings, providing a justification for using (i)–(v) as the defining properties of the solution concept in their settings. More broadly, our approach avoids imposing such structure a priori: these properties are derived from the definition of stable assignment under rigorous and transparent arguments.

[3] Ide and Talamàs (2025) provide a recent example of this approach. See also Amaral and Rivera-Padilla (2024) and Ritto (2024), among others.

[4] This exercise is related to Hornstein and Prescott (1993) and uses an aggregate production set similar to theirs in the representation of knowledge economies as a competitive economy.

Knowledge economies have been used to interpret different parts of empirical earnings distributions. Gabaix and Landier (2008) calibrate a model similar to Lucas (1978) and Rosen (1982) to the top of the U.S. earnings distribution and show that it tracks top-income dynamics over time. As Garicano and Rossi-Hansberg (2015) argue, explaining the lower tail may require a Garicano and Rossi-Hansberg (2004)-type framework, a view supported by Caicedo, Lucas, and Rossi-Hansberg (2019), who show that a generalized Garicano and Rossi-Hansberg's (2004) economy fits the U.S. wage distribution and its evolution over 1990–2010. We contribute to this quantitative literature by showing that the simpler economy in Garicano and Rossi-Hansberg (2004) can explain wage polarization and by introducing a flexible extension that jointly matches wage polarization and the establishment-size distribution. Jointly fitting these distributions typically requires heterogeneity on both the worker and firm sides and is thus quantitatively demanding (e.g., Macera and Tsujiyama (2024)). By contrast, our framework makes such exercises tractable and extends the quantitative reach of knowledge-economy models to important economic outcomes. We find that skill-biased technological change, task routinization, and advances in ICT are all important for explaining wage polarization, forces often highlighted in the literature (e.g. Autor, Levy, and Murnane (2003); Autor, Katz, and Kearney (2008); Acemoglu and Autor (2011); Garicano and Rossi-Hansberg (2015); Caicedo, Lucas, and Rossi-Hansberg (2019)).

2 Knowledge Economies

This section introduces a general framework of knowledge economies. The economy is populated by a large number of individuals, each of whom chooses to be a worker, a manager, or self-employed. Individuals match to form firms with one manager and possibly multiple workers, or stay unmatched as self-employed. The production of goods in each firm may depend on the knowledge of both the manager and the workers.

We describe the environment in Section 2.1 and introduce our solution concept in Section 2.2. Several special cases are considered in Section 2.3.

2.1 Model

There is a large number of individuals who are (potentially) heterogeneous in e.g. their knowledge or talent. This is captured by the *set Z of types* where Z is a compact metric space. The type distribution is denoted by ν and is a nonzero, finite, Borel measure on Z satisfying $Z = \mathrm{supp}(\nu)$. For a concrete example that we will use to illustrate the model and solution concept, let $Z = \{z_l, z_m, z_h\} \subseteq (0,1)$ and $\nu(Z) = 1$. In this example, there is a unit mass of individuals with one of three knowledge levels, and for each $z \in Z$, $\nu(z)$ is the fraction of individuals with knowledge level z.

Individuals can be workers, managers or self-employed. A *dummy type* $\emptyset \notin Z$ is used to represent the match of self-employed i.e. unmatched individuals, and we let $Z_\emptyset = Z \cup \{\emptyset\}$, with the assumption that $\emptyset$ is an isolated point in $Z_\emptyset$.

A manager chooses his workforce, i.e. the type, number, and wages of the workers he hires. We make the simplifying assumption that each workforce consists of workers of the same type who are paid a common wage. There are lower and upper bounds on the number of workers of type z' that a manager of type z can hire, described by continuous functions $\underline{n} : Z^2 \to \mathbb{R}_+$ and $\bar{n} : Z^2 \to \overline{\mathbb{R}}_+$ with $\underline{n} \leq \bar{n}$. Thus, a manager of type $z \in Z$ can be matched with a measure $n \in \mathbb{R}_+$ of workers of type $z' \in Z$ and pay wage $c \in \mathbb{R}_+$ if $\underline{n}(z, z') \leq n \leq \bar{n}(z, z')$. For our running example, let $\underline{n}(z, z') = 0$ and $\bar{n}(z, z') = 1/(1 - z')$, so that more knowledgeable workers can be hired in greater numbers (by any manager).

The output produced in such match (or firm) is $F(z, z', n)$, where $F : Z^2 \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and $F(z, z', 0) = 0$ for all $(z, z') \in Z^2$. For our running example, let $F(z, z', n) = zn$, so that output depends only on the manager's knowledge z and the number of workers n.

Individual preferences are defined as follows. A self-employed individual of type z earns $U_s(z)$, where $U_s : Z \to \mathbb{R}_+$ is continuous. A worker earns his wage, i.e. if a worker of type z' is matched with a manager (of type z) whose workforce is described by (z', n, c), then he earns c. A manager of type z whose workforce is described by (z', n, c) earns $F(z, z', n) - cn$. $F(z, z', n)$ is output produced and cn is the wage bill, so each manager earns his rent. We extend F to $Z \times Z_\emptyset \times \mathbb{R}_+$ by defining, for each $z \in Z$ and $n \in \mathbb{R}_+$, $F(z, \emptyset, n) = U_s(z)$. For our running example, let $U_s(z) = \frac{9}{10} z$.

In summary, a *knowledge economy* is $E = (Z, \nu, F, \underline{n}, \bar{n}, U_s)$. Thus, our running example completely specifies a knowledge economy once we fix the values of z_l, z_m, z_h and their measures. In Appendix A.1, we show that a knowledge economy (henceforth, an economy) is a particular case of the framework of large many-to-one matching markets with occupational choice introduced by Carmona and Laohakunakorn (2024b) and extended in Section 2 of Supplementary Material to allow the set of feasible matches of a manager to depend on his knowledge.

The definition of a knowledge economy here describes only the physical environment. Individuals' incentives and choices are specified in the next section as part of the solution concept.

2.2 Stable Outcomes

There are at least three natural choices for the solution concept of knowledge economies. First, one can embed the economy in a general equilibrium setting and use competitive equilibrium as the solution concept. Second, knowledge economies are a particular case of the framework of large many-to-one matching markets with occupational choice (see Section 2 of Supplementary Material); one can therefore use stable matching. Third, one can define a notion of stable outcomes, analogous to stability in transferable utility settings with one-to-one matching and no occupational choice. We use the latter approach since it yields the simplest definition. In Section 3, we show that the three approaches are equivalent.

We describe a solution of a knowledge economy by an assignment and an earnings function. Informally, the assignment specifies who matches with whom and the size of each workforce. In the three-type example of the previous section, the assignment specifies, for instance, the fraction of individuals with knowledge z_h who become managers in firms with n workers with knowledge z_l. An earnings function u specifies the earnings of each type, $(u(z_l), u(z_m), u(z_h))$. Each individual optimally chooses an occupation and, when applicable, whom to match with, taking as given the earnings of their match partner(s). For example, for the middle type, this means that

$$u(z_m) = \max\left\{ \underbrace{\max_{z\in Z, n\in[0,1/(1-z)]} (z_m - u(z))n,}_{\text{manager}} \underbrace{\frac{9}{10} z_m}_{\text{self-employed}}, \underbrace{\max_{z\in Z, n\in[0,1/(1-z_m)]} \frac{zn - u(z)}{n}}_{\text{worker}} \right\}.$$

That is, z_m cannot raise earnings by (i) becoming a manager and choosing (z, n) while paying workers their "equilibrium" earnings $u(z)$, (ii) becoming self-employed, or (iii) working for an optimally chosen manager z in an optimally chosen workforce size n, taking the manager's "equilibrium" earnings $u(z)$ as given.

We formalize this informal description. The set of types to match is $Z \times Z_\emptyset$ with the first coordinate describing the managers (respectively, self-employed) and the second the workers (resp. the dummy type $\emptyset$). Since many workers can be matched with a manager, an assignment is a measure over pairs of types (z, z') and positive numbers n, indicating roughly how many matches feature a manager of type z and n workers of type z'.

Let $\underline{n}(z, \emptyset) = \bar{n}(z, \emptyset) = 1$ for each $z \in Z$, $N = [0, \sup_{(z,z')\in Z\times Z_\emptyset} \bar{n}(z, z')] \cap \mathbb{R}_+$, and $\bar{N} : Z \times Z_\emptyset \rightrightarrows N$ be defined by setting, for each $(z, z') \in Z \times Z_\emptyset$, $\bar{N}(z, z') = [\underline{n}(z, z'), \bar{n}(z, z')] \cap \mathbb{R}_+$. An *assignment* is $\gamma \in M(Z \times Z_\emptyset \times N)$ such that $\text{supp}(\gamma) \subseteq \text{graph}(\bar{N})$ and, for each Borel subset B of Z,[5]

$$\gamma(B \times Z_\emptyset \times N) + \int_{Z\times B\times N} n \mathrm{d}\gamma(z, z', n) = \nu(B). \tag{1}$$

Equation (1) is simply an accounting identity: the first term on the left-hand side is the measure of those who are managers or self-employed with type in B, and the second is the measure of those who are workers with type in B.

An *earnings function* is $u \in C(Z_\emptyset)$ such that $u(\emptyset) = 0$, i.e. u specifies the earnings of each type.[6] An *outcome* is (γ, u) such that γ is an assignment and u is an earnings function.

An outcome (γ, u) is *stable* if

(i) $u(z) + nu(z') = F(z, z', n)$ for each $(z, z', n) \in \text{supp}(\gamma)$ and

(ii) $u(z) + nu(z') \geq F(z, z', n)$ for each $(z, z', n) \in \text{graph}(\bar{N})$.

[5] In general, given a metric space T, $M(T)$ denotes the set of finite, Borel measures on T endowed with the weak (narrow) topology (see Varadarajan (1958) for details).

[6] Given a metric space T, $C(T)$ denotes the space of bounded and continuous real-valued functions on T.

Moreover, γ is *stable* if there exists an earnings function u such that (γ, u) is stable. Condition (i) states that, in every match, the sum of the manager's earnings and the workers' earnings equals the total output they produce. Condition (ii) is no-blocking: no pair of types (z, z') can match with each other at a feasible firm size and produce a level of output that is greater than the sum of their earnings.

Conditions (i) and (ii) capture succinctly the optimality requirements in a knowledge economy. When $\sup_{(z,z')\in Z^2} \bar{n}(z, z') < \infty$, simple manipulations of (i) and (ii) imply that an outcome (γ, u) is stable if and only if

$$u(z) + nu(z') = F(z, z', n) \text{ for each } (z, z', n) \in \text{supp}(\gamma), \text{ and}$$
$$u(z) = \max\left\{\max_{\tilde{z}\in Z, \tilde{n}\in \bar{N}(z,\tilde{z})} [F(z, \tilde{z}, \tilde{n}) - \tilde{n}u(\tilde{z})], U_s(z), \max_{\hat{z}\in Z, \hat{n}\in \bar{N}(\hat{z},z)} \frac{F(\hat{z}, z, \hat{n}) - u(\hat{z})}{\hat{n}}\right\}$$

for each $z \in Z$. These conditions mean that, whenever there are matches with a manager of type z and n workers of type z' in the assignment γ, the manager earns $u(z) = F(z, z', n) - nu(z')$ and each worker earns $u(z') = \frac{F(z,z',n)-u(z)}{n}$. The earnings of each one of the two sides are thus determined by the earnings of the other. In addition, no type z can increase earnings: by (a) becoming a manager with an workforce $(\tilde{z}, \tilde{n})$ at worker pay $u(\tilde{z})$; (b) becoming self-employed; or (c) becoming part of the workforce of $\hat{n}$ workers hired by a manager of type $\hat{z}$ given that the latter earns $u(\hat{z})$.[7]

For our running example, let $z_l = 1/4$, $z_m = 1/2$, and $z_h = 7/8$, and suppose $\nu(z_l) = 2/5$, $\nu(z_m) = 3/10$, and $\nu(z_h) = 3/10$. Our results will imply a unique stable assignment γ with $\text{supp}(\gamma) = \{(z_h, z_l, 1/(1 - z_l)), (z_m, \emptyset, 1)\}$, so that all low-knowledge individuals are workers, all high-knowledge individuals are managers, and all middle-knowledge individuals are self-employed. Furthermore, workers are all hired up to capacity. Specifically, $\gamma(z_h, z_l, \frac{4}{3}) = \frac{3}{10}$ and $\gamma(z_m, \emptyset, 1) = \frac{3}{10}$, which is supported by any earnings function u such that $u(z_m) = \frac{9}{20}$, $\frac{9}{40} \leq u(z_l) \leq \frac{19}{80}$ and $u(z_h) = \frac{4}{3}(z_h - u(z_l))$.

[7]See Section 5 in Supplementary Material for a proof of the above characterization.

2.3 Special Cases

Several models in the literature are special cases of our framework of knowledge economies.[8]

2.3.1 Rosen (1982)

Rosen's (1982) setting is equivalent to an economy where $Z = [\underline{z}, \bar{z}]$, with $0 \leq \underline{z} < \bar{z} < \infty$, managers can hire any number of workers in $\mathbb{R}_+$, hence $\underline{n}(z, z') = 0$ and $\bar{n}(z, z') = \infty$, and production equals $F(z, z', n) = g(r(z))f(r(z), nq(z'))$ for each $z, z' \in Z$, where, in particular, f, g, q and r are continuous, $q(z) > 0$ and $r(z) > 0$ for each $z \in Z$, $g(r) > 0$ for each $r > 0$, and $f(z, 0) = f(0, z') = 0$ and $f(z, z') > 0$ for each $z, z' > 0$. The interpretation is that, for each $z \in Z$, $r(z)$ is the individual's ability as a manager and $q(z)$ is his ability as a worker; hence, production depends on the ability of the manager (through the functions g and f) and the aggregate ability $nq(z')$ of the workers. Self-employed individuals produce nothing, hence $U_s(z) = 0$.

2.3.2 Garicano and Rossi-Hansberg (2004)

Garicano and Rossi-Hansberg's (2004) setting corresponds to $Z = [0, \bar{z}]$ with $0 < \bar{z} < 1$, $\underline{n}(z, z') = \bar{n}(z, z') = \frac{1}{h(1-z')}$ with $0 < h < 1$ so a manager hires more workers the higher is worker knowledge, and $F(z, z', n) = zn$, i.e. the product of the manager knowledge and the number of workers. Self-employed individuals produce $U_s(z) = z$, i.e. the same production function with $n = 1$ since they are their workforce.[9]

An alternative specification of Garicano and Rossi-Hansberg's (2004) setting is obtained by letting production be $F(z, z', n) = \max\{z, z'\}n$, which corresponds more closely to the description in Garicano (2000). As we show in Supplementary Material, these two economies have the same stable outcomes.

Another variation is obtained by requiring only $\bar{z} > 0$ and introducing a function $G : Z \to \mathbb{R}_+$ which is the cumulative distribution function of a continuous and decreasing density. In this case, $\underline{n}(z, z') = \bar{n}(z, z') = \frac{1}{h(1-G(z'))}$ and $F(z, z', n) = G(z)n$. This formalization is

[8]We abstract from capital for simplicity, so the setting in Lucas (1978) is not nested in our framework. However, capital can be easily added as pointed out in Carmona and Laohakunakorn (2024b).

[9]Garicano and Rossi-Hansberg (2004) allow $\bar{z} = 1$; our results can be extended to this case.

only apparently more general. Indeed, if E denotes the economy just defined and E_u is such that $Z_u = G(Z)$, $\nu_u = \nu \circ G^{-1}$, $\underline{n}_u(z, z') = \bar{n}_u(z, z') = \frac{1}{h(1-z')}$ and $F_u(z, z', n) = zn$, then the stable outcomes of E_u are in a one-to-one relationship with those of E.[10]

The above result is implied by Theorem 6.1 in Supplementary Material which also implies that the knowledge distribution can be normalized to be uniform. Indeed, if E is the knowledge economy defined in the first paragraph of this section and ν has a continuous and strictly positive density, then let $Y = [0, \nu(Z)]$ and $\psi(z) = \nu([0, z])$ for each $z \in Z$, i.e. ψ is the cumulative distribution function of ν and is continuous, strictly increasing and satisfies $\psi(Z) = Y$. Then the stable outcomes of the economy E_ψ are in a one-to-one relationship with those of E, where E_ψ has knowledge set Y, a uniform knowledge distribution on Y, bounds $\underline{n}_\psi(y, y') = \bar{n}_\psi(y, y') = \frac{1}{h(1-\psi^{-1}(y'))}$, $U_{\psi,s}(y) = \psi^{-1}(y)$ and $F_\psi(y, y', n) = \psi^{-1}(y)n$.

2.3.3 Ide and Talamàs (2025)

Ide and Talamàs (2025) introduce artificial intelligence (AI henceforth) to the setting of Garicano and Rossi-Hansberg (2004). Their formalization of AI can be added to our setting by treating it as another type of knowledge as follows. Starting with the set of knowledge levels $[0, \bar{z}]$ of the previous section, where $z \in Z$ can be interpreted as the fraction of problems that someone can solve, let $z_{AI} \in [0, \bar{z}]$ be the fraction of problems AI can solve and define $Z = ([0, \bar{z}] \times \{0\}) \cup \{(z_{AI}, 1)\}$; thus, the first coordinate of $z \in Z$ indicates the fraction of problems someone can solve and the second whether that someone is a person or AI. Letting $p : Z \to [0, \bar{z}]$ denote the projection of Z onto $[0, \bar{z}]$, then let $\underline{n}(z, z') = \bar{n}(z, z') = \frac{1}{h(1-p(z'))}$ with $0 < h < 1$, $F(z, z', n) = p(z)n$ and $U_s(z) = p(z)$.

The knowledge distribution ν on Z can be described by a distribution G on $[0, \bar{z}]$ and a number α, which is the measure of AI, i.e. of $\{(z_{AI}, 1)\}$. Since whether knowledge is provided by a person or by AI does not matter for production and firm size, it follows that this economy can also be described by the Garicano and Rossi-Hansberg's (2004) economy of the previous section with a knowledge distribution $\hat{\nu}$ such that, for each Borel subset B of $[0, \bar{z}]$, $\hat{\nu}(B) = G(B)$ if $z_{AI} \notin B$ and $\hat{\nu}(B) = G(B) + \alpha$ otherwise.

Ide and Talamàs (2025) also consider the case where AI can only take a managerial role.

[10] This result provides a justification for the claim in footnote 4 of Fuchs, Garicano, and Rayo (2015).

This case can be represented in our setting by letting

$$F(z, z', n) = \begin{cases} p(z)n & \text{if } z' \in [0, \bar{z}] \times \{0\}, \\ 0 & \text{otherwise,} \end{cases}$$

and analogously for $U_s(z)$.[11]

2.3.4 Mak and Siow (2025)

The setting in Mak and Siow (2025) corresponds to the following case: Let Z be a compact subset of $\mathbb{R}^2_+$; the interpretation of $z = (z_1, z_2) \in Z$ is that z_1 is the individual's ability as a manager and z_2 as a worker. Production is $F(z, z', n) = R(p_1(z), p_2(z'))$, where $R : \mathbb{R}^2_+ \to \mathbb{R}$ is continuous and p_i is the projection of $\mathbb{R}^2$ onto its ith coordinate for each $i = 1, 2$; hence, production is a continuous function of the manager's and the worker's abilities. Matching is one-to-one, hence $\underline{n}(z, z') = \bar{n}(z, z') = 1$, and self-employed individuals produce nothing, $U_s(z) = 0$.[12]

This setting can be thought of as a one-to-one matching version of Rosen (1982) when for some $\underline{z}, \bar{z} \in \mathbb{R}_+$ with $\underline{z} < \bar{z}$, and continuous functions f, g, q and r, $Z = G([\underline{z}, \bar{z}])$ where $G = (r, q)$, and $R(r(z), q(z)) = g(r(z))f(r(z), q(z))$ for each $z \in [\underline{z}, \bar{z}]$.

2.3.5 Gabaix and Landier (2008) and Gola (2021)

Gabaix and Landier (2008) consider a two-sided one-to-one matching model, which can be represented as a knowledge economy as follows. Let M and W be disjoint compact subsets of $\mathbb{R}_+$, with the interpretation that M is the set of managers and W is the set of workers, fixed a priori. Let $Z = M \cup W$ and, for each $z, z' \in Z$, let $U_s(z) = 0$ (self-employed individuals do not produce), $\underline{n}(z, z') = \bar{n}(z, z') = 1$ (matching is one-to-one), $F(z, z', n) = 0$ if $z \in W$ or if $z' \in M$ (production requires a manager to take the managerial role and a worker to be the workforce) and $F(z, z', n) = f(z, z')n$ if $z \in M$ and $z' \in W$, where f is continuous and strictly positive.

[11] Alternatively, we can let $\underline{n}(z, z') = \bar{n}(z, z') = \frac{1}{h(1-p(z'))}$ if $z' \in [0, \bar{z}] \times \{0\}$ and $\underline{n}(z, z') = \bar{n}(z, z') = 0$ otherwise.

[12] Mak and Siow (2025) do not require Z to be compact; it seems possible to extend our results to this case.

Gola (2021) also considers a two-sided one-to-one matching model that can be written in our setting in a similar way to the one above. The main differences are that (i) W is a subset of $\mathbb{R}^L$ for some $L \in \mathbb{N}$, where $z \in W$ described a worker's skills, and (ii) M is a subset of $\mathbb{R} \times \{0, 1\}$, with the first coordinate describing a manager's productivity and the second indicating in which of two sectors the manager can operate.

3 Existence, Equivalence, and Characterization

In this section, we establish existence and characterize stable outcomes. We first define stable matchings and prove their existence. We then show that stable matchings and stable outcomes are equivalent, which implies existence of stable outcomes. We proceed in this order because stable matchings impose fewer restrictions than stable outcomes, so existence is easier to show. In particular, stable outcomes impose equal treatment—agents with the same knowledge earn the same—whereas stable matchings do not. We also show that these solution concepts coincide with solutions to the surplus-maximization optimal transport problem and its dual and with competitive equilibria.

Throughout, we assume an upper bound on firm size, $\sup_{(z,z') \in Z^2} \bar{n}(z, z') < \infty$. We focus on this case because bounded firm size is needed for stable matchings in knowledge economies to have empirically relevant properties (see also Section 4.2).[13]

3.1 Stable Matching

Using stable matching as the solution concept for knowledge economies has two advantages. First, the definition is a direct application of the general framework for large many-to-one matching markets with occupational choice (Supplementary Material, Section 2). Second, existence follows under general conditions. The latter arises because little is imposed in the definition of stable matchings and stable matchings are elements of a technically convenient metric space. A stable outcome specifies the allocation and earnings of individuals separately, through an assignment and an earnings function, whereas stable matching does not: wages

[13] Carmona and Laohakunakorn (2024b) consider the opposite case in the context of Rosen's (1982) economies.

are allowed to vary across matches even when the knowledge of the managers or the workers are the same across them.

In the definition of stable matchings, a workforce is represented as a measure over $Z \times C$, where $C = \mathbb{R}_+$ is the set of possible wages.[14] Specifically, if a manager of type $z \in Z$ hires $n \in N$ workers of type $z' \in Z$ at wage $c \in C$, then the manager is matched with $n1_{(z',c)} \in M(Z \times C)$, where $1_{(z',c)}$ denotes the probability measure degenerate on (z', c). Thus, in an economy E, a manager of type z can be matched with a measure of workers and wages in the set

$$P_m(z) = \{n1_{(z',c)} : (z', c) \in Z \times C \text{ and } n \in \mathbb{R}_+ \text{ such that } \underline{n}(z, z') \leq n \leq \bar{n}(z, z')\}.$$

By convention, we represent a self-employed individual of type z as matched with a measure in $P_s(z) = \{1_{(\emptyset,0)}\}$. Let $X = \cup_{z \in Z} P_m(z)$ and $X_\emptyset = X \cup \{1_{(\emptyset,0)}\}$, the latter being the *set of possible matches of managers and self-employed individuals*.

A matching in an economy E is a measure on $Z \times X_\emptyset$ that describes the occupational choices of individuals and matching patterns. Matches take the form (z, δ) and the occupational choices are described by the place in the match each individual occupies: if $\delta \in X$, then the first coordinate z refers to managers and δ encodes the associated workforce. If $\delta \in X_\emptyset \setminus X$, then z refers to a self-employed individual. Then μ roughly specifies how many matches described by each possible (z, δ) there are.

Formally, a *matching* is a Borel measure μ on $Z \times X_\emptyset$ such that (i) $\text{supp}(\mu) \subseteq \text{graph}(P_m) \cup \text{graph}(P_s)$, and (ii) $\nu_M + \nu_S + \nu_W = \nu$, where, for each Borel subset B of Z, $\nu_M(B) = \mu(B \times X)$, $\nu_S(B) = \mu(B \times (X_\emptyset \setminus X))$ and $\nu_W(B) = \int_{Z \times X} \delta(B \times C) \mathrm{d}\mu(z, \delta)$. Condition (i) requires that the matches that form (i.e. that belong to the support of μ) satisfy the feasibility constraints in P_m and P_s. Condition (ii) requires that everyone in the market is accounted for.

Somewhat informally, a matching μ is stable if, for each $(z, z', n, c) \in Z \times Z_\emptyset \times N \times C$ such that $(z, n1_{(z',c)}) \in \text{supp}(\mu)$, (a) neither z nor z' can gain by becoming self-employed, and (b) neither z nor z' can gain by hiring, as a manager, some feasible $\hat{n}1_{(\hat{z},\hat{c})}$ that benefits $\hat{z}$, where $\hat{z}$ may be a manager, a worker, or self-employed under the matching μ. See Appendix

[14] See Section 2 of Supplementary Material and Carmona and Laohakunakorn (2024b).

A.2 for the formal definition of stable matchings.

3.2 Existence

The main result of this section establishes existence of stable matchings. It builds on Carmona and Laohakunakorn (2024b) but requires additional arguments because, in our knowledge economies, the set of feasible workforces depends on manager knowledge and the set of wages that managers can pay is not bounded above.

We deal with the first difficulty by extending the framework of Carmona and Laohakunakorn (2024b) in Supplementary Material, Section 2. We provide an existence theorem and sufficient conditions for its assumptions (Theorem 3.1 and Lemma 3.8 in Supplementary Material, Section 3).

We deal with the second difficulty by considering a sequence of truncated economies that impose an upper bound on wages and then letting this bound go to infinity. Theorem 3.1 applies to each truncated economy. Limit results in Supplementary Material, Section 3, imply that any knowledge economy has a stable matching.

In addition to the assumptions in Section 2, we make the following assumption to guarantee the existence of stable matchings:

A $\sup_{(z,z')\in Z^2} \bar{n}(z,z') < \infty$ and either (i) $\nu(\{z \in Z : F(z,z,\bar{n}(z,z)) = 0 \text{ and } U_s(z) = 0\}) = 0$ or (ii) $\min_{(z,z')\in Z^2} \underline{n}(z,z') > 0$.

The role of condition A is as follows. Production is bounded above (since F is continuous and its domain is compact), and hence unbounded wages require negligible spans of control. Part (ii) rules this out directly. Part (i) is used to show that the sequence of stable matchings of the truncated economies is tight and hence wages are bounded except in a set of matches with small measure.[15]

Theorem 1 *If condition A holds in an economy E, then E has a stable matching.*

[15] Essentially, unbounded wages and corresponding negligible spans of control lead to zero rent to the manager since $F(z,z',0) = 0$ for each $z, z' \in Z$; but if the type of such manager is z and $F(z,z,\bar{n}(z,z)) > 0$ or $U_s(z) > 0$, then such manager can hire $\bar{n}(z,z)$ workers of type z at any wage $\varepsilon > 0$ such that $F(z,z,\bar{n}(z,z)) - \varepsilon\bar{n}(z,z) > 0$ or become self-employed to obtain a strictly positive rent, thus contradicting stability.

3.3 Equivalence between Stable Outcomes and Stable Matchings

The main difference between the notions of stable outcomes and stable matchings is that earnings are match-specific in the latter, whereas, in the former, all individuals with the same knowledge earn the same regardless of match and occupation, i.e. an equal treatment property holds. There is also a difference in the mathematical spaces in which the elements of the two concepts lie, but this difference is not essential. Indeed, a matching μ is a measure on $Z \times X_\emptyset$, and each element of $X_\emptyset$ has the form $n1_{(z',c)}$. Thus, a matching can be thought to be a measure on $Z \times Z_\emptyset \times N \times C$. Under this view, the marginal of μ on $Z \times Z_\emptyset \times N$ is an assignment. The substantive issue is therefore whether an equal treatment property holds in each stable matching, so that earnings can be described by an earnings function.

We establish equal treatment and the equivalence of stable outcomes and stable matchings under a strengthening of condition A.

B $\sup_{(z,z')\in Z^2} \bar{n}(z,z') < \infty$ and either (i) for each $z \in Z$, $F(z,z,\bar{n}(z,z)) > 0$ or $U_s(z) > 0$, or (ii) $\min_{(z,z')\in Z^2} \underline{n}(z,z') > 0$.

Condition B is useful because it rules out managers with zero span of control. Part (ii) does so directly. Under part (i), each type earns strictly positive payoffs in any stable matching, which implies that every manager hires a positive measure of workers.[16] Among the special cases in Section 2.3, only the Rosen's (1982) setting does not satisfy condition B. Nevertheless, any truncated version of the Rosen's (1982) setting (e.g. when there is $\bar{n} \in \mathbb{R}_+$ such that $\bar{n}(z,z') = \bar{n}$ for each $(z,z') \in Z^2$) satisfies part (i). The other applications considered in that section satisfy part (ii).

Given a matching μ of an economy, let

$$M = \{z \in Z : (z,\delta) \in \text{supp}(\mu) \text{ for some } \delta \in X\}, \tag{2}$$

$$S = \{z \in Z : (z,\delta) \in \text{supp}(\mu) \text{ for some } \delta \in X_\emptyset \setminus X\} \text{ and} \tag{3}$$

$$W = \{z \in Z : z \in \text{supp}(\delta_Z) \text{ for some } (\hat{z},\delta) \in \text{supp}(\mu)\} \tag{4}$$

[16] As in Footnote 15, if a type z earns zero in the support of a stable matching, then he can become a manager or self-employed and earn a strictly positive rent.

denote the set of manager, self-employed, and worker types, respectively.[17] Lemma 1 states the equal treatment property.[18]

Lemma 1 *If μ is a stable matching of an economy E satisfying condition B, then there exists a continuous function $u : Z \to \mathbb{R}$ such that*

1. $u(z) = F(z, z', n) - nc$ *for each* $z \in M$ *and* $(z', c, n) \in Z \times C \times \mathbb{R}_+$ *such that* $(z, n1_{(z',c)}) \in \operatorname{supp}(\mu)$,
2. $u(z) = U_s(z)$ *for each* $z \in S$, *and*
3. $u(z) = c$ *for each* $z \in W$ *and* $(\hat{z}, c, n) \in Z \times C \times \mathbb{R}_+$ *such that* $(\hat{z}, n1_{(z,c)}) \in \operatorname{supp}(\mu)$.

The equal treatment property holds because wage c is a transfer from the manager to workers. Suppose a type-z manager earns strictly less rent than another type-z manager matched with $n1_{(z',c)}$. The former can profitably deviate by hiring type-z' workers at wage $c + \varepsilon$, with small $\varepsilon > 0$, and thereby attain rent arbitrarily close to that of the latter and thus higher than his own. This is a contradiction to the stability of the matching.

Focusing on managers, the above argument shows that there exists a function u such that $u(z)$ is the rent of a manager of type z. An analogous argument implies that this function is continuous. If a type-z manager's rent were strictly below, and bounded away from, the rent of a nearby type $\tilde{z}$, then the type-z manager could attract the workers of the latter by paying slightly higher wage and obtain rent arbitrarily close to $u(\tilde{z})$ which is higher than $u(z)$. This again contradicts stability.

The equal treatment property implies that one can obtain a stable outcome from a stable matching. This requires a function to transform a stable matching into a stable assignment, as follows. Given a stable matching μ, let u be as in Lemma 1, extended to $Z_\emptyset$ by setting $u(\emptyset) = 0$. We call u the *earnings function of* μ. Define $g : \operatorname{supp}(\mu) \to Z \times Z_\emptyset \times N$ by setting, for each $(z, \delta) \in \operatorname{supp}(\mu)$, $g(z, \delta) = (z, z', n)$, where $(z', n) \in Z_\emptyset \times N$ is such that $\delta = n1_{(z',u(z'))}$. Then g is a homeomorphism between $\operatorname{supp}(\mu)$ and $g(\operatorname{supp}(\mu)) \subseteq Z \times Z_\emptyset \times N$.[19] Thus, $\mu \circ g^{-1}$

[17] Given two metric spaces T and Y and $\delta \in M(T \times Y)$, δ_T (resp. δ_Y) denotes the marginal of δ on T (resp. Y).

[18] See Greinecker and Kah (2021, Theorem 3) for an analogous result in the case of 1-1 matching without occupational choice.

[19] See Lemma 14 in Appendix.

is a measure on $Z \times Z_\emptyset \times N$ and $\text{supp}(\mu \circ g^{-1}) = g(\text{supp}(\mu))$. Theorem 2 below shows that $(\mu \circ g^{-1}, u)$ is a stable outcome.

Conversely, it is always possible to obtain a stable matching from a stable outcome, as follows. Given a stable outcome (γ, v), define $h : \text{supp}(\gamma) \to Z \times X_\emptyset$ by setting, for each $(z, z', n) \in \text{supp}(\gamma)$, $h(z, z', n) = (z, n1_{(z', v(z'))})$. Then h is a homeomorphism between $\text{supp}(\gamma)$ and $h(\text{supp}(\gamma)) \subseteq Z \times X_\emptyset$. Thus, $\gamma \circ h^{-1}$ is a measure on $Z \times X_\emptyset$ and $\text{supp}(\gamma \circ h^{-1}) = h(\text{supp}(\gamma))$. Theorem 2 shows that $\gamma \circ h^{-1}$ is a stable matching.

Theorem 2 *Let E be an economy satisfying condition B. If μ is a stable matching of E and u is its earnings function, then $(\mu \circ g^{-1}, u)$ is a stable outcome and*

$$M = \{z \in Z : (z, z', n) \in \text{supp}(\gamma) \textit{ for some } (z', n) \in Z \times N\},$$

$$S = \{z \in Z : (z, \emptyset, 1) \in \text{supp}(\gamma)\} \textit{ and}$$

$$W = \{z \in Z : (\hat{z}, z, n) \in \text{supp}(\gamma) \textit{ for some } (\hat{z}, n) \in Z \times N\}.$$

If (γ, v) is a stable outcome of E, then $\gamma \circ h^{-1}$ is a stable matching and v is its earnings function.

3.4 Optimal Transport

Theorems 1 and 2 imply that a stable outcome exists in any economy satisfying condition B. This result allows us to show that in any stable outcome, the assignment solves the surplus maximization problem and the earnings function solves its dual. Since the surplus of a match equals that match's production, stable assignments are the solutions to the production maximizing problem. Its dual is the earnings minimization problem. These are infinite-dimensional linear programming problems, whose solutions are often easier to characterize than stable assignments or stable matchings, and which form the core of the computational method in our quantitative analysis in Section 5.

For each assignment $\gamma \in M(Z \times Z_\emptyset \times N)$, let $\gamma_Z \in M(Z)$ be the marginal of γ on Z, $\gamma_{Z_\emptyset \times N} \in M(Z_\emptyset \times N)$ be the marginal of γ on $Z_\emptyset \times N$, and $\gamma_{Z,n} \in M(Z)$ be defined by setting, for each Borel subset B of Z, $\gamma_{Z,n}(B) = \int_{B \times N} n \mathrm{d}\gamma_{Z_\emptyset \times N}(z', n)$. Define the set of assignments:

$$\Gamma = \{\gamma \in M(Z \times Z_\emptyset \times N) : \gamma_Z + \gamma_{Z,n} = \nu \text{ and } \mathrm{supp}(\gamma) \subseteq \mathrm{graph}(\bar{N})\}.$$

The equation $\gamma_Z + \gamma_{Z,n} = \nu$ is just a succinct way of writing the feasibility condition (1). An assignment γ is *surplus maximizing* if it solves $\max_{\tau \in \Gamma} \int_{Z \times Z_\emptyset \times N} F \mathrm{d}\tau$.

Define the set of earnings functions that satisfy no blocking:

$$U = \{u \in C(Z_\emptyset) : u(z) + nu(z') \geq F(z, z', n) \text{ for each } (z, z', n) \in \mathrm{graph}(\bar{N}) \text{ and } u(\emptyset) = 0\}.$$

The dual of the surplus maximization problem is $\min_{u \in U} \int_Z u \mathrm{d}\nu$.

Theorem 3 *In an economy satisfying condition B, an outcome (γ, u) is stable if and only if the assignment γ is surplus maximizing and the earnings function u solves the dual of the surplus maximization problem.*

One way of deriving the properties of stable outcomes is to apply the definition of stability directly. Theorem 3 provides an alternative characterization. This approach is useful when solving the surplus maximization problem is easier than analyzing stable outcomes directly, and the following result is helpful in this regard by providing a necessary condition for its solutions.

A set $K \subseteq Z \times Z_\emptyset \times N$ is *F-monotone* if $\int_{Z \times Z_\emptyset \times N} F \mathrm{d}\zeta \geq \int_{Z \times Z_\emptyset \times N} F \mathrm{d}\tau$ for each finite measure ζ concentrated on finitely many points of K and for each finitely-supported measure τ on $Z \times Z_\emptyset \times N$ such that $\tau_Z + \tau_{Z,n} = \zeta_Z + \zeta_{Z,n}$ and $\mathrm{supp}(\tau) \subseteq \mathrm{graph}(\bar{N})$. In words, no finitely-supported measure on K can be improved, in the sense of yielding a higher surplus, by transporting mass in a feasible way. Theorem 4 shows that the support of surplus-maximizing assignment is F-monotone.[20]

Theorem 4 *If an assignment $\gamma \in \Gamma$ is surplus maximizing, then* $\mathrm{supp}(\gamma)$ *is F-monotone.*

[20] Theorem 4 is analogous to Beiglböck and Griessler (2019, Theorem 1.4), whose definition of F-monotonicity requires, in addition, that $\tau(Z \times Z_\emptyset \times N) = \zeta(Z \times Z_\emptyset \times N)$ and all elements of Γ to be probability measures. Theorem 4 dispenses with both requirements.

3.5 Competitive Equilibrium

An assignment describes the allocation of knowledge-differentiated labor in the economy and the earnings function its value. This suggests that stable outcomes are part of a competitive equilibrium. We establish such a result in this section.

Our definition of competitive equilibrium for a knowledge economy is obtained by representing the economy in a general equilibrium setting and then applying the corresponding notion of competitive equilibrium. The general equilibrium setting we consider is analogous to the classical one in Debreu (1959) with the following differences. First, in contrast to the finitely many individuals in Debreu (1959), knowledge economies are populated by non-atomic individuals described by a distribution, as in Hildenbrand (1974). Also in contrast to both Debreu (1959) and Hildenbrand (1974), a knowledge economy features infinitely many commodities since the set of knowledge levels (i.e. the type space Z) is allowed to be infinite and, correspondingly, there are infinitely many types of labor, i.e. labor is a differentiated commodity, as in Jones (1984).

We represent a knowledge economy in a general equilibrium setting as follows. The commodity space is $L = \mathbb{R} \times \text{ca}(Z)$, representing allocations of the consumption good and labor, where $\text{ca}(Z)$ denotes the space of countably additive set functions on Z. An element $(x, \delta) \in L$ is interpreted as follows: x is the amount of the consumption good, used as output if $x > 0$ and as input if $x < 0$; δ is a countably additive set function defined on the Borel subsets of Z, with $\delta(B)$ denoting the number of individuals with knowledge in the Borel subset B of Z, used as input if $\delta(B) < 0$ and as output if $\delta(B) > 0$.

Prices are in $L^* = \mathbb{R} \times C(Z)$. We normalize the price of the consumption good to one, and the prices of labor are non-negative across skill types. Let $L^*_+ = \{(p, c) \in \mathbb{R} \times C(Z) : p = 1 \text{ and } c(z) \geq 0 \text{ for each } z \in Z\}$. The value of $y = (x, \delta) \in L$ at price $q = (p, c) \in L^*$ is $q(y) = px + \int_Z c\mathrm{d}\delta$.

Consumers are characterized by their knowledge level $z \in Z$ and are described implicitly by the knowledge distribution ν. The consumption set is $\Omega = \mathbb{R}_+ \times M(Z)$, and preferences are represented by the utility function $u : \Omega \to \mathbb{R}$ defined by $u(x, \delta) = x$, for each $(x, \delta) \in \Omega$; individuals value only the consumption good. Endowments depend on the knowledge level z

since an individual with knowledge z is endowed with one unit of labor of type z: for each $z \in Z$, let $e(z) = (0, 1_z)$ be the endowment of those with knowledge z. Thus, the value of the endowment at price $(p, c) \in L^*$ is $c(z)$. We write $e_x(z) = 0$ and $e_\delta(z) = 1_z$ for each $z \in Z$.

The description of production follows Hornstein and Prescott (1993) in that only the aggregate production set is specified and in a way that the formation of firms, i.e. the matching of managers and workers, is part of the description of production. The aggregate production set is

$$\begin{aligned} Y = \{&(x, \delta) \in L : \text{ there exists } \gamma \in M(Z \times Z_\emptyset \times N) \text{ such that } \operatorname{supp}(\gamma) \subseteq \operatorname{graph}(\bar{N}), \\ &x \leq \int_{Z \times Z_\emptyset} F(z, z', n) \mathrm{d}\gamma(z, z', n) \text{ and } \delta(B) \leq -(\gamma_Z + \gamma_{Z,n})(B) \text{ for each Borel } B \subseteq Z\}. \end{aligned}$$

Production therefore requires the formation of firms described by the distribution γ. A commodity (x, δ) is feasible if consumption does not exceed total output and if, for each Borel set of types $B \subseteq Z$, the labor used as managerial or worker input does not exceed the amount available in those firms.

A competitive equilibrium consists of a measure τ over $Z \times \Omega$ describing the consumers' choices, a production vector y^* and a price q such that (i) the marginal of τ on Z is equal to the distribution ν of individuals in the economy, (ii) almost every consumer maximizes his utility subject to his budget constraint, (iii) production is profit maximizing, and (iv) consumption does not exceed the sum of endowments and production. Formally, a *competitive equilibrium* is a measure $\tau \in M(Z \times \Omega)$, a production vector $y^* \in Y$ and a price $q \in L^*_+$ such that

(i) $\tau_Z = \nu$,

(ii) $\tau(\{(z, x, \delta) \in Z \times \Omega : (x, \delta) \text{ solves } \max_{(x', \delta') \in \Omega} u(x, \delta) \text{ subject to } q(x', \delta') \leq q(e(z))\}) = \nu(Z)$,

(iii) $q(y) \leq q(y^*)$ for each $y \in Y$, and

(iv) $\int_\Omega i_\Omega \mathrm{d}\tau_\Omega \leq \int_Z e \mathrm{d}\nu + y^*$, where $i_\Omega : \Omega \to \Omega$ is the identity on Ω.

Theorem 5 below characterizes competitive equilibria. It shows that each individual consumes the amount of the consumption good affordable given their endowment and supplies all of their labor. Aggregate consumption equals firms' total output, and the amount of labor they use as inputs equals the amount available in them. As a result, once the matching is described by a measure γ and prices c are specified, all elements of a competitive equilibrium are pinned down. These outcomes (γ, c) coincide exactly with stable outcomes.

Theorem 5 *In an economy satisfying condition A, $(\tau, y^*, (1, c))$ is a competitive equilibrium if and only if there exists an assignment γ such that*

1. *(γ, c) is stable, where c is extended to $Z_\emptyset$ by setting $c(\emptyset) = 0$,*

2. *$\tau = \nu \circ f^{-1}$, where $f : Z \to Z \times \Omega$ is such that $f(z) = (z, c(z), 0)$ for each $z \in Z$, and*

3. *$y^* = (x^*, \delta^*)$, where $x^* = \int_{Z \times Z_\emptyset} F(z, z', n) \mathrm{d}\gamma(z, z', n)$ and $\delta^*(B) = -(\gamma_Z + \gamma_{Z,n})(B) = -\nu(B)$ for each Borel $B \subseteq Z$.*

4 Understanding Worker-Manager Matching and the Wage Distribution: Theory

In this section we use the general results of Section 3 to understand which properties of knowledge economies imply realistic properties for matching of workers to managers and the wage distribution. In Section 4.1, we establish sufficient conditions for the earnings function u to be strictly increasing (i.e. for more knowledgeable individuals to earn more) and for the matching of workers to managers to be positive assortative and characterized by occupational stratification. Putting all these results together then yields sufficient conditions for a detailed characterization of stable outcomes.

To illustrate the usefulness of this characterization, Section 4.2 applies it to compare the settings of Garicano and Rossi-Hansberg (2004) and Rosen (1982), identifying which differences between the two models account for their considerably distinct implications for worker-manager matching and the wage distribution.

4.1 A Characterization of Stable Outcomes

In order to obtain a detailed characterization of stable outcomes, the key simplification we make throughout this section is that the firm size constraint is always binding. This is guaranteed by the following strengthening of condition B:

C $\sup_{(z,z')\in Z^2} \bar{n}(z,z') < \infty$ and either (i) F is linear in n and, for each $z \in Z$, either $F(z,z,\bar{n}(z,z)) > 0$ or $U_s(z) > 0$, or (ii) $\bar{n}(z,z') = \underline{n}(z,z') > 0$ for each $(z,z') \in Z^2$.[21]

Part (i) of condition C adds the linearity of production in n to part (i) of condition B. Part (ii) strengthens part (ii) of condition B by requiring $\bar{n} = \underline{n}$. The main implication of condition C is that the optimal span of control is $\bar{n}$; more precisely, if (γ, v) is a stable outcome of an economy satisfying condition C and $(z,z',n) \in \mathrm{supp}(\gamma)$, then $n = \bar{n}(z,z')$. Thus, in this case, it makes sense to define $s : Z \times Z_\emptyset \to \mathbb{R}_+$ by setting, for each $(z,z') \in Z \times Z_\emptyset$,

$$s(z,z') = F(z,z',\bar{n}(z,z')),$$

so that s describes the production at each match.

Our goal is to understand when the stable outcome is unique, positive assortative, satisfies occupational stratification, and can be represented by an assignment function and a wage function, each characterized by a differential equation. This characterization was first proposed by Garicano and Rossi-Hansberg (2004) as the definition of equilibrium in their setting. Our main result provides sufficient conditions for it to hold well beyond their setting.

The above properties are formalized in the following definition. We say that an outcome (γ, u) is *represented by* (z_1, z_2, ϕ, c) if the following conditions hold:

1. $\underline{z} < z_1 \leq z_2 < \bar{z}$,
2. $W = [\underline{z}, z_1]$, $M = [z_2, \bar{z}]$, and $S = [z_1, z_2]$ if and only if $z_1 < z_2$; otherwise $S = \emptyset$,
3. the assignment function $\phi : [z_2, \bar{z}] \to [\underline{z}, z_1]$ is strictly increasing and differentiable and satisfies $\phi(z_2) = \underline{z}$, $\phi(\bar{z}) = z_1$, and for each $z \in [z_2, \bar{z}]$,

$$\phi'(z) = \frac{\bar{n}(z,\phi(z))\theta(z)}{\theta(\phi(z))},$$

[21] The function F is *linear in n* if $F(z,z',\lambda n) = \lambda F(z,z',n)$ for each $(z,z',n,\lambda) \in Z^2 \times \mathbb{R}^2_+$.

4. the wage function $c : [\underline{z}, z_1] \to [0, \max_{(z,z')\in Z^2} \frac{s(z,z')}{\bar{n}(z,z')}]$ is differentiable and, for each $z \in [\underline{z}, z_1]$,[22]

$$c'(z) = \frac{1}{\bar{n}(\phi^{-1}(z), z)} \left(\frac{\partial s(\phi^{-1}(z), z)}{\partial y} - c(z) \frac{\partial \bar{n}(\phi^{-1}(z), z)}{\partial y} \right),$$

5. the assignment γ is the pushforward measure of the knowledge distribution ν with respect to a function σ, i.e. $\gamma = \nu \circ \sigma^{-1}$, where $\sigma : [z_1, \bar{z}] \to Z \times Z_\emptyset \times N$ is defined by setting, for each $z \in [z_1, \bar{z}]$,

$$\sigma(z) = \begin{cases} (z, \phi(z), \bar{n}(z, \phi(z))) & \text{if } z \in [z_2, \bar{z}], \\ (z, \emptyset, 1) & \text{if } z \in [z_1, z_2), \end{cases}$$

6. the earnings function $u : Z \to \mathbb{R}_+$ is such that, for each $z \in Z$,

$$u(z) = \begin{cases} c(z) & \text{if } z \in [\underline{z}, z_1], \\ U_s(z) & \text{if } z \in [z_1, z_2], \\ s(z, \phi(z)) - \bar{n}(z, \phi(z)) c(\phi(z)) & \text{if } z \in [z_2, \bar{z}], \end{cases}$$

7. $s(z_2, \underline{z}) - \bar{n}(z_2, \underline{z}) c(\underline{z}) \geq U_s(z_2)$.

The meaning of each condition is as follows. When a stable outcome (γ, u) admits such a representation, there is a strictly positive measure of both workers and managers, and occupational stratification holds. The definitions of M, S and W imply that each of these sets is closed and their union is Z. Thus, they intersect. More precisely, if $S = \emptyset$, then $z_1 = z_2$ belongs to $M \cap W$ and, therefore, type z_2 is indifferent between being a worker and a manager, both of which are weakly preferred to self-employment:

$$c(z_2) = s(z_2, \underline{z}) - \bar{n}(z_2, \underline{z}) c(\underline{z}) \geq U_s(z_2).$$

If $S \neq \emptyset$, then z_1 belongs to $W \cap S$ and is indifferent between being a worker and self-

[22]For any function h defined on (a subset of) $Z \times Z_\emptyset$, we denote by $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$ the partial derivatives with respect to the first and second arguments respectively.

employed; furthermore, z_2 belongs to $S \cap M$ and is indifferent between being a manager and self-employed:

$$c(z_1) = U_s(z_1) \text{ and } U_s(z_2) = s(z_2, \underline{z}) - \bar{n}(z_2, \underline{z})c(\underline{z}).$$

Each manager type $z \in M$ is matched with only one worker type $\phi(z)$. The assignment function ϕ is strictly increasing and assigns the lowest manager type z_2 to the lowest worker type $\underline{z}$ and the highest manager type $\bar{z}$ to the highest worker type z_1. Thus, the matching is strictly positive assortative. Moreover, ϕ is differentiable and satisfies an initial value problem equivalent to feasibility: for each Borel subset B of types, the measure of types in B equals the measure of types in B that are either managers, self-employed, or workers.

The wage function c is differentiable and satisfies an ordinary differentiable equation derived from the first-order condition of the manager's maximization problem of finding the preferred type of worker. Specifically, a manager of type $\phi^{-1}(z)$ solves $\max_{z' \in W} s(\phi^{-1}(z), z') - \bar{n}(\phi^{-1}(z), z')c(z')$, of which z is the solution.

The outcome (γ, u) is then fully described by (z_1, z_2, ϕ, c), which means that the assignment γ is the pushforward measure of the type distribution ν with respect to σ, c is the restriction of u to $[\underline{z}, z_1]$, the earnings of those in (z_1, z_2) are $U_s(z)$, and the earnings of those in $[z_2, \bar{z}]$ are $F(\sigma(z)) - \bar{n}(z, \phi(z))c(\phi(z))$ and that σ is fully described by (z_1, z_2, ϕ, c).

We provide sufficient conditions for stable outcomes to be represented by (z_1, z_2, ϕ, c). The argument relies on intermediate results that give sufficient conditions for positive assortativeness and occupational stratification, Theorems 8 and 9 in Appendix.

The assumptions under which stable outcomes in knowledge economies admit a representation by (z_1, z_2, ϕ, c) combine condition C with slightly strengthened versions of the assumptions of Theorems 8 and 9 in Appendix, together with condition (ix) below. Condition (ix) requires the existence of a pair of types z and z' such that their match's surplus exceeds their self-employment payoffs: $s(z, z') > U_s(z) + \bar{n}(z, z')U_s(z')$.

D (i) Condition C holds, (ii) s and $\bar{n}$ are C^2, (iii) $Z = [\underline{z}, \bar{z}] \subseteq \mathbb{R}$, (iv) ν has a continuously differentiable and strictly positive density θ, (v) $\bar{n}$ is strictly positive and increasing in z and in z', (vi) for each stable outcome of E, $\hat{z}, z \in Z$ and $z' \in Z_\emptyset$ such that

$(\hat{z}, z), (z, z') \in p_{Z\times Z_\emptyset}(\mathrm{supp}(\gamma))$,[23]

$$\frac{\partial s(z,z')}{\partial x} + \left(\max\left\{\frac{U_s(z)}{\bar{n}(z,z')}, \frac{s(z,z)}{\bar{n}(z,z')(1+\bar{n}(z,z))}\right\} - \frac{s(z,z')}{\bar{n}(z,z')}\right)\frac{\partial \bar{n}(z,z')}{\partial x}$$
$$- \frac{1}{\bar{n}(\hat{z},z)}\frac{\partial s(\hat{z},z)}{\partial y} + \max\left\{\frac{U_s(z)}{\bar{n}(\hat{z},z)}, \frac{s(z,z)}{\bar{n}(\hat{z},z)(1+\bar{n}(z,z))}\right\}\frac{\partial \bar{n}(\hat{z},z)}{\partial y} > 0,$$ [24]

(vii) for each stable outcome of E and $z, z' \in Z$ such that $(z, z') \in p_{Z\times Z_\emptyset}(\mathrm{supp}(\gamma))$,

$$\frac{\partial s(z,z')}{\partial x} - U_s'(z) - \frac{s(z,z') - U_s(z)}{\bar{n}(z,z')}\frac{\partial \bar{n}(z,z')}{\partial x} > 0,$$

(viii) $\ln \bar{n}$ is supermodular and $\frac{\partial^2 \frac{s(x,y)\bar{n}(z,y)}{\bar{n}(x,y)}}{\partial x \partial y} > 0$ for each $x, y, z \in Z$,[25] and (ix) there exist $z, z' \in Z$ such that $U_s(z) < s(z, z') - \bar{n}(z, z')U_s(z')$.

Theorem 6 shows that an economy satisfying condition D has a unique stable outcome which is fully characterized by its representability by (z_1, z_2, ϕ, c).

Theorem 6 *Any economy E satisfying condition D has a unique stable outcome, and (γ, u) is a stable outcome of E if and only if (γ, u) is represented by (z_1, z_2, ϕ, c).*

As an easy consequence of Theorem 6 we obtain sufficient conditions for the earnings function to be strictly increasing.[26]

Corollary 1 *Let E be an economy satisfying condition D and (γ, u) be its unique stable outcome. Then u is strictly increasing if (i) $\frac{\partial s(\phi^{-1}(z),z)}{\partial y} > c(z)\frac{\partial \bar{n}(\phi^{-1}(z),z)}{\partial y}$ for each $z \in W$, (ii) $\frac{\partial s(z,\phi(z))}{\partial x} > c(\phi(z))\frac{\partial \bar{n}(z,\phi(z))}{\partial x}$ for each $z \in M$, and (iii) $U_s'(z) > 0$ for each $z \in S$.*

[23]The function $p_{Z\times Z_\emptyset} : Z \times Z_\emptyset \times N \to Z \times Z_\emptyset$ is the projection of $Z \times Z_\emptyset \times N$ onto $Z \times Z_\emptyset$.

[24]A sufficient condition for (vi) is for this inequality to hold for each $\hat{z}, z \in Z$ and $z' \in Z_\emptyset$. The usefulness of weakening this condition as in (vi) is that some pairs $(\hat{z}, z)$ or (z, z') can be easily excluded; this happens for instance when $S = \emptyset$, in which case one may assume that $(\hat{z}, z)$ and (z, z') belong to Z^2.

[25]The requirement that $\ln \bar{n}$ be supermodular plays an important role in the proof beyond being part of a set of sufficient conditions for positive assortativeness in Theorem 8. Assuming the other set of sufficient conditions for positive assortativeness in Theorem 8 together with $\ln \bar{n}$ being supermodular would amount to assume that $\bar{n}$ is a product and $\frac{\partial^2 \frac{s(x,y)}{\bar{n}(x,z)}}{\partial x \partial y} > 0$ for each $x, y, z \in Z$; as shown in the proof of Corollary 7, this is stronger than part (iii).

[26]See also Theorem 7.1 in Supplementary Material for alternative sufficient conditions for the earnings function to be increasing.

Theorem 6 and Corollary 1 require several assumptions. These assumptions can nonetheless easily be satisfied in the special cases of Section 2.3. We illustrate this point for selected cases.

First, consider a truncated Rosen's (1982) economy with production linear in n. Let E be such that $Z = [\underline{z}, \bar{z}]$ with $0 \le \underline{z} < \bar{z} < \infty$, ν has a C^1 and strictly positive density, $\underline{n} \equiv 0$, $\bar{n} \equiv \bar{n}$ for some $\bar{n} \in \mathbb{R}_+$, $U_s \equiv 0$ and, $F(z, z', n) = g(z)q(z')n$ for each $(z, z', n) \in Z^2 \times \mathbb{R}_+$. Assume that (a) $g : \mathbb{R}_+ \to \mathbb{R}_{++}$ and $q : Z \to \mathbb{R}_{++}$ are C^2, (b) g' and q' are strictly positive and (c) $\bar{n} > \max_{(z,z',\hat{z}) \in Z^3} \frac{g(\hat{z})q'(z)}{g'(z)q(z')}$. An economy satisfying this description and assumption is called a *bounded and linear Rosen economy.* Note that (c) holds provided that $\bar{n}$ is sufficiently large.

Corollary 2 *If E is a bounded and linear Rosen economy, then E has a unique stable outcome which is represented by (z_1, z_2, ϕ, c) and its earning function is strictly increasing.*

Next, analogously to the above terminology, we say that E is a *Garicano and Rossi-Hansberg economy* if $Z = [0, \bar{z}]$ with $\bar{z} < 1$, ν has a C^1 and strictly positive density, $\underline{n}(z, z') = \bar{n}(z, z') = \frac{1}{h(1-z')}$ with $0 < h < 1$, $F(z, z', n) = zn$, and $U_s(z) = z$ for each $(z, z', n) \in Z \times Z \times \mathbb{R}_+$.

Corollary 3 *If E is a Garicano and Rossi-Hansberg economy, then E has a unique stable outcome which is represented by (z_1, z_2, ϕ, c) and its earning function is strictly increasing.*

Lastly, a *simple Mak and Siow economy* is E such that $Z = [\underline{z}, \bar{z}]$ with $0 \le \underline{z} < \bar{z} < \infty$, ν has a C^1 and strictly positive density, $\underline{n} = \bar{n} \equiv 1$, $U_s \equiv 0$, and $F(z, z', 1) = f(z, z')$ for each $(z, z') \in Z^2$, where f is strictly positive, C^2 and satisfies $\frac{\partial f(x,y)}{\partial y} > 0$, $\frac{\partial^2 f(x,y)}{\partial x \partial y} > 0$ and $\frac{\partial f(z,y)}{\partial x} > \frac{\partial f(x,z)}{\partial y}$ for each $x, y, z \in Z$. Note that when $\frac{\partial^2 f(x,y)}{\partial x^2} \le 0$ and $\frac{\partial^2 f(x,y)}{\partial y^2} \le 0$, the latter condition is equivalent to $\frac{\partial f(z,\underline{z})}{\partial x} > \frac{\partial f(\bar{z},z)}{\partial y}$ for each $z \in Z$. In the case where $f(x, y) = x^\alpha y^{1-\alpha}$ for some $0 < \alpha < 1$, it holds if $\frac{\alpha}{1-\alpha} > \frac{\bar{z}}{\underline{z}}$.

Corollary 4 *If E is a simple Mak and Siow economy, then E has a unique stable outcome which is represented by (z_1, z_2, ϕ, c) and its earnings function is strictly increasing.*

4.2 Rosen Meets Garicano and Rossi-Hansberg

Building on the characterization in the previous section, this section formally compares the models of Rosen (1982) and Garicano and Rossi-Hansberg (2004) to understand why these two workhorse models in the organization literature deliver distinct implications for worker-manager matching and the wage distribution. We show that both can be represented as knowledge economies that differ in only three respects. We will show that their contrasting conclusions are driven primarily by differences in the number of workers each manager can hire.

To establish the above conclusions, we decompose the effect of the differences in the two settings by considering several economies. These include the economies of Rosen (1982) and Garicano and Rossi-Hansberg (2004), as well as intermediate economies that connect the two by changing one of the three elements at a time. In all the economies we consider in this section, we assume $Z = [\underline{z}, \bar{z}]$ with $0 < \underline{z} < \bar{z} < 1$, $\underline{n} \equiv 0$, and ν has a continuously differentiable and strictly positive density θ. Define $n^* : Z \to \mathbb{R}_+$ by setting, for each $z \in Z$, $n^*(z) = 1/[h(1-z)]$, and assume

$$h \leq \frac{1}{2\bar{z}} - 1 + \sqrt{1 + \frac{1}{4\bar{z}^2}}. \tag{5}$$

For example, if the most knowledgeable agent can solve 80% of the problems (i.e. $\bar{z} = 0.8$), then (5) is satisfied for $h \leq 0.8$. Furthermore, (5) is satisfied independently of $\bar{z}$ if $h \leq (\sqrt{5}-1)/2 \simeq 0.618$.[27]

The remaining elements are parameterized by continuous functions $f : Z \to \overline{\mathbb{R}}_+$ and $g : Z \to \mathbb{R}_+$, and $\alpha \in [0, 1)$. Specifically, let $E(f, g, \alpha)$ denote the economy where $\bar{n}(z, z') = f(z')$, $U_s(z) = g(z)$, and $F(z, z', n) = z^{1+\alpha} n^{1-\alpha}$ for each $(z, z', n) \in Z \times Z \times \mathbb{R}_+$. For each $\alpha \in (0, 1)$, the economy $E_{r,\alpha} = E(\infty, 0, \alpha)$ is a particular case of Rosen's (1982) setting, and Garicano and Rossi-Hansberg's (2004) setting can be represented by $E_{\text{grh}} = E(n^*, i_Z, 0)$, where i_Z denotes the identity on Z.[28] This formalization then makes clear that Garicano

[27]This assumption on h ensures occupational stratification in all the economies we consider.

[28]This representation of Garicano and Rossi-Hansberg's (2004) setting differs from the one in Section 2.3 in the specification of $\underline{n}$, which equals n^* in Section 2.3 and 0 in this section. Corollaries 3 and 5 show that this change makes no difference for the set of stable matchings, hence both are equivalent descriptions of Garicano and Rossi-Hansberg's (2004) model. Setting $\underline{n} \equiv 0$ has the advantage that then this element is common

and Rossi-Hansberg's (2004) setting differs from the above particular case of Rosen's (1982) setting only in three respects: the bound on the number of workers each manager can hire $\bar{n}$, the self-employed payoffs U_s, and the factor share of workers in the production function F.

We decompose the differences between E_{grh} and $E_{r,\alpha}$ as follows. First, we consider an intermediate economy $E(n^*, 0, 0)$, which differs from $E_{\text{grh}} = E(n^*, i_Z, 0)$ only by self-employed payoffs.[29] We show that the stable matchings in the two economies coincide when no agents are self-employed in E_{grh}, and are otherwise qualitatively similar.[30] Next, we consider another intermediate economy $E(n^*, 0, \alpha)$ which differs from $E(n^*, 0, 0)$ only by the factor share of workers in the production function. We show that the correspondence $\Phi(\alpha)$ of the set of stable matchings of $E(n^*, 0, \alpha)$ is continuous at $\alpha = 0$, so the stable matchings of the two markets are qualitatively similar. Thus, the qualitative differences between the stable matchings of E_{grh} and $E_{r,\alpha} = E(\infty, 0, \alpha)$ are driven by managers' hiring capacity: n^* in Garicano and Rossi-Hansberg (2004), but unbounded in Rosen (1982).

Formally, we first use Theorem 6 to characterize the stable matching of E_{grh}.

Corollary 5 *The economy E_{grh} has a unique stable matching which is represented by (z_1, z_2, ϕ, c), where c is strictly increasing.*

We also use Theorem 6 to characterize the stable matching of $E(n^*, 0, 0)$, which differs from $E_{\text{grh}} = E(n^*, i_Z, 0)$ only in self-employed payoffs.

Corollary 6 *The economy $E(n^*, 0, 0)$ has a unique stable matching which is represented by (z_1, z_2, ϕ, c), where $z_1 = z_2$ and c is strictly increasing.*

Corollaries 5 and 6 imply that the stable matchings of E_{grh} and $E(n, 0, 0)$ have the same properties. In fact, when the parameters $(\theta, h, \underline{z}, \bar{z})$ are such that the stable matching of E_{grh} has no self-employed individuals, then E_{grh} and $E(n^*, 0, 0)$ have the same stable matching.[31] When $(\theta, h, \underline{z}, \bar{z})$ are such that the stable matching of E_{grh} has self-employed

between the knowledge economies representing the models of Rosen (1982) and Garicano and Rossi-Hansberg (2004).

[29]We note that $E(n^*, 0, 0)$ is the setting of Antràs, Garicano, and Rossi-Hansberg (2006).

[30]We state the results in this section for matchings instead of outcomes for technical reasons. Specifically, it is more convenient to show convergence in the space of matchings.

[31]Indeed, if μ is the stable matching of E_{grh} and $S = \emptyset$, then μ is a stable matching of $E(n^*, 0, 0)$ since the incentive to become self-employed is not greater in $E(n^*, 0, 0)$ than in E_{grh}. The conclusion then follows because $E(n^*, 0, 0)$ has a unique stable matching.

individuals, then the stable matchings of E_{grh} and $E(n^*, 0, 0)$ are not the same but have the same properties: occupations are ordered in the same way, managers and workers are matched in a strictly increasing i.e. positive assortative way, wages are strictly increasing, and, despite different initial conditions, the wage and assignment functions c and ϕ satisfy the same differential equations. In the sense of this paragraph, the difference in the self-employed payoffs between E_{grh} and $E_{r,\alpha}$—the only difference between E_{grh} and $E(n^*, 0, 0)$—cannot explain the differences in their stable matchings.

We next argue that the difference in α cannot explain the differences in the stable matchings of E_{grh} and $E_{r,\alpha}$. To this end, consider the economy $E(n^*, 0, \alpha)$ which differs from $E(n^*, 0, 0)$ only in α. Theorem 7 below shows that the stable matchings of $E(n^*, 0, \alpha)$ are close to the stable matching of $E(n^*, 0, 0)$ when α is close to zero. For each $\alpha \in [0, 1)$, let $\Phi(\alpha)$ be the set of stable matchings of $E(n^*, 0, \alpha)$.

Theorem 7 *The correspondence Φ has nonempty values for each $\alpha \in [0, 1)$ and is continuous at $\alpha = 0$.*

In the sense of Theorem 7, the difference in the production function (i.e. α) between E_{grh} and $E_{r,\alpha}$—the only difference between $E(n^*, 0, 0)$ and $E(n^*, 0, \alpha)$—cannot explain the differences in their stable matchings. Thus, these must be caused by the differences in the number of workers a manager can hire, which can also be seen by comparing $E_{r,\alpha} = E(\infty, 0, \alpha)$ and $E(n^*, 0, \alpha)$ since they differ only in the bounds on firm size.

Theorem 11 in Appendix characterizes worker-manager matching and the wage distribution in the economy $E_{r,\alpha}$. It shows that, in any stable matching of $E_{r,\alpha}$, workers, who are those with a knowledge level in $[\underline{z}, z_1]$, receive wage $w > 0$ and that every manager with knowledge $z \in [z_1, \bar{z}]$ hires the same number $n(z, w)$ of workers. The actual matching between workers and managers is indeterminate and can be done in any way such that the feasibility condition holds. The marginal type is z_1 and is, therefore, indifferent between being a worker or a manager.

Thus, in contrast to the case of $E(n^*, 0, \alpha)$, $E(n^*, 0, 0)$ and E_{grh}, the stable matchings of $E_{r,\alpha}$ are not unique, the matching of managers and workers is indeterminate and, thus, need not be positive assortative, and wages are constant. The key difference between $E_{r,\alpha}$ and the other economies is that the number of workers a manager can hire is unbounded in $E_{r,\alpha}$.

5 Understanding Worker-Manager Matching and the Wage Distribution: Quantitative Analysis

In this section, we evaluate the ability of the knowledge economy model to match trends in earnings inequality in the United States. We focus on the wage polarization documented in Acemoglu and Autor (2011), the erosion of middle-distribution earnings relative to both tails in the period 1988-2008. Between 1988 and 2008, relative changes in earnings form a U-shaped pattern: percentiles below and above the median rise relative to the median, in stark contrast to the generally increasing pattern observed in the earlier period of 1974-1988.

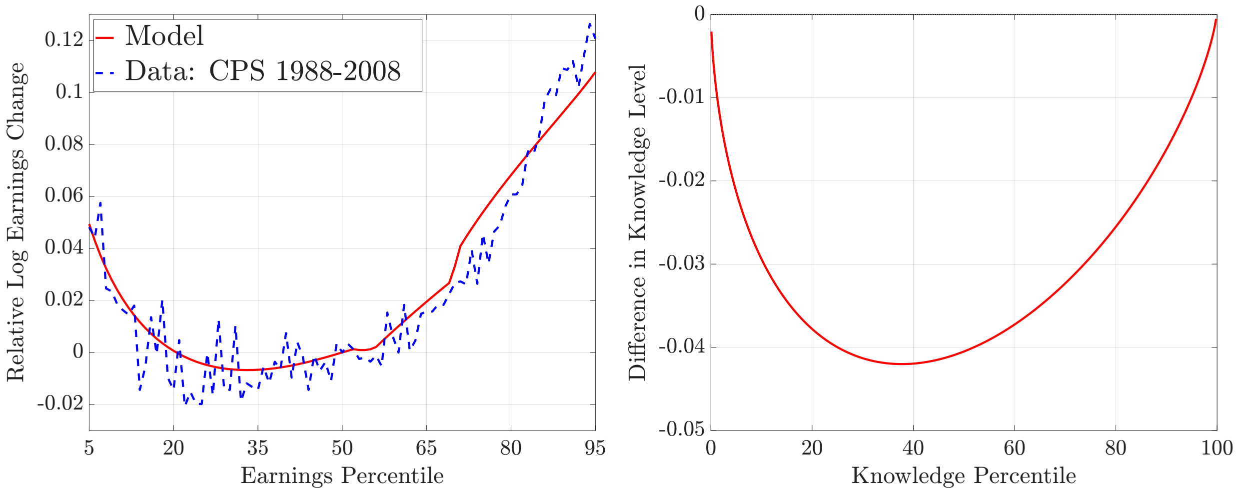
We consider several specifications of knowledge economies. We show that the knowledge economy similar to Garicano and Rossi-Hansberg (2004) accounts well for the wage polarization but fails to match the U.S. establishment-size distribution. We then consider a more flexible version of Garicano and Rossi-Hansberg (2004) and show that it can jointly match wage polarization and the establishment-size distribution.

5.1 Wage Polarization in Garicano and Rossi-Hansberg (2004)

We ask whether the knowledge economy of Garicano and Rossi-Hansberg (2004) can account for the wage polarization. This economy is specified by setting $Z = [\underline{z}, \bar{z}]$ with $\underline{z} = 0.001$ and $\bar{z} = 0.999$, $\underline{n}(z, z') = \bar{n}(z, z') = \frac{1}{h(1-z')}$, $U_s(z) = z$ and $F(z, z', n) = zn$. This leaves two elements to calibrate to the U.S. wage distributions in 1988 and 2008: the knowledge distribution and the communication costs parameter h. We specify the knowledge distribution via a Beta distribution, Beta(β_1, β_2), which is more flexible than the uniform distribution used in Garicano and Rossi-Hansberg (2004). The uniform case corresponds to $\beta_1 = \beta_2 = 1$.

We solve discrete versions of the surplus-maximization problem and its dual described in Section 3.4 to obtain the unique stable outcome.[32] We calibrate (h, β_1, β_2) by minimizing the sum of squared residuals between model and data earnings percentiles from the 5th to

[32]See Appendix C for computational details.



(A) Wage Polarization **(B)** Knowledge Difference, 1988-2008

Figure 1: Wage Polarization and Knowledge Distribution in Garicano and Rossi-Hansberg (2004). Panel A shows changes in log earnings percentiles relative to the median between 1988 and 2008. The data come from the Current Population Survey. Panel B shows the change in knowledge level at each percentile between 1988 and 2008.

the 95th. The data come from Acemoglu and Autor (2011), who use the Current Population Survey (CPS) to construct usual hourly wages for all wage workers employed during the reference week, drawing on the Outgoing Rotation Group samples. The sample includes workers aged 16–64 who work full-time, full-year, defined as at least 35 hours per week and 40 weeks per year. For each year, they compute the 5th through 95th percentiles of log hourly wages, excluding the self-employed and those in military occupations, and apply three-year moving averages. The log median wage is normalized to zero.

The calibration yields $(h, \beta_1, \beta_2) = (0.73, 1.33, 1.52)$ for 1988 and $(h, \beta_1, \beta_2) = (0.70, 1.16, 1.50)$ for 2008. Communication costs decrease by 4.3% over the period. The implied knowledge distribution has thinner tails than a uniform distribution and becomes more right-skewed, shifting mass toward low-knowledge workers. Panel A of Figure 1 shows the resulting changes in the earnings distribution. The model successfully replicates wage polarization, and the magnitude of the earnings changes is broadly consistent with the data.

Panel B of Figure 1 plots the change in knowledge across percentiles (i.e. the fraction of problems each percentile can solve) between 1988 and 2008. The 1988 knowledge distribution first-order stochastically dominates the 2008 distribution: individuals at pth percentile in

2008 can solve fewer problems than individuals at pth percentile in 1988. This change to the knowledge distribution can be interpreted, as described in Caicedo, Lucas, and Rossi-Hansberg (2019), as "a shift in the level of the complexity and profitability of technologies relative to the distribution of knowledge in the population," namely, between 1988 and 2008, technology has become more complex relative to knowledge available in the population.[33] Because knowledge at each percentile also corresponds to output per unit of labor, Panel B can be interpreted as technological change biased toward the most knowledgeable individuals, i.e. a skill-biased technological change.

In summary, the knowledge economy of Garicano and Rossi-Hansberg (2004) accounts reasonably well for the polarization of the US wage distribution in the period 1988-2008. Improvement of technology relative to available knowledge plays an important role, so does a decrease in communication costs which shifts the span-of-control function downwards at all knowledge levels. Relative to Caicedo, Lucas, and Rossi-Hansberg (2019), our result shows that a simpler model can match the evolution of the U.S. wage distribution. While it confirms their emphasis on the importance of improvements in technology relative to available knowledge, we also find that changes in communication costs are quantitatively important, as suggested by Garicano and Rossi-Hansberg (2015).

However, an important empirical challenge remains: the calibrated model implies a counterfactual establishment-size distribution. To assess this, we use the Business Dynamics Statistics, which provide annual information on all U.S. establishments since 1978, and compute the establishment-size distribution in 1988 using three-year moving averages.[34] In the knowledge economy model, each match corresponds to an establishment (firm), and establishment-size, excluding the manager, equals $1/[h(1-z)]$ when worker knowledge is z. Figure 2 shows the resulting establishment-size distribution in comparison to the 1988 U.S. distribution.

It is clear that the knowledge economy of Garicano and Rossi-Hansberg (2004) cannot

[33] Caicedo, Lucas, and Rossi-Hansberg (2019) consider the case, discussed in Section 2.3.2, where there is a problem distribution G in addition to the knowledge distribution ν. This case is, as we described in Section 2.3.2, equivalent to one where the problem distribution is uniform as in this section and the knowledge distribution is $\nu \circ G^{-1}$. If Γ denotes the cdf of the knowledge distribution ν, then the cdf of $\nu \circ G^{-1}$ is $\Gamma \circ G^{-1}$, which makes clear that changes to the knowledge distribution in this section are changes of Γ relative to G.

[34] The 2008 establishment-size distribution is similar to that in 1988.

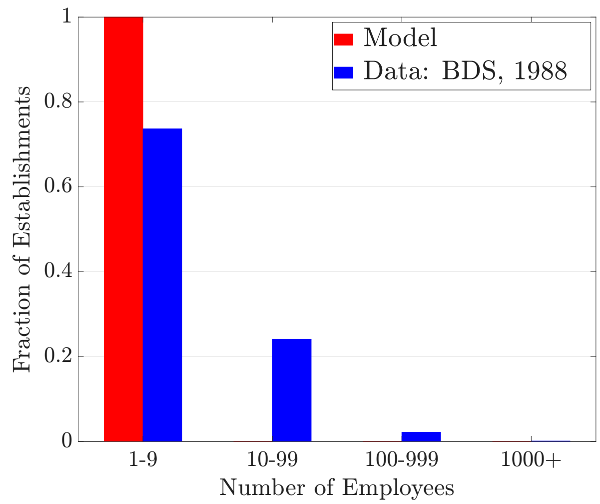


Figure 2: Establishment-Size Distribution in Garicano and Rossi-Hansberg (2004). The figure shows the establishment-size distribution in 1988. The data come from the Business Dynamics Statistics.

account for the 1988 US establishment-size distribution. The model generates a highly compressed size distribution: establishment size ranges from 1.36 to 2.59, so virtually all mass falls in a single bin. More generally, the span of control of the biggest establishment relative to the smallest is $\frac{[h(1-z_1)]^{-1}}{[h(1-\underline{z})]^{-1}} = \frac{1-\underline{z}}{1-z_1}$, where z_1 is the knowledge of the most knowledgeable worker. In the 1988 calibration, $z_1 = 0.47$, implying that the largest establishment is only about 1.9 times the smallest. Thus, the model delivers very limited dispersion in establishment size. Achieving a realistic establishment-size distribution in the the knowledge economy of Garicano and Rossi-Hansberg (2004) would require a higher z_1, but this would compress the range of manager knowledge and generate too little earnings inequality. This trade-off points to the need for more general technology functions, which we develop in the next section.

5.2 A Flexible Knowledge Economy

We consider a knowledge economy that relaxes the functional form assumptions used in Section 5.1. Specifically, we assume $\underline{n}(z, z') = \bar{n}(z, z') = \frac{1}{h(1-z')^\sigma}$ and $F(z, z', n) = z^\alpha n$. This specification introduces two new parameters: α, the elasticity of output with respect to manager knowledge, and σ, which governs how span of control varies with worker knowledge.[35]

[35]This generalization of Garicano's (2000) span of control function has also been considered by Kapicka and Slavik (2021), who argue that it is necessary to jointly match the wage distribution and the fraction of

The economy of Garicano and Rossi-Hansberg (2004) is nested as the special case $(\alpha, \sigma) = (1, 1)$.

The parameters $(\alpha, \sigma, h, \beta_1, \beta_2)$ shape the matching between managers and workers as well as the wage and establishment-size distributions. We first calibrate them to the 1988 economy by minimizing the maximum of the sums of squared residuals for the earnings distribution and the establishment-size distribution. We then hold the knowledge distribution parameters (β_1, β_2) fixed at their 1988 values and calibrate the remaining technology parameters to the 2008 economy.[36,37] This procedure yields $(\alpha, \sigma, h, \beta_1, \beta_2) = (0.80, 39.25, 0.27, 1.18, 22.38)$ for 1988 and $(\alpha, \sigma, h) = (0.92, 33.76, 0.24)$ for 2008.

Panel A of Figure 3 shows that this economy replicates the observed wage polarization between 1988 and 2008 very well.[38] A major improvement relative to the Garicano and Rossi-Hansberg's (2004) economy of Section 5.1 appears in Panel B, which reports the establishment-size distribution in 1988; although the model still understates the thin right tail, it successfully captures the wide variation in establishment sizes.[39] A further notable result is that the flexible model, calibrated to match the 5th to 95th percentiles of the income distribution and the establishment-size distribution, also replicates the top of the income distribution closely. In particular, the untargeted top 1% income share is 10.8% in 1988 and 12.5% in 2008, compared with 10.6% and 13.7% in recent U.S. estimates reported by Auten and Splinter (2024).[40] This contrasts sharply with the Garicano and Rossi-Hansberg's (2004) economy in Section 5.1, which implies much lower top income shares of 2.1% in 1988 and 2.4% in 2008. Establishment sizes are also much lower in the economy in Section 5.1

managers in the data.

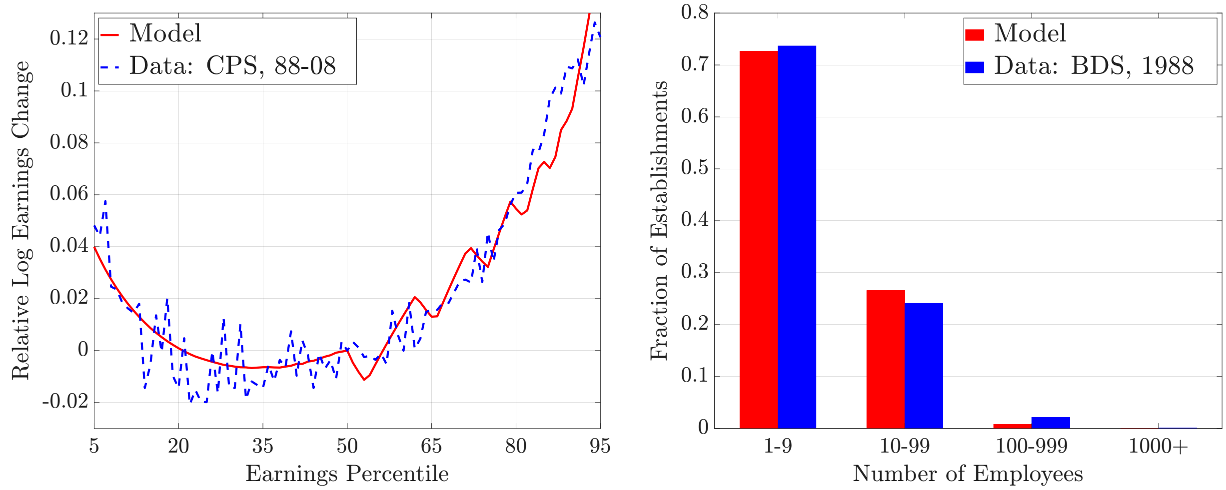
[36]As alluded in Footnote 33 and pointed out in more detail in Section 6 in Supplementary Material to this paper, changes to the technology elements of a knowledge economy are relative to its knowledge distribution; hence, it makes sense to change only one of the two. Changes to α and σ have the advantage of being easier to interpret than changes to β_1 and β_2.

[37]We employ a global optimization algorithm that combines local search with quasi-random exploration based on a Sobol sequence since the model is highly nonlinear; see Appendix C for details. The appendix also provides informal identification arguments based on numerical comparative statics.

[38]The wage distributions for each year are shown in Appendix Figure A2. They show that the earnings are an increasing and smooth function of the percentiles of the knowledge distribution; thus, the zig-zag pattern in Panel A of Figure 3 is not due to non-monotonic earning functions.

[39]The model matches the establishment-size distribution in 2008 well too.

[40]Estimates of top income shares are subject to ongoing methodological debate. Piketty, Saez, and Zucman (2024) re-evaluate Auten and Splinter's (2024) methodology and report higher top 1% shares (14.4% in 1988 and 17.7% in 2008), while Auten and Splinter (2025) respond and provide updated estimates that remain lower (10.7% and 13.8%). All numbers reported here are three-year moving averages.



(A) Wage Polarization

(B) Firm Size Distribution

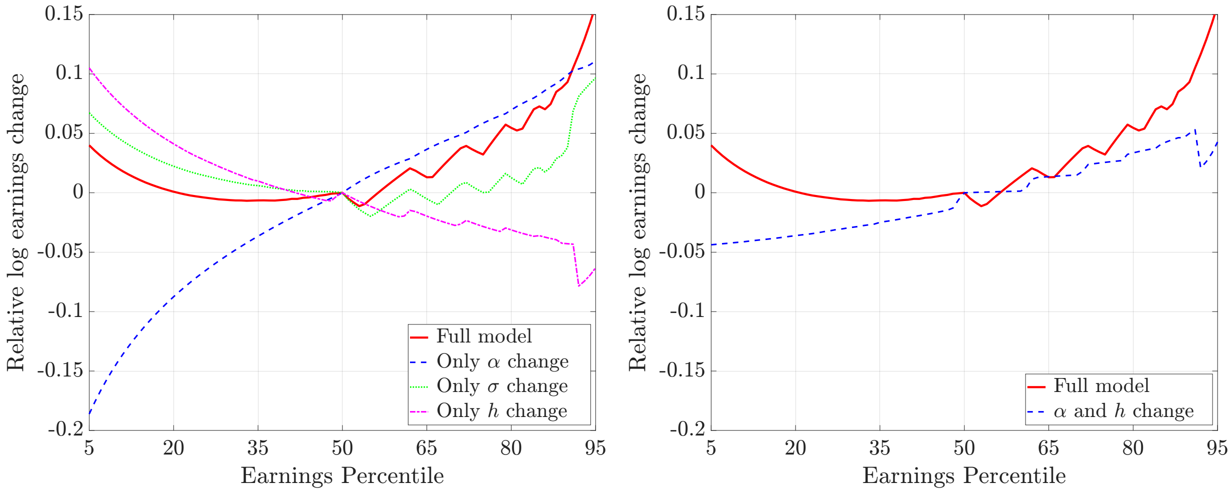
Figure 3: Wage Polarization and Establishment-Size Distribution in the Flexible Knowledge Economy Model. Panel A presents changes in log earnings percentiles relative to the median between 1988 and 2008. Panel B shows the establishment-size distribution in 1988.

relative to the data, which suggests that jointly matching the wage and the establishment-size distributions is important to account for the top 1% income shares.

The calibration implies a right-skewed knowledge distribution and yields $\alpha < 1$, so production is concave in manager knowledge. The model implies that 7.8% of agents are managers in both 1988 and 2008, compared with 10.9% in the data for 2008.[41] Because the knowledge distribution is highly concentrated on small values of z, the most knowledgeable worker z_1 is roughly 0.12, and the median type is roughly 0.038. The span-of-control function becomes very steep near $\bar{z}$ and reaches high values: a manager hiring workers of type z_1 can employ roughly 554 workers. The model can therefore generate relatively large establishments. The calibration implies no self-employment, so z_1 also coincides with the knowledge of the least knowledgeable manager, $z_2 = z_1$.

Wage polarization is explained by a 15% increase in α, a 14% decrease in σ, and a 11% decrease in h. To interpret these changes, Figure 4 reports changes in earnings relative to the median under counterfactuals in which only one parameter changes from its 1988 value to its 2008 value. The red solid line shows wage polarization in the calibrated model (Panel A of Figure 3), which matches the rise in both tails.

[41]Source: Labor Force Statistics from the CPS, U.S. Bureau of Labor Statistics, Table 9.



(A) One parameter changes

(B) Changes to α and h

Figure 4: Wage Polarization: Counterfactual Exercises. Panel A shows wage polarization implied by counterfactuals in which only one parameter changes between 1988 and 2008. Panel B shows wage changes implied by the counterfactual in which only α and h change between 1988 and 2008.

The blue dashed line shows a counterfactual in which only the elasticity of output with respect to manager knowledge, α, changes from 0.80 to 0.92. The rise in α increases the sensitivity of production to manager knowledge. Without any changes to wages, the payoff of z_1 as a manager would then decline, so the wage of the least knowledgeable individuals (who are hired by z_1) must decline. As Panel A shows, this decline is decreasing in knowledge and generates a large increase in upper-tail earnings and a large decrease in lower-tail earnings. As discussed in Section 5.1, this change can be interpreted both as an increase in technological complexity relative to available knowledge and as a skill-biased technological change, i.e. a technological change that raises the relative productivity of highly knowledgeable individuals.

The pink dash-dotted line in Panel A shows a counterfactual in which only the economy-wide communication costs, h, decline from 0.27 to 0.24. This change implies that managers can oversee larger teams, thus the worker-manager cutoff z_1 increases. In particular, the marginal manager before the change must be better off as a worker than as a manager, which means that the wage of the least knowledgeable worker increases.

The combined changes in α and h cannot generate the observed polarization, as shown in Panel B of Figure 4, since they imply wage changes that are increasing in percentiles. The change of σ helps explaining polarization since, as the green dotted line in Panel A shows, it alone leads to a polarized change in wages. The parameter σ governs how worker knowledge translates into organizational scale through the span-of-control function $1/[h(1-z)^{\sigma}]$. A fall in σ makes scale less sensitive to worker knowledge, broadly consistent with codification and routinization of tasks, collapsing the scalability advantage of high-z workers while affecting low-z workers relatively less. Wages decline, and this decline is more pronounced for high-z workers than for low-z workers. As a result, z_1 decreases from 0.120 to 0.115.

Putting these three channels together yields an account of wage polarization that, in addition to the standard effects—skill-biased technological change and decrease in communication costs—already present in the Garicano and Rossi-Hansberg's (2004) knowledge economy, adds a decline in the scalability advantage of the most knowledgeable workers.

6 Concluding Remarks

This paper provides a unified framework for knowledge economies and offers tools, in the form of both theoretical results and computational methods, to study them. Specifically, it shows that the standard solution concepts used to analyze them—competitive equilibrium and stable outcome—are equivalent to the solutions of (infinite-dimensional) linear programming problems which are useful both for characterization and for computation.

The above is used to give sufficient conditions for stable outcomes of knowledge economies to be characterized by strictly increasing wages, positive assortativeness and occupational stratification. It is also used to quantitatively evaluate their ability to account for the wage polarization during the 1988-2008 period and the establishment-size distribution in the United States. Accounting for both is not straightforward, but we show that a flexible version of the Garicano and Rossi-Hansberg's (2004) economy is able to do it. Wage polarization is explained via a combination of an improvement of technology relative to available knowledge and a change of the managers' span-of-control function whereby it increases for the least knowledgeable workers and is everywhere flatter. The main difference relative to Garicano and Rossi-Hansberg's (2004) economy is that this function grows much faster, which is key to its ability to match the US establishment-size distributions.

An important question concerns the impact of richer organizational forms, and of screening in particular, on the distribution of wages and establishment sizes. We plan in a future paper to embed the optimal organization setting of Carmona and Laohakunakorn (2024a) in the framework of this paper and use it to answer it.

References

Acemoglu, D., and D. Autor (2011): "Skills, Tasks and Technologies: Implications for Employment and Earnings," in *Handbook of Labor Economics, vol. 4*, ed. by D. Card, and O. Ashenfelter. North-Holland, Amsterdam.

Adenbaum, J. (2023): "Endogenous Firm Structure and Worker Specialization," University of Edinburgh.

Amaral, P., and A. Rivera-Padilla (2024): "Cross-Country Income Dispersion, Human Capital, and Technology Adoption," California State University Fullerton.

Antràs, P., L. Garicano, and E. Rossi-Hansberg (2006): "Offshoring in a Knowledge Economy," *Quarterly Journal of Economics*, 121, 31–77.

Auten, G., and D. Splinter (2024): "Income Inequality in the United States: Using Tax Data to Measure Long-Term Trends," *Journal of Political Economy*, 132, 2179–2227.

——— (2025): "Reply to Piketty, Saez, and Zucman (2024): Income Inequality in the United States," Office of Tax Analysis, U.S. Treasury and Joint Committee on Taxation.

Autor, D. H., L. F. Katz, and M. S. Kearney (2008): "Trends in U.S. Wage Inequality: Revising the Revisionists," *The Review of Economics and Statistics*, 90, 300–32.

Autor, D. H., F. Levy, and R. J. Murnane (2003): "The Skill Content of Recent Technological Change: An Empirical Exploration," *The Quarterly Journal of Economics*, 118, 1279–1333.

Becker, G., and K. Murphy (1992): "The Division of Labor, Coordination Costs, and Knowledge," *Quarterly Journal of Economics*, 107, 1137–1160.

Beiglböck, M., and C. Griessler (2019): "A Land of Monotone Plenty," *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze*, 19, 109–127.

Bogachev, V. (2007): *Measure Theory*, vol. II. Springer, Berlin.

Caicedo, S., R. Lucas, and E. Rossi-Hansberg (2019): "Learning, Career Paths, and the Distribution of Wages," *American Economic Journal: Macroeconomics*, 11, 49–88.

Carmona, G., and K. Laohakunakorn (2024a): "Improving the Organization of Knowledge in Production by Screening Problems," *Journal of Political Economy*, 132, 1290–1326.

——— (2024b): "Stable Matching in Large Markets with Occupational Choice," *Theoretical Economics*, 19, 1261–1304.

Chiappori, P.-A. (2017): *Matching with Transfers*. Princeton University Press, Princeton.

Chiappori, P.-A., R. McCann, and L. Nesheim (2010): "Hedonic Price Equilibria, Stable Matching, and Optimal Transport: Equivelence, Topology, and Uniqueness," *Economic Theory*, 42, 317–354.

Debreu, G. (1959): *The Theory of Value*. Wiley, New York.

DePree, J., and C. Swartz (1989): *Introduction to Real Analysis*. Wiley.

Freund, L. (2023): "Superstar Teams: The Micro Origins and Macro Implications of Coworker Complementarities," University of Cambridge.

Fuchs, W., L. Garicano, and L. Rayo (2015): "Optimal Contracting and the Organization of Knowledge," *Review of Economic Studies*, 82, 632–658.

Gabaix, X., and A. Landier (2008): "Why Has CEO Pay Increased so Much?," *Quarterly Journal of Economics*, 123, 49–100.

Garicano, L. (2000): "Hierarchies and the Organization of Knowledge in Production," *Journal of Political Economy*, 108, 874–904.

Garicano, L., and E. Rossi-Hansberg (2004): "Inequality and the Organization of Knowledge," *American Economic Review*, 94, 197–202.

——— (2006): "Organization and Inequality in a Knowledge Economy," *Quarterly Journal of Economics*, 121, 1383–1435.

——— (2015): "Knowledge-Based Hierarchies: Using Organizations to Understand the Economy," *Annual Review of Economics*, 7, 1–30.

Gola, P. (2021): "Supply and Demand in a Two-Sector Matching Model," *Journal of Political Economy*, 129, 940–978.

Greinecker, M., and C. Kah (2021): "Pairwise Stable Matching in Large Economies," *Econometrica*, 89, 2929–2974.

Gretsky, N., J. Ostroy, and W. Zame (1992): "The Nonatomic Assignment Model," *Economic Theory*, 2, 103–127.

——— (1999): "Perfect Competition in the Continuous Assignment Model," *Journal of Economic Theory*, 88, 60–118.

Hildenbrand, W. (1974): *Core and Equilibria of a Large Economy*. Princeton University Press, Princeton.

Hornstein, A., and E. C. Prescott (1993): "The Firm and the Plant in General Equilibrium Theory," in *General Equilibrium, Growth, and Trade II: The Legacy of Lionel Mckenzie*, ed. by R. Becker, M. Boldrin, R. Jones, and W. Thomson. Academic Press, San Diego.

Ide, E., and E. Talamàs (2025): "Artificial Intelligence in the Knowledge Economy," *Journal of Political Economy*, 133, 0–0.

Jones, L. E. (1984): "A Competitive Model of Commodity Differentiation," *Econometrica*, 52, 507–530.

Kapicka, M., and C. Slavik (2021): "Organization of Knowledge and Taxation," CERGE-EI Working Papers wp705, The Center for Economic Research and Graduate Education - Economics Institute, Prague.

Lucas, R. (1978): "On the Size Distribution of Business Firms," *Bell Journal of Economics*, 9, 508–523.

Macera, M., and H. Tsujiyama (2024): "Inequality and Technological Change," *Quantitative Economics*, 15, 427–451.

Mak, E., and A. Siow (2025): "Occupational Choice, Matching and Earnings Inequality," *Journal of Political Economy*, 133, 355–383.

McCann, R., and M. Trokhimtchouk (2010): "Optimal Partition of a Large Labor Force into Working Pairs," *Economic Theory*, 42, 375–395.

Parthasarathy, K. (1967): *Probability Measures on Metric Spaces*. Academic Press, New York.

Piketty, T., E. Saez, and G. Zucman (2024): "Income Inequality in the United States: A Comment," Paris School of Economics and UC Berkeley.

Ritto, J. (2024): "Doing Without Nominal Rigidities: Real Effects of Monetary Policy in a Monetary World," University of Toronto.

Rosen, S. (1982): "Authority, Control and the Distribution of Earnings," *Bell Journal of Economics*, 13, 311–323.

——— (2002): "Markets and Diversity," *American Economic Review*, 92, 1–15.

Tesch, G. (2012): *Ordinary Differential Equations and Dynamical Systems*. American Mathematical Society, Providence.

Varadarajan, V. S. (1958): "Weak Convergence of Measures on Separable Metric Spaces," *Sankhyā: The Indian Journal of Statistics*, 19, 15–22.

Villani, C. (2009): *Optimal Transport: Old and New*. Springer, Berlin.

Knowledge Economies

Appendix

Guilherme Carmona, Krittanai Laohakunakorn, and Hitoshi Tsujiyama

Contents

A Knowledge Economies as Large Many-to-One Matching Markets with Occupational Choice

In this section, we show how to represent a knowledge economy as a large many-to-one matching market with occupational choice, as defined in Section 2 of Supplementary Material, and define stable matchings for this model.

A.1 Knowledge Economy as a Matching Market

Consider a knowledge economy $E = (Z, \nu, \underline{n}, \bar{n}, F, U_s)$ and recall that a large many-to-one matching market with occupational choice is defined by (i) a set of types, (ii) a type distribution, (iii) a set of contracts, (iv) feasibility correspondences $P_m : Z \rightrightarrows M(Z \times C)$ and $P_s : Z \rightrightarrows \{1_{(\emptyset,c)} : c \in C\}$ for managers and self-employed, respectively, and (v) preferences $\succ_z$ for each type $z \in Z$. We use the elements of E to define these five elements of a large many-to-one matching market with occupational choice.

The set of types and the type distribution are as in E, namely Z and ν respectively. Contracts are wages and thus $C = \mathbb{R}_+$ as in Section 2. A manager of type z can choose the type of workers he hires, their wage and how many of them to hire within the bounds set by $\underline{n}$ and $\bar{n}$. Thus,

$$P_m(z) = \{n1_{(z',c)} : (z', c) \in Z \times C \text{ and } n \in \mathbb{R}_+ \text{ such that } \underline{n}(z, z') \leq n \leq \bar{n}(z, z')\}.$$

Self-employed individuals receive a wage of zero (as they receive a rent, not a wage), thus $P_s(z) = \{1_{(\emptyset,0)}\}$.

Individual preferences are defined by specifying payoff functions that depend on the individual's occupation and match. If an individual of type z chooses to be a worker, then his match is described by the type of the manager who hires him and his wage, thus, by a measure $1_{(\hat{z},c)}$ with $(\hat{z}, c) \in Z \times C$. Hence, the utility of those who choose occupation w is given by a function $(z, 1_{(\hat{z},c)}) \mapsto U_z(w, 1_{(\hat{z},c)})$ and we specify that

$$U_z(w, 1_{(\hat{z},c)}) = c \text{ for each } (z, \hat{z}, c) \in Z \times Z \times C$$

so that each worker's preferences are represented by his wage.

The match of a self-employed individual is, in general, described by the wage c or, equivalently, by a measure $1_{(\emptyset,c)}$ with $c \in C$. Hence, the utility of those who choose occupation

s is given by a function $(z, 1_{(\emptyset,c)}) \mapsto U_z(s, 1_{(\emptyset,c)})$ and we specify that

$$U_z(s, 1_{(\emptyset,c)}) = U_s(z) \text{ for each } (z, c) \in Z \times C.$$

Finally, the match of a manager is described by a measure $n1_{(z',c)}$ and the utility of those who choose occupation m is given by a function $(z, n1_{(z',c)}) \mapsto U_z(m, n1_{(z',c)})$. We specify that

$$U_z(m, n1_{(z',c)}) = F(z, z', n) - cn \text{ for each } (z, z', n, c) \in Z \times Z \times N \times C,$$

so that each manager's preferences are represented by his rent.

A.2 Stable Matchings in Knowledge Economies

We provide in this section a formal definition of stable matchings for a knowledge economy. This is done by applying the definition of a stable matching for general large many-to-one matching markets with occupational choice to the one defined in the previous section.

Recall that a matching in an economy E is a Borel measure μ on $Z \times X_\emptyset$ such that $\operatorname{supp}(\mu) \subseteq \operatorname{graph}(P_m) \cup \operatorname{graph}(P_s)$, and $\nu_M + \nu_S + \nu_W = \nu$. Roughly, a matching is stable if no one can increase his utility by changing his occupation or matching with his targets or both. Someone's targets are those who are better off with him than in their current match.

Given a matching μ, a type $z \in Z$ and an occupation $a \in \{w, m, s\}$, individuals of type z can target certain types and wages (z^*, c) when choosing occupation a provided that someone of type z^* is better off with someone of type z with occupation a and with wage c than in his current match. The set of such (z^*, c) is denoted by $T_z^a(\mu)$.

The targets for the prospective self-employed are the type-wage pairs that are feasible when someone is unmatched: For each $z \in Z$, let $T_z^s(\mu) = \{(\emptyset, 0)\}$.

The targets of prospective managers are as follows. Someone of type z^* can be targeted at wage c by anyone of type z who chooses to be a manager when c is bigger than the payment that such type z^* individual obtains in his current match. If such z^* individual is a worker, then

(a) there exists $(z', \delta', c') \in Z \times X \times C$ such that $(z', \delta') \in \operatorname{supp}(\mu)$, $(z^*, c') \in \operatorname{supp}(\delta')$ and $c > c'$,

i.e. z' is the type of the manager that hires him, c' is his wage and $\delta' = n1_{(z^*,c')}$ for some $n \in [\underline{n}(z', z^*), \bar{n}(z', z^*)]$ describes the workforce of his manager; since such z^* individual is a worker, his utility in the current match is c' and, hence, $c > c'$ for him to prefer to be matched with a type z at wage c. If such z^* individual is self-employed, then

(b) there exists $\delta' \in X_\emptyset \setminus X$ such that $(z^*, \delta') \in \text{supp}(\mu)$ and $c > U_s(z^*)$

since $(z^*, \delta') \in \text{supp}(\mu)$ and $\delta' \in X_\emptyset \setminus X$ indicate that he his self-employed and, hence, $c > U_s(z^*)$ for him to prefer to be matched with a type z at wage c. Finally, when such z^* individual is a manager,

(c) there exists $\delta' \in X$ such that $(z^*, \delta') \in \text{supp}(\mu)$ and $c > U_{z^*}(m, \delta')$.

Thus, for each $z \in Z$, let $T_z^m(\mu)$ be the set of $(z^*, c) \in Z \times C$ such that (a), (b) or (c) hold.[1]

Matches are pairs (z, δ) and consist, in particular, of a manager of type z and workers described by δ if $\delta \in X$ or a self-employed of type z if $\delta \in X_\emptyset \setminus X$. Stability requires that none of these individuals can increase his utility by changing his occupation or matching with his targets or both. We decompose this requirement by final occupation and note that it is enough to consider the case where such final occupation is m or s.[2]

The set $S_M(\mu)$ below describes matches (z, δ) such that no one in it gains by changing his occupation-match pair by becoming a manager (from being a worker as in condition (ii) below or self-employed as in condition (iii)) or his match while remaining a manager (as in condition (i) below) and matching with his targets (or his current employees in the latter case). Formally, let $S_M(\mu)$ be the set of $(z, \delta) \in Z \times X_\emptyset$ such that, if $\delta \in X$, then

(i) there does not exist $\delta' \in P_m(z)$ such that $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ and $U_z(m, \delta') > U_z(m, \delta)$,

(ii) for each $(z', c) \in \text{supp}(\delta)$, there does not exist $\delta' \in P_m(z')$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu)$ and $U_{z'}(m, \delta') > c$,

and, if $\delta \in X_\emptyset \setminus X$, then

(iii) there does not exist $\delta' \in P_m(z)$ such that $\text{supp}(\delta') \subseteq T_z^m(\mu)$ and $U_z(m, \delta') > U_s(z)$.

The set $IR(\mu)$ describes matches (z, δ) such that no one in it gains by changing his occupation to become self-employed. Formally, let $IR(\mu)$ be the set of $(z, \delta) \in Z \times X_\emptyset$ such that, if $\delta \in X$, then

(i) $U_z(m, \delta) \geq U_s(z)$ and

(ii) $c \geq U_s(z')$ for each $(z', c) \in \text{supp}(\delta)$.

A matching μ is *stable* if $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$.

[1]It follows by Theorem 2.1 in Section 2 of Supplementary Material that the targets $T_z^w(\mu)$ of prospective workers are not needed for the definition of a stable matching.

[2]This follows by Theorem 2.1 in Section 2 of Supplementary Material.

B Proofs

B.1 Proof of Theorem 1

Let E be an economy. For each $k \in \mathbb{N}$, let $P_{k,m}(z) = \{n1_{(z',c)} : (z',c) \in Z \times [0,k]$ and $\underline{n}(z,z') \leq n \leq \bar{n}(z,z')\}$ and E_k be equal to E except for this change to P_m. Letting $E_0 = E$ and $C_k = [0,k]$ for each $k \in \mathbb{N}$, it follows by Lemma 3.8 that E_k is rich for each $k \in \mathbb{N}_0$ since, for each $z \in Z$, every $\delta \in P_{k,m}(z)$ has finite support, hence condition (β) holds. As for condition (α), let $z \in Z$, $\delta = n1_{(z',c)} \in P_{k,m}(z)$, $n > 0$ and V_δ be a neighborhood of δ. If $\underline{n}(z,z') < \bar{n}(z,z')$, then there is $\delta' = n'1_{(z',c)} \in V_\delta \cap P_{k,m}(z)$ such that $\underline{n}(z,z') < n' < \bar{n}(z,z')$; hence, using the continuity of $\underline{n}$ and $\bar{n}$, there are neighborhoods V_z and $V_{z'}$ of z and z', respectively, such that, for each $(\hat{z},\tilde{z}) \in V_z \times V_{z'}$ and $c' \in C_k$, $n'1_{(\tilde{z},c')} \in V_\delta \cap P_{k,m}(\hat{z})$. If $\underline{n}(z,z') = \bar{n}(z,z')$, then, using the continuity of $\bar{n}$, there are neighborhoods V_z and $V_{z'}$ of z and z', respectively, such that, for each $(\hat{z},\tilde{z}) \in V_z \times V_{z'}$ and $c' \in C_k$, $\bar{n}(\hat{z},\tilde{z})1_{(\tilde{z},c')} \in V_\delta \cap P_{k,m}(\hat{z})$. Hence, condition ($\alpha$) holds. Since, for each $k \in \mathbb{N}$, E_k is clearly rational, continuous and bounded (the latter since $\max_{(z,z') \in Z^2} \bar{n}(z,z') \in \mathbb{R}$), it follows by Theorem 3.1 that there exists a stable matching μ_k of E_k for each $k \in \mathbb{N}$.

Note that E satisfies all the above properties except that P_m is not compact-valued. Nevertheless, P_m is closed-valued and, hence, Lemmas 3.2–3.4 apply. It is straightforward to check that all the conditions in these lemmas are satisfied, except perhaps conditions 3 and 4 of Lemma 3.4. Regarding condition 3 (the case of condition 4 is analogous), if $z_{k_j}, \delta_{k_j}, \mu_{k_j}, \delta'$ and $V_{\delta'}$ are as in its statement, there is $\delta'_{k_j} \in \Lambda(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \cap V_{\delta'}$ for each $j \geq J$ such that $\delta'_{k_j} \to \delta'$ because E is rich. Since $\delta' = n1_{(z',c)}$ and $\delta'_{k_j} = n_{k_j}1_{(z'_{k_j}, c_{k_j})}$ for some $(z,c,n) \in Z \times C \times \mathbb{R}_+$ and $(z'_{k_j}, c_{k_j}, n_{k_j}) \in Z \times C \times \mathbb{R}_+$, it follows that $n_{k_j} \to n$. If $n > 0$, then $(z'_{k_j}, c_{k_j}) \to (z',c)$ too and, letting $M \in \mathbb{N}$ be such that $M > c$, it follows that $\delta'_{k_j} \in P_{m,k_j}(z_{k_j})$ whenever j is such that $k_j \geq M$ and $c_{k_j} < M$; if $n = 0$, then $0 = \delta' \in P_{m,k_j}(z_{k_j}) \cap \Lambda(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \cap V_{\delta'}$ for each $j \in \mathbb{N}$. Hence, condition 3 of Lemma 3.4 holds.

It then follows by Lemmas 3.2–3.4 that, to show that E has a stable matching, it suffices to establish that $\{\mu_k\}_{k=1}^\infty$ has a convergent subsequence i.e. that the conclusion of Lemma 3.1 holds. As in the proof of Lemma 3.1, it suffices to show that $\{\mu_k\}_{k=1}^\infty$ is tight. Let $\varepsilon > 0$ and let $\eta > 0$ be such that $\nu(\{z \in Z : F(z,z,\bar{n}(z,z)) < \eta$ and $U_s(z) < \eta\}) < \varepsilon/2$ if part (i) of condition A holds; such η exists since $\nu(\{z \in Z : F(z,z,\bar{n}(z,z)) = 0$ and $U_s(z) = 0\}) = 0$. Otherwise, let $\eta = 0$. Let $\hat{Z} = \{z \in Z : F(z,z,\bar{n}(z,z)) \geq \eta$ or $U_s(z) \geq \eta\}$ and note that $\hat{Z}$ is compact and $\nu(\hat{Z}^c) < \varepsilon/2$. For each $M \in \mathbb{N}$, let $P_{m,M,\eta} : \hat{Z} \rightrightarrows M(Z \times C)$ be defined by setting, for each $z \in \hat{Z}$, $P_{m,M,\eta}(z) = \{n1_{(z',c)} : z' \in Z, c \in [0,M], \underline{n}(z,z') \leq n \leq \bar{n}(z,z')\}$. Then $L_M = \text{graph}(P_{m,M,\eta}) \cup (Z \times \{1_{(\emptyset,0)}\})$ is compact and $L_M^c = (\hat{Z}^c \times X) \cup B_M$, where $B_M = \{(z,\delta) \in \hat{Z} \times M(Z \times C) : \delta \notin P_{m,M,\eta}(z)\}$. Since, for each $k \in \mathbb{N}$, $\mu_k(\hat{Z}^c \times X) \leq \nu(\hat{Z}^c) < \varepsilon/2$, it suffices to show that there exist $K, M \in \mathbb{N}$ such that $\mu_k(B_M) < \varepsilon/2$ for each $k \geq K$.

Suppose that the above claim does not hold. Then, for each $j \in \mathbb{N}$, there exists $k_j \geq j$, $z_{k_j} \in \hat{Z}$, $z'_{k_j} \in Z$, $n_{k_j} \in [\underline{n}(z_{k_j}, z'_{k_j}), \bar{n}(z_{k_j}, z'_{k_j})]$ and $c_{k_j} \in (j, \infty)$ such that $(z_{k_j}, n_{k_j} 1_{(z'_{k_j}, c_{k_j})}) \in \operatorname{supp}(\mu_{k_j})$. Since μ_{k_j} is stable, we then have that

$$0 \leq U_{z_{k_j}}(m, n_{k_j} 1_{(z'_{k_j}, c_{k_j})}) \leq \max_{(z,z',n) \in Z^2 \times [0, \max_{(z,z') \in Z^2} \bar{n}(z,z')]} F(z, z', n) - c_{k_j} n_{k_j}.$$

Thus, $n_{k_j} \to 0$ as $c_{k_j} \to \infty$, which contradicts part (ii) of condition A.

Thus, assume that part (i) of condition A holds. Then,

$$U_{z_{k_j}}(m, n_{k_j} 1_{(z'_{k_j}, c_{k_j})}) \leq F(z_{k_j}, z'_{k_j}, n_{k_j}) \to 0$$

since F is uniformly continuous (as its domain is compact) and $F(z, z', 0) = 0$ for each $(z, z') \in Z^2$. Moreover $\eta > 0$ and, for each $j \in \mathbb{N}$, either $F(z_{k_j}, z_{k_j}, \bar{n}(z_{k_j}, z_{k_j})) \geq \eta$ or $U_{z_{k_j}}(s) \geq \eta$ since $z_{k_j} \in \hat{Z}$.

First suppose that $F(z_{k_j}, z_{k_j}, \bar{n}(z_{k_j}, z_{k_j})) \geq \eta$ for infinitely many js. Let $\zeta > 0$ be such that $\eta - \zeta \max_{z \in Z} \bar{n}(z, z) > 0$. Thus, for each j sufficiently large, $U_{z_{k_j}}(m, n_{k_j} 1_{(z'_{k_j}, c_{k_j})}) < \min\{\zeta, \eta - \zeta \max_{z \in Z} \bar{n}(z, z)\}$, $(z_{k_j}, \zeta) \in T^m_{z_{k_j}}(\mu_{k_j})$ and

$$\begin{aligned} U_{z_{k_j}}(m, \bar{n}(z_{k_j}, z_{k_j}) 1_{(z_{k_j}, \zeta)}) &= F(z_{k_j}, z_{k_j}, \bar{n}(z_{k_j}, z_{k_j})) - \zeta \bar{n}(z_{k_j}, z_{k_j}) \geq \eta - \zeta \max_{z \in Z} \bar{n}(z, z) \\ &> U_{z_{k_j}}(m, n_{k_j} 1_{(z'_{k_j}, c_{k_j})}); \end{aligned}$$

but this contradicts the stability of μ_{k_j}.

Next suppose that $U_{z_{k_j}}(s) \geq \eta$ for all but finitely many js. Then for each j sufficiently large, $U_{z_{k_j}}(m, n_{k_j} 1_{(z'_{k_j}, c_{k_j})}) < U_{z_{k_j}}(s)$, which contradicts the stability of μ_{k_j}.

Thus, part (i) of condition A cannot hold. This contradiction establishes the above claim and this concludes the proof of Theorem 1.

B.2 Proof of Lemma 1

We prove Lemma 1 in a series of lemmas, the first of which establishes that each manager hires a strictly positive number of workers in any stable matching.

Lemma 2 *If μ is a stable matching and $(z, \delta) \in \operatorname{supp}(\mu)$, then $\delta \neq 0$.*

Proof. The conclusion of the lemma is obvious when $\min_{(z,z') \in Z^2} \underline{n}(z, z') > 0$. Hence, assume that part (i) of condition B holds.

Suppose that there exists $z \in Z$ such that $(z, 0) \in \text{supp}(\mu)$. Then $U_z(m, 0) = 0$ since $F(z, z', 0) - 0c = 0$ for each $z' \in Z$ and $c \in \mathbb{R}_+$.

First suppose that $F(z, z, \bar{n}(z, z)) > 0$ and let $\varepsilon > 0$ be such that $F(z, z, \bar{n}(z, z)) - \bar{n}(z, z)\varepsilon > 0$. Then $(z, \varepsilon) \in T_z^m(\mu)$ and $U_z(m, \bar{n}(z, z)1_{(z,\varepsilon)}) = F(z, z, \bar{n}(z, z)) - \bar{n}(z, z)\varepsilon > 0 = U_z(m, 0)$, contradicting the stability of μ. If $U_s(z) > 0$, then $U_s(z) > 0 = U_z(m, 0)$, contradicting the stability of μ. ■

The following lemma is the equal treatment property for workers.

Lemma 3 *If $z, \hat{z}, z' \in Z$, $n, \hat{n} \in \mathbb{R}_+$ and $c, \hat{c} \in C$ are such that $(z, n1_{(z',c)}), (\hat{z}, \hat{n}1_{(z',\hat{c})}) \in \text{supp}(\mu)$, then $c = \hat{c}$.*

Proof. It follows by Lemma 2 that $n, \hat{n} > 0$. Hence, if $c > \hat{c}$, then managers of type z can gain by hiring workers of type z' at wage $c - \varepsilon$ for some $\varepsilon > 0$ such that $c - \varepsilon > \hat{c}$, a contradiction to the stability of μ. Thus, $c \leq \hat{c}$ and an analogous argument shows that $c \geq \hat{c}$; hence, $c = \hat{c}$. ■

Define $c : W \to \mathbb{R}_+$ by setting, for each $z \in W$, $c(z) = c$, where $c \in \mathbb{R}_+$ is such that $(\hat{z}, n1_{(z,c)}) \in \text{supp}(\mu)$ for some $\hat{z} \in Z$ and $n \in \mathbb{R}_+$. Lemma 3 implies that the function c is well-defined. Lemma 4 asserts that c is bounded.

Lemma 4 *c is bounded.*

Proof. Suppose not; then let $\{z_k\}_{k=1}^{\infty}$ be such that $z_k \in W$ for each $k \in \mathbb{N}$ and $c(z_k) \to \infty$. For each $k \in \mathbb{N}$, let $(\hat{z}_k, n_k) \in Z \times \mathbb{R}_+$ be such that $(\hat{z}_k, n_k 1_{(z_k, c(z_k))}) \in \text{supp}(\mu)$. Since $(\hat{z}_k, z_k, n_k) \in Z^2 \times N$ for each k and $Z^2 \times N$ is compact (recall that $N = [0, \max_{(z,z') \in Z^2} \bar{n}(z, z')]$), we may assume that $\{(\hat{z}_k, z_k, n_k)\}_{k=1}^{\infty}$ converges; let $(\hat{z}, z, n) = \lim_k (\hat{z}_k, z_k, n_k)$.

Since $U_{\hat{z}_k}(m, n_k 1_{(z_k, c(z_k))}) \geq U_s(\hat{z}_k) \geq 0$, it follows that

$$n_k \leq \frac{\max_{(z,z',n) \in Z^2 \times N} F(z, z', n)}{c(z_k)}.$$

Thus, $n_k \to 0$. Since $(\hat{z}_k, n_k 1_{(z_k, c(z_k))}) \in \text{supp}(\mu)$ for each k, $(\hat{z}_k, n_k 1_{(z_k, c(z_k))}) \to (\hat{z}, 0)$ and $\text{supp}(\mu)$ is closed, it follows that $(\tilde{z}, 0) \in \text{supp}(\mu)$. But this contradicts Lemma 2. ■

The following lemma asserts the continuity of c and is a simple consequence of the two previous lemmas.

Lemma 5 *c is continuous.*

Proof. Let $z \in W$ and $\{z_k\}_{k=1}^{\infty}$ be such that $z_k \to z$ and $z_k \in W$ for each $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $(\hat{z}_k, n_k) \in Z \times \mathbb{R}_+$ be such that $(\hat{z}_k, n_k 1_{(z_k, c(z_k))}) \in \text{supp}(\mu)$. Since

$(\hat{z}_k, n_k) \in Z \times N$ for each k and $Z \times N$ is compact, we may assume that $\{(\hat{z}_k, n_k)\}_{k=1}^{\infty}$ converges; let $(\hat{z}, n) = \lim_k(\hat{z}_k, n_k)$.

Note that $\{c(z_k)\}_{k=1}^{\infty}$ converges to $c(z)$ if and only if every subsequence of $\{c(z_k)\}_{k=1}^{\infty}$ has a further subsequence converging to $c(z)$. Since $\{c(z_k)\}_{k=1}^{\infty}$ is bounded by Lemma 4, each subsequence of $\{c(z_k)\}_{k=1}^{\infty}$ has a convergent subsequence; let $\{c(z_j)\}_{j=1}^{\infty}$ denote such subsequence and $c = \lim_j c(z_j)$. Since $(\hat{z}_j, n_j 1_{(z_j, c(z_j))}) \in \text{supp}(\mu)$ for each j, $(\hat{z}_j, n_j 1_{(z_j, c(z_j))}) \to (\hat{z}, n 1_{(z,c)})$ and $\text{supp}(\mu)$ is closed, it follows that $(\tilde{z}, n1_{(z,c)}) \in \text{supp}(\mu)$. Then $c = c(z)$ by Lemma 3. ■

The following is the equal treatment property for managers.

Lemma 6 *If $z, z', \hat{z} \in Z$ and $n', \hat{n} \in \mathbb{R}_+$ are such that $(z, n'1_{(z', c(z'))}), (z, \hat{n}1_{(\hat{z}, c(\hat{z}))}) \in \text{supp}(\mu)$, then $U_z(m, n'1_{(z', c(z'))}) = U_z(m, \hat{n}1_{(\hat{z}, c(\hat{z}))})$.*

Proof. If $U_z(m, n'1_{(z', c(z'))}) > U_z(m, \hat{n}1_{(\hat{z}, c(\hat{z}))})$, then, letting $\varepsilon > 0$ be such that $F(z, z', n') - (c(z') + \varepsilon)n' > U_z(m, \hat{z})$, it follows that $(z', c(z') + \varepsilon) \in T_z^m(\mu)$ and $U_z(m, n'1_{(z', c(z')+\varepsilon)}) > U_z(m, \hat{n}1_{(\hat{z}, c(\hat{z}))})$, a contradiction to the stability of μ. Thus, $U_z(m, n'1_{(z', c(z'))}) \leq U_z(m, \hat{n}1_{(\hat{z}, c(\hat{z}))})$ and an analogous argument shows that $U_z(m, n'1_{(z', c(z'))}) \geq U_z(m, \hat{n}1_{(\hat{z}, c(\hat{z}))})$; hence, $U_z(m, n'1_{(z', c(z'))}) = U_z(m, \hat{n}1_{(\hat{z}, c(\hat{z}))})$. ■

Define $u : M \to \mathbb{R}_+$ by setting, for each $z \in M$, $u(z) = U_z(m, n'1_{(z', c(z'))})$, where $(z', n') \in Z \times \mathbb{R}_+$ is such that $(z, n'1_{(z', c(z'))}) \in \text{supp}(\mu)$. Lemma 6 implies that the function u is well-defined.

The following lemma shows that the function $u : M \to \mathbb{R}$ is continuous; the proof is analogous to the one of Lemma 5 and relies on the closedness of $\text{supp}(\mu)$ and the equal treatment property for managers in Lemma 6.

Lemma 7 *$u : M \to \mathbb{R}$ is continuous.*

Proof. Let $z \in M$ and $\{z_k\}_{k=1}^{\infty}$ be such that $z_k \to z$ and $z_k \in M$ for each $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $(z'_k, n_k) \in Z \times \mathbb{R}_+$ be such that $(z_k, n_k 1_{(z'_k, c(z'_k))}) \in \text{supp}(\mu)$. Since $(z'_k, n_k) \in Z \times N$ for each k and $Z \times N$ is compact, we may assume that $\{(z'_k, n_k)\}_{k=1}^{\infty}$ converges; let $(z', n) = \lim_k(z'_k, n_k)$.

We first claim that $\{u(z_k)\}_{k=1}^{\infty}$ is bounded. Indeed, for each $k \in \mathbb{N}$,

$$0 \leq U_s(z_k) \leq U_{z_k}(m, n_k 1_{(z'_k, c(z'_k))}) \leq \max_{(z, z', n) \in Z^2 \times N} F(z, z', n).$$

We have that $\{u(z_k)\}_{k=1}^{\infty}$ converges to $u(z)$ if and only if every subsequence of $\{u(z_k)\}_{k=1}^{\infty}$ has a further subsequence converging to $u(z)$. Since $\{u(z_k)\}_{k=1}^{\infty}$ is bounded, each subsequence of $\{u(z_k)\}_{k=1}^{\infty}$ has a convergent subsequence; let $\{u(z_j)\}_{j=1}^{\infty}$ denote such subsequence and

$u = \lim_j u(z_j)$. Since $(z_j, n_j 1_{(z'_j, c(z'_j))}) \in \text{supp}(\mu)$ for each j, $(z_j, n_j 1_{(z'_j, c(z'_j))}) \to (z, n 1_{(z', c(z'))})$ and $\text{supp}(\mu)$ is closed, it follows that $(z, n 1_{(z', c(z'))}) \in \text{supp}(\mu)$. Then $u(z) = U_z(m, n 1_{(z', c(z'))})$ by Lemma 6 and

$$u(z) = U_z(m, n 1_{(z', c(z'))}) = F(z, z', n) - c(z')n = \lim_j (F(z_j, z'_j, n_j) - c(z'_j) n_j)$$
$$= \lim_j U_{z_j}(m, n_j 1_{(z'_j, c(z'_j))}) = \lim_j u(z_j) = u.$$

Thus, $u = u(z)$. ■

Self-employed individuals of type z receive a payoff of $U_s(z)$. The follows lemma shows that if, for some type z, there are type z individuals with different occupations, then the payoffs from such occupations are the same at z.

Lemma 8 *The following holds: (i) $u(z) = c(z)$ if $z \in M \cap W$, (ii) $u(z) = U_s(z)$ if $z \in M \cap S$ and (iii) $c(z) = U_s(z)$ if $z \in W \cap S$.*

Proof. Let $z \in M \cap W$. Suppose that $u(z) > c(z)$ and let $(z', n') \in Z \in \mathbb{R}_+$ be such that $(z, n' 1_{(z', c(z'))}) \in \text{supp}(\mu)$. Then $n' > 0$ by Lemma 2 and let $\varepsilon > 0$ be such that $F(z, z', n') - (c(z') + \varepsilon)n' > c(z)$. Then $(z', c(z') + \varepsilon) \in T^m_z(\mu)$ and $U_z(m, n(z') 1_{(z', c(z')+\varepsilon)}) > c(z)$, contradicting the stability of μ.

If $u(z) < c(z)$, then let $(\hat{z}, n) \in Z \times \mathbb{R}_+$ be such that $(\hat{z}, n 1_{(z, c(z))}) \in \text{supp}(\mu)$. Then $n > 0$ by Lemma 2 and let $\varepsilon > 0$ be such that $F(\hat{z}, z, n) - (u(z) + \varepsilon)n > F(\hat{z}, z, n) - c(z)n$. Then $(z, u(z) + \varepsilon) \in T^m_{\hat{z}}(\mu)$ and $U_{\hat{z}}(m, n 1_{(z, u(z)+\varepsilon)}) > U_{\hat{z}}(m, n 1_{(z, c(z))})$, contradicting the stability of μ. Thus, (i) follows.

Let $z \in M \cap S$. The stability of μ implies that $u(z) \geq U_s(z)$. If $u(z) > U_s(z)$, then let $(z', n') \in Z \times \mathbb{R}_+$ be such that $(z, n' 1_{(z', c(z'))}) \in \text{supp}(\mu)$ and $\varepsilon > 0$ be such that $F(z, z', n') - (c(z') + \varepsilon)n' > U_s(z)$. It then follows that $(z', c(z') + \varepsilon) \in T^m_z(\mu)$ and $U_z(m, n' 1_{(z', c(z')+\varepsilon)}) > U_s(z)$, contradicting the stability of μ. Thus, (ii) follows.

Let $z \in W \cap S$. The stability of μ implies that $c(z) \geq U_s(z)$. If $c(z) > U_s(z)$, then $(\hat{z}, n) \in Z \times \mathbb{R}_+$ be such that $(\hat{z}, n 1_{(z, c(z))}) \in \text{supp}(\mu)$. Then $n > 0$ by Lemma 2 and let $\varepsilon > 0$ be such that $F(\hat{z}, z, n) - (U_s(z) + \varepsilon)n > F(\hat{z}, z, n) - c(z)n$. Then $(z, U_s(z) + \varepsilon) \in T^m_{\hat{z}}(\mu)$ and $U_{\hat{z}}(m, n 1_{(z, U_s(z)+\varepsilon)}) > U_{\hat{z}}(m, n 1_{(z, c(z))})$, contradicting the stability of μ. Thus, (iii) follows. ■

The next step is to combine $c : W \to \mathbb{R}_+$, $u : M \to \mathbb{R}$ and $z \mapsto U_s(z)$. Before that, we establish that the sets M, S and W are closed, a property that is needed for the argument.

Lemma 9 *W is closed.*

Proof. Let $\{z_k\}_{k=1}^{\infty}$ be a convergent sequence in W and $z = \lim_k z_k$. For each k, let $(\hat{z}_k, n_k) \in Z \times N$ be such that $(\hat{z}_k, n_k 1_{(z_k, c(z_k))}) \in \text{supp}(\mu)$. Since $Z \times N$ is compact, we may assume that $\{(\hat{z}_k, n_k)\}_{k=1}^{\infty}$ converges; let $(\hat{z}, n) = \lim_k(\hat{z}_k, n_k)$ and note that $\underline{n}(\hat{z}, z) \leq n \leq \bar{n}(\hat{z}, z)$ since $\underline{n}(\hat{z}_k, z_k) \leq n_k \leq \bar{n}(\hat{z}_k, z_k)$ for each $k \in \mathbb{N}$.

Lemma 4 implies that, taking a subsequence if needed, we may assume that $\{c(z_k)\}_{k=1}^{\infty}$ converges. Let $c = \lim_k c(z_k)$. Then $n_k 1_{(z_k, c(z_k))} \to n 1_{(z,c)}$ and, since $\text{supp}(\mu)$ is closed, it follows that $(\hat{z}, n1_{(z,c)}) \in \text{supp}(\mu)$. Lemma 2 implies that $n > 0$ and, thus, $z \in W$. ■

Lemma 10 *M is closed.*

Proof. Let $\{z_k\}_{k=1}^{\infty}$ be a convergent sequence in M and $z = \lim_k z_k$. For each k, let δ_k be such that $(z_k, \delta_k) \in \text{supp}(\mu)$ and $(z'_k, n'_k) \in Z \times N$ be such that $\delta_k = n'_k 1_{(z'_k, c(z'_k))}$. Since $Z \times N$ is compact, we may assume that $\{(z'_k, n'_k)\}_{k=1}^{\infty}$ converges; let $(z', n') = \lim_k(z'_k, n'_k)$ and note that $\underline{n}(z, z') \leq n' \leq \bar{n}(z, z')$ since $\underline{n}(z_k, z'_k) \leq n'_k \leq \bar{n}(z_k, z'_k)$ for each $k \in \mathbb{N}$.

Lemma 9 implies that $z' \in W$ and Lemma 5 that $c(z') = \lim_k c(z'_k)$. Letting $\delta = n' 1_{(z', c(z'))}$, it follows that $\delta_k \to \delta$. Since $\text{supp}(\mu)$ is closed, it follows that $(z, \delta) \in \text{supp}(\mu)$ and, thus, $z \in M$. ■

Lemma 11 *S is closed.*

Proof. That S is closed follows because both $\text{supp}(\mu)$ and $Z \times \{1_{(\emptyset,0)}\}$ are closed and $\text{supp}(\mu) \cap (Z \times (X_\emptyset \setminus X)) = \text{supp}(\mu) \cap (Z \times \{1_{(\emptyset,0)}\})$. ■

The following lemma shows that M, S and W cover Z, i.e. any type has an occupation. The idea is that $\mu((M \cup S \cup W) \times X_\emptyset) = \mu(Z \times X_\emptyset)$ and $\delta((M \cup S \cup W) \times C) = \delta(Z \times C)$ mostly by the definition of M, S and W, and then feasibility implies that $\nu(M \cup S \cup W) = \nu(Z)$. Thus, $Z = \text{supp}(\nu) = M \cup S \cup W$.

Lemma 12 *$Z = M \cup S \cup W$.*

Proof. Let $K = M \cup S \cup W$ and note that we have that $K \subseteq Z$ by definition.

Conversely, note first that K is closed by Lemmas 9, 10 and 11. Furthermore, letting $p(\text{supp}(\mu))$ be the projection of $\text{supp}(\mu)$ in Z, we have that $\text{supp}(\mu) \subseteq p(\text{supp}(\mu)) \times X_\emptyset = (M \cup S) \times X_\emptyset \subseteq K \times X_\emptyset$ and, hence,

$$\mu(K \times X_\emptyset) \geq \mu(\text{supp}(\mu)) = \mu(\text{supp}(\mu) \cap (Z \times X_\emptyset)) = \mu(Z \times X_\emptyset).$$

Furthermore, for each $(z, \delta) \in \text{supp}(\mu) \cap (Z \times X)$, there is $(z', n') \in Z \times \mathbb{R}_+$ such that $\delta = n' 1_{(z', c(z'))}$ and $n' > 0$ by Lemma 2; hence, $z' \in W$. Thus, $\delta((Z \setminus W) \times C) = 0$,

$\delta(W \times C) = \delta(Z \times C)$ and $\delta(K \times C) = \delta(Z \times C)$. Hence,

$$\begin{aligned}
&\nu(K) = \mu(K \times X) + \mu(K \times (X_\emptyset \setminus X)) + \int_{Z \times X} \delta(K \times C) \mathrm{d}\mu(z, \delta) \\
&= \mu(K \times X_\emptyset) + \int_{(Z \times X) \cap \operatorname{supp}(\mu)} \delta(K \times C) \mathrm{d}\mu(z, \delta) \\
&\geq \mu(Z \times X_\emptyset) + \int_{(Z \times X) \cap \operatorname{supp}(\mu)} \delta(Z \times C) \mathrm{d}\mu(z, \delta) = \nu(Z).
\end{aligned}$$

It then follows by the definition of $\operatorname{supp}(\nu)$ that $Z = \operatorname{supp}(\nu) \subseteq K$. ■

To complete the proof of Lemma 1, define $u : Z \to \mathbb{R}$ by setting, for each $z \in Z$,

$$u(z) = \begin{cases} u(z) & \text{if } z \in M, \\ c(z) & \text{if } z \in W, \\ U_s(z) & \text{if } z \in S. \end{cases}$$

It follows by Lemma 8 that u is well-defined. It follows by $U_z(s, 1_{(\emptyset,c)}) = U_s(z)$ for each $(z, c) \in Z \times C$ and by Lemmas 3 and 6 that conditions 1–3 in the statement of the lemma hold. We have that u is continuous since M, W and S are closed (by Lemmas 10, 9, 11), $Z = M \cup W \cup S$ (by Lemma 12) and $u|_M$, $u|_W$ and $u|_S$ are continuous. Indeed, let K be a closed subset of $\mathbb{R}$ and, for each $A \in \{M, S, W\}$, let K_A be a closed subset of Z such that $A \cap K_A = u|_A^{-1}(K)$. Then $u^{-1}(K) = (M \cap K_M) \cup (S \cap K_S) \cup (W \cap K_W)$ is a closed subset of Z.

B.3 Proof of Theorem 2

We start the proof of Theorem 2 by showing that each manager hires a strictly positive number of workers in any stable assignment.

Lemma 13 *If γ is a stable assignment and $(z, z', n) \in \operatorname{supp}(\gamma)$, then $n > 0$.*

Proof. The conclusion of the lemma is obvious when $\min_{(z,z') \in Z^2} \underline{n}(z, z') > 0$, hence assume that part (i) of condition B holds. Let $v \in C(Z_\emptyset)$ be such that $v(\emptyset) = 0$ and (γ, v) is stable.

Suppose that there exists $z, z' \in Z$ such that $(z, z', 0) \in \operatorname{supp}(\gamma)$. Then $v(z) = v(z) + 0v(z') = F(z, z', 0) = 0$. If $F(z, z, \bar{n}(z, z)) > 0$, then $(z, z, \bar{n}(z, z)) \in \operatorname{graph}(\bar{N})$ and $F(z, z, \bar{n}(z, z)) > 0 = v(z) + \bar{n}(z, z)v(z)$, contradicting the stability of γ. If $U_s(z) > 0$, then $(z, \emptyset, \bar{n}(z, \emptyset)) \in \operatorname{graph}(\bar{N})$ and $F(z, \emptyset, \bar{n}(z, \emptyset)) = U_s(z) > 0 = v(z) + \bar{n}(z, \emptyset)v(\emptyset)$, contradicting the stability of γ. ■

The following lemma establishes the claims made about the functions needed to transform stable matchings in stable assignments and vice-versa.

Lemma 14 *If μ is a stable matching, then g is a homeomorphism between* $\mathrm{supp}(\mu)$ *and* $g(\mathrm{supp}(\mu))$*; if (γ, v) is stable, then h is a homeomorphism between* $\mathrm{supp}(\gamma)$ *and* $h(\mathrm{supp}(\gamma))$.

Proof. Let μ is a stable matching. Then g is 1-1 since $g(z,\delta) = g(\hat{z},\hat{\delta}) = (\tilde{z}, z', n)$ implies that $z = \hat{z} = \tilde{z}$ and $\delta = \hat{\delta} = n1_{(z',c(z'))}$.

To see that g is continuous, let $(z,\delta) \in \mathrm{supp}(\mu)$ and $\{(z_k,\delta_k)\}_{k=1}^{\infty}$ be such that $(z_k,\delta_k) \in \mathrm{supp}(\mu)$ for each $k \in \mathbb{N}$ and $(z_k,\delta_k) \to (z,\delta)$. For each $k \in \mathbb{N}$, let $(z'_k, n_k) \in Z_{\emptyset} \times N$ be such that $\delta_k = n_k 1_{(z'_k, c(z'_k))}$. Then $n_k = \int_{Z\times C} 1 \mathrm{d}\delta_k \to \int_{Z\times C} 1\mathrm{d}\delta = n$ and $n_k z'_k = \int_{Z\times C} i_Z \mathrm{d}\delta_k \to \int_{Z\times C} i_Z \mathrm{d}\delta = nz'$, where $i_Z : Z \to Z$ is the identity on Z. It follows by Lemma 2 that $\delta \neq 0$ and, hence, $n > 0$. Then $z'_k = \frac{n_k z'_k}{n_k} \to \frac{nz'}{n} = z'$. Hence, $g(z_k,\delta_k) = (z_k, z'_k, n_k) \to (z, z', n) = g(z,\delta)$.

To see that g^{-1} is continuous, let $(z, z', n) \in g(\mathrm{supp}(\mu))$ and $\{(z_k, z'_k, n_k)\}_{k=1}^{\infty}$ be such that $(z_k, z'_k, n_k) \in g(\mathrm{supp}(\mu))$ for each $k \in \mathbb{N}$ and $(z_k, z'_k, n_k) \to (z, z', n)$. Then $g^{-1}(z_k, z'_k, n_k) = (z_k, n_k 1_{(z'_k, c(z'_k))}) \to (z, n1_{(z',c(z'))}) = g^{-1}(z, z', n)$.

Conversely, let (γ, v) be stable. Then h is 1-1 by Lemma 13, h is continuous by an analogous argument to the one establishing the continuity of g^{-1} and h^{-1} is continuous by an analogous argument to the one establishing the continuity of g. ■

Let μ be a stable matching and u be its earnings function. We will show that $(\mu \circ g^{-1}, u)$ is a stable outcome.

Let $\gamma = \mu \circ g^{-1}$. We first show that γ is an assignment. Let $p_N : Z \times Z_{\emptyset} \times N \to N$ be the projection of $Z \times Z_{\emptyset} \times N$ onto N, $p_{Z_{\emptyset}} : Z \times Z_{\emptyset} \times N \to Z_{\emptyset}$ be the projection of $Z \times Z_{\emptyset} \times N$ onto $Z_{\emptyset}$ and B be a Borel subset of Z; then $g^{-1}(B \times Z_{\emptyset} \times N) = \mathrm{supp}(\mu) \cap (B \times X_{\emptyset})$, $p_N(g(z,\delta))1_B(p_{Z_{\emptyset}}(g(z,\delta))) = \delta(B \times C)$ for each $(z,\delta) \in \mathrm{supp}(\mu)$ and

$$\begin{aligned}
&\gamma(B \times Z_{\emptyset} \times N) + \int_{Z\times B\times N} n \mathrm{d}\gamma(z, z', n) = \\
&\gamma(B \times Z_{\emptyset} \times N) + \int_{Z\times Z\times N} n1_B(z') \mathrm{d}\gamma(z, z', n) = \\
&\mu(g^{-1}(B \times Z_{\emptyset} \times N)) + \int_{Z\times X} p_N(g(z,\delta))1_B(p_{Z_{\emptyset}}(g(z,\delta)))\mathrm{d}\mu(z,\delta) = \\
&\mu(B \times X_{\emptyset}) + \int_{Z\times X} \delta(B \times C)\mathrm{d}\mu(z,\delta) = \nu(B).
\end{aligned}$$

To complete the argument that γ is an assignment, we need to show that $n \in \bar{N}(z, z')$ for each $(z, z', n) \in \mathrm{supp}(\gamma)$. We will do this at the same time as showing that (γ, u) is stable.

Let $(z, z', n) \in \mathrm{supp}(\gamma)$. Since $\mathrm{supp}(\mu) = g^{-1}(\mathrm{supp}(\gamma))$ by Carmona and Laohakunakorn (2024b, Lemma 1), it follows that $(z, n1_{(z',u(z'))}) \in \mathrm{supp}(\mu)$. If $z' \in Z$, then $z \in M$ and

$z' \in W$. Thus, $\underline{n}(z,z') \leq n \leq \bar{n}(z,z')$ and

$$u(z) + nu(z') = F(z,z',n) - nu(z') + nu(z') = F(z,z',n).$$

If $z' = \emptyset$, then $z \in S$, $n = 1$ and $u(z) + nu(z') = U_s(z) + 0 = F(z,\emptyset,1)$. Thus, we have shown that γ is an assignment and that condition (i) of stable assignment is satisfied.

We now show condition (ii) of stable assignment. Let $(z,z',n) \in Z \times Z_\emptyset \times N$ be such that $n \in \bar{N}(z,z')$. If $(z',n) = (\emptyset,1)$, then $u(z) + nu(z') = u(z) \geq F(z,\emptyset,1)$, where the inequality holds since μ is stable. If $z' \neq \emptyset$ and $u(z) + nu(z') < F(z,z',n)$, then $F(z,z',n) - nu(z') > u(z)$. Letting $\varepsilon > 0$ be such that $F(z,z',n) - n(u(z') + \varepsilon) > u(z)$, it follows that $(z', u(z') + \varepsilon) \in T_z^m(\mu)$ and $U_z(m, n1_{(z',u(z')+\varepsilon)}) > u(z)$, a contradiction to the stability of μ. Thus, $u(z) + nu(z') \geq F(z,z',n)$.

It follows from Lemma 14 that $\text{supp}(\gamma) = g(\text{supp}(\mu))$ and $\text{supp}(\mu) = g^{-1}(\text{supp}(\gamma))$. Hence,

$$\begin{aligned}
&M = \{z \in Z : (z,z',n) \in \text{supp}(\gamma) \text{ for some } (z',n) \in Z \times N\},\\
&S = \{z \in Z : (z,\emptyset,1) \in \text{supp}(\gamma)\} \text{ and}\\
&W = \{z \in Z : (\hat{z},z,n) \in \text{supp}(\gamma) \text{ for some } (\hat{z},n) \in Z \times N\}.
\end{aligned}$$

This concludes the proof of the first part of Theorem 2.

Let (γ, v) be stable. We will show that $\mu = \gamma \circ h^{-1}$ is a stable matching.

We first show that μ is a matching. Let $p_{X_\emptyset}$ be the projection of $Z \times X_\emptyset$ onto $X_\emptyset$ and B be a Borel subset of Z; then $h^{-1}(B \times X_\emptyset) = \text{supp}(\gamma) \cap (B \times Z_\emptyset \times N)$, $p_{X_\emptyset}(h(z,z',n))(B \times C) = n1_B(z')$ for each $(z,z',n) \in \text{supp}(\gamma)$ and

$$\begin{aligned}
&\mu(B \times X_\emptyset) + \int_{Z \times X} \delta(B \times C)\text{d}\mu(z,\delta) =\\
&\gamma \circ h^{-1}(B \times X_\emptyset) + \int_{Z \times X} \delta(B \times C)\text{d}\gamma \circ h^{-1}(z,\delta) =\\
&\gamma(B \times Z_\emptyset \times N) + \int_{Z \times Z_\emptyset \times N} n1_B(z')\text{d}\gamma(z,z',n) =\\
&\gamma(B \times Z_\emptyset \times N) + \int_{Z \times B \times N} n\text{d}\gamma(z,z',n) = \nu(B).
\end{aligned}$$

Furthermore, if $(z,\delta) \in \text{supp}(\mu)$ and $h^{-1}(z,\delta) = (z,z',n)$, then $(z,z',n) \in \text{supp}(\gamma)$ since $h^{-1}(\text{supp}(\mu)) = \text{supp}(\gamma)$ by Carmona and Laohakunakorn (2024b, Lemma 1). Hence, $n \in \bar{N}(z,z')$ and $\text{supp}(\mu) \subseteq \text{graph}(P_m) \cup \text{graph}(P_s)$.

We next show that v is the earnings function of μ. Since $Z = W \cup S \cup M$ by Lemma 12, it is enough to show that the restriction to u nad v to each $B = W, S, M$ coincide. Let $z \in W$. Then $(\hat{z}, n1_{(z,c)}) \in \text{supp}(\mu)$ for some $\hat{z} \in Z$, $n \in N$ and $c \in C$ and $(\hat{z}, n1_{(z,c)}) = h(\hat{z},z,n)$.

Since $h(\hat{z}, z, n) = (\hat{z}, n1_{(z,v(z))})$, it follows that $c = v(z)$. By Lemma 1, $u(z) = U_z(w, 1_{(\hat{z},c)}) = c = v(z)$.

Let $z \in M$. Then $(z, n1_{(z',v(z'))}) \in \text{supp}(\mu)$ for some $(z', n) \in Z \times N$ such that $(z, z', n) \in \text{supp}(\gamma)$. Thus, by Lemma 1 and the stability of (γ, v), $u(z) = U_z(m, n1_{(z',v(z'))}) = F(z, z', n) - nv(z') = v(z)$. Finally, if $z \in S$, $u(z) = U_s(z) = F(z, \emptyset, 1) = v(z) + v(\emptyset) = v(z)$ by Lemma 1 and the stability of (γ, v). It then follows that v is the earnings function of μ.

We conclude this proof by showing that μ is stable. Let $(z, n1_{(z',v(z'))}) \in \text{supp}(\mu)$; then $U_z(m, n1_{(z',v(z'))}) = v(z)$ and $U_{z'}(w, 1_{(z,v(z'))}) = v(z')$ if $z' \in Z$ and $U_z(s, 1_{(\emptyset,0)}) = u(z)$ if $z' = \emptyset$. It is clear that $(z, n1_{(z',v(z'))}) \in IR(\mu)$ since $v(z) \geq U_s(z)$ and, when $z' \in Z$, $v(z') \geq U_s(z')$ by the stability of (γ, v). To see that $(z, n1_{(z',v(z'))}) \in S_M(\mu)$, let $\tilde{z} \in \{z, z'\}$ if $z' \in Z$ and $\tilde{z} = z$ if $z = \emptyset$. Note that $\delta' \in X$ and $\text{supp}(\delta') \subseteq T^m_{\tilde{z}}(\mu)$ implies that $\delta' = \hat{n}1_{(\hat{z},v(\hat{z})+\varepsilon)}$ for some $\hat{z} \in Z$, $\underline{n}(z, \hat{z}) \leq \hat{n} \leq \bar{n}(z, \hat{z})$ and $\varepsilon > 0$. Thus, $U_{\tilde{z}}(m, \delta') \leq F(\tilde{z}, \hat{z}, \hat{n}) - \hat{n}v(\hat{z}) \leq v(\tilde{z})$ and, hence, $(z, n1_{(z',v(z'))}) \in S_M(\mu)$. If $z' \in Z$ and $\text{supp}(\delta')$ is not contained in $T^m_z(\mu)$, then $\text{supp}(\delta') = \{(z', v(z'))\}$ and $U_z(m, \delta') = F(z, z', \hat{n}) - \hat{n}v(z')$ for some $\hat{n} \in N$ such that $\underline{n}(z, z') \leq \hat{n} \leq \bar{n}(z, z')$. The stability of (γ, v) implies that $F(z, z', \hat{n}) - \hat{n}v(z') \leq v(z)$ and, hence, $(z, n1_{(z',v(z'))}) \in S_M(\mu)$.

B.4 Proof of Theorem 3

Note that for each assignment γ and $u \in C(Z_\emptyset)$ such that $u(\emptyset) = 0$,

$$\int_Z u \mathrm{d}\nu = \int_{Z \times Z_\emptyset \times N} (u(z) + nu(z')) \mathrm{d}\gamma(z, z', n)$$

since

$$\begin{aligned}
&\int_{Z \times Z_\emptyset \times N} (u(z) + nu(z')) \mathrm{d}\gamma(z, z', n) = \\
&\int_{Z \times Z_\emptyset \times N} u(z) \mathrm{d}\gamma(z, z', n) + \int_{Z \times Z_\emptyset \times N} n(z') u(z') \mathrm{d}\gamma(z, z', n) = \\
&\int_Z u \mathrm{d}\gamma_Z + \int_Z u \mathrm{d}\gamma_{Z,n} = \int_Z u \mathrm{d}\nu.
\end{aligned}$$

If (γ, u) is stable, then

$$\int_Z u \mathrm{d}\nu = \int_{Z \times Z_\emptyset \times N} F \mathrm{d}\gamma \leq \sup_{\hat{\gamma} \in \Gamma} \int_{Z \times Z_\emptyset \times N} F \mathrm{d}\hat{\gamma} \leq \inf_{\hat{u} \in U} \int_Z \hat{u} \mathrm{d}\nu \leq \int_Z u \mathrm{d}\nu. \tag{A1}$$

Thus, γ is surplus maximizing and u solves the dual of the surplus maximization problem.

Conversely, let (γ', u') be such that γ' is surplus maximizing and u' solves the dual of the surplus maximization problem. Let (γ, u) be stable (which exists by Theorems 1 and 2).

We have that $\int_{Z\times Z_\emptyset\times N} F\mathrm{d}\gamma = \int_Z u\mathrm{d}\nu$ by (A1) and, hence, $\int_{Z\times Z_\emptyset\times N} F\mathrm{d}\gamma' = \int_{Z\times Z_\emptyset\times N} F\mathrm{d}\gamma = \int_Z u\mathrm{d}\nu = \int_Z u\mathrm{d}\nu'$. This then implies that $\int_{\mathrm{graph}(\bar{N})}(u'(z)+nu'(z')-F(z,z',n))\mathrm{d}\gamma'(z,z',n) = \int_{Z\times Z_\emptyset\times N}(u'(z)+nu'(z')-F(z,z',n))\mathrm{d}\gamma'(z,z',n) = 0$ and, since $u'(z)+nu'(z')-F(z,z',n)\geq 0$ for each $(z,z',n)\in \mathrm{graph}(\bar{N})$, $\mathrm{supp}(\gamma')\subseteq\{(z,z',n)\in Z\times Z_\emptyset\times N : u'(z)+nu'(z') = F(z,z',n)\}$. Thus, (γ',u') is stable.

B.5 Proof of Theorem 4

We start by establishing two preliminary lemmas. The first is a straightforward variation of Lemma 4.1 in Beiglböck and Griessler (2019). For each $l\in\mathbb{N}$, $1\leq i\leq l$ and Polish space D, $\pi_i : D^l\to D$ is the projection of D^l onto the ith coordinate.

Lemma 15 *If (D,m) is a Polish measure space, $0<m(D)<\infty$, $l\in\mathbb{N}$ and K is an analytic subset of D^l, then one of the following conditions holds:*

(i) There exist m-null sets $K_1,\ldots,K_l\subseteq D$ such that $K\subseteq\cup_{i=1}^l\pi_i^{-1}(K_i)$.

(ii) There is a measure η on D^l such that $\eta(K)>0$ and $\eta\circ\pi_i^{-1}\leq m$ for each $i=1,\ldots,l$.

Proof. Apply Beiglböck and Griessler (2019, Lemma 4.1) to (D,m') where $m' = m/m(D)$. If condition (i) of the lemma holds, then there exist m'-null sets, hence m-null sets, $K_1,\ldots,K_l\subseteq D$ such that $K\subseteq\cup_{i=1}^l\pi_i^{-1}(D_i)$. If condition (ii) of the lemma holds, there is a measure η' on D^l such that $\eta'(D)>0$ and $\eta'\circ\pi_i^{-1}\leq m'$ for each $i=1,\ldots,l$. Letting $\eta = m(D)\eta'$, it follows that condition (ii) of this claim holds. ■

For convenience, let $D = \mathrm{graph}(\bar{N})$ and $F(\tau) = \int_D F\mathrm{d}\tau$ for each $\tau\in M(D)$. For each $g\in C(Z)$, let $g(\emptyset) = 0$ and $f_g : D\to\mathbb{R}$ be defined by setting, for each $(z,z',n)\in D$, $f_g(z,z',n) = g(z)+ng(z')$. Then let $\mathcal{F} = \{f_g : g\in C(Z)\}$ and note that $\mathcal{F}\subseteq C(D)$, i.e. each element of $\mathcal{F}$ is continuous and bounded.

Lemma 16 *For each $\zeta,\tau\in M(D)$, $\tau_Z+\tau_{Z,n} = \zeta_Z+\zeta_{Z,n}$ if and only if $\int_D f\mathrm{d}\tau = \int_D f\mathrm{d}\zeta$ for each $f\in\mathcal{F}$.*

Proof. Let $\zeta,\tau\in M(D)$. If $\tau_Z+\tau_{Z,n} = \zeta_Z+\zeta_{Z,n}$ then, for each $f\in\mathcal{F}$, it follows that, letting $g\in C(Z)$ be such that $f = f_g$,

$$\int_D f\mathrm{d}\tau = \int_Z g\mathrm{d}\tau_Z + \int_Z g\mathrm{d}\tau_{Z,n} = \int_Z g\mathrm{d}\zeta_Z + \int_Z g\mathrm{d}\zeta_{Z,n} = \int_D f\mathrm{d}\zeta.$$

Conversely, if $\int_D f\mathrm{d}\tau = \int_D f\mathrm{d}\zeta$ for each $f\in\mathcal{F}$, then $\int_Z g\mathrm{d}(\tau_Z+\tau_{Z,n}) = \int_Z g\mathrm{d}(\zeta_Z+\zeta_{Z,n})$ for each $g\in C(Z)$; hence, by e.g. Parthasarathy (1967, Theorem 5.9, p. 39), $\tau_Z+\tau_{Z,n} = \zeta_Z+\zeta_{Z,n}$. ■

Let $\gamma \in M(D)$ be a surplus maximizing assignment. The following lemma is the core of the argument.

Lemma 17 *For each $l \in \mathbb{N}$, there is a subset S_l of D such that $\gamma(S_l^c) = 0$ and the following holds:*

$$\text{for each } \zeta \in M(D) \text{ with } \operatorname{supp}(\zeta) \subseteq S_l,\ |\operatorname{supp}(\zeta)| \leq l \text{ and } \zeta(D) \leq 1, \tag{A2}$$

$$\text{and each } \tau \in M(D) \text{ with } |\operatorname{supp}(\tau)| \leq l,\ \tau(D) \leq l \text{ and } \tau_Z + \tau_{Z,n} = \zeta_Z + \zeta_{Z,n}, \tag{A3}$$

$$F(\zeta) \geq F(\tau). \tag{A4}$$

Proof. Let $l \in \mathbb{N}$ and define

$$\begin{aligned} K = &\{(d_1, \ldots, d_l) \in D^l : \text{ there exist } \zeta, \tau \in M(D) \text{ such that} \\ &\zeta(D) \leq 1, \operatorname{supp}(\zeta) \subseteq \{d_1, \ldots, d_l\}, \\ &|\operatorname{supp}(\tau)| \leq l, \tau(D) \leq l, \tau_Z + \tau_{Z,n} = \zeta_Z + \zeta_{Z,n} \text{ and } F(\tau) > F(\zeta)\}. \end{aligned}$$

Note that K is the projection of the set

$$\begin{aligned} \hat{K} = &\{(d_1, \ldots, d_l, \zeta_1, \ldots, \zeta_l, d'_1, \ldots, d'_l, \tau_1, \ldots, \tau_l) \in D^l \times \mathbb{R}^l_+ \times D^l \times \mathbb{R}^l_+ : \\ &\sum_{i=1}^l \zeta_i \leq 1, \sum_{i=1}^l \tau_i \leq l, \sum_{i=1}^l \zeta_i f(d_i) = \sum_{i=1}^l \tau_i f(d'_i) \text{ for each } f \in \mathcal{F} \text{ and} \\ &\sum_{i=1}^l \zeta_i F(d_i) < \sum_{i=1}^l \tau_i F(d'_i)\}. \end{aligned}$$

onto the first l coordinates by Lemma 16. Since the set $\hat{K}$ is Borel, it follows that K is analytic.

We apply Lemma 15 to the space (D, γ) and the set K: if (i) holds, then define $S_l = \cap_{i=1}^l K_i^c$. Then $\gamma(S_l^c) = \gamma(\cup_{i=1}^l K_i) \leq \sum_{i=1}^l \gamma(K_i) = 0$. Furthermore, let ζ, τ be such that (A2) and (A3) hold and assume, to get a contradiction, that (A4) fails. Then letting $d_1, \ldots, d_l$ be such that $\operatorname{supp}(\zeta) = \{d_1, \ldots, d_l\}$ (repeating some elements if necessary), $d'_1, \ldots, d'_l$ be such that $\operatorname{supp}(\tau) \subseteq \{d'_1, \ldots, d'_l\}$ and, for each $i = 1, \ldots, l$, $\zeta_i = \zeta(d_i)$ and $\tau_i = \tau(d'_i)$, it follows that $(d_1, \ldots, d_l, \zeta_1, \ldots, \zeta_l, d'_1, \ldots, d'_l, \tau_1, \ldots, \tau_l) \in \hat{K}$. Thus, $(d_1, \ldots, d_l) \in K$ and, for some $1 \leq i \leq l$, $d_i \in K_i$ since $K \subseteq \cup_{i=1}^l \pi_i^{-1}(K_i)$. But this is a contradiction since, for each $1 \leq i \leq l$, $d_i \in \operatorname{supp}(\zeta) \subseteq S_l \subseteq K_i^c$. This contradiction shows that (A4) holds and this completes the proof of the lemma when condition (i) in Lemma 15 holds.

It remains to show that condition (ii) in Lemma 15 cannot hold. Suppose that there is a measure η on D^l as in condition (ii) in Lemma 15. We may assume that η is concentrated on

K and satisfies $\eta \circ \pi_i^{-1} \leq \gamma/l$ for each $i = 1, \ldots, l$; indeed, if not, use η' defined by setting $\eta'(B) = \eta(B \cap K)/l$ for each Borel $B \subseteq D^l$.

We apply the Jankow–von Neumann selection theorem (e.g. Bogachev (2007, Theorem 6.9.2, p. 35) to the set $\hat{K}$ to define a mapping $\phi : K \to \mathbb{R}_+^l \times D^l \times \mathbb{R}_+^l$ such that, letting

$$\phi(d) = (\zeta_1(d), \ldots, \zeta_l(d), d_1'(d), \ldots, d_l'(d), \tau_1(d), \ldots, \tau_l(d)),$$

$(d, \phi(d)) \in \hat{K}$ for each $d \in K$, and ϕ is measurable with respect to the σ-field generated by the analytic subsets of D^l. Setting

$$\zeta_d = \sum_{i=1}^{l} \zeta_i(d) 1_{d_i} \text{ and } \tau_d = \sum_{i=1}^{l} \tau_i(d) 1_{d_i'(d)},$$

we thus obtain kernels $d \mapsto \zeta_d$ and $d \mapsto \tau_d$ from D^l with the σ-field generated by its analytic subsets to D with its Borel sets. We use these kernels to define measures ω and ω' on the Borel subsets of D by setting, for each Borel subset $B \subseteq D$,

$$\omega(B) = \int \zeta_d(B) \mathrm{d}\eta(d) \text{ and } \omega'(B) = \int \tau_d(B) \mathrm{d}\eta(d).$$

We have that $\omega \leq \gamma$ since, for each Borel $B \subseteq D$, noting that $\zeta_i(d) \leq 1$ for each $d \in K$ and $1 \leq i \leq l$,

$$\begin{aligned}
\omega(B) &= \int \sum_{i=1}^{l} \zeta_i(d) 1_{d_i}(B) \mathrm{d}\eta(d) = \sum_{i=1}^{l} \int \zeta_i(d) 1_{d_i}(B) \mathrm{d}\eta(d) \\
&\leq \sum_{i=1}^{l} \int 1_{d_i}(B) \mathrm{d}\eta(d) = \sum_{i=1}^{l} \int 1_B(\pi_i(d)) \mathrm{d}\eta(d) \\
&= \sum_{i=1}^{l} \int 1_B \mathrm{d}\eta \circ \pi_i^{-1} = \sum_{i=1}^{l} \eta \circ \pi_i^{-1}(B) \leq \sum_{i=1}^{l} \frac{\gamma(B)}{l} = \gamma(B).
\end{aligned}$$

Moreover, for each $f \in \mathcal{F}$, we have that $\int f \mathrm{d}\tau_d = \int f \mathrm{d}\zeta_d$ for each $d \in K$ and, hence,

$$\int f \mathrm{d}\omega' = \int \left(\int f \mathrm{d}\tau_d \right) \mathrm{d}\eta(d) = \int \left(\int f \mathrm{d}\zeta_d \right) \mathrm{d}\eta(d) = \int f \mathrm{d}\omega,$$

where the first and last equality are justified since f is continuous and bounded, $\zeta_d(D) \leq 1$ and $\tau_d(D) \leq l$ for each $d \in K$. Similarly, since $\int F \mathrm{d}\tau_d = F(\tau_d) > F(\zeta_d) = \int F \mathrm{d}\zeta_d$ for each $d \in K$ and F is continuous and bounded, we obtain that

$$F(\omega') = \int F \mathrm{d}\omega' = \int \left(\int F \mathrm{d}\tau_d \right) \mathrm{d}\eta(d) > \int \left(\int F \mathrm{d}\zeta_d \right) \mathrm{d}\eta(d) = \int F \mathrm{d}\omega = F(\omega).$$

In conclusion, $\gamma' = \gamma - \omega + \omega'$ belongs to Γ because $\omega \le \gamma$, $\int f \mathrm{d}\omega' = \int f \mathrm{d}\omega$ for each $f \in \mathcal{F}$ and Lemma 16, and is such that $F(\gamma') = F(\gamma) - F(\omega) + F(\omega') > Fs(\gamma)$. But this is a contradiction since γ is a surplus maximizing assignment. ■

We use Lemma 17 to complete the proof of Theorem 4. Let $S = \cap_{l=1}^{\infty} S_l$. Then $\gamma(S^c) = 0$ and S is F-monotone. Indeed, $\gamma(S^c) \le \sum_{l=1}^{\infty} \gamma(S_l^c) = 0$ and let $\zeta \in M(D)$ be supported on finitely many points of S and $\tau \in M(D)$ be finitely-supported and such that $\tau_Z + \tau_{Z,n} = \zeta_Z + \zeta_{Z,n}$. Let $\zeta' = \zeta / \max\{1, \zeta(D)\}$, $\tau' = \tau / \max\{1, \zeta(D)\}$ and $l = \max\{|\mathrm{supp}(\zeta)|, |\mathrm{supp}(\tau)|, \lceil \tau'(D) \rceil\}$; then $|\mathrm{supp}(\zeta')| \le l$, $|\mathrm{supp}(\tau')| \le l$, $\mathrm{supp}(\zeta') \subseteq S \subseteq S_l$, $\zeta'(D) \le 1$, $\tau'(D) \le l$ and $\tau'_Z + \tau'_{Z,n} = \zeta'_Z + \zeta'_{Z,n}$. It then follows by the properties of S_l, i.e. (A2)–(A4) that $F(\zeta') \ge F(\tau')$ and, hence, $F(\zeta) \ge F(\tau)$. Thus, S is F-monotone.

Note that any subset of an F-monotone set is F-monotone and that the closure of an F-monotone set is also F-monotone since F is continuous; thus, any subset of $\bar{S}$ is F-monotone. Since $\mathrm{supp}(\gamma) \subseteq \bar{S}$, it follows that $\mathrm{supp}(\gamma)$ is F-monotone.

B.6 Proof of Theorem 5

(Necessity) Let (τ, y^*, q) be a competitive equilibrium. For each $y = (x, \delta) \in L$, let $Y_y = \{\gamma \in M(Z \times Z_\emptyset \times N) : x \le \int_{Z \times Z_\emptyset \times N} F(z, z', n) \mathrm{d}\gamma(z, z', n)$ and $\delta(B) \le -(\gamma_Z + \gamma_{Z,n})(B)$ for each Borel $B \subseteq Z\}$. Then, $y \in Y$ if and only if $Y_y \ne \emptyset$.

Lemma 18 $q(y^*) = 0$.

Proof. Note first that $0 \in Y$ since $\gamma = 0 \in Y_0$; hence, $q(y^*) \ge q(0) = 0$. If $q(y^*) > 0$, then let $\lambda > 1$ and note that $\lambda y^* \in Y$ since $\lambda \gamma \in Y_{\lambda y^*}$ for each $\gamma \in Y_{y^*}$. Thus, $q(\lambda y^*) = \lambda q(y^*) > q(y^*)$, a contradiction. ■

Extend c to $Z_\emptyset$ by letting $c(\emptyset) = 0$. Let $\gamma \in Y_{y^*}$. For each $(z, z', n) \in \mathrm{graph}(\bar{N})$, if $z' \in Z$, then $(F(z, z', n), -1_z - n1_{z'}) \in Y$ since $1_{(z,z',n)} \in Y_{(F(z,z',n), -1_z - n1_{z'})}$. Hence, $q(F(z, z', n), -1_z - n1_{z'}) = F(z, z', n) - c(z) - nc(z') \le 0$. Furthermore, if $z' = \emptyset$, $(F(z, \emptyset, 1), -1_z) \in Y$ since $1_{(z,\emptyset,1)} \in Y_{(F(z,\emptyset,1), -1_z)}$. Hence, $q(F(z, \emptyset, 1), -1_z) = F(z, \emptyset, 1) - c(z) \le 0$. Thus,

$$F(z, z', n) \le c(z) + n(z')c(z') \text{ for each } (z, z', n) \in \mathrm{graph}(\bar{N}) \qquad \text{(A5)}$$

and

$$0 = q(y^*) = x^* + \int_Z c(z) \mathrm{d}\delta^*(z) \le \int_{Z \times Z_\emptyset \times N} F(z, z', n) \mathrm{d}\gamma(z, z', n) - \int_Z c \mathrm{d}(\gamma_Z + \gamma_{Z,n})$$
$$= \int_{Z \times Z_\emptyset \times N} \Big(F(z, z', n) - c(z) - n(z')c(z') \Big) \mathrm{d}\gamma(z, z', n) \le 0.$$

This implies that

$$x^* = \int_{Z \times Z_\emptyset \times N} F(z, z', n) \mathrm{d}\gamma(z, z', n), \tag{A6}$$

$$\int_Z c(z) \mathrm{d}\delta^*(z) = -\int_Z c(z) \mathrm{d}(\gamma_Z + \gamma_{Z,n})(z), \tag{A7}$$

$$F(z, z', n) - c(z) - n(z')c(z') = 0 \text{ for each } (z, z', n) \in \operatorname{supp}(\gamma), \text{ and} \tag{A8}$$

$$x^* = \int_{Z \times Z_\emptyset \times N} F(z, z', n) \mathrm{d}\gamma(z, z', n) = \int_Z c(z) \mathrm{d}(\gamma_Z + \gamma_{Z,n})(z). \tag{A9}$$

Let $\hat{Z} = \{z \in Z : F(z, z, \bar{n}(z, z)) = 0\}$.

Lemma 19 *$c(z) > 0$ for each $z \in \hat{Z}^c$.*

Proof. Let $z \in \hat{Z}^c$; hence, $F(z, z, \bar{n}(z, z)) > 0$. Note that $(F(z, z, \bar{n}(z, z)), -(\bar{n}(z, z) + 1)1_z) \in Y$ since $1_{(z,z,\bar{n}(z,z))} \in Y_{(F(z,z,\bar{n}(z,z)),-(\bar{n}(z,z)+1)1_z)}$. Hence,

$$q(F(z, z, \bar{n}(z, z)), -(\bar{n}(z, z) + 1)1_z) = F(z, z, \bar{n}(z, z)) - c(z)(\bar{n}(z, z) + 1) \leq 0$$

and $c(z) \geq \frac{F(z,z,\bar{n}(z,z))}{\bar{n}(z,z)+1} > 0$. ■

It follows by Lemma 19 that

$$\{(z, x, \delta) : (x, \delta) \text{ solves } \max_{(x',\delta') \in \Omega} u_z(x, \delta) \text{ subject to } q(x', \delta') \leq q(e(z))\} =$$
$$\{(z, c(z)) : z \in Z\} \times M(\hat{Z}).$$

Thus, $\operatorname{supp}(\tau) \subseteq \{(z, c(z)) : z \in Z\} \times M(\hat{Z})$ since $\hat{Z}$ is closed and, hence, so is $\{(z, c(z)) : z \in Z\} \times M(\hat{Z})$.

It follows that

$$\int_\Omega i_\Omega \mathrm{d}\tau_\Omega = \left(\int_Z c(z) \mathrm{d}\nu(z), \pi \right)$$

where $\pi \in M(\hat{Z})$ is such that, for each Borel subset B of $\hat{Z}$, $\pi(B) = \int_\Omega \delta(B) \mathrm{d}\tau_\Omega(x, \delta)$. Then $\int_\Omega i_\Omega \mathrm{d}\tau_\Omega \leq \int_Z e \mathrm{d}\nu + y^*$ implies that

$$\int_Z c(z) \mathrm{d}\nu(z) \leq x^* = \int_Z c(z) \mathrm{d}(\gamma_Z + \gamma_{Z,n})(z) \text{ and}$$
$$\pi(B) \leq \nu(B) + \delta^*(B) \leq \nu(B) - (\gamma_Z + \gamma_{Z,n})(B)$$

for each Borel $B \subseteq Z$. Since $\nu(\hat{Z}) = 0$, it follows that $\pi = 0$. Hence, $\nu - (\gamma_Z + \gamma_{Z,n}) \geq 0$. Since $\hat{Z}^c \subseteq \{z : c(z) > 0\}$, it follows that $[\nu - (\gamma_Z + \gamma_{Z,n})](Z) = [\nu - (\gamma_Z + \gamma_{Z,n})](\{z : c(z) > 0\})$. Since $c \geq 0$ and $\int_Z c \mathrm{d}\nu \leq \int_Z c \mathrm{d}(\gamma_Z + \gamma_{Z,n})$, it follows that $\int_Z c \mathrm{d}[\nu - (\gamma_Z + \gamma_{Z,n})] = 0$ and

$[\nu-(\gamma_Z+\gamma_{Z,n})](\{z : c(z)>0\}) = 0$. Hence, for each Borel $B \subseteq Z$, $0 \le [\nu-(\gamma_Z+\gamma_{Z,n})](B) \le [\nu-(\gamma_Z+\gamma_{Z,n})](Z) = [\nu-(\gamma_Z+\gamma_{Z,n})](\{z : c(z)>0\}) = 0$. Thus, $\nu(B) = -\delta^*(B) = (\gamma_Z+\gamma_{Z,n})(B)$ for each Borel $B \subseteq Z$.

It then follows from $\gamma_Z+\gamma_{Z,n} = \nu$, (A5) and (A8) that (γ, c) is stable.

It follows from $\int_{Z\times\Omega} \delta(\hat{Z}) \mathrm{d}\tau(z,x,\delta) = \pi(\hat{Z}) = 0$ that $\mathrm{supp}(\tau) \subseteq \{(z, c(z), 0) : z \in Z\}$ since $\{(z, c(z), 0) : z \in Z\}$ is closed. Thus, for each bounded continuous function $g : Z \times \Omega \to \mathbb{R}$,

$$\int_{Z\times\Omega} g \mathrm{d}\tau = \int_{\mathrm{supp}(\tau)} g \mathrm{d}\tau = \int_{\mathrm{supp}(\tau)} g \circ f(z) \mathrm{d}\tau(z,x,\delta) = \int_{Z\times\Omega} g \circ f(z) \mathrm{d}\tau(z,x,\delta) =$$
$$\int_Z g \circ f \mathrm{d}\tau_Z = \int_Z g \circ f \mathrm{d}\nu = \int_{Z\times\Omega} g \mathrm{d}\nu \circ f^{-1};$$

hence, $\tau = \nu \circ f^{-1}$.

(Sufficiency) Let (γ, c) be stable and define $\tau = \nu \circ f^{-1}$, where $f : Z \to Z \times \Omega$ is such that $f(z) = (z, c(z), 0)$ for each $z \in Z$, and $y^* = (x^*, \delta^*)$, where $x^* = \int_{Z\times Z_\emptyset \times N} F(z,z',n) \mathrm{d}\gamma(z,z',n)$, $\delta^*(B) = -(\gamma_Z+\gamma_{Z,n})(B) = -\nu(B)$ for each Borel $B \subseteq Z$.

Conditions 1 and 2 of the definition of a competitive equilibrium hold by definition of τ. We have that

$$\int_\Omega i_\Omega \mathrm{d}\tau_\Omega = \left(\int_Z c(z) \mathrm{d}\nu(z), 0\right), \text{ and}$$
$$\int_Z e \mathrm{d}\nu + y^* = \left(\int_{Z\times Z_\emptyset\times N} F(z,z',n) \mathrm{d}\gamma(z,z',n), 0\right)$$

and

$$\int_{Z\times Z_\emptyset\times N} F(z,z',n) \mathrm{d}\gamma(z,z',n) - \int_Z c(z) \mathrm{d}\nu(z)$$
$$= \int_{Z\times Z_\emptyset\times N} F(z,z',n) \mathrm{d}\gamma(z,z',n) - \int_Z c(z) \mathrm{d}(\gamma_Z+\gamma_{Z,n})(z)$$
$$= \int_{Z\times Z_\emptyset\times N} \Big(F(z,z',n) - c(z) - nc(z')\Big) \mathrm{d}\gamma(z,z',n) = 0$$

since (γ, c) is stable. Thus, condition 4 holds.

The above argument shows that $q(y^*) = 0$. Since $F(z,z',n) - c(z) - nc(z') \le 0$ for each $(z,z',n) \in \mathrm{graph}(\bar{N})$ by the stability of (γ, c), it follows that $q(y) \le 0$ for each $y \in Y$. Thus, condition 3 also holds.

B.7 Characterizing the Managers' Span of Control

An implication of condition C is that managers hire as much workers they can, in the sense that $n = \bar{n}(z, z')$ if $(z, z', n) \in Z \times Z \times N$ are in the support of a stable assignment. This simplifies the description of stable assignments, since any stable assignment is fully described by its marginal of $Z \times Z_\emptyset$, i.e. it suffices to detail who matches with whom.

Lemma 20 *Let E be an economy satisfying condition C. If (γ, u) is a stable outcome in E, then letting $\pi = \gamma_{Z \times Z_\emptyset}$, (π, u) satisfies*

1. $\pi_Z + \pi_{Z,n} = \nu$,[3]
2. $u(z) + \bar{n}(z, z')u(z') = s(z, z')$ *for each* $(z, z') \in \operatorname{supp}(\pi)$ *and*
3. $u(z) + \bar{n}(z, z')u(z') \geq s(z, z')$ *for each* $(z, z') \in Z \times Z_\emptyset$.

and

$$\begin{aligned} M &= \{z \in Z : (z, z') \in \operatorname{supp}(\pi) \textit{ for some } z' \in Z\}, \\ S &= \{z \in Z : (z, \emptyset) \in \operatorname{supp}(\pi)\} \textit{ and} \\ W &= \{z \in Z : (\hat{z}, z) \in \operatorname{supp}(\pi) \textit{ for some } \hat{z} \in Z\}. \end{aligned}$$

Conversely, if (π, u) are such that $\pi \in M(Z \times Z_\emptyset)$, u is an earnings function and (π, u) satisfies 1-3, then $(\pi \circ t^{-1}, u)$ is stable in E, where $t : Z \times Z_\emptyset \to Z \times Z_\emptyset \times N$ is such that $t(z, z') = (z, z', \bar{n}(z, z'))$ for each $(z, z') \in Z \times Z_\emptyset$.

The proof of Lemma 20 uses the following result establishing that managers hire as much workers as possible in any stable assignment.

Lemma 21 *If γ is a stable assignment of an economy satisfying condition C and $(z, z', n) \in \operatorname{supp}(\gamma) \cap (Z \times Z \times N)$, then $n = \bar{n}(z, z')$.*

Proof. The conclusion of the lemma clearly holds if part (ii) of condition C holds. Hence, assume that part (i) holds. Let γ is a stable assignment, u be such that (γ, u) is stable and $(z, z', n) \in \operatorname{supp}(\gamma) \cap (Z \times Z \times N)$. We have that $u(z) > 0$ since, otherwise, either $u(z) + \bar{n}(z, z)u(z) = 0 < F(z, z, \bar{n}(z, z))$, which is a contradiction to the stability of γ since $(z, z, \bar{n}(z, z)) \in \operatorname{graph}(\bar{N})$, or $u(z) + \bar{n}(z, \emptyset)u(\emptyset) = 0 < U_s(z) = F(z, \emptyset, \bar{n}(z, \emptyset))$, which is a contradiction to the stability of γ since $(z, \emptyset, \bar{n}(z, \emptyset)) \in \operatorname{graph}(\bar{N})$.

[3] Analogously to the definition of $\gamma_{Z,n}$, $\pi_{Z,n} \in M(Z)$ is defined by setting, for each Borel subset B of Z, $\pi_{Z,n}(B) = \int_{Z \times B} \bar{n}(z, z')\mathrm{d}\pi(z, z')$.

Lemma 13 implies that $n > 0$. If $n < \bar{n}(z,z')$, then let $\lambda = \bar{n}(z,z')/n > 0$ and note that

$$\begin{aligned} F(z,z',\bar{n}(z,z')) &= (F(z,z',n) - u(z')n)\lambda + u(z')\bar{n}(z,z') \\ &> F(z,z',n) - u(z')n + u(z')\bar{n}(z,z') = u(z) + u(z')\bar{n}(z,z'). \end{aligned}$$

But this is a contradiction to the stability of γ since $(z,z',\bar{n}(z,z')) \in \text{graph}(\bar{N})$. This contradiction shows that $n = \bar{n}(z,z')$. ■

Turning to the proof of Lemma 20, we first show that if $\gamma = \pi \circ t^{-1}$, then $\gamma_{Z\times Z_\emptyset} = \pi$. Indeed, for each Borel subset B of $Z \times Z_\emptyset$,

$$\gamma_{Z\times Z_\emptyset}(B) = \gamma(B \times N) = \pi(t^{-1}(B \times N)) = \pi(B).$$

We next show that if γ and π are such that $\gamma_{Z\times Z_\emptyset} = \pi$, then $\pi_Z = \gamma_Z$ and $\pi_{Z,n} = \gamma_{Z,n}$. Let B be a Borel subset of Z. Then

$$\pi_Z(B) = \gamma_{Z\times Z_\emptyset}(B \times Z_\emptyset) = \gamma(B \times Z_\emptyset \times N) = \gamma_Z(B),$$

and, letting $1_B : Z_\emptyset \to \{0,1\}$ be such that $1_B(z') = 1$ if $z' \in B$ and 0 otherwise and $1_B(z,z',n) = 1_B(z,z') = 1_B(z')$,

$$\begin{aligned} \pi_{Z,n}(B) &= \int \bar{n}1_B \mathrm{d}\pi = \int \bar{n}1_B \mathrm{d}\gamma = \int n1_B(z')\mathrm{d}\gamma(z,z',n) = \int_{Z\times B\times N} n\mathrm{d}\gamma(z,z',n) \\ &= \int_{B\times N} n\mathrm{d}\gamma_{Z_\emptyset\times N}(z',n) = \gamma_{Z,n}(B). \end{aligned}$$

It then follows from what has been shown above that if γ is an assignment and $\pi = \gamma_{Z\times Z_\emptyset}$, then π satisfies 1. It also follows that if π satisfies 1 and $\gamma = \pi \circ t^{-1}$, then γ is an assignment.

Suppose that (γ, v) is stable and let $p : \text{supp}(\gamma) \to Z \times Z_\emptyset$ be defined by setting, for each $(z,z',n) \in \text{supp}(\gamma)$, $p(z,z',n) = (z,z')$; then p is a homeomorphism between $\text{supp}(\mu)$ and $p(\text{supp}(\mu))$. If $(z,z') \in \text{supp}(\pi)$, then (z,z',n) for some $n \in N$ such that $(z,z',n) \in \text{supp}(\gamma)$ by Carmona and Laohakunakorn (2024b, Lemma 1) and, hence, $n = \bar{n}(z,z')$ by Lemma 21. Thus, $v(z) + \bar{n}(z,z')v(z') = F(z,z',\bar{n}(z,z')) = s(z,z')$. Furthermore, for each $(z,z') \in Z \times Z_\emptyset$, $(z,z',\bar{n}(z,z')) \in \text{graph}(\bar{N})$ and, hence, $v(z) + \bar{n}(z,z')v(z') \geq F(z,z',\bar{n}(z,z')) = s(z,z')$.

The characterization of the sets M, S and W follows from $\text{supp}(\pi) = p(\text{supp}(\gamma))$ and $\text{supp}(\gamma) = p^{-1}(\text{supp}(\pi))$, which in turn follows from Carmona and Laohakunakorn (2024b, Lemma 1).

Finally, suppose that (π, v) is stable. If $(z,z',n) \in \text{supp}(\gamma)$, then $(z,z',n) = t(z,z')$ by Carmona and Laohakunakorn (2024b, Lemma 1) and, hence, $n = \bar{n}(z,z')$. Then $v(z) + nv(z') = v(z) + \bar{n}(z,z')v(z') = s(z,z') = F(z,z',\bar{n}(z,z')) = F(z,z',n)$.

If $(z, z', n) \in \text{graph}(\bar{N})$ is such that $n = \bar{n}(z, z')$, then

$$v(z) + nv(z') = v(z) + \bar{n}(z, z')v(z') \geq s(z, z') = F(z, z', \bar{n}(z, z')) = F(z, z', n).$$

In particular, this completes the proof when part (ii) of condition C holds. Hence, assume that part (i) holds and that $n < \bar{n}(z, z')$ and let $\lambda = \bar{n}(z, z')/n > 1$. Part (i) of condition C implies that $v(z) > 0$ for each $z \in Z$ since in case $s(z, z) > 0$, $v(z) + \bar{n}(z, z)v(z) \geq s(z, z) > 0$, and in case $s(z, \emptyset) > 0$, $v(z) + \bar{n}(z, \emptyset)v(\emptyset) \geq s(z, \emptyset) > 0$; thus, $v(z) > F(z, z', n) - nv(z')$ if $F(z, z', n) - nv(z') \leq 0$. If $F(z, z', n) - nv(z') > 0$, then

$$\begin{aligned} v(z) &\geq s(z, z') - \bar{n}(z, z')v(z') = F(z, z, \bar{n}(z, z')) - \bar{n}(z, z')v(z') \\ &= \lambda(F(z, z', n) - nv(z')) > F(z, z', n) - nv(z'). \end{aligned}$$

B.8 Positive Assortativeness

In this section we give sufficient conditions for the matching of managers and workers to be positive assortative, i.e. for more knowledgeable managers to hire more knowledgeable workers. These condition involve the upper bound $\bar{n}$, i.e. the optimal span of control, both separately and in combination with the (optimal) production function s.

Defining the notion of "more knowledgeable" is easiest in the case where the set of knowledge levels Z is a subset of $\mathbb{R}$ since then we can use the standard $\geq$ order. We thus focus on this case and let $p_{Z \times Z_\emptyset}$ denote the projection of $Z \times Z_\emptyset \times N$ onto $Z \times Z_\emptyset$. An assignment γ is *positive assortive* if $z' \geq \hat{z}'$ whenever $(z, z'), (\hat{z}, \hat{z}') \in p_{Z \times Z_\emptyset}(\text{supp}(\gamma)) \cap Z^2$ and $z > \hat{z}$.

The sufficient conditions for positive assortativeness in Theorem 8 below include some super or submodularity conditions and that $\bar{n}$ be increasing in z. We say that $\bar{n}$ is *increasing in* z if $\bar{n}(z, z') \geq \bar{n}(\hat{z}, z')$ whenever $z, \hat{z}, z' \in Z$ and $z \geq \hat{z}$ i.e. more knowledgeable managers can hire more workers. A function $g : Z^2 \to \mathbb{R}$ is *supermodular (resp. submodular)* if $g(x', y') - g(x, y') \geq g(x', y) - g(x, y)$ (resp. $g(x', y') - g(x, y') \leq g(x', y) - g(x, y)$) for each $x, x', y, y' \in Z$ such that $x' \geq x$ and $y' \geq y$; it is *strictly supermodular (weak form)* if $g(x', y') - g(x, y') > g(x', y) - g(x, y)$ for each $x, x', y, y' \in Z$ such that $x' > x$ and $y' > y$.

Theorem 8 *If E is an economy satisfying condition C and $Z \subseteq \mathbb{R}$, and (γ, u) is stable in E, then γ is positive assortative if $\bar{n}$ is increasing in z and strictly positive, and either (i) $\ln \bar{n}$ is submodular and $(x, y) \mapsto \frac{s(x,y)}{\bar{n}(x,z)}$ is strictly supermodular (weak form) for each $z \in Z$ or (ii) $\ln \bar{n}$ is supermodular and $(x, y) \mapsto \frac{s(x,y)\bar{n}(z,y)}{\bar{n}(x,y)}$ is strictly supermodular (weak form) for each $z \in Z$.*

A special case that arises often is that $\bar{n}$ is independent of z such as in Garicano and Rossi-Hansberg (2004) (since $\bar{n}(z,z') = \frac{1}{h(1-z')}$ for each $(z,z') \in Z^2$) and in Mak and Siow (2025) (since $\bar{n} \equiv 1$). In this case, the conditions of Theorem 8 are satisfied provided that s is strictly supermodular (weak form). The intuition for this conclusion is easy. Let $(z,z'),(\hat{z},\hat{z}') \in p_{Z\times Z_\emptyset}(\text{supp}(\gamma)) \cap Z^2$ be such that $z > \hat{z}$ and assume that $z' < \hat{z}'$. Then $(z,z',\bar{n}(z,z'))$ and $(\hat{z},\hat{z}',\bar{n}(\hat{z},\hat{z}'))$ belong to $\text{supp}(\gamma)$ and Theorem 4 implies that $\text{supp}(\gamma)$ is F-monotone, hence it is not possible to transport mass in a feasible way from $\xi = 1_{(z,z',\bar{n}(z,z'))} + 1_{(\hat{z},\hat{z}',\bar{n}(\hat{z},\hat{z}'))}$ to obtain τ such that $\int F\mathrm{d}\tau > \int F\mathrm{d}\xi$. However, when $\bar{n}$ is independent of z and s is strictly supermodular (weak form), such τ exists, namely $\tau = 1_{(z,\hat{z}',\bar{n}(\hat{z}'))} + 1_{(\hat{z},z',\bar{n}(z'))}$. This contradiction then implies that it must be $z' \geq \hat{z}'$.

The conditions of Theorem 8 do not require that $\bar{n}$ be independent of z and are there to make the above argument go through. Such argument is now more complicated since e.g., in ξ, there is a measure 1 of individuals of knowledge z and a measure $\bar{n}(\hat{z},\hat{z}')$ of individuals of knowledge $\hat{z}'$ but managers with knowledge z hire $\bar{n}(z,\hat{z}')$ workers of knowledge $\hat{z}'$; when $\bar{n}(z,\hat{z}') \neq \bar{n}(\hat{z},\hat{z}')$, there will be some individuals of knowledge z or $\hat{z}'$ left unmatched.

The following corollary considers a slightly more general case than the one where $\bar{n}$ is independent of z, namely it assumes that $\ln\bar{n}$ is both super and submodular. We say that a function $f : Z^2 \to \mathbb{R}_{++}$ is a *product* if there exists $a \in \mathbb{R}_{++}$ and functions $f_1, f_2 : Z \to \mathbb{R}_{++}$ such that $f(x,y) = af_1(x)f_2(y)$ for each $(x,y) \in Z^2$; it turns out that $\ln f$ is both both super and submodular if and only if f is a product.

Corollary 7 *In an economy satisfying condition C and $Z \subseteq \mathbb{R}$, any stable assignment is positive assortative if $\bar{n}$ is increasing in z and strictly positive, $\ln\bar{n}$ is a product and $(x,y) \mapsto \frac{s(x,y)}{\bar{n}(x,\underline{z})}$ is strictly supermodular (weak form).*

It is possible to modify Theorem 8 to obtain a strict version of positive assortativeness. An assignment γ is *strictly positive assortive* if $z' > \hat{z}'$ whenever $(z,z'),(\hat{z},\hat{z}') \in p_{Z\times Z_\emptyset}(\text{supp}(\gamma)) \cap Z^2$ and $z > \hat{z}$. A function $g : Z^2 \to \mathbb{R}$ is *strictly supermodular (strong form)* if $g(x',y') - g(x,y') > g(x',y) - g(x,y)$ for each $x,x',y,y' \in Z$ such that $x' \geq x$ and $y' \geq y$ and $(x',y') \neq (x,y)$. Then, for each stable outcome (γ,u) in an economy satisfying condition C and $Z \subseteq \mathbb{R}$, γ is strictly positive assortative if $\bar{n}$ is increasing in z, and either (i) $\ln\bar{n}$ is submodular and $(x,y) \mapsto \frac{s(x,y)}{\bar{n}(x,z)}$ is strictly supermodular (strong form) for each $z \in Z$, or (ii) $\ln\bar{n}$ is supermodular and $(x,y) \mapsto \frac{s(x,y)\bar{n}(z,y)}{\bar{n}(x,y)}$ is strictly supermodular (strong form) for each $z \in Z$.[4]

[4]This result is established at the end of the proof of Theorem 8.

B.8.1 Proof of Theorem 8

Let (γ, u) be stable in an economy satisfying condition C and $Z \subseteq \mathbb{R}$, and suppose that γ is not positive assortative. Then, there are $(z, z'), (\hat{z}, \hat{z}') \in Z^2$ such that $(z, z'), (\hat{z}, \hat{z}') \in p_{Z \times Z_\emptyset}(\text{supp}(\gamma))$, $z > \hat{z}$ and $\hat{z}' > z'$. By Lemma 21, we have that $(z, z', \bar{n}(z, z')), (\hat{z}, \hat{z}', \bar{n}(\hat{z}, \hat{z}')) \in \text{supp}(\gamma)$ and by Theorem 4, $\text{supp}(\gamma)$ is F-monotone,

Let

$$\begin{aligned}
&\zeta = 1_{(z,z',\bar{n}(z,z'))} + 1_{(\hat{z},\hat{z}',\bar{n}(\hat{z},\hat{z}'))},\\
&m = \min\left\{\frac{\bar{n}(z,\hat{z}') - \bar{n}(\hat{z},\hat{z}')}{\bar{n}(z,\hat{z}')}, \frac{\bar{n}(z,z') - \bar{n}(\hat{z},z')}{\bar{n}(z,z')}\right\}, \text{ and}\\
&\tau = \frac{\bar{n}(\hat{z},\hat{z}')}{\bar{n}(z,\hat{z}')} 1_{(z,\hat{z}',\bar{n}(z,\hat{z}'))} + 1_{(\hat{z},z',\bar{n}(\hat{z},z'))} + m 1_{(z,z',\bar{n}(z,z'))}\\
&\quad + \left(\frac{\bar{n}(z,\hat{z}') - \bar{n}(\hat{z},\hat{z}')}{\bar{n}(z,\hat{z}')} - m\right) 1_{(z,\emptyset,1)} + \bar{n}(z,z')\left(\frac{\bar{n}(z,z') - \bar{n}(\hat{z},z')}{\bar{n}(z,z')} - m\right) 1_{(z',\emptyset,1)}.
\end{aligned}$$

Then $\tau_Z + \tau_{Z,n} = \zeta_Z + \zeta_{Z,n}$.

Consider first the case where $\ln \bar{n}$ is submodular. Then $m = 1 - \frac{\bar{n}(\hat{z},\hat{z}')}{\bar{n}(z,\hat{z}')}$ since $\ln \bar{n}$ is submodular. Indeed, $m = 1 - \frac{\bar{n}(\hat{z},\hat{z}')}{\bar{n}(z,\hat{z}')}$ if and only if

$$\frac{\bar{n}(\hat{z},z')}{\bar{n}(z,z')} \leq \frac{\bar{n}(\hat{z},\hat{z}')}{\bar{n}(z,\hat{z}')}.$$

This holds if and only if $\bar{n}(z,\hat{z}')\bar{n}(\hat{z},z') \leq \bar{n}(z,z')\bar{n}(\hat{z},\hat{z}')$, which is equivalent to

$$\ln \bar{n}(z,\hat{z}') + \ln \bar{n}(\hat{z},z') \leq \ln \bar{n}(z,z') + \ln \bar{n}(\hat{z},\hat{z}');$$

this inequality holds since $\ln \bar{n}$ is submodular.

Furthermore,

$$\begin{aligned}
&\int_{Z \times Z_\emptyset \times N} F \mathrm{d}\zeta = s(z,z') + s(\hat{z},\hat{z}') \text{ and}\\
&\int_{Z \times Z_\emptyset \times N} F \mathrm{d}\tau = \frac{\bar{n}(\hat{z},\hat{z}')}{\bar{n}(z,\hat{z}')} s(z,\hat{z}') + s(\hat{z},z') + \left(1 - \frac{\bar{n}(\hat{z},\hat{z}')}{\bar{n}(z,\hat{z}')}\right) s(z,z') +\\
&\qquad \bar{n}(z,z')\left(\frac{\bar{n}(\hat{z},\hat{z}')}{\bar{n}(z,\hat{z}')} - \frac{\bar{n}(\hat{z},z')}{\bar{n}(z,z')}\right) s(z',\emptyset) \geq\\
&\qquad \frac{\bar{n}(\hat{z},\hat{z}')}{\bar{n}(z,\hat{z}')} s(z,\hat{z}') + s(\hat{z},z') + \left(1 - \frac{\bar{n}(\hat{z},\hat{z}')}{\bar{n}(z,\hat{z}')}\right) s(z,z').
\end{aligned}$$

Hence, $\int_{Z\times Z_\emptyset\times N} F\mathrm{d}\tau > \int_{Z\times Z_\emptyset\times N} F\mathrm{d}\zeta$ since

$$\begin{aligned}
&\frac{\bar{n}(\hat{z},\hat{z}')}{\bar{n}(z,\hat{z}')}s(z,\hat{z}') + s(\hat{z},z') + \left(1-\frac{\bar{n}(\hat{z},\hat{z}')}{\bar{n}(z,\hat{z}')}\right)s(z,z') > s(z,z') + s(\hat{z},\hat{z}')\\
&\Leftrightarrow \frac{\bar{n}(\hat{z},\hat{z}')}{\bar{n}(z,\hat{z}')}s(z,\hat{z}') + s(\hat{z},z') > \frac{\bar{n}(\hat{z},\hat{z}')}{\bar{n}(z,\hat{z}')}s(z,z') + s(\hat{z},\hat{z}')\\
&\Leftrightarrow \frac{\bar{n}(\hat{z},\hat{z}')}{\bar{n}(z,\hat{z}')}s(z,\hat{z}') + \frac{\bar{n}(\hat{z},\hat{z}')}{\bar{n}(\hat{z},\hat{z}')}s(\hat{z},z') > \frac{\bar{n}(\hat{z},\hat{z}')}{\bar{n}(z,\hat{z}')}s(z,z') + \frac{\bar{n}(\hat{z},\hat{z}')}{\bar{n}(\hat{z},\hat{z}')}s(\hat{z},\hat{z}')\\
&\Leftrightarrow \frac{s(z,\hat{z}')}{\bar{n}(z,\hat{z}')} + \frac{s(\hat{z},z')}{\bar{n}(\hat{z},\hat{z}')} > \frac{s(z,z')}{\bar{n}(z,\hat{z}')} + \frac{s(\hat{z},\hat{z}')}{\bar{n}(\hat{z},\hat{z}')}
\end{aligned}$$

and the last inequality holds because $(x,y)\mapsto \frac{s(x,y)}{\bar{n}(x,z)}$ is strictly supermodular (weak form) for each $z\in Z$. But $\int_{Z\times Z_\emptyset\times N} F\mathrm{d}\tau > \int_{Z\times Z_\emptyset\times N} F\mathrm{d}\zeta$ is a contradiction to the F-monotonicity of $\mathrm{supp}(\gamma)$. This contradiction shows that γ is positive assortative.

To see that the strict version of Theorem 8 holds, simply note that the above argument requires only to change $\hat{z}' > z'$ to $\hat{z}' \geq z'$ and to appeal to the strong form of the strict supermodularity of $(x,y)\mapsto \frac{s(x,y)}{\bar{n}(x,z)}$ to conclude that $\int_{Z\times Z_\emptyset\times N} F\mathrm{d}\tau > \int_{Z\times Z_\emptyset\times N} F\mathrm{d}\zeta$.

Consider next the case where $\ln\bar{n}$ is supermodular. In this case, $m = 1-\frac{\bar{n}(\hat{z},z')}{\bar{n}(z,z')}$ and $\int_{Z\times Z_\emptyset\times N} F\mathrm{d}\tau > \int_{Z\times Z_\emptyset\times N} F\mathrm{d}\zeta$ holds since this inequality is implied by

$$\begin{aligned}
&\frac{\bar{n}(\hat{z},\hat{z}')}{\bar{n}(z,\hat{z}')}s(z,\hat{z}') + s(\hat{z},z') > \frac{\bar{n}(\hat{z},z')}{\bar{n}(z,z')}s(z,z') + s(\hat{z},\hat{z}')\\
&\Leftrightarrow \frac{\bar{n}(\hat{z},\hat{z}')}{\bar{n}(z,\hat{z}')}s(z,\hat{z}') + \frac{\bar{n}(\hat{z},z')}{\bar{n}(\hat{z},z')}s(\hat{z},z') > \frac{\bar{n}(\hat{z},z')}{\bar{n}(z,z')}s(z,z') + \frac{\bar{n}(\hat{z},\hat{z}')}{\bar{n}(\hat{z},\hat{z}')}s(\hat{z},\hat{z}'),
\end{aligned}$$

which holds because $(x,y)\mapsto \frac{\bar{n}(z,y)}{\bar{n}(x,y)}s(x,y)$ is strictly supermodular (weak form) for each $z\in Z$. As above, this contradicts the F-monotonicity of $\mathrm{supp}(\gamma)$ and shows that γ is positive assortative.

Assuming that $(x,y)\mapsto \frac{\bar{n}(z,y)}{\bar{n}(x,y)}s(x,y)$ is strictly supermodular (strong form) for each $z\in Z$ then implies that γ is strictly positive assortative as above.

B.8.2 Proof of Corollary 7

We first show that a function $f: Z^2\to\mathbb{R}_{++}$ is such that $\ln f$ is both super and submodular if and only if f is a product. Sufficiency follows since if $f(x,y) = af_1(x)f_2(y)$ for each

$(x,y) \in Z^2$, then, for each $x, x', y, y' \in Z$,

$$\ln f(x', y') - \ln f(x, y') = \ln\left(\frac{f(x', y')}{f(x, y')}\right) = \ln\left(\frac{f_1(x')}{f_1(x)}\right)$$
$$= \ln\left(\frac{f(x', y)}{f(x, y)}\right) = \ln f(x', y) - \ln f(x, y).$$

To show the necessity part, consider first a function $M : Z^2 \to \mathbb{R}$ and assume that M is both super and submodular. Fix $(x,y) \in Z^2$ and note that $M(x,y) - M(\underline{z}, y) = M(x, \underline{z}) - M(\underline{z}, \underline{z})$. Hence, letting $b = M(\underline{z}, \underline{z})$, $g_1(x) = M(x, \underline{z})$ and $g_2(y) = M(\underline{z}, y)$, it follows that $M(x,y) = b + g_1(x) + g_2(y)$.

It then follows by that above that, for each $(x,y) \in Z^2$, $\ln f(x,y) = b + g_1(x) + g_2(y)$. Hence, $f(x,y) = \mathrm{e}^{\ln \bar{n}(x,y)} = \mathrm{e}^b \mathrm{e}^{g_1(x)} \mathrm{e}^{g_2(y)} = a f_1(x) f_2(y)$ where $a = \mathrm{e}^b$, $f_1(x) = \mathrm{e}^{g_1(x)}$ and $f_2(y) = \mathrm{e}^{g_2(y)}$. Thus, f is a product.

We next show that when $\bar{n}$ is a product, the conditions (a) $(x,y) \mapsto \frac{s(x,y)}{\bar{n}(x,z)}$ is strictly supermodular (weak form) for each $z \in Z$ and (b) $(x,y) \mapsto \frac{s(x,y)\bar{n}(z,y)}{\bar{n}(x,y)}$ is strictly supermodular (weak form) for each $z \in Z$ both reduce to (c) $(x,y) \mapsto \frac{s(x,y)}{\bar{n}(x,\underline{z})}$ is strictly supermodular (weak form).

Write $\bar{n}(x,y) = a f_1(x) f_2(y)$ for each $(x,y) \in Z^2$ and let $x, y, z \in Z$. Then $\frac{1}{\bar{n}(x,z)} = \frac{1}{a f_1(x) f_2(z)}$ and $\frac{\bar{n}(z,y)}{\bar{n}(x,y)} = \frac{f_1(z)}{f_1(x)}$. Since $f_1(z)$ and $a f_2(z)$ are constants for fixed z, (a) holds if and only if $(x,y) \mapsto \frac{s(x,y)}{f_1(x)}$ is strictly supermodular (weak form) and same for (b). Since $a f_2(\underline{z})$ is a constant, then $(x,y) \mapsto \frac{s(x,y)}{f_1(x)}$ is strictly supermodular (weak form) holds if and only if (c) holds.

When $\bar{n}$ is independent of z, it follows that $f_1(x) = f_1(\underline{z})$ for each $x \in Z$ and hence, $(x,y) \mapsto \frac{s(x,y)}{f_1(x)}$ is strictly supermodular (weak form) if and only if s is strictly supermodular (weak form).

B.9 Occupational Stratification

The main result of this section provides sufficient conditions for occupational stratification in any stable assignment, namely that the least knowledgeable individuals are workers, followed by self-employed and then by managers. A stable assignment satisfies *occupational stratification* if $\sup W \leq \inf S$, $\sup S \leq \inf M$ and $\sup W \leq \inf M$.[5] Theorem 9 provides sufficient conditions for occupational stratification.

Theorem 9 *Let γ be a stable assignment in an economy satisfying condition C and (1) s and $\bar{n}$ are continuously differentiable, (2) $Z = [\underline{z}, \bar{z}] \subseteq \mathbb{R}$, (3) ν is atomless, (4) $\bar{n}$ increasing*

[5]The requirement that $\sup W \leq \inf M$ is implied by the other two when $S \neq \emptyset$ but not otherwise.

in z and in z', (5) $\bar{n} > 0$, (6) for each $\hat{z}, z \in Z$ and $z' \in Z_\emptyset$ such that $(\hat{z}, z), (z, z') \in p_{Z\times Z_\emptyset}(\text{supp}(\gamma))$,

$$\frac{\partial s(z,z')}{\partial x} + \left(\max\left\{\frac{U_s(z)}{\bar{n}(z,z')}, \frac{s(z,z)}{\bar{n}(z,z')(1+\bar{n}(z,z))}\right\} - \frac{s(z,z')}{\bar{n}(z,z')}\right)\frac{\partial \bar{n}(z,z')}{\partial x}$$
$$-\frac{1}{\bar{n}(\hat{z},z)}\frac{\partial s(\hat{z},z)}{\partial y} + \max\left\{\frac{U_s(z)}{\bar{n}(\hat{z},z)}, \frac{s(z,z)}{\bar{n}(\hat{z},z)(1+\bar{n}(z,z))}\right\}\frac{\partial \bar{n}(\hat{z},z)}{\partial y} > 0,$$

and, (7) for each $z, z' \in Z$ such that $(z, z') \in p_{Z\times Z_\emptyset}(\text{supp}(\gamma))$,

$$\frac{\partial s(z,z')}{\partial x} - U_s'(z) - \frac{s(z,z') - U_s(z)}{\bar{n}(z,z')}\frac{\partial \bar{n}(z,z')}{\partial x} > 0.$$

Then γ satisfies occupational stratification.

Occupational stratification is essentially about connectedness of the sets $M\setminus(S\cup W)$ and $S\setminus W$, which translate to the convexity of these sets since Z is a subset of $\mathbb{R}$. The requirement that the knowledge distribution ν is atomless implies that W, S and M have no isolated points and this allows us to show that a local property is sufficient for the connectedness of the above sets. In the case of $M\setminus(S\cup W)$, such local property is that, for each z in M but different from its maximum, there is $\varepsilon > 0$ such that $(z, z+\varepsilon) \subseteq M\setminus(S\cup W)$, i.e. if there are managers with knowledge z, then all individuals with a slightly higher knowledge will be managers. For this conclusion to hold, we consider small perturbations to the support of a stable assignment (hence the requirement that s and $\bar{n}$ be C^1 and that $\bar{n} > 0$ to compute the resulting change in aggregate output): when $\bar{n}$ is increasing in z and z' and (7) holds, it is possible to change the support of γ in a feasible way and increase aggregate output if, contrary to the desired conclusion that $(z, z+\varepsilon) \subseteq M\setminus(S\cup W)$ for some $\varepsilon > 0$, there is some sequence $\{z_k\}_{k=1}^\infty$ in S converging to z from above; the same is true if (6) holds and there is some sequence $\{z_k\}_{k=1}^\infty$ in W converging to z from above.

B.9.1 Proof of Theorem 9

In this proof, we let $\pi = \gamma_{Z\times Z_\emptyset}$ and note that by Lemma 21, $(z,z') \in \text{supp}(\pi)$ if and only if $(z, z', \bar{n}(z,z')) \in \text{supp}(\gamma)$. For a measure $\zeta \in M(Z\times Z_\emptyset)$, we let $F(\zeta) = \int_{Z\times Z_\emptyset\times N} F \mathrm{d}(\zeta\circ t^{-1})$, where $t : Z\times Z_\emptyset \to Z\times Z_\emptyset\times N$ is such that $t(z,z') = (z, z', \bar{n}(z,z'))$ for each $(z,z') \in Z\times Z_\emptyset$.

Consider first the case where $\nu(M) = 0$. Then $0 = \nu(M) \geq \pi(M\times Z) = \pi(\text{supp}(\pi)\cap(Z\times Z)) = \pi(Z\times Z)$ and, hence,

$$\pi(Z\times Z_\emptyset) = \pi(Z\times Z) + \pi(Z\times\{\emptyset\}) = \pi(Z\times\{\emptyset\}) = \pi(\text{supp}(\pi)\cap(Z\times\{\emptyset\})) = \pi(S\times\{\emptyset\}).$$

Since $S \times \{\emptyset\}$ is closed, it follows that $\text{supp}(\pi) \subseteq S \times \{\emptyset\}$ and that $M = W = \emptyset$. Then occupational stratification holds since $\sup W = -\infty$, $\inf S = \underline{z}$, $\sup S = \bar{z}$ and $\inf M = \infty$.

It follows by the above argument that we may assume that $\nu(M) > 0$. Then $\pi(Z \times Z) > 0$ since otherwise $\text{supp}(\pi) \subseteq Z \times \{\emptyset\}$ and, hence, $M = \emptyset$, i.e. $\nu(M) = 0$. Then $\nu(W) \geq \int_{Z \times W} \bar{n} \text{d}\pi = \int_{\text{supp}(\pi) \cap (Z \times Z)} \bar{n} \text{d}\pi = \int_{Z \times Z} \bar{n} \text{d}\pi > 0$.

Let $z_2 = \min M$ and $\bar{z}_2 = \max M$. Note that both z_2 and $\bar{z}_2$ exist, and $z_2 < \bar{z}_2$ since M is compact (by Lemma 10) and $\nu(M) > 0$. Let $\underline{z}_1 = \min W$ and $z_1 = \max W$. Thus, both z_1 and $\underline{z}_1$ exist, and $\underline{z}_1 < z_1$ since W is compact (by Lemma 9) and $\nu(W) > 0$.

The following lemmas carry out the computations needed in Lemma 25 below, which is a key part of the argument showing that M, S and W are intervals. It considers a match (z, z') where the manager is of type z and the workers of type z' (or type z' is self-employed) and a sequence of matches $(\hat{z}_k, z_k)$ such that $z_k \to z$.

Lemma 22 *Let $z' \in Z_\emptyset$, $z, \hat{z} \in Z$, $\{z_k\}_{k=1}^\infty \subseteq Z$, $\{\hat{z}_k\}_{k=1}^\infty \subseteq Z$ and $\{\zeta_k\}_{k=1}^\infty \subseteq M(Z \times Z_\emptyset)$ be such that $z < z_k$ for each $k \in \mathbb{N}$, $z_k \to z$, $\hat{z}_k \to \hat{z}$ and $\zeta_k = 1_{(z,z')} + 1_{(\hat{z}_k, z_k)}$. Then there is a subsequence $\{\zeta_{k_j}\}_{j=1}^\infty$ of $\{\zeta_k\}_{k=1}^\infty$ and a corresponding sequence $\{\tau_{k_j}\}_{j=1}^\infty$ such that $\tau_{k_j,Z} + \tau_{k_j,Z,n} = \zeta_{k_j,Z} + \zeta_{k_j,Z,n}$ and*

$$\lim_j \frac{F(\tau_{k_j}) - F(\zeta_{k_j})}{z_{k_j} - z} > 0 \Leftrightarrow \frac{\partial s(z,z')}{\partial x} - \frac{s(z,z')}{\bar{n}(z,z')} \frac{\partial \bar{n}(z,z')}{\partial x} - \frac{1}{\bar{n}(\hat{z}, z)} \frac{\partial s(\hat{z}, z)}{\partial y} + \frac{U_s(z)}{\bar{n}(\hat{z}, z)} \frac{\partial \bar{n}(\hat{z}, z)}{\partial y} + \frac{U_s(z)}{\bar{n}(z,z')} \frac{\partial \bar{n}(z,z')}{\partial x} > 0.$$

Proof. Note that

$$\bar{n}(\hat{z}_k, z) \leq \bar{n}(\hat{z}_k, z_k) \frac{\bar{n}(z_k, z')}{\bar{n}(z, z')}$$

since $\bar{n}$ is increasing in z and z'. We consider three cases.

Case 1: $\bar{n}(\hat{z}_k, z) \geq 1$ for infinitely many k. In this case, taking a subsequence if necessary, assume that $\bar{n}(\hat{z}_k, z) \geq 1$ for each $k \in \mathbb{N}$ and let

$$\begin{aligned}\tau_k =& \frac{\bar{n}(z,z')}{\bar{n}(z_k,z')} 1_{(z_k,z')} + \frac{1}{\bar{n}(\hat{z}_k, z)} 1_{(\hat{z}_k, z)} + \left(1 - \frac{1}{\bar{n}(\hat{z}_k, z)}\right) 1_{(\hat{z}_k, z_k)} \\ &+ \left(\frac{\bar{n}(\hat{z}_k, z_k) - \bar{n}(\hat{z}_k, z)}{\bar{n}(\hat{z}_k, z)} + \frac{\bar{n}(z_k, z') - \bar{n}(z,z')}{\bar{n}(z_k, z')}\right) 1_{(z_k, \emptyset)}.\end{aligned}$$

Note that, for each $k \in \mathbb{N}$, all terms in τ_k are positive since $\bar{n}(\hat{z}_k, z) \geq 1$, $\bar{n}(\hat{z}_k, z_k) \geq \bar{n}(\hat{z}_k, z)$ and $\bar{n}(z_k, z') \geq \bar{n}(z, z')$ and that

$$\tau_{k,Z} + \tau_{k,Z,n} = \zeta_{k,Z} + \zeta_{k,Z,n} = 1_z + \bar{n}(z,z') 1_{z'} + 1_{\hat{z}_k} + \bar{n}(\hat{z}_k, z_k) 1_{z_k}.$$

Furthermore, for each $k \in \mathbb{N}$,

$$
\begin{aligned}
F(\zeta_k) =& s(z,z') + s(\hat{z}_k, z_k), \\
F(\tau_k) =& \frac{\bar{n}(z,z')}{\bar{n}(z_k,z')} s(z_k,z') + \frac{1}{\bar{n}(\hat{z}_k,z)} s(\hat{z}_k,z) + \left(1 - \frac{1}{\bar{n}(\hat{z}_k,z)}\right) s(\hat{z}_k,z_k) \\
&+ \left(\frac{\bar{n}(\hat{z}_k,z_k) - \bar{n}(\hat{z}_k,z)}{\bar{n}(\hat{z}_k,z)} + \frac{\bar{n}(z_k,z') - \bar{n}(z,z')}{\bar{n}(z_k,z')}\right) U_s(z_k) \text{ and} \\
\frac{F(\tau_k) - F(\zeta_k)}{z_k - z} =& \frac{s(z_k,z') - s(z,z')}{z_k - z} - \frac{s(z_k,z')}{\bar{n}(z_k,z')} \frac{\bar{n}(z_k,z') - \bar{n}(z,z')}{z_k - z} \\
&- \frac{1}{\bar{n}(\hat{z}_k,z)} \frac{s(\hat{z}_k,z_k) - s(\hat{z}_k,z)}{z_k - z} + \frac{U_s(z_k)}{\bar{n}(\hat{z}_k,z)} \frac{\bar{n}(\hat{z}_k,z_k) - \bar{n}(\hat{z}_k,z)}{z_k - z} \\
&+ \frac{U_s(z_k)}{\bar{n}(z_k,z')} \frac{\bar{n}(z_k,z') - \bar{n}(z,z')}{z_k - z}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\lim_k \frac{F(\tau_k) - F(\zeta_k)}{z_k - z} =& \frac{\partial s(z,z')}{\partial x} - \frac{s(z,z')}{\bar{n}(z,z')} \frac{\partial \bar{n}(z,z')}{\partial x} - \frac{1}{\bar{n}(\hat{z},z)} \frac{\partial s(\hat{z},z)}{\partial y} \\
&+ \frac{U_s(z)}{\bar{n}(\hat{z},z)} \frac{\partial \bar{n}(\hat{z},z)}{\partial y} + \frac{U_s(z)}{\bar{n}(z,z')} \frac{\partial \bar{n}(z,z')}{\partial x}.
\end{aligned}
$$

Case 2: $\bar{n}(\hat{z}_k, z_k)\frac{\bar{n}(z_k,z')}{\bar{n}(z,z')} < 1$ for infinitely many k. In this case, taking a subsequence if necessary, assume that $\bar{n}(\hat{z}_k, z_k)\frac{\bar{n}(z_k,z')}{\bar{n}(z,z')} < 1$ for each $k \in \mathbb{N}$ and let

$$
\begin{aligned}
\tau_k =& \bar{n}(\hat{z}_k,z_k) 1_{(z_k,z')} + 1_{(\hat{z}_k,z)} + \left(1 - \bar{n}(\hat{z}_k,z_k)\frac{\bar{n}(z_k,z')}{\bar{n}(z,z')}\right) 1_{(z,z')} \\
&+ \left(\bar{n}(\hat{z}_k,z_k)\frac{\bar{n}(z_k,z')}{\bar{n}(z,z')} - \bar{n}(\hat{z}_k,z)\right) 1_{(z,\emptyset)}.
\end{aligned}
$$

Note that, for each $k \in \mathbb{N}$, all terms in τ_k are positive since $\bar{n}(\hat{z}_k, z_k)\frac{\bar{n}(z_k,z')}{\bar{n}(z,z')} < 1$, $\bar{n}(\hat{z}_k, z_k) \geq \bar{n}(\hat{z}_k, z)$ and $\bar{n}(z_k, z') \geq \bar{n}(z, z')$ and that

$$
\tau_{k,Z} + \tau_{k,Z,n} = \zeta_{k,Z} + \zeta_{k,Z,n} = 1_z + \bar{n}(z,z')1_{z'} + 1_{\hat{z}_k} + \bar{n}(\hat{z}_k,z_k)1_{z_k}.
$$

Furthermore, for each $k \in \mathbb{N}$,

$$\begin{aligned}
F(\zeta_k) =& s(z,z') + s(\hat{z}_k, z_k),\\
F(\tau_k) =& \bar{n}(\hat{z}_k, z_k) s(z_k, z') + s(\hat{z}_k, z) + \left(1 - \bar{n}(\hat{z}_k, z_k)\frac{\bar{n}(z_k, z')}{\bar{n}(z,z')}\right) s(z,z')\\
&+ \left(\bar{n}(\hat{z}_k, z_k)\frac{\bar{n}(z_k, z')}{\bar{n}(z,z')} - \bar{n}(\hat{z}_k, z)\right) U_s(z)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\bar{n}(\hat{z}_k, z_k)}\frac{F(\tau_k) - F(\zeta_k)}{z_k - z} = \frac{s(z_k, z') - s(z,z')}{z_k - z} - \frac{s(z,z')}{\bar{n}(z,z')}\frac{\bar{n}(z_k, z') - \bar{n}(z,z')}{z_k - z}\\
&- \frac{1}{\bar{n}(\hat{z}_k, z_k)}\frac{s(\hat{z}_k, z_k) - s(\hat{z}_k, z)}{z_k - z} + \frac{U_s(z)}{\bar{n}(\hat{z}_k, z_k)}\frac{\bar{n}(\hat{z}_k, z_k) - \bar{n}(\hat{z}_k, z)}{z_k - z}\\
&+ \frac{U_s(z)}{\bar{n}(z,z')}\frac{\bar{n}(z_k, z') - \bar{n}(z,z')}{z_k - z}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\lim_k \frac{F(\tau_k) - s(\zeta_k)}{z_k - z} > 0 \Leftrightarrow& \frac{\partial s(z,z')}{\partial x} - \frac{s(z,z')}{\bar{n}(z,z')}\frac{\partial \bar{n}(z,z')}{\partial x} - \frac{1}{\bar{n}(\hat{z}, z)}\frac{\partial s(\hat{z}, z)}{\partial y}\\
&+ \frac{U_s(z)}{\bar{n}(\hat{z}, z)}\frac{\partial \bar{n}(\hat{z}, z)}{\partial y} + \frac{U_s(z)}{\bar{n}(z,z')}\frac{\partial \bar{n}(z,z')}{\partial x} > 0.
\end{aligned}$$

Case 3: $\bar{n}(\hat{z}_k, z) < 1 \le \bar{n}(\hat{z}_k, z_k)\frac{\bar{n}(z_k, z')}{\bar{n}(z,z')}$ for all k sufficiently large. In this case, $\bar{n}(\hat{z}, z) = 1$. Taking a subsequence if necessary, assume that $\bar{n}(\hat{z}_k, z) < 1 \le \bar{n}(\hat{z}_k, z_k)\frac{\bar{n}(z_k, z')}{\bar{n}(z,z')}$ for each $k \in \mathbb{N}$ and let

$$\tau_k = \frac{\bar{n}(z,z')}{\bar{n}(z_k, z')} 1_{(z_k, z')} + 1_{(\hat{z}_k, z)} + \left(\bar{n}(\hat{z}_k, z_k) - \frac{\bar{n}(z,z')}{\bar{n}(z_k, z')}\right) 1_{(z_k, \emptyset)} + (1 - \bar{n}(\hat{z}_k, z))\, 1_{(z, \emptyset)}.$$

Note that, for each $k \in \mathbb{N}$, all terms in τ_k are positive and that

$$\tau_{k,Z} + \tau_{k,Z,n} = \zeta_{k,Z} + \zeta_{k,Z,n} = 1_z + \bar{n}(z,z') 1_{z'} + 1_{\hat{z}_k} + \bar{n}(\hat{z}_k, z_k) 1_{z_k}.$$

Then, for each $k \in \mathbb{N}$,

$$\begin{aligned}
F(\zeta_k) &= s(z,z') + s(\hat{z}_k, z_k),\\
F(\tau_k) &= \frac{\bar{n}(z,z')}{\bar{n}(z_k, z')} s(z_k, z') + s(\hat{z}_k, z) + \left(\bar{n}(\hat{z}_k, z_k) - \frac{\bar{n}(z,z')}{\bar{n}(z_k, z')}\right) U_s(z_k)\\
&\quad + (\bar{n}(\hat{z}, z) - \bar{n}(\hat{z}_k, z)) U_s(z)
\end{aligned}$$

and

$$\frac{F(\tau_k)-F(\zeta_k)}{z_k-z}=\frac{s(z_k,z')-s(z,z')}{z_k-z}-\frac{s(z_k,z')}{\bar{n}(z_k,z')}\frac{\bar{n}(z_k,z')-\bar{n}(z,z')}{z_k-z}$$
$$-\frac{s(\hat{z}_k,z_k)-s(\hat{z}_k,z)}{z_k-z}$$
$$+U_s(z_k)\frac{\bar{n}(\hat{z}_k,z_k)(\bar{n}(z_k,z')-n(z,z'))}{\bar{n}(z_k,z')(z_k-z)}+U_s(z_k)\frac{\bar{n}(z,z')}{\bar{n}(z_k,z')}\frac{\bar{n}(\hat{z}_k,z_k)-\bar{n}(\hat{z}_k,z)}{\bar{n}(\hat{z},z)(z_k-z)}$$
$$+U_s(z_k)\frac{\bar{n}(z,z')}{\bar{n}(z_k,z')}\frac{\bar{n}(\hat{z}_k,z)-\bar{n}(\hat{z},z)}{z_k-z}+\frac{(\bar{n}(\hat{z},z)-\bar{n}(\hat{z}_k,z))}{z_k-z}U_s(z).$$

Note that

$$\frac{\bar{n}(\hat{z}_k,z)-\bar{n}(\hat{z},z)}{z_k-z}\left(\frac{U_s(z_k)\bar{n}(z,z')}{\bar{n}(z_k,z')}-U_s(z)\right)$$
$$=\frac{\bar{n}(\hat{z}_k,z)-\bar{n}(\hat{z},z)}{\bar{n}(z_k,z'_k)}\left(-\frac{U_s(z_k)(\bar{n}(z_k,z')-\bar{n}(z,z'))}{z_k-z}+\frac{\bar{n}(z_k,z')(U_s(z_k)-U_s(z))}{z_k-z}\right).$$

Since $\lim_k \bar{n}(\hat{z}_k,z)=\bar{n}(\hat{z},z)=1$, it follows that

$$\lim_k \frac{\bar{n}(\hat{z}_k,z)-\bar{n}(\hat{z},z)}{z_k-z}\left(\frac{U_s(z_k)\bar{n}(z,z')}{\bar{n}(z_k,z')}-U_s(z)\right)=0.$$

Hence,

$$\lim_k \frac{F(\tau_k)-F(\zeta_k)}{z_k-z}=\frac{\partial s(z,z')}{\partial x}-\frac{\partial \bar{n}(z,z')}{\partial x}\frac{s(z,z')}{\bar{n}(z,z')}-\frac{\partial s(\hat{z},z)}{\partial y}\frac{1}{\bar{n}(\hat{z},z)}$$
$$+\frac{U_s(z)}{\bar{n}(z,z')}\frac{\partial \bar{n}(z,z')}{\partial x}+\frac{U_s(z)}{\bar{n}(\hat{z},z)}\frac{\partial \bar{n}(\hat{z},z)}{\partial y}.$$

■

The following lemma considers an alternative feasible matching which does not create self-employed individuals when $z'\neq\emptyset$.

Lemma 23 *Let $z'\in Z_\emptyset$, $z,\hat{z}\in Z$, $\{z_k\}_{k=1}^\infty\subseteq Z$ and $\{\hat{z}_k\}_{k=1}^\infty\subseteq Z$ be such that $z<z_k$ for each $k\in\mathbb{N}$, $z_k\to z$ and $\hat{z}_k\to\hat{z}$. Then there is a subsequence $\{\zeta_{k_j}\}_{j=1}^\infty$ of $\{\zeta_k\}_{k=1}^\infty$ and a*

corresponding sequence $\{\tau_{k_j}\}_{j=1}^{\infty}$ *such that* $\tau_{k_j,Z} + \tau_{k_j,Z,n} = \zeta_{k_j,Z} + \zeta_{k_j,Z,n}$ *and*

$$\lim_k \frac{F(\tau_k) - F(\zeta_k)}{z_k - z} > 0 \iff \frac{\partial s(z,z')}{\partial x} - \frac{\partial \bar{n}(z,z')}{\partial x}\frac{s(z,z')}{\bar{n}(z,z')} - \frac{1}{\bar{n}(\hat{z},z)}\frac{\partial s(\hat{z},z)}{\partial y} + \frac{s(z,z)}{\bar{n}(z,z')(1+\bar{n}(z,z))}\frac{\partial \bar{n}(z,z')}{\partial x} + \frac{s(z,z)}{\bar{n}(\hat{z},z)(1+\bar{n}(z,z))}\frac{\partial \bar{n}(\hat{z},z)}{\partial y} > 0.$$

Proof. As in the proof of Lemma 22, we consider three cases.

Case 1: $\bar{n}(\hat{z}_k, z) \geq 1$ for infinitely many k. In this case, taking a subsequence if necessary, assume that $\bar{n}(\hat{z}_k, z) \geq 1$ for each $k \in \mathbb{N}$ and let

$$\begin{aligned}
\zeta_k =& 1_{(z,z')} + 1_{(\hat{z}_k,z_k)} \text{ and} \\
\tau_k =& \frac{\bar{n}(z,z')}{\bar{n}(z_k,z')}1_{(z_k,z')} + \frac{\bar{n}(z_k,z)\bar{n}(z,z') + \bar{n}(z_k,z')}{\bar{n}(z_k,z')(\bar{n}(\hat{z}_k,z) + \bar{n}(z_k,z)\bar{n}(\hat{z}_k,z_k))}1_{(\hat{z}_k,z)} \\
&+ \left(1 - \frac{\bar{n}(z_k,z)\bar{n}(z,z') + \bar{n}(z_k,z')}{\bar{n}(z_k,z')(\bar{n}(\hat{z}_k,z) + \bar{n}(z_k,z)\bar{n}(\hat{z}_k,z_k))}\right)1_{(\hat{z}_k,z_k)} \\
&+ \frac{\bar{n}(\hat{z}_k,z_k)\bar{n}(z_k,z') - \bar{n}(z,z')\bar{n}(\hat{z}_k,z)}{\bar{n}(z_k,z')(\bar{n}(\hat{z}_k,z) + \bar{n}(z_k,z)\bar{n}(\hat{z}_k,z_k))}1_{(z_k,z)}.
\end{aligned}$$

Note that, for each $k \in \mathbb{N}$, all terms in τ_k are positive since $\bar{n}(\hat{z}_k, z) \geq 1$, $\bar{n}(\hat{z}_k, z_k) \geq \bar{n}(\hat{z}_k, z)$ and $\bar{n}(z_k, z') \geq \bar{n}(z, z')$ and that

$$\tau_{k,Z} + \tau_{k,Z,n} = \zeta_{k,Z} + \zeta_{k,Z,n} = 1_z + \bar{n}(z,z')1_{z'} + 1_{\hat{z}_k} + \bar{n}(\hat{z}_k,z_k)1_{z_k}.$$

For each $k \in \mathbb{N}$,

$$
\begin{aligned}
F(\zeta_k) &= s(z,z') + s(\hat{z}_k, z_k), \\
F(\tau_k) &= \frac{\bar{n}(z,z')}{\bar{n}(z_k,z')} s(z_k,z') + \frac{\bar{n}(z_k,z)\bar{n}(z,z') + \bar{n}(z_k,z')}{\bar{n}(z_k,z')(\bar{n}(\hat{z}_k,z) + \bar{n}(z_k,z)\bar{n}(\hat{z}_k,z_k))} s(\hat{z}_k,z) \\
&\quad + \left(1 - \frac{\bar{n}(z_k,z)\bar{n}(z,z') + \bar{n}(z_k,z')}{\bar{n}(z_k,z')(\bar{n}(\hat{z}_k,z) + \bar{n}(z_k,z)\bar{n}(\hat{z}_k,z_k))}\right) s(\hat{z}_k,z_k) \\
&\quad + \frac{\bar{n}(\hat{z}_k,z_k)\bar{n}(z_k,z') - \bar{n}(z,z')\bar{n}(\hat{z}_k,z)}{\bar{n}(z_k,z')(\bar{n}(\hat{z}_k,z) + \bar{n}(z_k,z)\bar{n}(\hat{z}_k,z_k))} s(z_k,z) \text{ and} \\
\frac{F(\tau_k) - F(\zeta_k)}{z_k - z} &= \frac{s(z_k,z') - s(z,z')}{z_k - z} - \frac{\bar{n}(z_k,z') - \bar{n}(z,z')}{z_k - z} \frac{s(z_k,z')}{\bar{n}(z_k,z')} \\
&\quad - \frac{\bar{n}(z_k,z)\bar{n}(z,z') + \bar{n}(z_k,z')}{\bar{n}(z_k,z')(\bar{n}(\hat{z}_k,z) + \bar{n}(z_k,z)\bar{n}(\hat{z}_k,z_k))} \frac{s(\hat{z}_k,z_k) - s(\hat{z}_k,z)}{z_k - z} \\
&\quad + \frac{\bar{n}(\hat{z}_k,z_k)}{\bar{n}(z_k,z')(\bar{n}(\hat{z}_k,z) + \bar{n}(z_k,z)\bar{n}(\hat{z}_k,z_k))} \frac{\bar{n}(z_k,z') - \bar{n}(z,z')}{(z_k - z)} s(z_k,z) \\
&\quad + \frac{\bar{n}(z,z')}{\bar{n}(z_k,z')(\bar{n}(\hat{z}_k,z) + \bar{n}(z_k,z)\bar{n}(\hat{z}_k,z_k))} \frac{\bar{n}(\hat{z}_k,z_k) - \bar{n}(\hat{z}_k,z)}{(z_k - z)} s(z_k,z).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\lim_k \frac{F(\tau_k) - F(\zeta_k)}{z_k - z} =& \frac{\partial s(z,z')}{\partial x} - \frac{\partial \bar{n}(z,z')}{\partial x} \frac{s(z,z')}{\bar{n}(z,z')} - \frac{1}{\bar{n}(\hat{z},z)} \frac{\partial s(\hat{z},z)}{\partial y} \\
&+ \frac{s(z,z)}{\bar{n}(z,z')(1 + \bar{n}(z,z))} \frac{\partial \bar{n}(z,z')}{\partial x} \\
&+ \frac{s(z,z)}{\bar{n}(\hat{z},z)(1 + \bar{n}(z,z))} \frac{\partial \bar{n}(\hat{z},z)}{\partial y}.
\end{aligned}
$$

Case 2: $\bar{n}(\hat{z}_k, z_k)\frac{\bar{n}(z_k,z')}{\bar{n}(z,z')} < 1$ for infinitely many k. In this case, taking a subsequence if necessary, assume that $\bar{n}(\hat{z}_k, z_k)\frac{\bar{n}(z_k,z')}{\bar{n}(z,z')} < 1$ for each $k \in \mathbb{N}$ and let

$$
\begin{aligned}
\tau_k =& \bar{n}(\hat{z}_k,z_k) 1_{(z_k,z')} + 1_{(\hat{z}_k,z)} + \left(1 - \bar{n}(\hat{z}_k,z_k)\frac{\bar{n}(z_k,z')}{\bar{n}(z,z')}\right) 1_{(z,z')} \\
&+ \left(\frac{\bar{n}(\hat{z}_k,z_k)\frac{\bar{n}(z_k,z')}{\bar{n}(z,z')} - \bar{n}(\hat{z}_k,z)}{1 + \bar{n}(z,z)}\right) 1_{(z,z)}.
\end{aligned}
$$

Note that, for each $k \in \mathbb{N}$, all terms in τ_k are positive since $\bar{n}(\hat{z}_k, z_k)\frac{\bar{n}(z_k,z')}{\bar{n}(z,z')} < 1$, $\bar{n}(\hat{z}_k, z_k) \geq \bar{n}(\hat{z}_k, z)$ and $\bar{n}(z_k, z') \geq \bar{n}(z, z')$ and that

$$
\tau_{k,Z} + \tau_{k,Z,n} = \zeta_{k,Z} + \zeta_{k,Z,n} = 1_z + \bar{n}(z,z')1_{z'} + 1_{\hat{z}_k} + \bar{n}(\hat{z}_k,z_k)1_{z_k}.
$$

Furthermore, for each $k \in \mathbb{N}$,

$$\begin{aligned}
F(\zeta_k) =& s(z, z') + s(\hat{z}_k, z_k), \\
F(\tau_k) =& \bar{n}(\hat{z}_k, z_k) s(z_k, z') + s(\hat{z}_k, z) + \left(1 - \bar{n}(\hat{z}_k, z_k) \frac{\bar{n}(z_k, z')}{\bar{n}(z, z')}\right) s(z, z') \\
&+ \left(\frac{\bar{n}(\hat{z}_k, z_k) \frac{\bar{n}(z_k, z')}{\bar{n}(z, z')} - \bar{n}(\hat{z}_k, z)}{1 + \bar{n}(z, z)} \right) s(z, z)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\bar{n}(\hat{z}_k, z_k)} \frac{F(\tau_k) - F(\zeta_k)}{z_k - z} =& \frac{s(z_k, z') - s(z, z')}{z_k - z} - \frac{s(z, z')}{\bar{n}(z, z')} \frac{\bar{n}(z_k, z') - \bar{n}(z, z')}{z_k - z} \\
&- \frac{1}{\bar{n}(\hat{z}_k, z_k)} \frac{s(\hat{z}_k, z_k) - s(\hat{z}_k, z)}{z_k - z} + \frac{s(z, z)}{\bar{n}(\hat{z}_k, z_k)(1 + \bar{n}(z, z))} \frac{\bar{n}(\hat{z}_k, z_k) - \bar{n}(\hat{z}_k, z)}{z_k - z} \\
&+ \frac{s(z, z)}{\bar{n}(z, z')(1 + \bar{n}(z, z))} \frac{\bar{n}(z_k, z') - \bar{n}(z, z')}{z_k - z}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\lim_k \frac{F(\tau_k) - s(\zeta_k)}{z_k - z} > 0 \Leftrightarrow & \frac{\partial s(z, z')}{\partial x} - \frac{s(z, z')}{\bar{n}(z, z')} \frac{\partial \bar{n}(z, z')}{\partial x} - \frac{1}{\bar{n}(\hat{z}, z)} \frac{\partial s(\hat{z}, z)}{\partial y} \\
&+ \frac{s(z, z)}{\bar{n}(\hat{z}, z)(1 + \bar{n}(z, z))} \frac{\partial \bar{n}(\hat{z}, z)}{\partial y} \\
&+ \frac{s(z, z)}{\bar{n}(z, z')(1 + \bar{n}(z, z))} \frac{\partial \bar{n}(z, z')}{\partial x} > 0.
\end{aligned}$$

Case 3: $\bar{n}(\hat{z}_k, z) < 1 \leq \bar{n}(\hat{z}_k, z_k) \frac{\bar{n}(z_k, z')}{\bar{n}(z, z')}$ for all k sufficiently large. In this case, $\bar{n}(\hat{z}, z) = 1$. Taking a subsequence if necessary, assume that $\bar{n}(\hat{z}_k, z) < 1 \leq \bar{n}(\hat{z}_k, z_k) \frac{\bar{n}(z_k, z')}{\bar{n}(z, z')}$ for each $k \in \mathbb{N}$ and let

$$\tau_k = \frac{\bar{n}(z, z')}{\bar{n}(z_k, z')} 1_{(z_k, z')} + 1_{(\hat{z}_k, z)} + \left(\frac{\bar{n}(\hat{z}_k, z_k) - \frac{\bar{n}(z, z')}{\bar{n}(z_k, z')}}{1 + \bar{n}(z_k, z_k)} \right) 1_{(z_k, z_k)} + \left(\frac{1 - \bar{n}(\hat{z}_k, z)}{1 + \bar{n}(z, z)} \right) 1_{(z, z)}.$$

Note that, for each $k \in \mathbb{N}$, all terms in τ_k are positive and that

$$\tau_{k,Z} + \tau_{k,Z,n} = \zeta_{k,Z} + \zeta_{k,Z,n} = 1_z + \bar{n}(z, z') 1_{z'} + 1_{\hat{z}_k} + \bar{n}(\hat{z}_k, z_k) 1_{z_k}.$$

Then, for each $k \in \mathbb{N}$,

$$
\begin{aligned}
F(\zeta_k) &= s(z,z') + s(\hat{z}_k, z_k),\\
F(\tau_k) &= \frac{\bar{n}(z,z')}{\bar{n}(z_k,z')} s(z_k,z') + s(\hat{z}_k, z) + \left(\bar{n}(\hat{z}_k, z_k) - \frac{\bar{n}(z,z')}{\bar{n}(z_k,z')}\right) \frac{s(z_k,z_k)}{1+\bar{n}(z_k,z_k)}\\
&\quad + (\bar{n}(\hat{z},z) - \bar{n}(\hat{z}_k, z)) \frac{s(z,z)}{1+\bar{n}(z,z)}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{F(\tau_k) - F(\zeta_k)}{z_k - z} &= \frac{s(z_k,z') - s(z,z')}{z_k - z} - \frac{s(z_k,z')}{\bar{n}(z_k,z')} \frac{\bar{n}(z_k,z') - \bar{n}(z,z')}{z_k - z}\\
&- \frac{s(\hat{z}_k, z_k) - s(\hat{z}_k, z)}{z_k - z} + \frac{s(z_k,z_k)}{1+\bar{n}(z_k,z_k)} \frac{\bar{n}(\hat{z}_k, z_k)(\bar{n}(z_k,z') - n(z,z'))}{\bar{n}(z_k,z')(z_k - z)}\\
&+ \frac{s(z_k,z_k)}{1+\bar{n}(z_k,z_k)} \frac{\bar{n}(z,z')}{\bar{n}(z_k,z')} \frac{\bar{n}(\hat{z}_k, z_k) - \bar{n}(\hat{z}_k, z)}{\bar{n}(\hat{z},z)(z_k - z)}\\
&+ \frac{s(z_k,z_k)}{1+\bar{n}(z_k,z_k)} \frac{\bar{n}(z,z')}{\bar{n}(z_k,z')} \frac{\bar{n}(\hat{z}_k, z) - \bar{n}(\hat{z}, z)}{z_k - z}\\
&+ \frac{(\bar{n}(\hat{z},z) - \bar{n}(\hat{z}_k, z))}{z_k - z} \frac{s(z,z)}{1+\bar{n}(z,z)}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\frac{\bar{n}(\hat{z}_k, z) - \bar{n}(\hat{z}, z)}{z_k - z} \left(\frac{s(z_k,z_k)\bar{n}(z,z')}{(1+\bar{n}(z_k,z_k))\bar{n}(z_k,z')} - \frac{s(z,z)}{1+\bar{n}(z,z)}\right)\\
&= \frac{\bar{n}(\hat{z}_k, z) - \bar{n}(\hat{z}, z)}{\bar{n}(z_k, z_k')} \Bigg(- \frac{s(z_k,z_k)}{1+\bar{n}(z_k,z_k)} \frac{\bar{n}(z_k,z') - \bar{n}(z,z')}{z_k - z}\\
&+ \bar{n}(z_k,z') \frac{\frac{s(z_k,z_k)}{1+\bar{n}(z_k,z_k)} - \frac{s(z,z)}{1+\bar{n}(z,z)}}{z_k - z} \Bigg).
\end{aligned}
$$

Since $\lim_k \bar{n}(\hat{z}_k, z) = \bar{n}(\hat{z}, z) = 1$, it follows that

$$
\lim_k \frac{\bar{n}(\hat{z}_k, z) - \bar{n}(\hat{z}, z)}{z_k - z} \left(\frac{s(z_k,z_k)}{1+\bar{n}(z_k,z_k)} \frac{\bar{n}(z,z')}{\bar{n}(z_k,z')} - \frac{s(z,z)}{1+\bar{n}(z,z)}\right) = 0.
$$

Hence,

$$
\begin{aligned}
\lim_k \frac{F(\tau_k) - F(\zeta_k)}{z_k - z} &= \frac{\partial s(z,z')}{\partial x} - \frac{\partial \bar{n}(z,z')}{\partial x} \frac{s(z,z')}{\bar{n}(z,z')} - \frac{\partial s(\hat{z}, z)}{\partial y} \frac{1}{\bar{n}(\hat{z}, z)}\\
&+ \frac{s(z,z)}{\bar{n}(z,z')(1+\bar{n}(z,z))} \frac{\partial \bar{n}(z,z')}{\partial x} + \frac{s(z,z)}{\bar{n}(\hat{z},z)(1+\bar{n}(z,z))} \frac{\partial \bar{n}(\hat{z}, z)}{\partial y}.
\end{aligned}
$$

■

Note that when condition 6 of Theorem 9 holds, either one of the following must hold depending on whether $U_s(z) \geq \frac{s(z,z)}{1+\bar{n}(z,z)}$ holds or not; if yes, then we get (A10), if not we get (A11): For each $\hat{z}, z \in Z$ and $z' \in Z_\emptyset$ such that $(\hat{z}, z), (z, z') \in \operatorname{supp}(\pi)$,

$$\begin{aligned} &\frac{\partial s(z,z')}{\partial x} - \frac{s(z,z')}{\bar{n}(z,z')}\frac{\partial \bar{n}(z,z')}{\partial x} - \frac{1}{\bar{n}(\hat{z},z)}\frac{\partial s(\hat{z},z)}{\partial y} \\ &+ \frac{U_s(z)}{\bar{n}(\hat{z},z)}\frac{\partial \bar{n}(\hat{z},z)}{\partial y} + \frac{U_s(z)}{\bar{n}(z,z')}\frac{\partial \bar{n}(z,z')}{\partial x} > 0, \end{aligned} \tag{A10}$$

or for each $\hat{z}, z \in Z$ and $z' \in Z_\emptyset$ such that $(\hat{z}, z), (z, z') \in \operatorname{supp}(\pi)$,

$$\begin{aligned} &\frac{\partial s(z,z')}{\partial x} - \frac{\partial \bar{n}(z,z')}{\partial x}\frac{s(z,z')}{\bar{n}(z,z')} - \frac{1}{\bar{n}(\hat{z},z)}\frac{\partial s(\hat{z},z)}{\partial y} \\ &+ \frac{s(z,z)}{\bar{n}(z,z')(1+\bar{n}(z,z))}\frac{\partial \bar{n}(z,z')}{\partial x} + \frac{s(z,z)}{\bar{n}(\hat{z},z)(1+\bar{n}(z,z))}\frac{\partial \bar{n}(\hat{z},z)}{\partial y} > 0. \end{aligned} \tag{A11}$$

The following lemma is analogous to the previous ones and considers a match (z, z') where the manager is of type z and the workers of type z' and a sequence of matches $(z_k, \emptyset)$ such that $z_k \to z$.

Lemma 24 *Let $z' \in Z$, $z \in Z$ and $\{z_k\}_{k=1}^{\infty} \subseteq Z$ be such that $z < z_k$ for each $k \in \mathbb{N}$ and $z_k \to z$. If, for each $k \in \mathbb{N}$,*

$$\begin{aligned} \zeta_k &= 1_{(z,z')} + 1_{(z_k,\emptyset)} \text{ and} \\ \tau_k &= \frac{\bar{n}(z,z')}{\bar{n}(z_k,z')}1_{(z_k,z')} + \left(1 - \frac{\bar{n}(z,z')}{\bar{n}(z_k,z')}\right)1_{(z_k,\emptyset)} + 1_{(z,\emptyset)}, \end{aligned}$$

then $\tau_{k,Z} + \tau_{k,Z,n} = \zeta_{k,Z} + \zeta_{k,Z,n}$ and

$$\lim_k \frac{F(\tau_k) - F(\zeta_k)}{z_k - z} = \frac{\partial s(z,z')}{\partial x} - U_s'(z) - \frac{s(z,z') - U_s(z)}{\bar{n}(z,z')}\frac{\partial \bar{n}(z,z')}{\partial x}.$$

Proof. We have that $1 - \frac{\bar{n}(z,z')}{\bar{n}(z_k,z')} \geq 0$ since $\bar{n}(z_k, z') \geq \bar{n}(z, z')$ and that

$$\tau_{k,Z} + \tau_{k,Z,n} = \zeta_{k,Z} + \zeta_{k,Z,n} = 1_z + 1_{z_k} + \bar{n}(z,z')1_{z'}.$$

Furthermore,

$$\lim_k \frac{F(\tau_k)-F(\zeta_k)}{z_k-z} = \lim_k \Big(\frac{\bar{n}(z,z')}{\bar{n}(z_k,z')} \frac{s(z_k,z')-s(z,z')}{z_k-z} - \frac{\bar{n}(z,z')}{\bar{n}(z_k,z')} \frac{U_s(z_k)-U_s(z)}{z_k-z}$$
$$- \frac{\bar{n}(z_k,z')-\bar{n}(z,z')}{z_k-z} \frac{s(z,z')-U_s(z)}{\bar{n}(z_k,z')} \Big)$$
$$= \frac{\partial s(z,z')}{\partial x} - U_s'(z) - \frac{s(z,z')-U_s(z)}{\bar{n}(z,z')} \frac{\partial \bar{n}(z,z')}{\partial x}.$$

■

Lemma 25 shows that if there are managers of type z, then all individuals slightly more knowledgeable than z must be managers.

Lemma 25 *For each $z \in M \setminus \{\bar{z}_2\}$, there exists $\varepsilon > 0$ such that $(z, z+\varepsilon) \subseteq M \setminus (S \cup W)$.*

Proof. Suppose not; then there exists a sequence $\{z_k\}_{k=1}^{\infty}$ such that, for each $k \in \mathbb{N}$, $z_k > z$, $z_k \in (M \setminus (S \cup W))^c = M^c \cup (S \cup W)$ and $z_k \to z$; thus, $z_k \in S \cup W$ by Lemma 12. Let $z' \in Z$ be such that $(z, z') \in \text{supp}(\pi)$.

Suppose that $z_k \in S$ for infinitely many $k \in \mathbb{N}$ and, for such k, let $\zeta_k = 1_{(z,z')} + 1_{(z_k,\emptyset)}$ and $\tau = \frac{\bar{n}(z,z')}{\bar{n}(z_k,z')} 1_{(z_k,z')} + \left(1 - \frac{\bar{n}(z,z')}{\bar{n}(z_k,z')}\right) 1_{(z_k,\emptyset)} + 1_{(z,\emptyset)}$. Then ζ_k and τ_k are finitely-supported, $\text{supp}(\zeta_k) \subseteq \text{supp}(\pi)$, $\tau_{k,Z} + \tau_{k,Z,n} = \zeta_{k,Z} + \zeta_{k,Z,n}$ and $\lim_k \frac{F(\tau_k)-F(\zeta_k)}{z_k-z} > 0$ by Lemma 24. Thus, $F(\tau_k) - F(\zeta_k) > 0$ for each k sufficiently large, contradicting Theorem 4.

Thus, $z_k \notin S$ for each k sufficiently large and, hence, $z_k \in W$ for such k by Lemma 12. For each k sufficiently large, let $\hat{z}_k \in Z$ be such that $(\hat{z}_k, z_k) \in \text{supp}(\pi)$. Since Z is compact, we may assume that $\{\hat{z}_k\}_{k=1}^{\infty}$ converges; let $\hat{z} = \lim_k \hat{z}_k$. For each $k \in \mathbb{N}$, let $\zeta_k = 1_{(z,z')} + 1_{(\hat{z}_k,z_k)}$. If (A10) holds, then let $\{\tau_{k_j}\}_{j=1}^{\infty}$ be given by Lemma 22 (the argument when (A11) holds is analogous using Lemma 23 instead of Lemma 22). Then ζ_{k_j} and τ_{k_j} are finitely-supported, $\text{supp}(\zeta_{k_j}) \subseteq \text{supp}(\pi)$, $\tau_{k_j,Z} + \tau_{k_j,Z,n} = \zeta_{k_j,Z} + \zeta_{k_j,Z,n}$ and $\lim_j \frac{F(\tau_{k_j})-F(\zeta_{k_j})}{z_{k_j}-z} > 0$ by Lemma 22. Thus, $F(\tau_{k_j}) - F(\zeta_{k_j}) > 0$ for each j sufficiently large, contradicting Theorem 4. ■

Lemma 25 implies that $M = [z_2, \bar{z}]$ and $W \cup S = [\underline{z}, z_2]$. This argument requires the following result showing that M, S and W are perfect, i.e. have no isolated points.

Lemma 26 *M, S and W are perfect.*

Proof. We first show that M is perfect. Suppose that M has an isolated point z. Then $\text{supp}(\pi) \cap (\{z\} \times Z) \neq \emptyset$ and there is $\varepsilon > 0$ such that $B_\varepsilon(z) \cap M = \{z\}$. But this is a contradiction to the definition of $\text{supp}(\pi)$ since

$$\pi(B_\varepsilon(z) \times Z) = \pi(\text{supp}(\pi) \cap (B_\varepsilon(z) \times Z)) \leq$$
$$\pi((M \cap B_\varepsilon(z)) \times Z) = \pi(\{z\} \times Z) \leq \nu(\{z\}) = 0,$$

and $\text{supp}(\pi) \setminus (B_\varepsilon(z) \times Z)$ is closed and strictly contained in $\text{supp}(\pi)$. Thus, M has no isolated points and is, therefore, perfect.

We next show that S is perfect. Suppose that S has an isolated point z. Then $\text{supp}(\pi) \cap \{(z, \emptyset)\} \neq \emptyset$ and there is $\varepsilon > 0$ such that $B_\varepsilon(z) \cap S = \{z\}$. But this is a contradiction to the definition of $\text{supp}(\pi)$ since

$$\pi(B_\varepsilon(z) \times \{\emptyset\}) = \pi(\text{supp}(\pi) \cap (B_\varepsilon(z) \times \{\emptyset\})) \leq$$
$$\pi((S \cap B_\varepsilon(z)) \times \{\emptyset\}) = \pi(\{(z, \emptyset)\}) \leq \nu(\{z\}) = 0,$$

and $\text{supp}(\pi) \setminus (B_\varepsilon(z) \times \{\emptyset\})$ is closed and strictly contained in $\text{supp}(\pi)$. Thus, S has no isolated points and is, therefore, perfect.

We finally show that W is perfect. Suppose that W has an isolated point z. Then $\text{supp}(\pi) \cap (Z \times \{z\}) \neq \emptyset$ and there is $\varepsilon > 0$ such that $B_\varepsilon(z) \cap W = \{z\}$. Then

$$\pi(Z \times B_\varepsilon(z)) = \pi(\text{supp}(\pi) \cap (Z \times B_\varepsilon(z))) \leq$$
$$\pi(Z \times (W \cap B_\varepsilon(z))) = \pi(Z \times \{z\}) = 0,$$

since $0 \leq \int_{Z \times \{z\}} \bar{n} \mathrm{d}\pi \leq \nu(\{z\}) = 0$ and $\bar{n} > 0$. But this is a contradiction to the definition of $\text{supp}(\pi)$ since $\text{supp}(\pi) \setminus (Z \times B_\varepsilon(z))$ is closed and strictly contained in $\text{supp}(\pi)$. Thus, W has no isolated points and is, therefore, perfect. ■

The following lemma shows that M and $S \cup W$ are intervals and that, except for z_2, any workers and self-employed are less knowledgeable than managers. It proof simply extends globally the local conclusion of Lemma 25.

Lemma 27 *$M = [z_2, \bar{z}]$ and $S \cup W = [\underline{z}, z_2]$.*

Proof. Let $\bar{\varepsilon} = \sup\{\varepsilon > 0 : (z_2, z_2 + \varepsilon) \subseteq M \setminus (S \cup W)\}$. Such $\bar{\varepsilon}$ exists because $\{\varepsilon > 0 : (z_2, z_2 + \varepsilon) \subseteq M \setminus (S \cup W)\}$ is nonempty by Lemma 25 and is bounded above by $\bar{z} - z_2$. We then have that $(z_2, z_2 + \bar{\varepsilon}) \subseteq M \setminus (S \cup W)$ by the definition of $\bar{\varepsilon}$. Indeed, each $z \in (z_2, z_2 + \bar{\varepsilon})$ belongs to $M \setminus (S \cup W)$ since, letting $\varepsilon > 0$ be such that $z < z_2 + \varepsilon$ and $\varepsilon < \bar{\varepsilon}$, it follows that there exists $\varepsilon' > \varepsilon$ such that $(z_2, z_2 + \varepsilon') \subseteq M \setminus (S \cup W)$; hence, $z \in (z_2, z_2 + \varepsilon') \subseteq M \setminus (S \cup W)$.

We have that $z_2 + \bar{\varepsilon} \in M$ since M is closed by Lemma 10 and every $z < z_2 + \bar{\varepsilon}$ belongs to M. We claim that $\bar{\varepsilon} = \bar{z} - z_2$. If $\bar{\varepsilon} < \bar{z} - z_2$, then there exists $\eta > 0$ such that $(z_2 + \bar{\varepsilon}, z_2 + \bar{\varepsilon} + \eta) \subseteq M \setminus (S \cup W)$ by Lemma 25. This then implies that $z_2 + \bar{\varepsilon} \in (S \cup W)^c$ since otherwise $z_2 + \bar{\varepsilon}$ would be an isolated point of S or W, contradicting Lemma 26. Thus, $(z_2, z_2 + \bar{\varepsilon}] \subseteq M \setminus (S \cup W)$ and $(z_2, z_2 + \bar{\varepsilon} + \eta) \subseteq M \setminus (S \cup W)$, contradicting the definition of $\bar{\varepsilon}$.

Thus, it follows that $\bar{\varepsilon} = \bar{z} - z_2$ and $\bar{z} = z_2 + \bar{\varepsilon} \in M$. Then $\bar{z} \in (S \cup W)^c$ since otherwise $\bar{z}$ would be an isolated point of S or W, contradicting Lemma 26. Thus, $(z_2, \bar{z}] \subseteq M \setminus (S \cup W)$ and $S \cup W \subseteq [\underline{z}, z_2]$; in fact, $S \cup W = [\underline{z}, z_2]$ and $M = [z_2, \bar{z}]$ since $M \cup S \cup W = Z$ by Lemma 12, $z_2 = \min M$ by definition, M is closed by Lemma 10 and $S \cup W$ is closed by Lemmas 9 and 11. ■

Let $z_1 = \min S$ when S is nonempty and $z_1 = z_2$ otherwise.

Lemma 28 *$z_1 \leq z_2$ and $S \neq \emptyset$ if and only if $z_1 < z_2$.*

Proof. It follows by Lemma 27 that $z_1 \leq z_2$.

The definition of z_1 implies that $S \neq \emptyset$ if $z_1 < z_2$. For the converse, suppose that $S \neq \emptyset$ and $z_1 = z_2$. Then $S = \{z_2\}$ and $(z_2, \emptyset) \in \operatorname{supp}(\pi)$ by the definition of S. Since $\pi(\{(z_2, \emptyset)\}) \leq \nu(\{z_2\}) = 0$, it follows that $\operatorname{supp}(\pi) \cap (Z \times Z)$ is closed and a strict subset of $\operatorname{supp}(\pi)$ with $\pi(\operatorname{supp}(\pi) \cap (Z \times Z)) = \pi(\operatorname{supp}(\pi))$. This contradicts the definition of $\operatorname{supp}(\pi)$ and shows that $z_1 < z_2$. ■

Lemma 29 shows that if some individuals of type z are self-employed, then all individuals slightly more knowledgeable than z must be self-employed.

Lemma 29 *If $S \neq \emptyset$, then, for each $z \in S \setminus \{z_2\}$, there exists $0 < \varepsilon < z_2 - z$ such that $(z, z + \varepsilon) \subseteq S \setminus W$.*

Proof. Let $z \in S \setminus \{z_2\}$ and suppose that there is no $0 < \varepsilon < z_2 - z$ such that $(z, z + \varepsilon) \subseteq S \setminus W$. Then there exists a sequence $\{z_k\}_{k=1}^{\infty}$ such that, for each $k \in \mathbb{N}$, $z < z_k < z_2$, $z_k \in S^c \cup W$ and $z_k \to z$. Thus, $z_k \in W$ for each $k \in \mathbb{N}$ since $M = [z_2, \bar{z}]$ and $Z = M \cup S \cup W$ by Lemmas 12 and 27.

Let $z' = \emptyset$ and, for each $k \in \mathbb{N}$, let $\hat{z}_k \in Z$ be such that $(\hat{z}_k, z_k) \in \operatorname{supp}(\gamma)$. Since Z is compact, we may assume that $\{\hat{z}_k\}_{k=1}^{\infty}$ converges; let $\hat{z} = \lim_k \hat{z}_k$. For each $k \in \mathbb{N}$, let $\zeta_k = 1_{(z,z')} + 1_{(\hat{z}_k, z_k)}$; if (A10) holds, then let $\{\tau_{k_j}\}_{j=1}^{\infty}$ be given by Lemma 22 (the argument when (A11) holds is analogous using Lemma 23 instead of Lemma 22). Then ζ_{k_j} and τ_{k_j} are finitely-supported, $\operatorname{supp}(\zeta_{k_j}) \subseteq \operatorname{supp}(\pi)$, $\tau_{k_j,Z} + \tau_{k_j,Z,n} = \zeta_{k_j,Z} + \zeta_{k_j,Z,n}$ and $\lim_j \frac{F(\tau_{k_j}) - s(\zeta_{k_j})}{z_{k_j} - z} > 0$ by Lemma 22. Thus, $F(\tau_{k_j}) - F(\zeta_{k_j}) > 0$ for each j sufficiently large, contradicting Theorem 4. This contradiction shows that there exists $0 < \varepsilon < z_2 - z$ such that $(z, z + \varepsilon) \subseteq S \setminus W$. ■

The local conclusion of Lemma 29 extends globally and this shows that, when S is nonempty and with the exception of z_1, workers are less knowledgeable than self-employed.

Lemma 30 *$W = [\underline{z}, z_1]$ and, if $S \neq \emptyset$, then $S = [z_1, z_2]$.*

Proof. We can assume that S is nonempty since otherwise the conclusion follows from Lemma 27. Then $z_1 < z_2$ by Lemma 28.

Let $\bar{\varepsilon} = \sup\{\varepsilon > 0 : (z_1, z_1 + \varepsilon) \subseteq S \setminus W\}$. Such $\bar{\varepsilon}$ exists because $\{\varepsilon > 0 : (z_1, z_1 + \varepsilon) \subseteq S \setminus W\}$ is nonempty by Lemma 29 and is bounded above by $z_2 - z_1$. We then have that $(z_1, z_1 + \bar{\varepsilon}) \subseteq S \setminus W$ by the definition of $\bar{\varepsilon}$. Indeed, each $z \in (z_1, z_1 + \bar{\varepsilon})$ belongs to $S \setminus W$ since, letting $\varepsilon > 0$ be such that $z < z_1 + \varepsilon$ and $\varepsilon < \bar{\varepsilon}$, it follows that there exists $\varepsilon' > \varepsilon$ such that $(z_1, z_1 + \varepsilon') \subseteq S \setminus W$; hence, $z \in (z_1, z_1 + \varepsilon) \subseteq S \setminus W$.

We next claim that $\bar{\varepsilon} = z_2 - z_1$. Suppose not; then $\bar{\varepsilon} < z_2 - z_1$. We have that $z_1 + \bar{\varepsilon} \in S$ since S is closed; hence, there is $\eta > 0$ such that $(z_1 + \bar{\varepsilon}, z_1 + \bar{\varepsilon} + \eta) \subseteq S \setminus W$. If $z_1 + \bar{\varepsilon} \in W$, then $z_1 + \bar{\varepsilon}$ is an isolated point of W, contradicting Lemma 9. Thus, $(z_1, z_1 + \bar{\varepsilon}] \subseteq S \setminus W$. But then $(z_1, z_1 + \bar{\varepsilon} + \eta) \subseteq S \setminus W$, contradicting the definition of $\bar{\varepsilon}$. Thus, it follows that $\bar{\varepsilon} = z_2 - z_1$.

It follows from $\bar{\varepsilon} = z_2 - z_1$ that $(z_1, z_2) \subseteq S \setminus W$. We have that $z_1, z_2 \in S$ since S is closed, hence $S = [z_1, z_2]$ since $z_1 = \min S$ and $(z_2, \bar{z}] \subseteq S^c$. Then $[\underline{z}, z_1) \subseteq W$ by Lemma 27 and that $z_1 \in W$ since W is closed. Since $(z_1, z_2) \subseteq W^c$ and W is perfect, it follows that $z_2 \notin W$. Thus, $W = [\underline{z}, z_1]$ and $S = [z_1, z_2]$. ■

B.10 Proof of Theorem 6

It is technically convenient to establish this result for stable matchings and then use Theorem 2 to prove Theorem 6.

We say that a measure $\mu \in M(Z \times X_\emptyset)$ is *represented by* (z_1, z_2, ϕ, c) if conditions 1–4 and 7 in the definition of representability of an outcome by (z_1, z_2, ϕ, c) hold and

5'. $\mu = \nu \circ \hat{\sigma}^{-1}$ where $\hat{\sigma} : [z_1, \bar{z}] \to Z \times X_\emptyset$ is defined by setting, for each $z \in [z_1, \bar{z}]$,

$$\hat{\sigma}(z) = \begin{cases} (z, \bar{n}(z, \phi(z)) 1_{(\phi(z), c(\phi(z)))}) & \text{if } z \in [z_2, \bar{z}], \\ (z, 1_{(\emptyset, 0)}) & \text{if } z \in [z_1, z_2), \end{cases}$$

6' $c(z_1) = U_s(z_1)$ and $U_s(z_2) = s(z_2, \underline{z}) - \bar{n}(z_2, \underline{z}) c(\underline{z})$ if $S \neq \emptyset$ and $c(z_2) = s(z_2, \underline{z}) - \bar{n}(z_2, \underline{z}) c(\underline{z})$ if $S = \emptyset$.

We say that an economy E satisfies condition D' if the conclusion of condition 6 (resp. 7) in Theorem 9 holds for each $(z, z') \in Z \times Z_\emptyset$ (resp. $(z, z') \in Z \times Z$) such that $(z, n 1_{(z', c)}) \in \text{supp}(\mu)$ for some $(n, c) \in N \times C$ and each stable matching μ of E, and all the remaining conditions in D hold.

Theorem 10 *An economy E satisfying condition D' has a unique stable matching and μ is a stable matching of E if and only if μ is represented by (z_1, z_2, ϕ, c).*

We establish Theorem 10 as follows. We start by showing in Section B.10.1 that if μ is a stable matching of E, then there exists (z_1, z_2, ϕ, c) that represents it. Then we establish in Section B.10.2 a lemma that will be used to show both the converse of this statement in Section B.10.3 and the uniqueness of μ in Section B.10.4.

B.10.1 Proof of Theorem 10: Necessity

Let μ be a stable matching of E. Then (γ, u) is a stable assignment, where $u : Z \to \mathbb{R}$ is as in Lemma 1, $u(\emptyset) = 0$ and γ is as in Theorem 2. Furthermore, letting $\pi = \gamma_{Z \times Z_\emptyset}$, $M = \{z \in Z : (z, z') \in \text{supp}(\pi) \text{ for some } z' \in Z\}$, $S = \{z \in Z : (z, \emptyset) \in \text{supp}(\pi)\}$ and $W = \{z \in Z : (\hat{z}, z) \in \text{supp}(\pi) \text{ for some } \hat{z} \in Z\}$.

It follows by Theorem 9 that occupational stratification holds. Furthermore, it follows by Theorem 8 that γ is positive assortative.

Matching may be trivial if everyone is self-employed so that, effectively, no one matches. This happens if $U_s(z) \geq s(z, z') - \bar{n}(z, z')U_s(z')$ for each $z, z' \in Z$. Conversely, if this condition fails, then the sets M and W of managers and workers, respectively, have strictly positive measure.

Lemma 31 *$z_1 > \underline{z}$ and $z_2 < \bar{z}$.*

Proof. We first show that $\pi(Z \times Z) > 0$; suppose not, i.e. $\pi(Z \times Z) = 0$. Then $\text{supp}(\pi) \subseteq Z \times \{\emptyset\}$ and, hence, $v(z) = v(z) + \bar{n}(z, \emptyset)v(\emptyset) = s(z, \emptyset) = U_s(z)$ for each $z \in Z$. But the existence of $z, z' \in Z$ such that $U_s(z) < s(z, z') - \bar{n}(z, z')U_s(z')$ is then a contradiction to the stability of (π, v). This contradiction shows that $\pi(Z \times Z) > 0$.

It then follows from $\pi(Z \times Z) > 0$ that $\pi(M \times Z) = \pi(\text{supp}(\pi) \cap (Z \times Z)) = \pi(Z \times Z) > 0$ and, hence, $\nu(M) \geq \pi(Z \times Z) > 0$. Furthermore, $\nu(W) \geq \int_{Z \times W} \bar{n} \mathrm{d}\pi = \int_{\text{supp}(\pi) \cap (Z \times Z)} \bar{n} \mathrm{d}\pi = \int_{Z \times Z} \bar{n} \mathrm{d}\pi > 0$. Hence, if $z_1 = \underline{z}$, then $W \subseteq \{\underline{z}\}$ and $\nu(W) = 0$, a contradiction; thus, $z_1 > \underline{z}$. Similarly, if $z_2 = \bar{z}$, then $M \subseteq \{\bar{z}\}$ and $\nu(M) = 0$, a contradiction; thus, $z_2 < \bar{z}$. ∎

Define $\phi : M \rightrightarrows W$ by setting, for each $z \in M$,

$$\phi(z) = \{z' \in Z : (z, z') \in \text{supp}(\pi)\}.$$

Then ϕ is nonempty-valued by the definition of M, $\phi(M) = W$ by the definition of W and ϕ has a closed graph since $\text{supp}(\pi)$ is closed.

Let $Q = \{z \in M : \phi(z) \text{ is not a singleton}\}$. The correspondence ϕ is increasing due to positive assortativeness and, hence, the set Q of (lower hemi) discontinuities of ϕ is countable.

Lemma 32 *Q is countable.*

Proof. For each $z \in Q$, let $r(z) \in \mathbb{Q}$ be such that $\min \phi(z) < r(z) < \max \phi(z)$. This defines a function $r : Q \to \mathbb{Q}$ which, as we now claim, is strictly increasing. Indeed, if $z, \hat{z} \in Q$ are such that $z < \hat{z}$, then $(z, \max \phi(z)) \in \text{supp}(\pi)$, $(\hat{z}, \min \phi(\hat{z})) \in \text{supp}(\pi)$ and, hence, $\max \phi(z) \leq \min \phi(\hat{z})$ since π is positive assortative. Thus, $r(z) < \max \phi(z) \leq \min \phi(\hat{z}) < r(\hat{z})$. Thus, r maps Q in a one-to-one way to a subset of $\mathbb{Q}$, implying that Q is countable. ■

We next show that each worker type is matched with a unique manager type. The reason is roughly that if worker type z' were matched with manager types z^* and $\tilde{z}$, then positive assortativeness implies that type z' is matched with all types in $[z^*, \tilde{z}]$; furthermore, those types in $[z^*, \tilde{z}] \setminus Q$ are only matched with z', which means that a zero measure of workers (those with type z') is matched with a positive measure of managers (those with type in $[z^*, \tilde{z}] \setminus Q$).

Lemma 33 *For each $z \in W$, there exists $z^* \in Z$ such that $\{\hat{z} \in Z : (\hat{z}, z) \in \text{supp}(\pi)\} = \{z^*\}$.*

Proof. The definition of W implies that $\{\hat{z} \in Z : (\hat{z}, z) \in \text{supp}(\pi)\}$ is nonempty. Suppose that the conclusion of the lemma fails; then let $z' \in W$ and $z^*, \tilde{z} \in Z$ be such that $z^*, \tilde{z} \in \{\hat{z} \in Z : (\hat{z}, z') \in \text{supp}(\pi)\}$ and $z^* < \tilde{z}$. Then $z^*, \tilde{z} \in M$ and $[z^*, \tilde{z}] \subseteq M$ by occupational stratification. Furthermore, $[z^*, \tilde{z}] \subseteq \{\hat{z} \in Z : (\hat{z}, z') \in \text{supp}(\pi)\}$ since if $z \in (z^*, \tilde{z})$ and $\tilde{z}' \in Z$ is such that $(z, \tilde{z}') \in \text{supp}(\pi)$, then positive assortativeness implies that $z' \leq \tilde{z}' \leq z'$, hence $\tilde{z}' = z'$.

We have that $(z^*, \tilde{z}) \setminus Q \subseteq M \setminus (W \cup S)$ by occupational stratification and $\phi(z) = \{z'\}$ for each $z \in (z^*, \tilde{z}) \setminus Q$. Thus,

$$\nu([z^*, \tilde{z}] \setminus Q) = \nu((z^*, \tilde{z}) \setminus Q) = \pi(((z^*, \tilde{z}) \setminus Q) \times Z) = \pi(((z^*, \tilde{z}) \setminus Q) \times \{z'\}).$$

Letting $\alpha = \min_{(x,y) \in Z^2} \bar{n}(x, y)$, it follows that $\alpha > 0$ since $\bar{n} > 0$. Since Q is countable by Lemma 32,

$$\begin{aligned} 0 = \nu(\{z'\}) &\geq \int_{Z \times \{z'\}} \bar{n} \mathrm{d}\pi \geq \alpha \pi(Z \times \{z'\}) \geq \alpha \pi(((z^*, \tilde{z}) \setminus Q) \times \{z'\}) \\ &= \alpha \nu([z^*, \tilde{z}] \setminus Q) = \alpha \nu([z^*, \tilde{z}]) > 0, \end{aligned}$$

a contradiction. ■

It then follows that matching is strictly positive assortative in the sense the better managers have better workers.

Lemma 34 *If $(z, z'), (\hat{z}, \hat{z}') \in Z^2$, $(z, z'), (\hat{z}, \hat{z}') \in \text{supp}(\pi)$ and $z > \hat{z}$, then $z' > \hat{z}'$.*

Proof. We have that $z' \geq \hat{z}'$ since π is positive assortative and that $z' \neq \hat{z}'$ by Lemma 33. Thus, $z' > \hat{z}'$. ■

Strict positive assortativeness then implies that ϕ is a function, i.e. $Q = \emptyset$. This happens because if manager type z were matched with worker types z^* and $\tilde{z}$, then strict positive assortativeness implies that type z is matched with all types in $[z^*, \tilde{z}]$ and that these types are not matched with any other manager type. But then a zero measure of managers (those with type z) is matched with a positive measure of workers (those with type in $[z^*, \tilde{z}]$).

Lemma 35 *ϕ is a continuous and strictly increasing function. Furthermore, $\phi(z_2) = \underline{z}$ and $\phi(\bar{z}) = z_1$.*

Proof. We first show that $\phi(z)$ is a singleton for each $z \in M$, i.e. $Q = \emptyset$. Suppose not; then let $z \in M$ and $z^*, \tilde{z} \in \phi(z)$ be such that $z^* < \tilde{z}$. Then $[z^*, \tilde{z}] \subseteq W$ by occupational stratification. Furthermore, $[z^*, \tilde{z}] \subseteq \phi(z)$ since if $z' \in (z^*, \tilde{z})$ and $\hat{z} \in M$ is such that $z' \in \phi(\hat{z})$, then, by Lemma 34, $z' > \tilde{z}$ if $\hat{z} > z$ and $z' < z^*$ if $\hat{z} < z$; thus, $\hat{z} = z$ and $z' \in \phi(z)$.

We have that $[z^*, \tilde{z}] \cap \phi(x) = \emptyset$ for each $x \in M \setminus \{z\}$. Indeed, Lemma 34 implies that $\min \phi(x) > \tilde{z}$ for each $x > z$ and that $\max \phi(x) < z^*$ for each $x < z$. Since $(z^*, \tilde{z}) \subseteq W \setminus (M \cup S)$, it follows that

$$\nu([z^*, \tilde{z}]) = \pi([z^*, \tilde{z}] \times Z_\emptyset) + \int_{Z \times [z^*, \tilde{z}]} \bar{n} \mathrm{d}\pi = 0 + \int_{(Z \setminus \{z\}) \times [z^*, \tilde{z}]} \bar{n} \mathrm{d}\pi = 0,$$

a contradiction to $\nu([z^*, \tilde{z}]) > 0$. This contradiction shows that $\phi(z)$ is a singleton for each $z \in M$.

It then follows that ϕ is a function. Since the graph of ϕ is closed, it follows that ϕ is continuous. Lemma 34 implies that ϕ is strictly increasing.

It follows from $\phi(M) = W$ that ϕ is onto. This then implies that $\phi(z_2) = \underline{z}$ and $\phi(\bar{z}) = z_1$ since ϕ is strictly increasing. ■

The properties of ϕ above are then used to show that the wage function c is differentiable.

Lemma 36 *c is differentiable and, for each $z \in W$,*

$$c'(z) = \frac{1}{\bar{n}(\phi^{-1}(z), z)} \left(\frac{\partial s(\phi^{-1}(z), z)}{\partial y} - c(z) \frac{\partial \bar{n}(\phi^{-1}(z), z)}{\partial y} \right).$$

Proof. Let $z \in W$ and $\{z_k\}_{k=1}^{\infty}$ be such that, for each $k \in \mathbb{N}$, $z_k \in W$, $z_k \neq z$ and $z_k \to z$. Let $\{\hat{z}_k\}_{k=1}^{\infty}$ be such that $\hat{z}_k = \phi^{-1}(z_k)$ for each $k \in \mathbb{N}$. We have that ϕ^{-1} exists and is continuous by Lemma 35 and by the compactness of M. Thus, $\hat{z}_k \to \phi^{-1}(z)$.

The stability of μ implies that, for each $k \in \mathbb{N}$, $s(\phi^{-1}(z), z) - \bar{n}(\phi^{-1}(z), z)c(z) \geq s(\phi^{-1}(z), z_k) - \bar{n}(\phi^{-1}(z), z_k)c(z_k)$. Thus, a simple manipulation of this expression implies that

$$\frac{c(z_k)-c(z)}{z_k-z} \geq \frac{1}{\bar{n}(\phi^{-1}(z),z)}\left(\frac{s(\phi^{-1}(z),z_k)-s(\phi^{-1}(z),z)}{z_k-z}\right.$$
$$\left.-c(z_k)\frac{\bar{n}(\phi^{-1}(z),z_k)-\bar{n}(\phi^{-1}(z),z)}{z_k-z}\right);$$

hence, $\liminf_k \frac{c(z_k)-c(z)}{z_k-z} \geq \frac{1}{\bar{n}(\phi^{-1}(z),z)}\left(\frac{\partial s(\phi^{-1}(z),z)}{\partial y} - c(z)\frac{\partial \bar{n}(\phi^{-1}(z),z)}{\partial y}\right)$ since $c = u|_W$ is continuous.

The stability of μ also implies that, for each $k \in \mathbb{N}$, $s(\hat{z}_k, z_k) - \bar{n}(\hat{z}_k, z_k)c(z_k) \geq s(\hat{z}_k, z) - \bar{n}(z_k, z)c(z)$. Thus,

$$\frac{c(z_k)-c(z)}{z_k-z} \geq \frac{1}{\bar{n}(\hat{z}_k,z)}\left(\frac{s(\hat{z}_k,z_k)-s(\hat{z}_k,z)}{z_k-z} - c(z_k)\frac{\bar{n}(\hat{z}_k,z_k)-\bar{n}(\hat{z}_k,z)}{z_k-z}\right);$$

hence, $\limsup_k \frac{c(z_k)-c(z)}{z_k-z} \geq \frac{1}{\bar{n}(\phi^{-1}(z),z)}\left(\frac{\partial s(\phi^{-1}(z),z)}{\partial y} - c(z)\frac{\partial \bar{n}(\phi^{-1}(z),z)}{\partial y}\right)$ since s and $\bar{n}$ are continuously differentiable. It thus follows that

$$\lim_k \frac{c(z_k)-c(z)}{z_k-z} = \frac{1}{\bar{n}(\phi^{-1}(z),z)}\left(\frac{\partial s(\phi^{-1}(z),z)}{\partial y} - c(z)\frac{\partial \bar{n}(\phi^{-1}(z),z)}{\partial y}\right).$$

Hence, c is differentiable and, for each $z \in W$, $c'(z) = \lim_k \frac{c(z_k)-c(z)}{z_k-z}$. ■

We have shown so far that types in $[z_1, z_2]$ are self-employed and, thus, matched with $1_{(\emptyset,0)}$ and types in $[z_2, \bar{z}]$ are managers and matched with workers of type $\phi(z)$, thus, with $n(\phi(z))1_{(\phi(z),c(\phi(z)))}$. Hence, the matching μ is fully described by the distribution ν of types and the function σ.

Lemma 37 $\mu = \nu \circ \sigma^{-1}$.

Proof. Let B be a Borel subset of $Z \times X_\emptyset$. Then

$$\begin{aligned}\nu \circ \sigma^{-1}(B) &= \nu(\{z \in Z : \sigma(z) \in B\}) \\ &= \nu(\{z \in [z_1, z_2) : \sigma(z) \in B\}) + \nu(\{z \in [z_2, \bar{z}] : \sigma(z) \in B\}).\end{aligned}$$

Let $\hat{D} = \{z \in (z_1, z_2) : \sigma(z) \in B\}$ and note that

$$\begin{aligned}
&\nu(\{z \in [z_1, z_2) : \sigma(z) \in B\}) = \nu(\hat{D}) \\
&= \mu(\hat{D} \times X) + \mu(\{z \in (z_1, z_2) : \sigma(z) \in B\} \times (X_\emptyset \setminus X)) + \int_{Z\times X} \delta(\hat{D} \times C)\mathrm{d}\mu(z, \delta) \\
&= 0 + \mu(\{z \in (z_1, z_2) : \sigma(z) \in B\} \times \{1_{(\emptyset,0)}\}) + 0 \\
&= \mu(\{z \in [z_1, z_2] : \sigma(z) \in B\} \times \{1_{(\emptyset,0)}\}).
\end{aligned}$$

Let $D = \{z \in (z_2, \bar{z}] : \sigma(z) \in B\}$ and note that

$$\begin{aligned}
&\nu(\{z \in [z_2, \bar{z}] : \sigma(z) \in B\}) = \nu(D) \\
&= \mu(D \times X) + \mu(D \times (X_\emptyset \setminus X)) + \int_{Z\times X} \delta(D \times C)\mathrm{d}\mu(z, \delta) \\
&= \mu(D \times X) + 0 + 0 = \mu(\{z \in [z_2, \bar{z}] : \sigma(z) \in B\} \times X).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mu(B) = \mu(\mathrm{supp}(\mu) \cap B) \\
&= \mu(\mathrm{supp}(\mu) \cap B \cap (Z \times \{1_{(\emptyset,0)}\})) + \mu(\mathrm{supp}(\mu) \cap B \cap (Z \times X)) \\
&= \mu(\{z \in [z_1, z_2] : \sigma(z) \in B\} \times \{1_{(\emptyset,0)}\}) + \mu(\{z \in [z_2, \bar{z}] : \sigma(z) \in B\} \times X) \\
&= \nu(\{z \in [z_1, z_2) : \sigma(z) \in B\}) + \nu(\{z \in [z_2, \bar{z}] : \sigma(z) \in B\}) \\
&= \nu \circ \sigma^{-1}(B).
\end{aligned}$$

Since B is arbitrary, $\nu \circ \sigma^{-1} = \mu$. ■

Let $\nu(z') = \nu([\underline{z}, z'])$ for each $z' \in Z$. For each $z \geq z_2$, individuals of knowledge up to $\phi(z)$ are workers and are matched with managers of knowledge in $[z_2, z]$, hence their measure $\nu(\phi(z))$ equals the measure of workers hired by managers of knowledge between z_2 and z, which is $\int_{[z_2,z]} \bar{n}(x, \phi(x))\mathrm{d}\nu(x) = \int_{z_2}^{z} \bar{n}(x, \phi(x))\theta(x)\mathrm{d}x$.

Lemma 38 *For each $z \in [z_2, \bar{z}]$, $\nu(\phi(z)) = \int_{z_2}^{z} \bar{n}(x, \phi(x))\theta(x)\mathrm{d}x$.*

Proof. Let $z \in [z_2, \bar{z}]$ and let $\tau : Z \times X \to \mathbb{R}$ be defined by setting, for each $(z, \delta) \in Z \times X$, $\tau(z, \delta) = \delta([\underline{z}, \phi(z)] \times C)$. It follows by Lemma 37 that

$$\int_{Z\times X} \delta([\underline{z}, \phi(z)] \times C)\mathrm{d}\mu(x, \delta) = \int_{[z_2,\bar{z}]} \tau(\sigma(x))\mathrm{d}\nu(x) = \int_{[z_2,z]} \bar{n}(x, \phi(x))\mathrm{d}\nu(x).$$

Thus,

$$\nu([\underline{z}, \phi(z)]) = \int_{[z_2,z]} \bar{n}(x, \phi(x))\mathrm{d}\nu(x)$$

since $\nu([\underline{z},\phi(z)]) = \nu([\underline{z},\phi(z)))$ and $[\underline{z},\phi(z)) \subseteq W \setminus (S \cup M)$. Since ν has a continuous density θ, it follows that $\nu(\phi(z)) = \nu([\underline{z},\phi(z)]) = \int_{z_2}^{z} \bar{n}(x,\phi(x))\theta(x)\mathrm{d}x$ for each $z \in [z_2,\bar{z}]$. ■

The feasibility of the matching μ is fully captured by the equality $\nu(\phi(z)) = \int_{z_2}^{z} \bar{n}(x,\phi(x))\theta(x)\mathrm{d}x$ for each $z \in [z_2,\bar{z}]$ as stated in the previous lemma. It can be equivalently stated in terms of the derivative of ϕ as the following lemma shows.

Lemma 39 *ϕ is differentiable and, for each $z \in [z_2,\bar{z}]$,*

$$\phi'(z) = \frac{\bar{n}(z,\phi(z))\theta(z)}{\theta(\phi(z))}.$$

Proof. The function $z' \mapsto \nu(z')$ is strictly increasing; let $\lambda : [0,\nu(\bar{z})] \to Z$ be its inverse. It then follows by Lemma 38 that, for each $z \in [z_2,\bar{z}]$,

$$\phi(z) = \lambda\left(\int_{z_2}^{z} \bar{n}(x,\phi(x))\theta(x)\mathrm{d}x\right).$$

We have that $z \mapsto \nu(z)$ is differentiable and that its derivative at $z \in Z$ is $\theta(z)$. Then λ is differentiable and $\lambda'(x) = \frac{1}{\theta(\lambda(x))}$ for each $x \in [0,\nu(\bar{z})]$. Let $\zeta : [z_2,\bar{z}] \to \mathbb{R}$ be defined by setting, for each $z \in [z_2,\bar{z}]$, $\zeta(z) = \int_{z_2}^{z} \bar{n}(x,\phi(x))\theta(x)\mathrm{d}x$. Then ζ is differentiable with $\zeta'(z) = \bar{n}(z,\phi(z))\theta(z)$ for each $z \in [z_2,\bar{z}]$. Since $\phi = \lambda \circ \zeta$, it follows that ϕ is differentiable and that, for each $z \in [z_2,\bar{z}]$, $\phi'(z) = \frac{\bar{n}(z,\phi(z))\theta(z)}{\theta(\lambda(\zeta(z)))}$. Since $\zeta(z) = \nu(\phi(z))$ by Lemma 38, we obtain that $\lambda(\zeta(z)) = \phi(z)$ and, hence, $\phi'(z) = \frac{\bar{n}(z,\phi(z))\theta(z)}{\theta(\phi(z))}$. ■

The following two results show that individuals who belong to two of the sets M, S and W must be indifferent between the corresponding occupations. Lemma 40 considers the case where there are self-employed individuals, in which case $z_1 < z_2$, $z_1 \in W \cap S$ and $z_2 \in S \cap M$. Consequently, those with knowledge z_1 are indifferent between being a worker or self-employed and those with knowledge z_2 are indifferent between being a manager or self-employed.

Lemma 40 *If $S \neq \emptyset$, then $c(z_1) = U_s(z_1)$ and $U_s(z_2) = s(z_2,\underline{z}) - \bar{n}(z_2,\underline{z})c(\underline{z})$.*

Proof. Recall that $\phi(z_2) = \underline{z}$ and that $c = u|_W$. Then $c(z_1) = u(z_1) = u(z_1) + u(\emptyset) = s(z_1,\emptyset) = U_s(z_1)$, $u(z_2) = u(z_2) + u(\emptyset) = s(z_2,\emptyset) = U_s(z_2)$ and $u(z_2) + \bar{n}(z_2,\underline{z})c(\underline{z}) = s(z_2,\underline{z})$. Hence, $c(z_1) = U_s(z_1)$ and $U_s(z_2) = s(z_2,\underline{z}) - \bar{n}(z_2,\underline{z})c(\underline{z})$. ■

The following lemma considers the case where there are no self-employed individuals, in which case $z_1 = z_2$ and $z_2 \in W \cap M$. Consequently, those with knowledge z_2 are indifferent between being a worker or a manager.

Lemma 41 *If $S = \emptyset$, then $c(z_2) = s(z_2,\underline{z}) - \bar{n}(z_2,\underline{z})c(\underline{z}) \geq U_s(z_2)$.*

Proof. Recall that $\phi(z_2) = \underline{z}$ and that $c = u|_W$. Then $u(z_2) + \bar{n}(z_2, \underline{z})c(\underline{z}) = s(z_2, \underline{z})$ and, hence, $c(z_2) = u(z_2) = s(z_2, \underline{z}) - \bar{n}(z_2, \underline{z})c(\underline{z})$. Also, $c(z_2) = u(z_2) + u(\emptyset) \geq s(z_2, \emptyset) = U_s(z_2)$. ■

The necessity part of Theorem 10 then follows by occupational stratification and Lemmas 31, 35–37 and 39–41.

B.10.2 Proof of Theorem 10: A Lemma

Lemma 42 forms the core of the argument showing that representation by (z_1, z_2, ϕ, c) is sufficient for stable matchings. It shows that there can only be one (z_1, z_2, ϕ, c) representing a measure $\mu \in M(Z \times X_\emptyset)$. This then implies that if μ is represented by (z_1, z_2, ϕ, c) and $\hat{\mu}$ is a stable matching whose existence is guaranteed by Theorem 1, then μ must equal $\hat{\mu}$ since the latter is also represented by (z_1, z_2, ϕ, c); hence, μ is a stable matching.

Lemma 42 *If (z_1, z_2, ϕ, c) and $(\hat{z}_1, \hat{z}_2, \hat{\phi}, \hat{c})$ are such that each represents a measure in $M(Z \times X_\emptyset)$, then $(z_1, z_2, \phi, c) = (\hat{z}_1, \hat{z}_2, \hat{\phi}, \hat{c})$.*

We prove Lemma 42 in what follows. Let (z_1, z_2, ϕ, c) and $(\hat{z}_1, \hat{z}_2, \hat{\phi}, \hat{c})$ be such that each represents a measure in $M(Z \times X_\emptyset)$.

Recall that $\nu(z) = \nu([\underline{z}, z]) = \int_{\underline{z}}^{z} \theta(x)\mathrm{d}x$ for each $z \in Z$. Thus, $\nu'(z) = \theta(z)$ for each $z \in Z$. The following lemma applies the fundamental theorem of calculus to the function $\nu \circ \phi$.

Lemma 43 *For each $z \in [z_2, \bar{z}]$, $\nu(\phi(z)) = \int_{z_2}^{z} \bar{n}(x, \phi(x))\theta(x)\mathrm{d}x$.*

Proof. For each $z \in [z_2, \bar{z}]$, $[\nu \circ \phi]'(z) = \theta(\phi(z))\phi'(z) = \bar{n}(z, \phi(z))\theta(z)$. Since $\nu(\phi(z_2)) = \nu(\underline{z}) = 0$, it follows that $\nu(\phi(z)) = \int_{z_2}^{z} [\nu \circ \phi]'(x)\mathrm{d}x = \int_{z_2}^{z} \bar{n}(x, \phi(x))\theta(x)\mathrm{d}x$ for each $z \in [z_2, \bar{z}]$. ■

The following is the key technical lemma in the proof of Lemma 42. It considers two solutions of initial value problems that differ (at most) on the initial conditions, and shows that if they coincide at some point, then they must coincide everywhere.

Lemma 44 *Let $a, b, \hat{a}, \hat{b} \in \mathbb{R}$ be such that $a < b$ and $\hat{a} < \hat{b}$, $G : [a, b] \times [\hat{a}, \hat{b}] \to \mathbb{R}$ be continuous and such that $(z, x) \mapsto \frac{\partial G(z,x)}{\partial x}$ is continuous, $g : [a, b] \to [\hat{a}, \hat{b}]$ be a solution to the initial value problem $x' = G(z, x)$ and $x(a) = g(a)$ and $\hat{g} : [a, b] \to [\hat{a}, \hat{b}]$ be a solution to the initial value problem $x' = G(z, x)$ and $x(a) = \hat{g}(a)$. If there exists $z_0 \in [a, b]$ such that $g(z_0) = \hat{g}(z_0)$, then $g = \hat{g}$.*

Proof. Let $z_0 \in [a, b]$ be such that $g(z_0) = \hat{g}(z_0)$. We first show that $g(z) = \hat{g}(z)$ for each $z \geq z_0$. If $z_0 = b$, then this conclusion holds; hence, we may assume that $z_0 < b$.

Suppose that $\{z \in [z_0, b] : g(z) \neq \hat{g}(z)\} \neq \emptyset$ and let $z^* = \inf\{z \in [z_0, b] : g(z) \neq \hat{g}(z)\}$. Then $g(z^*) = \hat{g}(z^*)$: this is clear if $z^* = z_0$ and, if $z^* > z_0$, then $z^* \in \{z \in [a, b] : g(z) = \hat{g}(z)\}$ since this set is closed (due to the continuity of g and $\hat{g}$, which follows from the fact that they are solutions to an initial value problem) and $g(z) = \hat{g}(z)$ for each $z_0 \leq z < z^*$. Furthermore, $z^* < b$ since otherwise the definition of z^* implies that $\{z \in [z_0, b] : g(z) \neq \hat{g}(z)\} = \emptyset$.

Let $\eta > 0$ be such that $g(z^*) + \eta \leq \hat{b}$ and $g(z^*) - \eta \geq \hat{a}$ if $g(z^*) \in (\hat{a}, \hat{b})$, $g(z^*) - \eta \geq \hat{a}$ if $g(z^*) = \hat{a}$ and $g(z^*) + \eta \leq \hat{b}$ if $g(z^*) = \hat{b}$. Define $R = [g(z^*) - \eta, g(z^*) + \eta] \cap [\hat{a}, \hat{b}]$ and let, by the continuity of g and $\hat{g}$, $\varepsilon > 0$ be such that $g(z) \in R$ and $\hat{g}(z) \in R$ for each $z \in [z^*, z^* + \varepsilon]$. Thus, both g and $\hat{g}$ are solutions of the initial value problem $x' = \tilde{G}(z, x)$ and $x(z^*) = g(z^*) = \hat{g}(z^*)$, where $\tilde{G}$ is the restriction of G to $[z^*, z^* + \varepsilon] \times R$. The definition of z^* implies that $g(z) \neq \hat{g}(z)$ for some $z \in (z^*, z^* + \varepsilon)$. But this contradicts the Picard-Lindelöf Theorem.[6] This contradiction shows that $\{z \in [z_0, b] : g(z) \neq \hat{g}(z)\} = \emptyset$, i.e. $g(z) = \hat{g}(z)$ for each $z \geq z_0$.

We next show that $g(z) = \hat{g}(z)$ for each $z \leq z_0$. If $z_0 = a$, then this conclusion holds; hence, we may assume that $z_0 > a$.

Suppose that $\{z \in [a, z_0] : g(z) \neq \hat{g}(z)\} \neq \emptyset$ and let $z^* = \sup\{z \in [a, z_0] : g(z) \neq \hat{g}(z)\}$. Then $g(z^*) = \hat{g}(z^*)$: this is clear if $z^* = z_0$ and, if $z^* < z_0$, then $z^* \in \{z \in [a, b] : g(z) = \hat{g}(z)\}$ since this set is closed and $g(z) = \hat{g}(z)$ for each $z^* < z \leq z_0$. Furthermore, $z^* > a$ since otherwise the definition of z^* implies that $\{z \in [a, z_0] : g(z) \neq \hat{g}(z)\} = \emptyset$.

Let $\eta > 0$ be such that $g(z^*) + \eta \leq \hat{b}$ and $g(z^*) - \eta \geq \hat{a}$ if $g(z^*) \in (\hat{a}, \hat{b})$, $g(z^*) - \eta \geq \hat{a}$ if $g(z^*) = \hat{a}$ and $g(z^*) + \eta \leq \hat{b}$ if $g(z^*) = \hat{b}$. Define $R = [g(z^*) - \eta, g(z^*) + \eta] \cap [\hat{a}, \hat{b}]$ and let, by the continuity of g and $\hat{g}$, $\varepsilon > 0$ be such that $g(z) \in R$ and $\hat{g}(z) \in R$ for each $z \in [z^* - \varepsilon, z^*]$. Thus, both g and $\hat{g}$ are solutions of the initial value problem $x' = \tilde{G}(z, x)$ and $x(z^*) = g(z^*) = \hat{g}(z^*)$, where $\tilde{G}$ is the restriction of G to $[z^*, z^* - \varepsilon] \times R$. The definition of z^* implies that $g(z) \neq \hat{g}(z)$ for some $z \in (z^* - \varepsilon, z^*)$. But this contradicts the Picard-Lindelöf Theorem. This contradiction shows that $\{z \in [a, z_0] : g(z) \neq \hat{g}(z)\} = \emptyset$, i.e. $g(z) = \hat{g}(z)$ for each $z \leq z_0$. ■

The following lemma applies Lemma 44 to conclude that it suffices to show that $z_2 = \hat{z}_2$.

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[6] See Tesch (2012, Theorem 2.2, p. 38) for a statement of this result. Since this statement differs slightly from the version we are using, here is a sketch of the proof of this result in the case where $R = [g(z^*), g(z^*) + \eta]$ i.e. $g(z^*) = \hat{a}$, based on DePree and Swartz (1989, Example 3, p. 285): Let $\Gamma = [z^*, z^* + \varepsilon] \times [g(z^*), g(z^*) + \eta]$ for convenience and $M, L > 0$ be such that $|G(z, x)| \leq M$ and $|G(z, x) - G(z, x')| \leq L|x - x'|$ for each $(z, x), (z, x') \in \Gamma$; since Γ is compact, M exists because G is continuous and L exists because $\frac{\partial G}{\partial x}$ is continuous. Let $\delta > 0$ be such that $[z^*, z^* + \delta] \times [g(z^*), g(z^*) + M\delta] \subseteq \Gamma$ and $\delta L < 1$, i.e. $\delta < \min\{\varepsilon, \eta/M, 1/L\}$. Let $\mathcal{C}$ be the space of continuous functions on $[z^*, z^* + \delta]$ whose range is contained in $[g(z^*), g(z^*) + M\delta]$; then $\mathcal{C}$, endowed with the sup norm, is a complete metric space. Finally, define $\Lambda : \mathcal{C} \to \mathcal{C}$ by setting, for each $\lambda \in \mathcal{C}$ and $z \in [z^*, z^* + \delta]$, $\Lambda(\lambda)(z) = g(z^*) + \int_{z^*}^{z} G(y, \lambda(y))\mathrm{d}y$. Then, indeed $\Lambda(\lambda) \in \mathcal{C}$ and Λ is a contraction. Thus, Λ has a unique fixed point.

Lemma 45 *If $z_2 = \hat{z}_2$, then $\phi = \hat{\phi}$, $z_1 = \hat{z}_1$ and $c = \hat{c}$.*

Proof. We divide the proof of this lemma in four parts.

Part 1: If $z_2 = \hat{z}_2$, then $\phi = \hat{\phi}$ and $z_1 = \hat{z}_1$.

Let $\tilde{z}_1 = \max\{z_1, \hat{z}_1\}$ and $G : [z_2, \bar{z}] \times [\underline{z}, \tilde{z}_1] \to \mathbb{R}$ be such that $G(z, x) = \frac{\bar{n}(z,x)\theta(z)}{\theta(x)}$ for each $(z, x) \in [z_2, \bar{z}] \times [\underline{z}, \tilde{z}_1]$. Then the conditions of Lemma 44 hold with ϕ and $\hat{\phi}$ being solutions to the initial value problems and $z_0 = z_2$, the latter since $\phi(z_2) = \hat{\phi}(z_2) = \underline{z}$. Then $\phi = \hat{\phi}$ and $z_1 = \phi(\bar{z}) = \hat{\phi}(\bar{z}) = \hat{z}_1$.

Part 2: If $z_2 = \hat{z}_2$ and $z_1 < z_2$, then $c = \hat{c}$.

By part 1, $\phi = \hat{\phi}$ and $z_1 = \hat{z}_1$. Let $b = \max_{(z,z')\in Z^2} \frac{s(z,z')}{\bar{n}(z,z')}$ and note that $0 \leq c(z) \leq b$ for each $z \in Z$ since $c(z) \geq U_s(z) \geq 0$ and $c(z) \leq \max_{(\hat{z},z)\in Z^2} \frac{s(\hat{z},z)}{\bar{n}(\hat{z},z)} = b$. Then let $G : [\underline{z}, z_1] \times [0, b] \to \mathbb{R}$ be such that

$$G(z, x) = \frac{1}{\bar{n}(\phi^{-1}(z), z)} \left(\frac{\partial s(\phi^{-1}(z), z)}{\partial y} - x\frac{\partial \bar{n}(\phi^{-1}(z), z)}{\partial y} \right)$$

for each $(z, x) \in [\underline{z}, z_1] \times [0, b]$. Then the conditions of Lemma 44 hold with c and $\hat{c}$ being solutions to the initial value problems and $z_0 = z_1$, the latter since $c(z_1) = \hat{c}(z_1) = U_s(z_1)$. Hence, $c = \hat{c}$.

Part 3: If $z_2 = \hat{z}_2$ and $z_1 = z_2$, then $c = \hat{c}$.

The argument for part 2 applies provided that $\{z \in [\underline{z}, z_2] : c(z) = \hat{c}(z)\} \neq \emptyset$, which we establish in what follows. Suppose that $\{z \in [\underline{z}, z_2] : c(z) = \hat{c}(z)\} = \emptyset$. Then $c(\underline{z}) \neq \hat{c}(\underline{z})$ and, since c and $\hat{c}$ are arbitrary, we may assume that $c(\underline{z}) > \hat{c}(\underline{z})$. Then $c(z_2) > \hat{c}(z_2)$ by the intermediate value theorem since c and $\hat{c}$ are continuous. Since $\bar{n} > 0$, it follows that

$$c(z_2) > \hat{c}(z_2) = s(z_2, \underline{z}) - \bar{n}(z_2, \underline{z})\hat{c}(\underline{z}) > s(z_2, \underline{z}) - \bar{n}(z_2, \underline{z})c(\underline{z}) = c(z_2),$$

a contradiction. This contradiction then shows that $\{z \in [\underline{z}, z_2] : c(z) = \hat{c}(z)\} \neq \emptyset$.

Part 4: If $z_2 = \hat{z}_2$, then $\phi = \hat{\phi}$, $z_1 = \hat{z}_1$ and $c = \hat{c}$.

Part 4 follows by parts 1, 2 and 3. ■

The function $\phi : [z_2, \bar{z}] \to [\underline{z}, z_1]$ is strictly increasing (condition 4). Hence, let $\varphi : [\underline{z}, z_1] \to [z_2, \bar{z}]$ be the inverse of ϕ. Then φ is strictly increasing and differentiable and, for each $z \in [\underline{z}, z_1]$,

$$\varphi'(z) = \frac{\theta(z)}{\bar{n}(\varphi(z), z)\theta(\varphi(z))}.$$

As noted, Lemma 45 implies that it remains to establish that $z_2 = \hat{z}_2$. Lemma 46 derives some consequences of the assumption that $z_2 > \hat{z}_2$. Namely it shows that if there are less managers in $[z_2, \bar{z}]$ than in $[\hat{z}_2, \bar{z}]$ (i.e. $[z_2, \bar{z}] \subset [\hat{z}_2, \bar{z}]$), then it must be less workers in $[\underline{z}, z_1]$

than in $[\underline{z}, \hat{z}_1]$ (i.e. $[\underline{z}, z_1] \subset [\underline{z}, \hat{z}_1]$) and that every worker in $[\underline{z}, z_1]$ is assigned to a more knowledgeable manager by φ as compared to $\hat{\varphi}$.

Lemma 46 *If $z_2 > \hat{z}_2$, then $z_1 < \hat{z}_1$ and $\varphi(z) > \hat{\varphi}(z)$ for each $z \in [\underline{z}, z_1]$.*

Proof. We have that $\varphi(\underline{z}) = z_2 > \hat{z}_2 = \hat{\varphi}(\underline{z})$. Let $\tilde{z}_1 = \min\{z_1, \hat{z}_1\}$ and assume that $\{z \in [\underline{z}, \tilde{z}_1] : \varphi(z) = \hat{\varphi}(z)\} \neq \emptyset$. Let $z_0 = \inf\{z \in [\underline{z}, \tilde{z}_1] : \varphi(z) = \hat{\varphi}(z)\}$. Then $\varphi(z_0) = \hat{\varphi}(z_0)$ since φ and $\hat{\varphi}$ are continuous. Thus, $z_0 > \underline{z}$.

The definition of z_0, the continuity of both φ and $\hat{\varphi}$ and the intermediate value theorem imply that $\varphi(z) > \hat{\varphi}(z)$ for each $z \in [\underline{z}, z_0)$. This then implies that $\phi(z) < \hat{\phi}(z)$ for each $z \in [z_2, \varphi(z_0))$. Indeed, $\phi(z_2) = \underline{z} < \hat{\phi}(z_2)$ since otherwise $\hat{\varphi}(\underline{z}) = z_2$ and, hence, $\hat{z}_2 = \hat{\varphi}(\underline{z}) = z_2$. Let $z \in (z_2, \varphi(z_0))$ and let x, x' be such that $z = \varphi(x) = \hat{\varphi}(x')$. Since $\varphi(x) > \hat{\varphi}(x)$, it follows that $\hat{\varphi}(x') > \hat{\varphi}(x)$. Then $x' > x$ since $\hat{\varphi}$ is strictly increasing and, thus, $\hat{\phi}(z) = x' > x = \phi(z)$.

It then follows by the above that, for each $z \in [z_2, \varphi(z_0))$, $\bar{n}(z, \phi(z))\theta(z) \leq \bar{n}(z, \hat{\phi}(z))\theta(z)$. Thus, by Lemma 43,

$$\begin{aligned}
\nu(z_0) = \nu(\phi(\varphi(z_0))) &= \int_{z_2}^{\varphi(z_0)} \bar{n}(x, \phi(x))\theta(x)\mathrm{d}x \\
&< \int_{\hat{z}_2}^{z_2} \bar{n}(x, \hat{\phi}(x))\theta(x)\mathrm{d}x + \int_{z_2}^{\varphi(z_0)} \bar{n}(x, \hat{\phi}(x))\theta(x)\mathrm{d}x \\
&= \int_{\hat{z}_2}^{\hat{\varphi}(z_0)} \bar{n}(x, \hat{\phi}(x))\theta(x)\mathrm{d}x \\
&= \nu(\hat{\phi}(\hat{\varphi}(z_0))) = \nu(z_0),
\end{aligned}$$

a contradiction. This contradiction shows that $\{z \in [\underline{z}, \tilde{z}_1] : \varphi(z) = \hat{\varphi}(z)\} = \emptyset$.

It then follows from $\{z \in [\underline{z}, \tilde{z}_1] : \varphi(z) = \hat{\varphi}(z)\} = \emptyset$, together with the continuity of φ and $\hat{\varphi}$ and $\varphi(\underline{z}) > \hat{\varphi}(\underline{z})$, that $\varphi(z) > \hat{\varphi}(z)$ for each $z \in [\underline{z}, \tilde{z}_1]$ by the intermediate value theorem. Hence, $\varphi(\tilde{z}_1) > \hat{\varphi}(\tilde{z}_1)$. If $\tilde{z}_1 = \hat{z}_1$, then $\varphi(\hat{z}_1) > \hat{\varphi}(\hat{z}_1) = \bar{z}$, a contradiction. Thus, $\tilde{z}_1 < \hat{z}_1$. It then follows that $z_1 = \tilde{z}_1 < \hat{z}_1$ and $\varphi(z) > \hat{\varphi}(z)$ for each $z \in [\underline{z}, z_1]$. ■

To conclude the argument, we consider three cases: (i) $z_1 = z_2$ and $\hat{z}_1 = \hat{z}_2$, (ii) $z_1 < z_2$ and $\hat{z}_1 < \hat{z}_2$, and (iii) $z_1 < z_2$ and $\hat{z}_1 = \hat{z}_2$. The case $z_1 = z_2$ and $\hat{z}_1 < \hat{z}_2$ is covered by case (iii) since (z_1, z_2) and $(\hat{z}_1, \hat{z}_2)$ are arbitrary, case (i) is considered in Lemma 47 and cases (ii) and (iii) in Lemma 49.

Lemma 47 *If $z_1 = z_2$ and $\hat{z}_1 = \hat{z}_2$, then $z_2 = \hat{z}_2$.*

Proof. Suppose not and assume that $z_2 > \hat{z}_2$. Then Lemma 46 implies that $z_2 = z_1 < \hat{z}_1 = \hat{z}_2 < z_2$, a contradiction. ■

The following is a technical lemma that will be used in the proof of Lemma 49. It uses the following consequences of part (iii) in condition D': Since $\ln \bar{n}$ is supermodular, then, for each $(z, z') \in Z^2$, $\frac{\partial^2 \ln \bar{n}(z,z')}{\partial x \partial y} \geq 0$ and, hence,

$$\frac{\partial^2 \bar{n}(z, z')}{\partial x \partial y} \geq \frac{1}{\bar{n}(z, z')} \frac{\partial \bar{n}(z, z')}{\partial x} \frac{\partial \bar{n}(z, z')}{\partial y}. \tag{A12}$$

Since $\frac{\partial^2 \frac{s(x,y)\bar{n}(z,y)}{\bar{n}(x,y)}}{\partial x \partial y} > 0$ holds for each $x, y, z \in Z$, then, in particular, by computing this derivative and then setting $z = x$, it follows that, for each $(x, y) \in Z^2$,

$$\begin{aligned} \frac{\partial^2 s(x, y)}{\partial x \partial y} > & \frac{1}{\bar{n}(x, y)} \left(\frac{\partial s(x, y)}{\partial y} \frac{\partial \bar{n}(x, y)}{\partial x} + s(x, y) \frac{\partial^2 \bar{n}(x, y)}{\partial x \partial y} \right) \\ & - \frac{s(x, y)}{\bar{n}(x, y)^2} \frac{\partial \bar{n}(x, y)}{\partial x} \frac{\partial \bar{n}(x, y)}{\partial y}. \end{aligned} \tag{A13}$$

Lemma 48 *If $z_1 < z_2$ and $z_2 > \hat{z}_2$, then $c(z_1) > \hat{c}(z_1)$.*

Proof. It follows from $z_2 > \hat{z}_2$ that $z_1 < \hat{z}_1$ by Lemma 46. Let $\tilde{z} \in Z$ and $f_{\tilde{z}} : Z \to \mathbb{R}$ be defined by setting, for each $z \in Z$, $f_{\tilde{z}}(z) = \frac{s(z,\tilde{z}) - U_s(z)}{\bar{n}(z,\tilde{z})}$; then $f_{\tilde{z}}$ is strictly increasing since

$$\bar{n}(z, \tilde{z})^2 f'_{\tilde{z}}(z) = \bar{n}(z, \tilde{z}) \left(\frac{\partial s(z, \tilde{z})}{\partial x} - U'_s(z) - \frac{\partial \bar{n}(z, \tilde{z})}{\partial x} \frac{s(z, \tilde{z}) - U_s(z)}{\bar{n}(z, \tilde{z})} \right) > 0$$

by condition 7 in Theorem 9 and $\bar{n} > 0$. Since $z_2 > \hat{z}_2$, it follows that

$$c(\underline{z}) = \frac{s(z_2, \underline{z}) - U_s(z_2)}{\bar{n}(z_2, \underline{z})} = f_{\underline{z}}(z_2) > f_{\underline{z}}(\hat{z}_2) = \frac{s(\hat{z}_2, \underline{z}) - U_s(\hat{z}_2)}{\bar{n}(\hat{z}_2, \underline{z})} \geq \hat{c}(\underline{z}).$$

We claim that $\{z \in [\underline{z}, z_1] : c(z) = \hat{c}(z)\} = \emptyset$. Suppose not and let $z_0 = \inf\{z \in [\underline{z}, z_1] : c(z) = \hat{c}(z)\}$. Then $z_0 > \underline{z}$ since $c(\underline{z}) > \hat{c}(\underline{z})$, $c(z_0) = \hat{c}(z_0)$ by the continuity of c and $\hat{c}$, and $c(z) \neq \hat{c}(z)$ for each $z \in [\underline{z}, z_0)$ by the definition of z_0. Then $c(z) > \hat{c}(z)$ for each $z \in [\underline{z}, z_0)$ by the intermediate value theorem since $c(\underline{z}) > \hat{c}(\underline{z})$.

Lemma 46 implies that $\varphi(z_0) > \hat{\varphi}(z_0)$. Let $f : Z \to \mathbb{R}$ be defined by setting, for each $z \in Z$,

$$f(x) = \frac{1}{\bar{n}(x, z_0)} \left(\frac{\partial s(x, z_0)}{\partial y} - c(z_0) \frac{\partial \bar{n}(x, z_0)}{\partial y} \right).$$

Then, for each $x \in (\hat{\varphi}(z_0), \varphi(z_0))$,

$$\bar{n}(x,z_0)^2 f'(x) = \bar{n}(x,z_0)\left(\frac{\partial^2 s(x,z_0)}{\partial x \partial y} - c(z_0)\frac{\partial^2 \bar{n}(x,z_0)}{\partial x \partial y}\right.$$
$$\left. - \frac{1}{\bar{n}(x,z_0)}\frac{\partial \bar{n}(x,z_0)}{\partial x}\left(\frac{\partial s(x,z_0)}{\partial y} - c(z_0)\frac{\partial \bar{n}(x,z_0)}{\partial y}\right)\right)$$
$$> (s(x,z_0) - \bar{n}(x,z_0)c(z_0))\left(\frac{\partial^2 \bar{n}(x,z_0)}{\partial x \partial y} - \frac{1}{\bar{n}(x,z_0)}\frac{\partial \bar{n}(x,z_0)}{\partial x}\frac{\partial \bar{n}(x,z_0)}{\partial y}\right),$$

where the inequality follows from (A13). We have that

$$\frac{\partial^2 \bar{n}(x,z_0)}{\partial x \partial y} - \frac{1}{\bar{n}(x,z_0)}\frac{\partial \bar{n}(x,z_0)}{\partial x}\frac{\partial \bar{n}(x,z_0)}{\partial y} \geq 0$$

by (A12). Furthermore, $c(z_0) \geq U_s(z_0) \geq 0$ and $s(\hat{\varphi}(z_0), z_0) - \bar{n}(\hat{\varphi}(z_0), z_0)c(z_0) = s(\hat{\varphi}(z_0), z_0) - \bar{n}(\hat{\varphi}(z_0), z_0)\hat{c}(z_0) \geq U_s(\hat{\varphi}(z_0))$; then

$$c(z_0) \leq \frac{s(\hat{\varphi}(z_0), z_0) - U_s(\hat{\varphi}(z_0))}{\bar{n}(\hat{\varphi}(z_0), z_0)} = f_{z_0}(\hat{\varphi}(z_0)) < f_{z_0}(x) = \frac{s(x,z_0) - U_s(x)}{\bar{n}(x,z_0)}$$

since f_{z_0} is strictly increasing and, hence, $s(x,z_0) - \bar{n}(x,z_0)c(z_0) > U_s(x) \geq 0$. Thus, it follows that $f'(x) > 0$ and, hence, f is strictly increasing.

Thus,

$$c'(z_0) = f(\varphi(z_0)) > f(\hat{\varphi}(z_0)) = \hat{c}'(z_0)$$

and, hence, there is $0 < \varepsilon < z_0$ such that $c'(z) > \hat{c}'(z)$ for each $z \in (z_0 - \varepsilon, z_0)$. Thus, by the mean value theorem, there is $z \in (z_0 - \varepsilon, z_0)$ such that

$$0 = c(z_0) - \hat{c}(z_0) = c(z_0 - \varepsilon) - \hat{c}(z_0 - \varepsilon) + (c'(z) - \hat{c}'(z))\varepsilon > 0,$$

a contradiction. This contradiction shows that $\{z \in [\underline{z}, z_1] : c(z) = \hat{c}(z)\} = \emptyset$.

It then follows that $c(z) > \hat{c}(z)$ for each $z \in [\underline{z}, z_1]$ by the intermediate value theorem since $c(\underline{z}) > \hat{c}(\underline{z})$. Thus, $c(z_1) > \hat{c}(z_1)$. ■

Lemma 49 considers cases (ii) and (iii) above, i.e. shows that $z_2 = \hat{z}_2$ whenever $z_1 < z_2$ and $\hat{z}_1 \leq \hat{z}_2$.

Lemma 49 *If $z_1 < z_2$ and $\hat{z}_1 \leq \hat{z}_2$, then $z_2 = \hat{z}_2$.*

Proof. Note first that $\hat{z}_2 > z_2$ is not possible when $\hat{z}_1 = \hat{z}_2$. Indeed, if $z_1 < z_2$, $\hat{z}_1 \leq \hat{z}_2$ and $\hat{z}_2 > z_2$ then $\hat{z}_1 < z_1$ by Lemma 46 and, hence, $\hat{z}_1 < z_1 < z_2 < \hat{z}_2 = \hat{z}_1$, a contradiction.

Thus, it is enough to show that $z_2 > \hat{z}_2$ is not possible by this argument in the case of $\hat{z}_1 = \hat{z}_2$ and by symmetry in the case of $\hat{z}_1 < \hat{z}_2$.

Suppose that $z_2 > \hat{z}_2$. Then $z_1 < \hat{z}_1$ by Lemma 46 and $c(z_1) > \hat{c}(z_1)$ by Lemma 48. Furthermore, $z_1 < \hat{z}_1$ implies that $z_1 \in \hat{W}$. Hence, $U_s(z_1) \leq \hat{c}(z_1) < c(z_1) = U_s(z_1)$, a contradiction. ■

Since (z_1, z_2) and $(\hat{z}_1, \hat{z}_2)$ are arbitrary in Lemma 49, this lemma also shows that if $z_1 = z_2$ and $\hat{z}_1 < \hat{z}_2$, then $z_2 = \hat{z}_2$. It then follows from Lemmas 47 and 49 that $z_2 = \hat{z}_2$ and by Lemma 45 that $z_1 = \hat{z}_1$, $\phi = \hat{\phi}$ and $c = \hat{c}$. Hence, Lemma 42 follows.

B.10.3 Proof of Theorem 10: Sufficiency

Let $\mu \in M(Z \times X_\emptyset)$ be represented by (z_1, z_2, ϕ, c); then $\mu = \nu \circ \sigma^{-1}$. Let, by Theorem 1, $\hat{\mu}$ be a stable matching and, by the necessity part of Theorem 6 just established, let $(\hat{z}_1, \hat{z}_2, \hat{\phi}, \hat{c})$ represent $\hat{\mu}$; then $\hat{\mu} = \nu \circ \hat{\sigma}^{-1}$, where $\hat{\sigma} : [\hat{z}_1, \bar{z}] \to Z \times X_\emptyset$ is defined as σ is but with $(\hat{z}_1, \hat{z}_2, \hat{\phi}, \hat{c})$ in place of (z_1, z_2, ϕ, c). It then follows by Lemma 42 that $(z_1, z_2, \phi, c) = (\hat{z}_1, \hat{z}_2, \hat{\phi}, \hat{c})$ and, hence, $\sigma = \hat{\sigma}$. Thus $\mu = \nu \circ \sigma^{-1} = \hat{\mu}$ and μ is a stable matching.

B.10.4 Proof of Theorem 10: Uniqueness

Let μ and $\hat{\mu}$ be stable matchings and, by Theorem 6, let μ be represented by (z_1, z_2, ϕ, c) and $\hat{\mu}$ by $(\hat{z}_1, \hat{z}_2, \hat{\phi}, \hat{c})$; in particular, $\mu = \nu \circ \sigma^{-1}$ and $\hat{\mu} = \nu \circ \hat{\sigma}^{-1}$, where $\hat{\sigma}$ is defined as σ is but with $(\hat{z}_1, \hat{z}_2, \hat{\phi}, \hat{c})$ in place of (z_1, z_2, ϕ, c). It then follows by Lemma 42 that $(z_1, z_2, \phi, c) = (\hat{z}_1, \hat{z}_2, \hat{\phi}, \hat{c})$ and, hence, $\sigma = \hat{\sigma}$. Thus $\mu = \nu \circ \sigma^{-1} = \hat{\mu}$.

B.10.5 Proof of Theorem 6: Necessity

Let (γ, u) be a stable outcome of E. By Theorem 2, $\mu = \gamma \circ h^{-1}$ is a stable matching of E and, by Theorem 10, there is (z_1, z_2, ϕ, c) that represents μ. Thus, to establish the necessity part of Theorem 6, it remains to prove the following lemma.

Lemma 50 *If (γ, u) is a stable outcome of an economy E, $h : \text{supp}(\gamma) \to Z \times X_\emptyset$ is as in Theorem 2 and $\gamma \circ h^{-1}$ is represented by (z_1, z_2, ϕ, c), then (γ, u) is represented by (z_1, z_2, ϕ, c).*

Proof. Let (γ, u) be a stable outcome of E and (z_1, z_2, ϕ, c) be such that $\mu = \gamma \circ h^{-1}$ is represented by it. To show that (γ, u) is represented by (z_1, z_2, ϕ, c) it suffices to show that conditions 5 and 6 of this definition holds.

To show the above, note first that $\sigma = g \circ \hat{\sigma}$ and that $g \circ h$ is the identity in $Z \times Z_\emptyset \times N$. Hence,

$$\gamma = \gamma \circ (g \circ h)^{-1} = (\gamma \circ h^{-1}) \circ g^{-1} = \mu \circ g^{-1} = (\nu \circ \hat{\sigma}^{-1}) \circ g^{-1} = \nu \circ (g \circ \hat{\sigma})^{-1} = \nu \circ \sigma^{-1}.$$

Next, we show that $u(z) = c(z)$ for each $z \in W$. Let $z \in W$; hence $(\hat{z}, z, n) \in \text{supp}(\gamma)$ for some $\hat{z} \in Z$ and $n \in N$. Thus, $\hat{z} \in M$ and $(\hat{z}, n1_{(z,u(z))}) \in \text{supp}(\mu)$, the latter since $\text{supp}(\mu) = h(\text{supp}(\gamma))$. Note that

$$\overline{\text{graph}(\hat{\sigma})} = \Big(\text{graph}(\hat{\sigma}|_{[z_1,z_2)}) \cup \{(z_2, 1_{(\emptyset,0)})\}\Big) \cup \text{graph}(\hat{\sigma}|_M)$$

and $\text{supp}(\mu) \subseteq \overline{\text{graph}(\hat{\sigma})}$. This then implies that $(\hat{z}, n1_{(z,u(z))}) \in \text{graph}(\hat{\sigma}|_M)$ and, hence, $u(z) = c(z)$. Note that this also shows that $z = \phi(\hat{z})$ and $n = \bar{n}(\hat{z}, \phi(\hat{z}))$.

Let $z \in M$. Then $(z, z', n) \in \text{supp}(\gamma)$ for some $z' \in Z$ and $n \in N$ and, since $z' \in W$, $z' = \phi(z)$, $n = \bar{n}(z, \phi(z))$ and $u(z') = c(\phi(z))$ by the above argument. The stability of (γ, u) then implies that

$$\begin{aligned} u(z) &= F(z, z', n) - nu(z') = F(z, \phi(z), \bar{n}(z, \phi(z))) - \bar{n}(z, \phi(z))c(\phi(z)) \\ &= F(\sigma(z)) - \bar{n}(z, \phi(z))c(\phi(z)). \end{aligned}$$

Finally, if $z \in S$, then $(z, \emptyset, 1) \in \text{supp}(\gamma)$ and $u(z) = u(z) + u(\emptyset) = F(z, \emptyset, 1) = U_s(z)$ by the stability of (γ, u). ∎

B.10.6 Proof of Theorem 6: Sufficiency

Let (γ, u) be an outcome of E which is represented by (z_1, z_2, ϕ, c). It suffices to show that $\mu = \gamma \circ h^{-1}$ is represented by (z_1, z_2, ϕ, c); indeed, if so, then μ is stable by Theorem 10, u is its earnings function and $(\gamma, u) = (\mu \circ g^{-1}, u)$ is a stable outcome.

To show that μ is represented by (z_1, z_2, ϕ, c) it suffices to show that conditions 5' and 6' of this definition holds. Note that $\hat{\sigma} = h \circ \sigma$. Then

$$\mu = \gamma \circ h^{-1} = (\nu \circ \sigma^{-1}) \circ h^{-1} = \nu \circ (h \circ \sigma)^{-1} = \nu \circ \hat{\sigma}^{-1}$$

and condition 5' holds. Condition 6' is implied by condition 6 since $u(z_1) = c(z_1) = U_s(z_1)$ and $u(z_2) = U_s(z_2) = s(z_2, \underline{z}) - \bar{n}(z_2, \underline{z})c(\underline{z})$ if $z_1 < z_2$ and $u(z_2) = c(z_2) = s(z_2, \underline{z}) - \bar{n}(z_2, \underline{z})c(\underline{z})$ if $z_1 = z_2$.

B.10.7 Proof of Theorem 6: Uniqueness

Let (γ, u) and $(\hat{\gamma}, \hat{u})$ be stable outcomes of E. Theorem 2 implies that $\mu = \gamma \circ h^{-1}$ and $\hat{\mu} = \hat{\gamma} \circ \hat{h}^{-1}$ are stable matchings of E and Theorem 10 that $\mu = \hat{\mu}$ and that μ is represented by (z_1, z_2, ϕ, c). Lemma 50 then implies that (γ, u) and $(\hat{\gamma}, \hat{u})$ are both represented by (z_1, z_2, ϕ, c), hence $\gamma = \hat{\gamma}$ and $u = \hat{u}$ by conditions 5 and 6 of the definition of representability.

B.11 Proof of Corollary 1

Let E be an economy satisfying condition D and (γ, u) be its unique stable outcome. Then, for some $0 < z_1 < \bar{z}$ and $z_2 \in [z_1, \bar{z})$, $W = [\underline{z}, z_1]$, $S = [z_1, z_2]$ or $S = \emptyset$ and $M = [z_2, \bar{z}]$. Since $u|_W = c$ and $u|_S = U_s$, it follows that u is strictly increasing on $[\underline{z}, z_2]$ by condition (i) together with

$$c'(z) = \frac{1}{\bar{n}(\phi^{-1}(z), z)} \left(\frac{\partial s(\phi^{-1}(z), z)}{\partial y} - c(z) \frac{\partial \bar{n}(\phi^{-1}(z), z)}{\partial y} \right) \tag{A14}$$

for each $z \in W$ and by condition (iii).

Let $z \in M$. Then $u(z) = s(z, \phi(z)) - \bar{n}(z, \phi(z))c(\phi(z))$ and, hence, using (A14),

$$\begin{aligned} u'(z) = &\left(\frac{\partial s(\phi^{-1}(z), z)}{\partial y} - c(z) \frac{\partial \bar{n}(\phi^{-1}(z), z)}{\partial y} - c'(z)\bar{n}(z, \phi(z)) \right) \phi'(z) \\ &+ \frac{\partial s(z, \phi(z))}{\partial x} - c(\phi(z)) \frac{\partial \bar{n}(z, \phi(z))}{\partial x} \\ &= \frac{\partial s(z, \phi(z))}{\partial x} - c(\phi(z)) \frac{\partial \bar{n}(z, \phi(z))}{\partial x}. \end{aligned}$$

It then follows by condition (ii) that $u'(z) > 0$ for each $z \in [z_2, \bar{z}]$ and this implies that u is strictly increasing.

B.12 Proof of Corollary 2

Let E be a linear Rosen economy. We show that condition D is satisfied in E. Condition C holds since $\bar{n} \in \mathbb{R}_+$, $F(z, z', n) = g(z)q(z')n$ and $F(z, z, n) = g(z)q(z)\bar{n} > 0$ for each $(z, z', n) \in Z^2 \times \mathbb{R}_+$, the latter inequality holding since g and q are strictly positive.

Part (ii) of condition D follows by the definition of $\bar{n}$ and because g and q are C^2. Part (iii) holds by the definition of Z, part (iv) by the definition of ν, and part (v) by the definition of $\bar{n}$.

Note that $\frac{\partial \bar{n}}{\partial x} = \frac{\partial \bar{n}}{\partial y} = 0$ and that $p_{Z \times Z_\emptyset}(\text{supp}(\gamma)) \subseteq Z^2$ for each stable γ since $U_s \equiv 0$ and $F(z, z, \bar{n}) > 0$ for each $z \in Z$. Hence, part (vi) holds since, for each $z, z', \hat{z} \in Z$,

$$\begin{aligned} &\frac{\partial s(z, z')}{\partial x} - \frac{1}{\bar{n}(\hat{z}, z)} \frac{\partial s(\hat{z}, z)}{\partial y} > 0 \\ &\Leftrightarrow g'(z)q(z')\bar{n} - g(\hat{z})q'(z) > 0 \Leftrightarrow \bar{n} > \max_{(z,z',\hat{z}) \in Z^2} \frac{g(\hat{z})q'(z)}{g'(z)q(z')}. \end{aligned}$$

Part (vii) holds since, for each $z, z' \in Z$, $\frac{\partial s(z,z')}{\partial x} = g'(z)q(z')\bar{n} > 0$ since g' and q are strictly positive.

Part (viii) of condition D holds since $\bar{n}$ is constant and, hence, $\ln \bar{n}$ is supermodular; moreover, $\frac{s(x,y)\bar{n}(z,y)}{\bar{n}(x,y)} = s(x,y)$ and

$$\frac{\partial^2 s(x,y)}{\partial x \partial y} = g'(x)q'(y)\bar{n} > 0$$

since $\bar{n} > 0$ and g' and q' are strictly positive. Finally, part (ix) holds since $U_s \equiv 0$ and $s(z,z') = g(z)q(z')\bar{n} > 0$.

We conclude the proof by showing that conditions (i)–(iii) in Corollary 1 hold. Condition (iii) holds trivially since $S = \emptyset$. For the remaining ones, note that, for each $z, z' \in Z$,

$$\frac{\partial \bar{n}(z,z')}{\partial x} = \frac{\partial \bar{n}(z,z')}{\partial y} = 0.$$

Hence, condition (ii) holds since

$$\frac{\partial s(z,z')}{\partial x} = g'(z)q(z')\bar{n} > 0$$

and condition (i) holds since

$$\frac{\partial s(z,z')}{\partial y} = g(z)q'(z')\bar{n} > 0.$$

B.13 Proof of Corollary 3

Let E be a Garicano and Rossi-Hansberg economy. We show that condition D is satisfied in E. Condition C holds since $\sup_{(z,z')\in Z^2} \bar{n}(z,z') = \frac{1}{h(1-\bar{z})} < \infty$ since $\bar{z} < 1$ and $\bar{n} = \underline{n} > 0$. Part (ii) of Condition D follows by the definition of $\bar{n}$ and s, part (iii) by the definition of Z, part (iv) by the by the assumption on ν and part (v) since $\min_{(z,z')\in Z^2} \bar{n}(z,z') = \frac{1}{h} > 0$ and $\bar{n}$ is constant in z and strictly increasing in z'.

Note that $\frac{\partial \bar{n}}{\partial x} = 0$, $\frac{\partial s(z,z')}{\partial x} = \bar{n}(z') > 1 = \frac{\partial s(z,\emptyset)}{\partial x}$, writing $\bar{n}(z')$ instead of $\bar{n}(z,z')$. Hence part (vi) holds if, for each $z, \hat{z} \in Z$,

$$\frac{\partial s(z,\emptyset)}{\partial x} - \frac{1}{\bar{n}(\hat{z},z)}\frac{\partial s(\hat{z},z)}{\partial y} + \frac{U_s(z)}{\bar{n}(\hat{z},z)}\frac{\partial \bar{n}(\hat{z},z)}{\partial y} > 0 \Leftrightarrow 1 - (\hat{z} - z)\frac{\bar{n}'(z)}{\bar{n}(z)} > 0.$$

Since $\frac{\bar{n}'(z)}{\bar{n}(z)} = \frac{1}{1-z}$ and $\hat{z} \le \bar{z} < 1$, it follows that $1 - (\hat{z} - z)\frac{\bar{n}'(z)}{\bar{n}(z)} = 1 - \frac{\hat{z}-z}{1-z} > 0$ and part (vi) holds.

Part (vii) holds if, for each $z, z' \in Z$,

$$\frac{\partial s(z,z')}{\partial x} - U_s'(z) > 0 \Leftrightarrow \bar{n}(z') - 1 > 0.$$

Since $\bar{n}(z') \geq \bar{n}(0) = \frac{1}{h} > 1$, it follows that part (vii) holds.

Part (viii) holds since $\bar{n}$ is constant in z and, hence, $\ln \bar{n}$ is supermodular; moreover, $\frac{s(x,y)\bar{n}(z,y)}{\bar{n}(x,y)} = s(x,y)$ and

$$\frac{\partial^2 s(x,y)}{\partial x \partial y} = \bar{n}'(y) > 0.$$

Finally, part (ix) holds by taking $z = \bar{z}$ and $z' = 0$ since $U_s(0) = 0$, $U_s(\bar{z}) = \bar{z}$ and $s(\bar{z}, 0) = \frac{\bar{z}}{h} > \bar{z} = U_s(\bar{z})$.

We conclude the proof by showing that conditions (i)–(iii) in Corollary 1 hold. Condition (iii) holds $U_s'(z) = 1$ for each $z \in Z$. For condition (ii), we have that

$$\frac{\partial \bar{n}(z,z')}{\partial x} = 0 \text{ and}$$
$$\frac{\partial s(z,z')}{\partial x} = \frac{1}{h(1-z')} > 0.$$

Regarding condition (i), note first that $u(z) > 0$ for each $z \in M$ since if $z \in M$, then $z > \underline{z} = 0$ and, hence, $F(z, z, \bar{n}(z,z)) > 0$ (see the proof of Lemma 2. Then, for each $z \in W$, letting $\hat{z} = \phi^{-1}(z)$,

$$\frac{\partial s(\hat{z},z)}{\partial y} - c(z)\frac{\partial \bar{n}(\hat{z},z)}{\partial y} = \frac{u(\hat{z})}{1-z} > 0.$$

B.14 Proof of Corollary 4

This proof is analogous to the one of Corollary 2. Let E be a simple Mak and Siow economy. We show that condition D is satisfied in E. Condition C holds since $\sup_{(z,z')\in Z^2} \bar{n}(z,z') = 1$ and its part (ii) holds. Part (ii) of condition D follows by the definition of $\bar{n}$ and because f is C^2. Part (iii) holds by the definition of Z, part (iv) follows by the assumption on ν and part (v) by the definition of $\bar{n}$.

Note that $\frac{\partial \bar{n}}{\partial x} = \frac{\partial \bar{n}}{\partial y} = 0$ and that $p_{Z \times Z_\emptyset}(\text{supp}(\gamma)) \subseteq Z^2$ for each stable γ since $U_s \equiv 0$ and $F(z,z,1) = f(z,z) > 0$ for each $z \in Z$. Hence part (vi) holds if, for each $z, z', \hat{z} \in Z$,

$$\frac{\partial s(z,z')}{\partial x} - \frac{1}{\bar{n}(\hat{z},z)}\frac{\partial s(\hat{z},z)}{\partial y} > 0 \Leftrightarrow \frac{\partial f(z,z')}{\partial x} - \frac{\partial f(\hat{z},z)}{\partial y} > 0;$$

thus, the assumptions on f imply that part (vi) holds.

Part (vii) holds if, for each $z, z' \in Z$, $\frac{\partial s(z,z')}{\partial x} = \frac{\partial f(z,z')}{\partial x} > 0$, which holds by the assumptions on f. Part (viii) of condition D holds since $\bar{n}$ is constant and, hence, $\ln \bar{n}$ is supermodular; moreover, $\frac{s(x,y)\bar{n}(z,y)}{\bar{n}(x,y)} = s(x,y) = f(x,y)$ and

$$\frac{\partial^2 s(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial x \partial y} > 0$$

by the assumptions on f. Finally, part (ix) holds since $U_s \equiv 0$ and $s(z,z') = f(z,z') > 0$.

We conclude the proof by showing that conditions (i)–(iii) in Corollary 1 hold. Condition (iii) holds trivially since $S = \emptyset$. For the remaining ones, note that, for each $z, z' \in Z$,

$$\frac{\partial \bar{n}(z,z')}{\partial x} = \frac{\partial \bar{n}(z,z')}{\partial y} = 0.$$

Hence, condition (ii) holds since

$$\frac{\partial s(z,z')}{\partial x} = \frac{\partial f(z,z')}{\partial x} > 0$$

and condition (i) holds since

$$\frac{\partial s(z,z')}{\partial y} = \frac{\partial f(z,z')}{\partial y} > 0.$$

B.15 Proof of Corollaries 5 and 6

The proof of Corollary 5 is analogous to the proof of Corollary 3 and thus omitted (the only change is that part (i) of Condition C holds instead of part (ii) and this follows from $\underline{z} > 0$).

For Corollary 6, we show that condition D is satisfied in $E(n^*, 0, 0)$. We only need to show that parts (vi) and (vii) of condition D hold since the other ones hold by exactly the same arguments as for E_{grh}.

Claim 1 *Let* (γ, u) *be stable in* $E(n^*, 0, 0)$*. Then* $(z, \emptyset) \notin p_{Z \times Z_\emptyset}(\text{supp}(\gamma))$.

Proof. Suppose that $(z, \emptyset) \in p_{Z \times Z_\emptyset}(\text{supp}(\gamma))$; then $u(z) = 0 < s(z,z)$, a contradiction to the stability of (γ, u). ■

Part (vi) of condition D holds if, for each $z, z', \hat{z} \in Z$,

$$\begin{aligned}
& \frac{\partial s(z,z')}{\partial x} - \frac{1}{\bar{n}(\hat{z},z)} \frac{\partial s(\hat{z},z)}{\partial y} + \frac{s(z,z)}{\bar{n}(\hat{z},z)(1+\bar{n}(z,z))} \frac{\partial \bar{n}(\hat{z},z)}{\partial y} > 0 \\
\iff\ & n^*(z') - \frac{\hat{z} n^{*\prime}(z)}{n^*(z)} + \frac{z n^{*\prime}(z)}{1+n^*(z)} > 0 \\
\iff\ & \frac{1}{h(1-z')} - \frac{\hat{z}}{1-z} + \frac{z}{(1-z)(1+h(1-z))} > 0.
\end{aligned}$$

Note that:

$$\begin{aligned}
&\frac{1}{h(1-z')} - \frac{\hat{z}}{1-z} + \frac{z}{(1-z)(1+h(1-z))} \\
&= \frac{1}{h(1-z')} - \frac{\hat{z}h}{1+h(1-z)} - \frac{\hat{z}-z}{(1-z)(1+h(1-z))} \\
&\geq \frac{1}{h} - \left(\bar{z}h + \frac{\bar{z}-z}{1-z}\right)\frac{1}{1+h(1-z)} \\
&> \frac{1}{h} - (\bar{z}h + \bar{z})\frac{1}{1+h(1-z)} \\
&= \frac{1}{h} - \frac{\bar{z}(1+h)}{1+h(1-\bar{z})}.
\end{aligned}$$

Thus, part (vi) holds since $\frac{1}{h} - \frac{\bar{z}(1+h)}{1+h(1-\bar{z})} \geq 0 \iff h \leq \frac{1}{2\bar{z}} - 1 + \sqrt{1 + \frac{1}{4\bar{z}^2}}$.

Part (vii) of condition D holds if, for each $z, z' \in Z$,

$$\frac{\partial s(z,z')}{\partial x} > 0 \iff n^*(z') > 0.$$

Since $n^*(z') \geq n^*(\underline{z}) > 0$, it follows that part (vii) holds.

B.16 Proof of Theorem 7

To see that Φ is nonempty valued, note that, for each $\alpha \in [0,1)$, $E(n^*, 0, \alpha)$ satisfies part (i) of condition B; hence a stable matching exists by Theorem 1.

Next, we show that it makes no difference to stable matchings if we replace $E(n^*, 0, \alpha)$ with an economy where wages can be at most $\max\{1, \bar{z}h\}$.

For each $\alpha \in [0,1)$, let $\hat{E}(n^*, 0, \alpha)$ be exactly as $E(n^*, 0, \alpha)$ except with $\hat{C} = [0, \max\{1, \bar{z}h\}]$ and $\hat{P}_m(z) = \{n^*(z')1_{(z',c)} : (z',c) \in Z \times \hat{C}\}$.

Lemma 51 *If $\alpha \in [0,1)$ and μ is a stable matching of $\hat{E}(n^*, 0, \alpha)$, then* $\text{supp}(\mu) \subseteq Z \times \hat{X}$.

Proof. Suppose not; then let $(z, \delta) \in \text{supp}(\mu) \cap (Z \times (\hat{X}_\emptyset \setminus \hat{X}))$. Let $\varepsilon > 0$ be such that $(zh(1-z))^\alpha z - \varepsilon > 0$, which exists since $z \in (0,1)$. Then $(z, \varepsilon) \in T_z^m(\mu)$ since $(z, \delta) \in \text{supp}(\mu)$ and $U_z(w, 1_{(z,\varepsilon)}) = \varepsilon > 0 = U_z(s, \delta)$. Thus, letting $\delta' = n^*(z)1_{(z,\varepsilon)}$, it follows that $\text{supp}(\delta') \subseteq T_z^m(\mu)$ and $U_z(m, \delta') = \Big((zh(1-z))^\alpha z - \varepsilon\Big) n^*(z) > 0 = U_z(s, \delta)$. Hence, $(z, \delta) \notin S(\mu)$, a contradiction to the stability of μ. ■

Lemma 52 *For each $\alpha \in [0,1)$, μ is a stable matching of $\hat{E}(n^*, 0, \alpha)$ if and only if μ is a stable matching of $E(n^*, 0, \alpha)$.*

Proof. Let μ be a stable matching of $\hat{E}(n^*, 0, \alpha)$. Note that, for each $(z, \delta) \in \text{supp}(\mu)$, $\delta \in \hat{X}$ and thus $\delta = n^*(z')1_{(z',c)}$ for some $(z', c) \in Z \times Z$ by Lemma 51. Furthermore, $U_z(m, n^*(z')1_{(z',c)}) \geq 0$ and $U_{z'}(w, 1_{(z,c)}) \geq 0$ since μ is individually rational in $\hat{E}(n^*, 0, \alpha)$.

We claim that μ is a stable matching of $E(n^*, 0, \alpha)$. Indeed, if μ is not a stable matching of $E(n^*, 0, \alpha)$, then there is $(z, z', c) \in Z^2 \times C$ such that $(z, n^*(z')1_{(z',c)}) \in \text{supp}(\mu)$ and $(\hat{z}, \tilde{z}, \tilde{c}) \in Z^2 \times C$ such that $\hat{z} \in \{z, z'\}$, $\tilde{c} > \max\{1, \bar{z}h\} \geq (\bar{z}h)^\alpha$ and $U_{\hat{z}}(m, n^*(\tilde{z})1_{(\tilde{z},\tilde{c})}) > U_{\hat{z}}(a, \hat{\delta})$ with $a = m$ and $\hat{\delta} = n^*(z')1_{(z',c)}$ if $\hat{z} = z$ and $a = w$ and $\hat{\delta} = 1_{(z,c)}$ if $\hat{z} = z'$. But $\tilde{c} > (\bar{z}h)^\alpha$ implies that $0 > U_{\hat{z}}(m, n^*(\tilde{z})1_{(\tilde{z},\tilde{c})})$ since $U_{\hat{z}}(m, n^*(\tilde{z})1_{(\tilde{z},\tilde{c})}) \geq 0$ is equivalent to $\tilde{c} \leq (\hat{z}h(1 - \tilde{z}))^\alpha \hat{z}$ and, thus, implies that $\tilde{c} \leq (\bar{z}h)^\alpha$; hence, $0 > U_{\hat{z}}(a, \hat{\delta})$. But this contradicts $U_z(m, n^*(z')1_{(z',c)}) \geq 0$ and $U_{z'}(w, 1_{(z,c)}) \geq 0$. Thus, it follows that μ is a stable matching of $E(n^*, 0, \alpha)$.

Let μ be a stable matching of $E(n^*, 0, \alpha)$. Then for each $(z, \delta) \in \text{supp}(\mu)$ and $(z', c) \in \text{supp}(\delta)$, individual rationality implies that $c \in [0, \max\{1, \bar{z}h\}]$; hence μ is a stable matching of $\hat{E}(n^*, 0, \alpha)$. ■

To prove continuity, we first show that Φ is upper hemi-continuous at $\alpha = 0$. By Lemma 52, $\Phi(\alpha)$ equals the set of stable matching of $\hat{E}(n^*, 0, \alpha)$. Let $\{\alpha_k\}_{k=1}^\infty$ be such that $\alpha_k \in (0, 1)$ for each $k \in \mathbb{N}$ and $\alpha_k \to 0$ and let $\{\mu_k\}_{k=1}^\infty$ be such that $\mu_k \in \Phi(\alpha_k)$ for each $k \in \mathbb{N}$. Since μ_k is a stable matching of $\hat{E}(n^*, 0, \alpha_k)$, $\mu_k \in M(Z \times \hat{X})$ for each $k \in \mathbb{N}$ by Lemma 51. Since $\hat{X}$ is compact due to $1/h(1 - z) \leq 1/h(1 - \bar{z}) < \infty$ for each $z \in Z$, we may assume that $\{\mu_k\}_{k=1}^\infty$ converges. Let $\mu = \lim_k \mu_k$.

We claim that $\mu \in \Phi(0)$. This claim follows by Corollary 3.1 (and its proof) in Supplementary Material. To see this, let $E_k = \hat{E}(n^*, 0, \alpha_k)$ and note that $Z_k = Z$, $\nu_k = \nu$, $C_k = \hat{C}$, $P_{a,k}(z) = \hat{P}_a(z)$ for each $z \in Z_k$ and $a \in \{m, s\}$. Conditions (a) and (b) of part 4 of Corollary 3.1 are satisfied since $P_{m,k}(z_k) = \hat{P}_m(z_k)$ for each $k \in \mathbb{N}$ and $z_k \in Z_k = Z$. The assumption of Corollary 3.1 that $U_z^k(m, \delta) = U_z(m, \delta)$ for each $(z, \delta) \in Z \times \hat{X}$ does not hold but the proofs of parts 3 and 4 go through provided that $\sup_{(z,z',c) \in Z^2 \times C} |U_z^k(m, n^*(z')1_{(z',c)}) - U_z(m, n^*(z')1_{(z',c)})| \to 0$. This condition holds since

$$\begin{aligned}
\left|U_z^k(m, n^*(z')1_{(z',c)}) - U_z(m, n^*(z')1_{(z',c)})\right| &= |(zh(1 - z'))^{\alpha_k} - 1| \frac{z}{h(1 - z')} \\
&\leq \max\{1 - (\underline{z}h(1 - \bar{z}))^{\alpha_k}, (\bar{z}h(1 - \underline{z}))^{\alpha_k} - 1\} \frac{\bar{z}}{h(1 - \bar{z})} \to 0.
\end{aligned}$$

Thus, it follows that $\mu \in \Phi(0)$ and, hence, Φ is upper hemi-continuous at $\alpha = 0$.

Finally, since $\Phi(0)$ is a singleton by Corollary 6, it follows that Φ is lower hemi-continuous at $\alpha = 0$. Thus, Φ is continuous at $\alpha = 0$.

B.17 Stable Matchings of a Rosen Economy

The following theorem characterizes the stable matchings of the economy $E_{r,\alpha}$ considered in Section 4.2.[7]

Theorem 11 *A matching μ is stable in $E_{r,\alpha}$ if and only if there exists $\gamma \in M(Z^2)$, $z_1 \in (\underline{z}, \bar{z})$ and $w > 0$ such that*

1. $\mu = \gamma \circ \tilde{g}^{-1}$, *where, for each $(z, z') \in Z^2$,*

$$\tilde{g}(z, z') = (z, n(z, w)1_{(z', w)}), \text{ and}$$
$$n(z, w) = \left(\frac{1-\alpha}{w}\right)^{\frac{1}{\alpha}} z^{\frac{1+\alpha}{\alpha}},$$

2. $\gamma(B \times Z) + \int_{Z \times B} n(z, w) \mathrm{d}\gamma(z, z') = \nu(B)$ *for each Borel $B \subseteq Z$,*

3. $W = [\underline{z}, z_1]$, $M = [z_1, \bar{z}]$, *and*

4. $z_1^{1+\alpha} n(z_1, w)^{1-\alpha} - wn(z_1, w) = w$.

Furthermore, z_1 and w are unique.

Proof. Theorem 11 follows in part from Theorem 3 in Carmona and Laohakunakorn (2024b). The latter result applies to a general production function $g(r(z))\psi(r(z), nq(z'))$ and thus set, for each $z \in Z$, $r \in \mathbb{R}_+$ and $(x, y) \in \mathbb{R}^2_+$, $r(z) = z$, $q(z) = 1$, $g(r) = r$ and $\psi(x, y) = x^\alpha y^{1-\alpha}$ to obtain $z^{1+\alpha} n^{1-\alpha}$ as in $E_{r,\alpha}$. In this case and with a constant wage $w > 0$, the optimal number $n(z, w)$ of workers for a manager of ability z, and the manager's rent $R(z, w) = z^{1+\alpha} n(z, w)^{1-\alpha} - wn(z, w)$ equal

$$n(z, w) = z^{\frac{1+\alpha}{\alpha}} \left(\frac{(1-\alpha)}{w}\right)^{\frac{1}{\alpha}},$$
$$R(z, w) = \alpha z^{\frac{1+\alpha}{\alpha}} \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}}.$$

It follows by Carmona and Laohakunakorn (2024b, Theorem 3) that μ is a stable matching of $E_{r,\alpha}$ if and only if there exists $\gamma \in M(Z^2)$ and $w > 0$ such that

$$\gamma(B \times Z) + \int_{Z \times B} n(z, w) \mathrm{d}\gamma(z, z') = \nu(B) \text{ for each Borel } B \subseteq Z, \tag{A15}$$
$$\operatorname{supp}(\gamma) \subseteq \{z \in Z : R(z, w) \geq w\} \times \{z \in Z : w \geq R(z, w)\}, \text{ and} \tag{A16}$$
$$\mu = \gamma \circ \tilde{g}^{-1}, \tag{A17}$$

[7] See Supplementary Material for the case where $\theta \equiv 1$, in which closed forms for z_1 and w can be obtained.

where $\tilde{g} : Z^2 \to Z \times X$ is defined by setting, for each $(z, z') \in Z^2$,

$$\tilde{g}(z, z') = (z, n(z, w)1_{(z',w)}).$$

Thus, it remains to show that (A16) is equivalent to conditions 3 and 4 in the statement of Theorem 11, and that w and z_1 are unique.

Note that $\tilde{g}$ is continuous and that its restriction to $\mathrm{supp}(\gamma)$ is 1-1 since $n(z, w) > 0$. Thus, $\tilde{g}$ is a homeomorphism between $\mathrm{supp}(\gamma)$ and $\tilde{g}(\mathrm{supp}(\gamma))$. Then $M = \{z \in Z : (z, z') \in \mathrm{supp}(\gamma)$ for some $z' \in Z\}$ and $W = \{z \in Z : (\hat{z}, z) \in \mathrm{supp}(\gamma)$ for some $\hat{z} \in Z\}$ since $\mathrm{supp}(\mu) = \tilde{g}(\mathrm{supp}(\gamma))$ by Carmona and Laohakunakorn (2024b, Lemma 1). We have that $\mathrm{supp}(\gamma)$ is compact since it is a closed subset of Z^2 and, hence, M and W are compact. Then $M \cup W = Z$ since $\mathrm{supp}(\gamma) \cap (M \times Z) = \mathrm{supp}(\gamma) = \mathrm{supp}(\gamma) \cap (Z \times W)$,

$$\begin{aligned}\nu(M \cup W) =& \gamma((M \cup W) \times Z) + \int_{Z \times (M \cup W)} n(z, w) \mathrm{d}\gamma(z, z') \\ &= \gamma(Z \times Z) + \int_{Z \times Z} n(z, w) \mathrm{d}\gamma(z, z') = \nu(Z)\end{aligned}$$

and, hence, $Z = \mathrm{supp}(\nu) \subseteq M \cup W \subseteq Z$. Furthermore, $\gamma(Z \times Z) > 0$ since otherwise $\gamma(Z \times Z) = 0$, $\int_{Z \times Z} n(z, w) \mathrm{d}\gamma(z, z') = 0$ and, hence, $0 = \gamma(Z \times Z) + \int_{Z \times Z} n(z, w) \mathrm{d}\gamma(z, z') = \nu(Z) > 0$, a contradiction.

It follows by (A16) that $M \subseteq \{z \in Z : R(z, w) \geq w\}$ and $W \subseteq \{z \in Z : w \geq R(z, w)\}$. Let $\lambda : Z \to \mathbb{R}$ be defined by setting, for each $z \in Z$, $\lambda(z) = R(z, w) - w$. Then λ is continuous and strictly increasing. We have that $\lambda(\underline{z}) < 0$ since otherwise $W \subseteq \{z \in Z : w \geq R(z, w)\} \subseteq \{\underline{z}\}$ and, hence, $\int_{Z \times W} n(z, w) \mathrm{d}\gamma(z, z') \leq \nu(\{\underline{z}\}) = 0$; but this is a contradiction since $n(z, w) > 0$ for each (z, w) and $\gamma(Z \times Z) > 0$ imply that $\int_{Z \times W} n(z, w) \mathrm{d}\gamma(z, z') = \int_{\mathrm{supp}(\gamma)} n(z, w) \mathrm{d}\gamma(z, z') = \int_{Z \times Z} n(z, w) \mathrm{d}\gamma(z, z') > 0$. Furthermore, $\lambda(\bar{z}) > 0$ since otherwise $M \subseteq \{z \in Z : R(z, w) \geq w\} \subseteq \{\bar{z}\}$ and hence $\gamma(Z \times Z) = \gamma(M \times Z) \leq \nu(\{\bar{z}\}) = 0$; but this contradicts $\gamma(Z \times Z) > 0$. It then follows that there is a unique $z_1 \in (\underline{z}, \bar{z})$ such that $\lambda(z_1) = 0$. Thus, $M \subseteq \{z \in Z : R(z, w) \geq w\} = [z_1, \bar{z}]$ and $W \subseteq \{z \in Z : w \geq R(z, w)\} = [\underline{z}, z_1]$, which, together with $M \cup W = Z$, implies that $W = [\underline{z}, z_1]$ and $M = [z_1, \bar{z}]$. The definition of z_1 implies that $R(z_1, w) = w$ and, hence $z_1^{1+\alpha} n(z_1, w)^{1-\alpha} - wn(z_1, w) = w$, since $R(z_1, w) = z_1^{1+\alpha} n(z_1, w)^{1-\alpha} - wn(z_1, w)$.

Conversely, suppose that conditions 3 and 4 in the statement of the Theorem hold. Thus, $\lambda(z_1) = 0$ and, since λ is strictly increasing, $\{z \in Z : R(z, w) \geq w\} = [z_1, \bar{z}] = M$ and $\{z \in Z : R(z, w) \leq w\} = [\underline{z}, z_1] = W$. Hence, for each $(z, z') \in \mathrm{supp}(\gamma)$, it follows that $(z, z') \in M \times W$ and, thus, $R(z, w) \geq w$ and $R(z', w) \leq w$, i.e. (A16) holds.

We conclude the proof of Theorem 11 by establishing the uniqueness of z_1 and w. Let

(γ, z_1, w) and $(\hat{\gamma}, \hat{z}_1, \hat{w})$ be such that conditions 1–4 hold. Let λ be as above and define $\hat{\lambda}(z) = R(z, \hat{w}) - \hat{w}$. Note that $w = \hat{w}$ implies that $z_1 = \hat{z}_1$ since then $\lambda = \hat{\lambda}$.

Suppose that $w > \hat{w}$. Then $R(z, w) < R(z, \hat{w})$ for each $z \in Z$ and, hence, $\lambda(z) < \hat{\lambda}(z)$ for each $z \in Z$. Thus, $z_1 > \hat{z}_1$ since z_1 (resp. $\hat{z}_1$) is the unique $z \in (\underline{z}, \bar{z})$ such that $\lambda(z_1) = 0$ (resp. $\hat{\lambda}(\hat{z}_1) = 0$).

Let γ_1 denote the marginal of γ on the first coordinate. For each Borel $B \subseteq [z_1, \bar{z}]$, condition 2 implies that $\gamma_1(B) = \gamma(B \times Z) = \nu(B)$ since $\int_{Z \times B} n(z, w) \mathrm{d}\gamma(z, z') = 0$ due to $W = [\underline{z}, z_1]$. Thus, γ_1 is the restriction of ν to $M = [z_1, \bar{z}]$. Analogously, $\hat{\gamma}_1$ is the restriction of ν to $[\hat{z}_1, \bar{z}]$.

It then follows that

$$\int_{Z \times Z} n(z, w) \mathrm{d}\gamma(z, z') = \int_Z n(z, w) \mathrm{d}\gamma_1(z) = \int_{[z_1, \bar{z}]} n(z, w) \mathrm{d}\nu(z).$$

Analogously, $\int_{Z \times Z} n(z, \hat{w}) \mathrm{d}\hat{\gamma}(z, z') = \int_{[\hat{z}_1, \bar{z}]} n(z, \hat{w}) \mathrm{d}\nu(z)$.

Since $W = [\underline{z}, z_1]$, condition 2 implies that

$$\int_{Z \times Z} n(z, w) \mathrm{d}\gamma(z, z') = \gamma([\underline{z}, z_1] \times Z) + \int_{Z \times [\underline{z}, z_1]} n(z, w) \mathrm{d}\gamma(z, z') = \nu([\underline{z}, z_1])$$

and, analogously, $\int_{Z \times Z} n(z, \hat{w}) \mathrm{d}\hat{\gamma}(z, z') = \nu([\underline{z}, \hat{z}_1])$. Thus, $z_1 > \hat{z}_1$ implies that $\nu([\underline{z}, z_1]) > \nu([\underline{z}, \hat{z}_1])$ and, hence,

$$\int_{Z \times Z} n(z, w) \mathrm{d}\gamma(z, z') > \int_{Z \times Z} n(z, \hat{w}) \mathrm{d}\hat{\gamma}(z, z'). \tag{A18}$$

Since $n(z, w) < n(z, \hat{w})$ for each $z \in Z$, it follows that

$$\begin{aligned} \int_{Z \times Z} n(z, w) \mathrm{d}\gamma(z, z') &= \int_{[z_1, \bar{z}]} n(z, w) \mathrm{d}\nu(z) < \int_{[z_1, \bar{z}]} n(z, \hat{w}) \mathrm{d}\nu(z) \\ &< \int_{[\hat{z}_1, \bar{z}]} n(z, \hat{w}) \mathrm{d}\nu(z) = \int_{Z \times Z} n(z, \hat{w}) \mathrm{d}\hat{\gamma}(z, z'), \end{aligned}$$

a contradiction to (A18). This contradiction shows that $w = \hat{w}$ and, hence, $z_1 = \hat{z}_1$. ■

C Numerical Appendix

C.1 Computational Method

This section describes the computational method used to compute stable outcomes. We focus on knowledge economies in which the type space is an interval $Z = [\underline{z}, \bar{z}]$ with a continuous and strictly positive density κ on Z and managers' span-of-control bounds coincide, $\bar{n} = \underline{n}$.

We summarize the wage distribution by reporting wages (which equals earnings in the general equilibrium perspective of Theorem 5) at percentiles of the knowledge distribution. In the economies we study, earnings are strictly increasing in knowledge, so knowledge percentiles coincide with wage percentiles. The density representation makes these percentiles easy to compute: for each $p \in \{0, \ldots, 100\}$, the pth percentile is $z_p \in Z$ such that $\int_{\underline{z}}^{z_p} \kappa(z)\mathrm{d}z = \frac{p}{100}$.

We compute the assignment and the earnings function by discretizing the type space $[\underline{z}, \bar{z}]$ and the knowledge distribution. After discretization, an allocation is $\gamma = (\gamma(z, z'))_{z \in Z, z' \in Z_\emptyset}$, where we write $\gamma(z, z') = \gamma\big(z, z', \bar{n}(z, z')\big)$, and it is obtained by solving the following linear programming problem:

$$\begin{aligned} \max_{\gamma} \quad & \sum_{z \in Z} \sum_{z' \in Z_\emptyset} s(z, z')\gamma(z, z') \\ \text{subject to} \quad & \sum_{z' \in Z_\emptyset} \gamma(z, z') + \sum_{\hat{z} \in Z} \bar{n}(\hat{z}, z)\gamma(\hat{z}, z) = \nu(z) \text{ for each } z \in Z, \text{ and} \\ & \gamma(z, z') \geq 0 \text{ for each } z \in Z, z' \in Z_\emptyset. \end{aligned}$$

In the discrete economy, the earnings function $u : Z_\emptyset \to \mathbb{R}$ is a vector $u = (u(z))_{z \in Z_\emptyset}$ and is computed by solving the following linear programming problem:

$$\begin{aligned} \min_{u} \quad & \sum_{z \in Z} u(z)\nu(z) \\ \text{subject to} \quad & u(z) + \bar{n}(z, z')u(z') \geq s(z, z') \text{ for each } z \in Z, z' \in Z_\emptyset \text{ and} \\ & u(\emptyset) = 0. \end{aligned}$$

In discretizing Z, we include the percentiles $z_p : 0 \leq p \leq 100$. The vector $(u(z_p))_{p=0}^{100}$ then summarizes the wage (earnings) distribution, and we focus on $p \in 5, \ldots, 95$. For a given parameter vector, we compute $y = (y_5, \ldots, y_{95})$, where $y_p = \ln(u(z_p)) - \ln(u(z_{50}))$, $p = 5, \cdots, 95$. We compare y to its empirical counterpart $x = (x_5, \ldots, x_{95})$ by the squared Euclidean distance $\sum_{p=5}^{95}(x_p - y_p)^2$, and choose parameters that minimize it.

In Section 5.2, we employ a global optimization algorithm that combines local search with quasi-random exploration based on a Sobol sequence. To ensure that the calibrated parameter vector θ^* corresponds to a global optimum, we proceed as follows:

Step 1: Initialize $i = 1$. Choose an initial guess θ_i.

Step 2: Apply the Nelder-Mead simplex method to obtain a candidate solution $\tilde{\theta}_i$.

Step 3: Draw a quasi-random parameter vector $\hat{\theta}_i$ generated by a Sobol sequence.

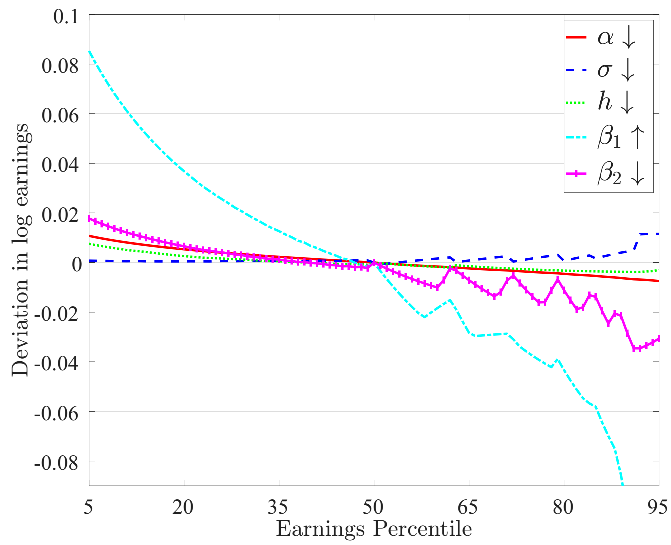

Figure A1: Comparative Statics: Changes in the Earnings Distribution. The figure shows deviations in log earnings at each percentile following a 1% change in a given parameter, relative to the 1988 model economy.

Step 4: Update the initial guess as $\theta_{i+1} = f(i)\tilde{\theta}_i^* + (1 - f(i))\hat{\theta}_i$, where $\tilde{\theta}_i^*$ is the best solution obtained up to iteration i, and $f : \mathbb{N} \to [0, 1]$ is an increasing weighting function.

Step 5: Stop if converged. Otherwise, set $i = i + 1$ and return to Step2.

C.2 Numerical Comparative Statics

In this section, we provide an informal identification argument of the structural parameters using numerical comparative statics.

We conduct numerical comparative statics around the calibrated 1988 economy in Section 5.2. Specifically, we consider a 1% decrease in $(\alpha, \sigma, h, \beta_2)$ and a 1% increase in β_1, where (β_1, β_2) parameterize the Beta distribution of knowledge. In these perturbations, earnings in the lower half of the distribution rise relative to the median; we focus on parameter changes that generate this empirically relevant pattern. Figure A1 reports deviations in log earnings at each percentile, and Table A1 reports changes in the establishment-size distribution (basis points), both relative to the 1988 model economy.

In our simulations, decreasing the worker-knowledge span-of-control elasticity σ is the only perturbation that leads to an increase in the upper tail of earnings (i.e., higher earnings above the median relative to the baseline). Lower σ also shifts the firm size distribution toward larger size bins because the span-of-control function becomes more sensitive to worker knowledge. In contrast, a decrease in the communication-cost parameter h shifts the establishment-size

Table A1: Comparative Statics: Changes in Establishment-Size Distribution (bpt)

	$\alpha\downarrow$	$\sigma\downarrow$	$h\downarrow$	$\beta_1\uparrow$	$\beta_2\downarrow$
Establishment Size					
1-9	0.0	−17.6	42.3	42.8	18.2
10-99	0.0	15.6	−42.1	−41.5	−16.2
100-999	0.0	1.9	−0.2	−1.3	−2.0
1000+	0.0	0.0	0.0	0.0	0.0

Note: The table reports the basis point change in the establishment-size distribution following a 1% change in a given parameter, relative to the 1988 model economy.

distribution toward smaller establishments. Changes in (β_1, β_2) affect the size distribution indirectly by shifting the mass of worker knowledge toward or away from high-z types; in our comparative statics these shifts tend to move firms toward smaller size bins. The opposing effects of σ and h on the establishment-size distribution are particularly helpful in separating these parameters.

The establishment-size distribution is relatively insensitive to the managerial returns parameter α, so α is mainly disciplined by earnings inequality moments. The remaining parameters are disciplined jointly by earnings and establishment-size moments. For example, a 1% decrease in α and in h generates qualitatively similar shifts in the earnings distribution, but only h is tightly constrained by the empirical establishment-size distribution. In addition, β_1 governs the density of knowledge near the lower endpoint, while β_2 governs the density near the upper endpoint. These distributional shifts can have nonlocal effects on earnings through equilibrium sorting and occupational choice. In our simulations, lower β_2 shifts the upper half of the earnings distribution more than the lower half, whereas lower β_1 generates sizable movements in both tails. Finally, establishment-size moments alone do not cleanly separate h from β_1 or β_2, but the earnings responses do. Our simulations indicate that higher β_1 generates much larger upper-tail movements than lower h, while lower h generates much smaller lower-tail movements than lower β_2, relative to the median. Overall, these comparative statics indicate that the parameters move key earnings and size moments in distinct directions, providing discipline for identification in the calibration.

C.3 Wage Distribution

Figure A2 shows the wage distribution in 1988 and 2008.

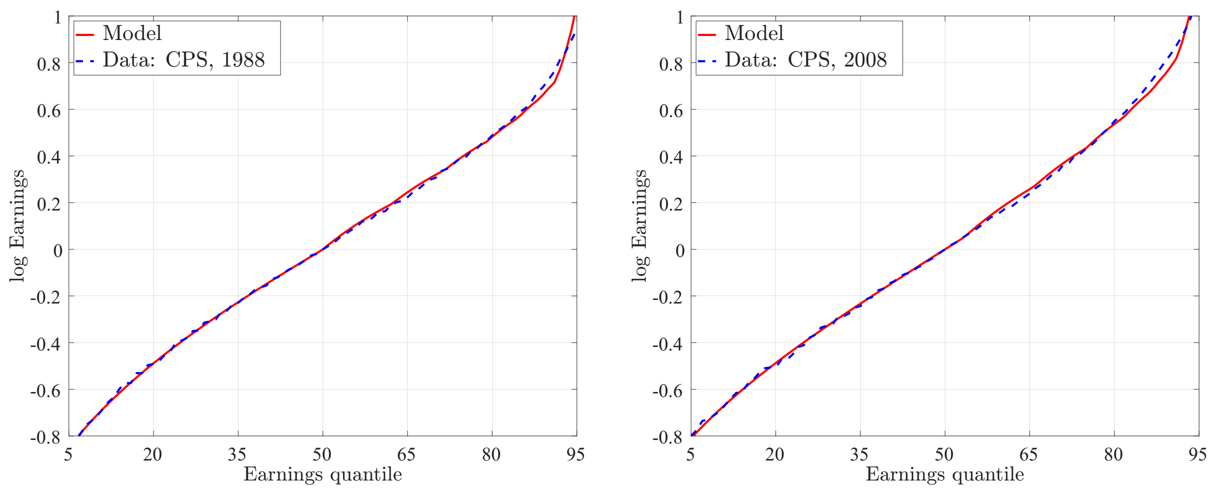

Figure A2: Wage Distributions in 1988 and 2008. Panels A and B report log earnings relative to the median in 1988 and 2008, respectively.