

Knowledge Economies^{*}

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Abstract

We develop a general framework for knowledge economies which feature the matching of individuals into firms whose production depends on members' knowledge. The framework is flexible and nests several existing models as special cases. We establish existence of a stable outcome and show that, in our setting, stability is equivalent to competitive equilibrium and surplus maximization. As a theoretical application, we characterize the stable outcomes in a canonical knowledge economy featuring multi-layer firms. As a quantitative application, we use U.S. wage and establishment-size distributions as benchmarks to compare the effects of different organizational technologies on equilibrium outcomes.

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1 Introduction

Several authors have emphasized that understanding firm sizes and the income distribution requires a more detailed view of production than the one provided by a standard neoclassical production function. Who works with whom, who supplies managerial knowledge, and how many workers that knowledge can support jointly determine organizational scale and the distribution of earnings. Organization theory therefore matters both for the evaluation of policies affecting firm sizes, as in Lucas (1978), and for understanding superstar managers and their effects on workers' earnings, as in Rosen (1981, 1982) and Garicano and Rossi-Hansberg (2004, 2006, 2015). The challenge is to provide an equilibrium theory of how productive resources are organized into firms.

Existing models have made progress in several important applications, but they are typically formulated for their specific settings. For example, Garicano and Rossi-Hansberg (2004) and Antràs, Garicano, and Rossi-Hansberg (2006) incorporate properties such as positive assortative matching directly into their equilibrium conditions. Extending these models therefore requires either establishing that these properties continue to hold in the new setting or providing a more general definition of equilibrium. The cost of developing these arguments can restrict the range of organizational structures that are studied. A unified framework with general results on existence and equilibrium characterization would allow different models to be studied under a common solution concept using the same tools. In this paper, we provide such a framework, thus contributing to the agenda of incorporating organization into economic models emphasized by Rosen (2002) and Garicano and Rossi-Hansberg (2015).

The economies in our framework are such that production depends explicitly on the knowledge (or talent or ability) of the individuals involved; we therefore refer to them as *knowledge economies*. They are defined succinctly by (i) a set of knowledge levels, (ii) a distribution over these levels, (iii) a set of feasible firms specifying their members' types, roles (or organizational layer) and numbers, and (iv) a production function that determines the output of each feasible firm and which allows for externalities. The allocation of individuals to firms and their earnings are determined in equilibrium. We establish existence of an

equilibrium and show that the allocation solves a surplus maximization problem, with the earnings function solving its dual. We characterize solutions to this problem and provide a computational method for finding them.

Our framework is flexible enough to encompass existing models as special cases.¹ This flexibility allows economically significant features of organization to be incorporated in an equilibrium model. For example, the type space can capture different dimensions of knowledge or ability, and the set of feasible firms can accommodate different organizational roles, numbers of layers and spans of control. Most importantly, for specifications satisfying our assumptions, these features can be studied using our tools instead of developing the equilibrium analysis from scratch. This reduces the work required to extend existing models and study how changes in organizational possibilities affect productivity, firm sizes and the distribution of earnings.

Our solution concept generalizes stable matching in transferable utility settings (see, e.g., Chiappori, 2017) to allow individuals to match into arbitrary feasible firms (instead of restricting to two-sided, one-to-one matching). We define a stable outcome as a distribution of firms and a continuous earnings function such that the allocation is feasible, active firms make zero profits and all feasible firms earn nonpositive profits when their members are compensated according to the earnings function. We show that stable outcomes exist under mild assumptions and are equivalent to competitive equilibria, justifying our focus on them. In the absence of externalities, equilibrium allocations maximize surplus and can be characterized by a condition ruling out gains from finite reallocations. The earnings function solves the dual problem and determines the price of knowledge. With externalities, these results hold fixing the externality at its equilibrium value.

The result that equilibrium allocations maximize surplus underpins our computational method. For an economy without externalities, we discretize the type space and the set of feasible firms and solve the resulting linear program. Its solution and dual variables give equilibrium allocations and earnings for the discretized economy. Occupational choices, matching patterns and organizational structures are determined jointly, without having

¹In the Supplementary Material, we represent in our framework the models of Garicano and Rossi-Hansberg (2004), Garicano and Rossi-Hansberg (2006), Ide and Talamàs (2025), Mak and Siow (2025), Gabaix and Landier (2008), Gola (2021) and a truncated version of Rosen (1982).

to assume any sorting or stratification properties in advance. For a class of multi-layer knowledge economies, we further reduce this problem to a network-flow formulation, from which equilibrium allocations and earnings can be recovered. Because it does not enumerate every feasible firm, this formulation has a number of variables that grows linearly rather than exponentially in the maximum number of layers.

Our general framework is sufficiently tractable to yield economic insights through theoretical analysis. We characterize stable outcomes in a multi-layer extension of Garicano and Rossi-Hansberg (2004). In this economy, the stable outcome is unique and satisfies the following properties: active firms have either a common number of layers or two adjacent numbers of layers, knowledge levels are stratified by occupation, and matching is positively assortative. In deriving these properties, we show that the structure of optimal organizations is restricted by the supermodularity of output in the knowledge of the top two layers and the substitutability between these layers. In particular, a more knowledgeable second-in-command raises the marginal value of the top manager’s knowledge, leading to positive sorting. On the other hand, the value that the top manager adds beyond what the second-in-command already provides falls as the second-in-command becomes more knowledgeable, leading to stratification and at most two adjacent numbers of layers. This latter substitutability property can fail under a more flexible span-of-control technology, and we construct an economy in which one-, two- and three-layer organizations coexist. Thus, our analysis identifies a technological restriction that is responsible for the limited variation in organizational depth in the original economy.

As another illustration of our framework, we compare alternative knowledge economies using the U.S. wage and establishment-size distributions as quantitative benchmarks. Under a calibration that jointly targets both distributions, the baseline manager–worker model of Garicano and Rossi-Hansberg (2004) generates too little dispersion in both. A flexible extension, which allows the sensitivity of span of control to worker knowledge and the elasticity of output with respect to manager knowledge to vary, generates greater dispersion in both wages and establishment sizes, improving the fit to the quantitative benchmarks.

Our quantitative exercise also illustrates how the calibrated economies differ in the relationship between establishment size and top earnings concentration. Allowing an additional

managerial layer improves the baseline model’s wage and establishment-size fit, but raises its top 1% earnings share only from 1.9% to 2.5%. The flexible model instead generates 12.5%, considerably closer to the 13.7% income-share benchmark from Auten and Splinter (2024). It is calibrated to wage percentiles from the 5th to the 95th and the establishment-size distribution, without targeting the top 1% share. These comparisons illustrate that replicating the firm size distribution and, in particular, generating large firms is important but not sufficient to obtain substantial concentration at the top of the earnings distribution. As these comparisons also illustrate, this happens because greater organizational depth (i.e. larger number of layers) need not generate a commensurate increase in top earnings concentration.²

In summary, our framework provides a coherent way to study which firms form in equilibrium and the implications for productivity, firm sizes and the distribution of earnings. Its broader purpose is to facilitate the incorporation of “serious organization theory” into a unified framework, and to provide tools that remain applicable as the restrictions on organizations and technology change.³ Ultimately, we hope that the framework will reduce the need to assume away important organizational features for tractability.

Related Literature

Our framework, solution concept and results on its existence and characterization extend previous ones to a level of generality that allows them to be used to address a broad range of economic questions. Our existence result builds on Carmona and Laohakunakorn (2024); the main step in our argument is to show that every stable matching as defined in that paper satisfies the additional restrictions imposed by our notion of stable outcomes, the most important being an equal treatment property. Greinecker and Kah (2021) obtain an analogous result for one-to-one matching without occupational choice. We then show that existence implies strong duality; hence, the equivalence of stable outcomes to solutions of an optimal

²We also apply the framework to compare how communication costs and the knowledge distribution affect organizational depth and wages. Lower communication costs favor deeper organizations over the range considered, whereas the fraction of three-layer firms first rises and then falls as high-knowledge individuals become more abundant. Changes in communication costs affect the entire wage distribution, whereas changes in the knowledge distribution operate primarily through the upper tail.

³In Lucas’s (1978) seminal paper, he writes “what we may hope for is not serious organization theory, but perhaps some insights into why organization theory matters economically.” These insights have since been firmly established, leaving the door open to the next step.

transport problem and to competitive equilibrium follows. These results extend previous ones in Shapley and Shubik (1971), Gretskey, Ostroy, and Zame (1992, 1999), Chiappori, McCann, and Nesheim (2010), McCann and Trokhimtchouk (2010) and Carlier and Ekeland (2010) by allowing for the simultaneous presence of many-to-one matching, occupational choice and externalities. In addition, we establish a condition on the support of feasible firm distributions that characterizes the solutions to our optimal transport problem, generalizing the usual cyclical monotonicity condition; a related condition appears in Beiglböck and Griessler (2019).

We illustrate our framework primarily in the context of knowledge hierarchies introduced by Garicano (2000) and embedded in equilibrium models by Garicano and Rossi-Hansberg (2004) and Garicano and Rossi-Hansberg (2006).⁴ Our characterization in Section 4 is closely related to Garicano and Rossi-Hansberg (2006), who study an economy with endogenous knowledge acquisition. However, our approach differs in two respects. First, we derive the implications of optimal organization directly from stability conditions, dispensing with the need to make certain mathematical assumptions (e.g. differentiability of earnings and single-valuedness of the assignment) from the outset. Second, and more importantly, the key step in our argument is to establish what Chen and Suen (2019) call the *accordion property*, an ordering of the layers of any two active firms relative to one another.⁵ The accordion property implies both occupational stratification and the restriction that active firms have L or $L - 1$ layers. This latter restriction is needed to prove occupational stratification in Garicano and Rossi-Hansberg (2006), but it is not independently established in their paper.⁶

A large literature develops quantitative models of the earnings distribution (e.g., Castañeda, Díaz-Giménez, and Ríos-Rull 2003) and the firm-size distribution (e.g., Luttmer 2007). A smaller literature connects the two through firm heterogeneity and frictional hiring (Helpman, Itskhoki, Muendler, and Redding 2017, Engbom and Moser 2022, Macera and Tsujiyama 2024). Accounting for concentration at the top of the earnings distribution

⁴Important applications of these models include Antràs, Garicano, and Rossi-Hansberg (2006), Caliendo and Rossi-Hansberg (2012), Caliendo, Monte, and Rossi-Hansberg (2015) and Caicedo, Lucas, and Rossi-Hansberg (2019).

⁵Chen and Suen’s (2019) monotone methods cannot be applied in the full generality of our setting – see Section 10 in the Supplementary Material for details.

⁶In the proof of Proposition 4, they argue that positive sorting, no self-matching and connectedness of the set of top managers imply occupational stratification. This implication requires the result that all active firms must have L or $L - 1$ layers.

poses a further challenge. Work on income and wealth concentration emphasizes high-productivity states (Castañeda, Díaz-Giménez, and Ríos-Rull 2003), entrepreneurial investment (Cagetti and De Nardi 2006), or both (Brüggemann 2021), as well as heterogeneous income growth rates (Gabaix, Lasry, Lions, and Moll 2016). Our framework offers a complementary mechanism through which occupational choice and sorting of individuals into firms can jointly generate dispersion in firm sizes and earnings, together with concentration at the top, under a flexible technology linking worker knowledge to organizational scale and a relative scarcity of highly knowledgeable individuals. More broadly, our contribution is to provide a common framework for comparing how alternative organizational technologies shape the relationship between organizational scale and top earnings concentration.

Organizational adjustments also shape the wage distribution. Garicano and Rossi-Hansberg (2006) show how communication technology can compress wage differences among production workers while widening them among managers. Caicedo, Lucas, and Rossi-Hansberg (2019) relate changes in the U.S. earnings distribution to increasingly complex and profitable problems relative to available knowledge. We build on these insights to specify the flexible economy we calibrate to the U.S. economy. Our quantitative analysis compares it with alternative knowledge economies along the lines of those papers and shows that greater establishment-size dispersion can accompany markedly different degrees of top earnings concentration across these economies.

2 Knowledge Economies

This section introduces a general framework of knowledge economies. The economy is populated by a large number of individuals, each of whom chooses either to work alone or in a firm. Firm production depends on the knowledge and occupational role of its members and on the number of individuals in each role, i.e. on its organizational structure.

We describe the environment in Section 2.1 and introduce our solution concept in Section 2.2.

2.1 Model

There is a large number of individuals who may differ, for example, in their knowledge or talent. These characteristics are captured by a type space Z , which is a compact metric space. The type distribution is a nonzero, finite Borel measure ν on Z satisfying $Z = \text{supp}(\nu)$. For a concrete example that we will use to illustrate the model and solution concept, let $Z = \{z_l, z_m, z_h\} \subseteq (0, 1)$ and $\nu(Z) = 1$. In this example, there is a unit mass of individuals with one of three knowledge levels, and for each $z \in Z$, $\nu(z)$ is the fraction of individuals with knowledge level z .

Individuals can match with each other in firms. A firm is described by its organizational characteristics, which consist of the number L of its layers, and the number n_i and type z_i of members of each layer $i \in \{1, \dots, L\}$. We assume that the number of layers is bounded by $\bar{L} \in \mathbb{N}$ and the number of members of each layer is both bounded and bounded away from zero, i.e. $n_i \in N = [\underline{n}, \bar{n}]$ for each $1 \leq i \leq L$ with $0 < \underline{n} \leq \bar{n} < \infty$ and $1 \in N$. The economic meaning of this assumption is that the relative sizes of layers are bounded. We normalize the size of layer L so that $n_L = 1$.⁷ The members of layer L are called managers, and managers of a firm with one layer, $L = 1$, are called self-employed. It will often be convenient to think loosely of one manager hiring the workers in the lower layers of each firm.⁸

A *firm* is then $(z_i, n_i)_{i=1}^L$ where $L \in \{1, \dots, \bar{L}\}$ and $(z_i, n_i) \in Z \times N$ for each $1 \leq i \leq L$. Let Y be the set of feasible firms, which we assume to be a closed subset of $\cup_{L=1}^{\bar{L}} (Z \times N)^L$.⁹ We assume that self-employment is feasible, i.e. $(Z \times \{1\}) \subseteq Y$, and that, for each $2 \leq L \leq \bar{L}$, the “span-of-control” correspondence $N_L : Z^L \rightrightarrows N^{L-1}$ defined by setting, for each $(z_1, \dots, z_L) \in Z^L$, $N_L(z_1, \dots, z_L) = \{(n_1, \dots, n_{L-1}) \in N^{L-1} : (z_1, n_1, \dots, z_{L-1}, n_{L-1}, z_L, 1) \in Y\}$ is lower hemicontinuous. Note that if $N = \{1\}$ and $\bar{L} = 2$, we have *one-to-one* matching. In addition, a *two-sided* matching market can be represented by assuming $Z = W \cup M$, with

⁷In our model, only the relative sizes of layers have economic meaning; the normalization that $n_L = 1$ is convenient for interpretation.

⁸A more precise interpretation is that, in each firm, there is a measure one of managers and a measure n_i of workers in layer $1 \leq i \leq L - 1$.

⁹Letting, for each $L \in \{1, \dots, \bar{L}\}$, $Y_L = (Z \times N)^L$, we endow Y_L with the product topology and $\cup_{L=1}^{\bar{L}} Y_L$ with the sum topology, i.e. $V \subseteq \cup_{L=1}^{\bar{L}} Y_L$ is open if and only if $V \cap Y_L \subseteq Y_L$ is open in Y_L for all $1 \leq L \leq \bar{L}$. With this topology, $\cup_{L=1}^{\bar{L}} Y_L$ is then a metrizable compact space; see the supplementary material to this paper for details.

$W \cap M = \emptyset$, and letting $Y = (Z \times \{1\}) \cup (W \times N \times M \times \{1\})$. Beyond these standard matching settings, we allow individuals to form general firms.¹⁰ For our running example, let $Y = (Z \times \{1\}) \cup \{(z_i, n_i)_{i=1}^2 \in (Z \times \mathbb{R}_+)^2 : n_1 = (1 - z_1)^{-1}, n_2 = 1\}$; thus, more knowledgeable workers can be hired in greater numbers (by any manager in a two-layer firm).

The output produced by a firm depends on the composition of its workforce and also on the distribution of firms, thereby allowing for externalities. The latter is represented by $\gamma \in M(Y)$ satisfying some properties below.¹¹ Hence, the output of a firm y equals $F(y, \gamma)$, where $F : Y \times M(Y) \rightarrow \mathbb{R}_+$ is continuous.¹² Note that the externalities depend on the entire distribution of firms; in particular, a single firm does not affect the externality which will be taken as given by individuals. Conceptually, allowing for externalities is important since they capture interactions between the individual organization and the rest of the economy. For notational convenience, the dependence of F on γ will be dropped when there are no externalities. For our running example, let $F((z_i, n_i)_{i=1}^2) = z_2 n_1$ so that, for two-layer firms, output depends only on the manager's knowledge z_2 and the number of workers n_1 , and let $F(z_1, 1) = 0.9z_1$ so that output of the self-employed is equal to 90% of their knowledge z_1 .

We consider the frictionless benchmark of perfectly transferable utility: payments in a firm $y = (z_i, n_i)_{i=1}^L$ consist of $(t_i)_{i=1}^L$ such that $\sum_{i=1}^L n_i t_i = F(y, \gamma)$ when the distribution of firms is equal to γ . Each individual in layer $1 \leq i \leq L$ receives t_i .

In summary, a *knowledge economy* is $E = (Z, \nu, Y, F)$. Thus, our running example completely specifies a knowledge economy once we fix the values of z_l, z_m, z_h and their measures.

Our framework is sufficiently general to nest a range of prominent models as special cases. Section 5 of the supplementary material establishes this for Gabaix and Landier (2008), Garicano and Rossi-Hansberg (2004), Garicano and Rossi-Hansberg (2006), Gola (2021), Ide and Talamàs (2025), Mak and Siow (2025) and a truncated version of Rosen (1982).¹³

The definition of a knowledge economy here describes only the physical environment.

¹⁰The only substantive restriction we are making is that firms consist of a finite number of worker types.

¹¹In general, given a metric space T , $M(T)$ denotes the set of finite, Borel measures on T endowed with the weak (narrow) topology (see Varadarajan (1958) for details).

¹²The nonnegativity of F is imposed only to interpret it as an output function and can be dispensed with from Theorems 1–3 in Section 3. See Footnote 7 in the supplementary material for details.

¹³We abstract from capital for simplicity, so the setting in Lucas (1978) is not nested in our framework. However, capital can be easily added as pointed out in Carmona and Laohakunakorn (2024).

Individuals' incentives and choices are specified in the next section as part of the solution concept.

2.2 Stable Outcomes

We first illustrate the solution concept using the running example and then give its formal definition. We describe a solution of a knowledge economy by an assignment and an earnings function. Informally, the assignment specifies a firm distribution. In the three-type, two-layer example of the previous section, the assignment specifies, for instance, the fraction of two-layer firms managed by type- z_h individuals and employing $(1 - z_l)^{-1}$ workers of type z_l . An earnings function u specifies the earnings of each type, $(u(z_l), u(z_m), u(z_h))$. Each individual optimally chooses an occupation and, when applicable, whom to match with, taking as given the earnings of their match partner(s). For example, for the middle type, this means that

$$u(z_m) = \max \left\{ \underbrace{\max_{z \in Z} \frac{z_m - u(z)}{1 - z}}_{\text{manager}}, \underbrace{\frac{9}{10} z_m}_{\text{self-employed}}, \underbrace{\max_{z \in Z} (z - u(z)(1 - z_m))}_{\text{worker}} \right\}.$$

That is, z_m cannot raise earnings by (i) becoming a manager of a firm $(z, (1 - z)^{-1}, z_m, 1)$ while paying workers their “equilibrium” earnings $u(z)$, (ii) becoming self-employed, or (iii) becoming a worker in a firm $(z_m, (1 - z_m)^{-1}, z, 1)$, taking the manager’s “equilibrium” earnings $u(z)$ as given.

We formalize this informal description. The set of feasible firms is Y and an assignment is a measure over Y indicating roughly how many firms with characteristics $y \in Y$ there are. For each $i \in \{1, \dots, \bar{L}\}$, let $Y_i = Y \cap (Z \times N)^i$, a closed subset of Y (see footnote 9). Note that $Y = \cup_{i=1}^{\bar{L}} Y_i$ and $Y_i \cap Y_j = \emptyset$ for $i \neq j$. Then define a continuous function $\ell : Y \rightarrow \{1, \dots, \bar{L}\}$ by setting, for each $y \in Y$, $\ell(y) = i$ if $y \in Y_i$. For each $j \in \{1, \dots, \bar{L}\}$, define continuous functions $z_j : \cup_{i \geq j} Y_i \rightarrow Z$ and $n_j : \cup_{i \geq j} Y_i \rightarrow N$ by setting, for each $y = (z_k, n_k)_{k=1}^l \in \cup_{i \geq j} Y_i$, $z_j(y) = z_j$ and $n_j(y) = n_j$.

Given $\gamma \in M(Y)$, let ν_γ be the measure of individuals according to γ : for each Borel

$B \subseteq Z$,¹⁴

$$\nu_\gamma(B) = \int_Y \sum_{i=1}^{\ell(y)} 1_B(z_i(y)) n_i(y) d\gamma(y).$$

The integrand is the mass of members of firm y whose types lie in B ; integrating across firms therefore gives the economy-wide mass of such individuals. An *assignment* is $\gamma \in M(Y)$ such that $\nu_\gamma = \nu$, i.e. the measure of individuals who are part of a firm is the same as the measure of individuals in the population. Denote the set of assignments by:

$$\Gamma = \{\gamma \in M(Y) : \nu_\gamma = \nu\}.$$

An *earnings function* is $u \in C(Z)$, i.e. u specifies the earnings of each type.¹⁵ An *outcome* is (γ, u) such that γ is an assignment and u is an earnings function.

An outcome (γ, u) is *stable* if

- (i) $\sum_{i=1}^{\ell(y)} n_i(y) u(z_i(y)) = F(y, \gamma)$ for each $y \in \text{supp}(\gamma)$ and
- (ii) $\sum_{i=1}^{\ell(y)} n_i(y) u(z_i(y)) \geq F(y, \gamma)$ for each $y \in Y$.

Moreover, γ is *stable* if there exists an earnings function u such that (γ, u) is stable. Condition (i) states that, in every active firm, the sum of its members' earnings equals the total output they produce. Condition (ii) is no-blocking: no feasible firm can form and produce a level of output that is greater than the sum of its members' earnings.

Conditions (i) and (ii) capture succinctly the optimality requirements in a knowledge economy. Simple manipulations of (i) and (ii) imply that an outcome (γ, u) is stable if and only if

$$\sum_{i=1}^{\ell(y)} n_i(y) u(z_i(y)) = F(y, \gamma) \text{ for each } y \in \text{supp}(\gamma), \text{ and}$$

$$u(z) = \max_{y \in Y, i \in \{1, \dots, \ell(y)\} : z_i(y) = z} \frac{F(y, \gamma) - \sum_{j \in \{1, \dots, \ell(y)\} \setminus \{i\}} n_j(y) u(z_j(y))}{n_i(y)} \text{ for each } z \in Z.$$

¹⁴Note that the function $y \mapsto \sum_{i=1}^{\ell(y)} 1_B(z_i(y)) n_i(y)$ is measurable for each Borel subset B of Z ; see Section B.1 in the appendix for a proof.

¹⁵Given a metric space T , $C(T)$ denotes the space of bounded and continuous real-valued functions on T .

The first of these conditions is the same as condition (i) in the definition of a stable outcome. The second condition means that no type z can increase earnings by optimally setting up a firm with ℓ layers in which they are in layer i given that they have to pay $u(z_j)$ to those who occupy layer j .¹⁶

An alternative interpretation of conditions (i) and (ii) is as follows. If we define the profit of firm $y \in Y$ as $\pi(y) = F(y, \gamma) - \sum_{i=1}^{\ell(y)} n_i(y)u(z_i(y))$, then (i) requires that $\pi(y) = 0$ for each $y \in \text{supp}(\gamma)$ and (ii) that $\pi(y) \leq 0$ for each $y \in Y$, i.e. the firms that form are profit maximizing and earn zero profits.

For our running example, let $z_l = 1/4$, $z_m = 1/2$, and $z_h = 7/8$, and suppose $\nu(z_l) = 2/5$, $\nu(z_m) = 3/10$, and $\nu(z_h) = 3/10$. Our results will imply a unique stable assignment γ with $\text{supp}(\gamma) = \{y_1, y_2\}$, where $y_1 = (z_m, 1)$ and $y_2 = (z_l, (1 - z_l)^{-1}, z_h, 1)$, so that all low-knowledge individuals are workers, all high-knowledge individuals are managers, and all middle-knowledge individuals are self-employed. Specifically, $\gamma(y_1) = \gamma(y_2) = \frac{3}{10}$, which is supported by any earnings function u such that $u(z_m) = \frac{9}{20}$, $\frac{9}{40} \leq u(z_l) \leq \frac{19}{80}$ and $u(z_h) = \frac{4}{3}(z_h - u(z_l))$.

3 General Properties of Stable Outcomes

In this section, we establish existence of stable outcomes. We then characterize their two components through a primal-dual pair of linear programs and show that the support of each stable assignment satisfies a generalization of the c -cyclical monotonicity condition from optimal transport. We also show that stable outcomes are equivalent to competitive equilibria.

3.1 Existence, Duality and Characterization

The results in this section hold for every knowledge economy. We therefore fix an arbitrary knowledge economy throughout.

Our existence result builds on Carmona and Laohakunakorn (2024), which provides a framework for large many-to-one matching markets with occupational choice and establishes the existence of stable matchings. We represent knowledge economies as matching markets and show that their stable matchings are equivalent to stable outcomes; hence existence of

¹⁶See Section 6 in Supplementary Material for a proof of the above characterization.

one implies the other.¹⁷ Two additional arguments are required.

First, in our knowledge economies, whether a firm is feasible depends on the knowledge of all individuals in the firm. We therefore extend the framework of Carmona and Laohakunakorn (2024) in Supplementary Material, Section 2, and provide an existence theorem and sufficient conditions for its assumptions (Theorem 3.1 and Lemma 3.8 in Supplementary Material, Section 3).

Second, the notion of stable matching in Carmona and Laohakunakorn (2024) is defined for general nontransferable utility matching models where equal treatment does not necessarily hold. Thus, it imposes fewer restrictions than stable outcomes as defined here and its existence is easier to show. To establish the equivalence, we show that an equal treatment property holds for every stable matching in our environment; as a consequence, the law of one price imposed in the definition of stable outcomes becomes a result, not an assumption, in our setting.

Theorem 1 *Stable outcomes exist.*

In our economy, since individuals are negligible, existence holds despite the presence of externalities and without any substitutes assumption on firm production.

Next, we show that stable outcomes solve a primal-dual pair of linear programs. Due to the presence of externalities, stable assignments do not necessarily maximize surplus, but instead maximize production for a fixed level of externality. For each $\gamma \in \Gamma$, define $F^\gamma : Y \rightarrow \mathbb{R}_+$ by setting, for each $y \in Y$, $F^\gamma(y) = F(y, \gamma)$. It will be useful to think of the surplus maximization problem in an economy without externalities, where production is given by F^γ . This is the linear program $\max_{\tau \in \Gamma} \int_Y F^\gamma d\tau$ which chooses an assignment to maximize production in this auxiliary economy. Define the constraint set for the dual problem of maximizing surplus in this auxiliary economy:

$$U^\gamma = \left\{ u \in C(Z) : \sum_{i=1}^{\ell(y)} n_i(y) u(z_i(y)) \geq F^\gamma(y) \text{ for each } y \in Y \right\}.$$

¹⁷Our definition of stable outcomes can thus be viewed as a characterization of stable matchings in this environment.

The dual is $\min_{v \in U^\gamma} \int_Z v d\nu$ which chooses an earnings function that satisfies no-blocking in the auxiliary economy to minimize total earnings. At a stable outcome, the values of these two programs coincide.

Theorem 2 *For each $\gamma \in \Gamma$:*

$$\max_{\tau \in \Gamma} \int_Y F^\tau d\tau = \min_{v \in U^\gamma} \int_Z v d\nu. \quad (1)$$

An outcome $(\gamma, u) \in \Gamma \times C(Z)$ is stable if and only if γ solves $\max_{\tau \in \Gamma} \int_Y F^\tau d\tau$ and u solves $\min_{v \in U^\gamma} \int_Z v d\nu$.

Since two-sided one-to-one matching is a special case of our model, Theorem 2 implies the standard optimal transport duality: when Y is restricted to capture two-sided one-to-one matching and there are no externalities, equation (1) is exactly the optimal transport duality.¹⁸ Theorem 2 also quantifies the inefficiency due to the presence of externalities: the surplus maximizing assignment solves $\max_\tau \int_Y F^\tau d\tau$ and hence the efficiency loss at a stable assignment γ is $\max_\tau \int_Y F^\tau d\tau - \int_Y F^\gamma d\gamma$. In general, a surplus maximizing assignment γ^* may not solve $\max_\tau \int_Y F^\tau d\tau$; thus, surplus maximizing assignments may not be stable. Finally, when there are no externalities, Theorem 2 implies that stable outcomes can be found numerically by discretizing Z and Y and solving a linear programming problem.¹⁹

The following result generalizes the c -cyclical monotonicity condition from optimal transport and provides a characterization of solutions to the surplus maximization problem (for a fixed externality). A set $S \subseteq Y$ is F^γ -monotone if

$$\int_Y F^\gamma d\zeta \geq \int_Y F^\gamma d\tau$$

for each finitely-supported ζ on S and for each finitely-supported τ on Y such that $\nu_\tau = \nu_\zeta$. In words, no finitely-supported measure on S can be improved, in the sense of yielding a higher surplus, by transporting mass in a feasible way. Theorem 3 shows that, given the

¹⁸See, for example, Villani (2009).

¹⁹In Section 14 of the Supplementary Material, we give a sufficient condition for the surplus maximization problem to have an equivalent network-flow formulation that avoids enumerating every feasible firm. This sufficient condition is satisfied by the knowledge economies we consider in Sections 4 and 5.

production function F^γ , the support of surplus maximizing assignments is F^γ -monotone, and any assignment with F^γ -monotone support is surplus maximizing.

Theorem 3 *For each $\gamma \in \Gamma$: γ solves $\max_{\tau \in \Gamma} \int_Y F^\gamma d\tau$ if and only if $\text{supp}(\gamma)$ is F^γ -monotone.*

3.2 Competitive Equilibrium

An assignment specifies how knowledge-differentiated labor is allocated across firms, while the earnings function gives the corresponding price of each labor type. This suggests that stable outcomes are part of a competitive equilibrium, which we develop below.²⁰

We describe a knowledge economy in a general equilibrium setting as follows. Consumers are characterized by their knowledge level $z \in Z$ and are described implicitly by the knowledge distribution ν . Individuals value only the consumption good. Endowments depend on the knowledge level z since an individual with knowledge z is endowed with one unit of labor of type z . We normalize the price of the consumption good to one, and the prices of labor are described by a continuous nonnegative function c .

The description of production follows Hornstein and Prescott (1993) in that only the aggregate production set is specified in such a way that the formation of firms, i.e. the distribution of firms, is part of the description of production. The aggregate production set is indexed by $\gamma^* \in M(Y)$, which describes the production externalities in the economy.

A competitive equilibrium consists of the consumers' choices, an externality γ^* , a production vector and prices such that almost every consumer maximizes utility subject to the budget constraint, production maximizes profit taking the externality as given, the production vector is supported by the firm distribution γ^* , and aggregate consumption does not exceed the sum of aggregate endowments and production.

In a knowledge economy with productive types,²¹ Theorem 8.1 in the Supplementary Material establishes the equivalence between competitive equilibria and stable outcomes. Individuals consume the value of their endowment and supply all of their labor. Aggregate consumption equals firms' total output, and firms' labor demand equals the population's labor supply type by type. As a result, once the distribution of firms is described by a

²⁰See the Supplementary Material for detailed statements and proofs for the results of this section.

²¹For every firm distribution, almost every type can produce positive output in some feasible firm composed entirely of that type.

measure γ and prices c are specified, all elements of a competitive equilibrium are pinned down. These outcomes (γ, c) coincide exactly with stable outcomes.

4 Theoretical Application: Stable Outcomes in Knowledge Economies with Multi-Layer Firms

We characterize the set of stable outcomes of the knowledge economy of Garicano and Rossi-Hansberg (2004) with multi-layered firms. This exercise illustrates the applicability of our framework and is interesting in its own right as it establishes several properties of the firm distribution in a canonical knowledge economy. The most striking of these properties is the result that all active firms must have at most two adjacent numbers of layers, L and $L - 1$, i.e. organizational depth is tightly controlled by the market.

We first specify the economy. We then explain the accordion property underlying the above result and show how relaxing the technology can allow firms with more than two numbers of layers to coexist.

4.1 The Canonical Economy and Its Stable Outcome

We consider the knowledge economy E^* defined as follows. The type space is $Z = [\underline{z}, \bar{z}]$ with $0 < \underline{z} < \bar{z} < 1$ such that

$$\frac{\bar{z} - \underline{z}}{h(1 - \underline{z})} > \bar{z}, \quad (2)$$

where $0 < h < 1$.²²

The type distribution is ν and has a continuous and strictly positive density $\theta : Z \rightarrow \mathbb{R}_{++}$. Let $\bar{L} \in \mathbb{N}$ be such that $\bar{L} \geq 2$. The set of feasible firms is Y^* defined by $Y^* = \cup_{L=1}^{\bar{L}} Y_L^*$, $Y_1^* = Z \times \{1\}$, $N = [(1 - \bar{z})/(1 - \underline{z}), (h(1 - \bar{z}))^{-1}]$ and, for each $1 < L \leq \bar{L}$,²³

$$Y_L^* = \{(z_1, n_1, \dots, z_L, n_L) \in (Z \times N)^L : n_L = 1, n_1 = (h(1 - z_{L-1}))^{-1} \\ \text{and } n_i = (1 - z_{i-1})(1 - z_{L-1})^{-1} \text{ for each } 1 < i < L\}.$$

²²Note that (2) holds when $\underline{z} = 0$, hence, for each $\bar{z} > 0$, there is $\underline{z} > 0$ such that it holds. This condition implies that an outcome where everyone is self-employed is not stable.

²³Note that for each $1 < L \leq \bar{L}$, the span-of-control correspondence N_L is lower hemicontinuous since, for each $z \in Z^L$, $N_L(z_1, \dots, z_L) = \left\{ \left(\frac{1}{h(1 - z_{L-1})}, \frac{1 - z_1}{1 - z_{L-1}}, \dots, \frac{1 - z_{L-2}}{1 - z_{L-1}} \right) \right\}$, i.e. N_L is a continuous function.

Then let the output function be $F : Y^* \rightarrow \mathbb{R}$ defined by setting, for each $y \in Y^*$, $F(y) = z_{\ell(y)}(y)n_1(y)$. See Garicano and Rossi-Hansberg (2004) and Garicano and Rossi-Hansberg (2006) for the motivation behind the defining elements of E^* . For our purposes, it suffices to note that the parameter h represents communication costs and determines the layer sizes consistent with the knowledge of the members of the firm.

For each firm in this economy, the size of each layer is determined by the knowledge profile $(z_1, \dots, z_L)$. Hence, we simplify (and abuse) the notation by regarding firms as elements of $\cup_{L=1}^{\bar{L}} Z^L$ and, thus, $Y^* = \cup_{L=1}^{\bar{L}} Z^L$.

Theorem 9.1 in the supplementary material characterizes the stable outcomes of this economy; here we focus on the main economic lessons.

The knowledge economy E^* has a unique stable outcome. First, there are at most two numbers of layers among active firms; specifically, there is $2 \leq L \leq \bar{L}$ such that either all active firms have L layers or all active firms have either L or $L - 1$ layers.

Second, there is occupational stratification, i.e. types are ordered according to occupation. If all firms have L layers, the least knowledgeable individuals are in layer 1, followed by those in layer 2 and so on until those in layer L , who are the most knowledgeable individuals in the economy. If some firms have L layers while others have $L - 1$, then the same holds: those in layer i of any firm are more knowledgeable than those in layer $i - 1$; in addition, those who are part of layer i of a firm with L layers are less knowledgeable than those in layer i of a firm with $L - 1$ layers.

Among firms with the same number of layers, the matching is strictly positive assortative: more knowledgeable top managers work with more knowledgeable individuals in every lower layer. Matching is then described by differentiable functions whose slopes are determined by market clearing. The wage function is strictly increasing in knowledge and differentiable within each occupational interval, where it satisfies a differential equation implied by profit maximization.

Thus, stability tightly controls organizational depth, the allocation of knowledge across firms and occupations, and the distribution of earnings.

4.2 Discussion

4.2.1 The Accordion Property

The key step in the proof is to establish a relation between the layers of any two active firms $(z_1, \dots, z_L)$ and $(x_1, \dots, x_K)$ when $z_L > x_K$. The accordion property says that

$$z_L > x_K \geq z_{L-1} \geq x_{K-1} \geq z_{L-2} \geq \dots \geq x_1 \geq z_{L-K},$$

so that, in particular, $L \geq K$.

The inequalities of the form $x_{K-l} \leq z_{L-l}$ reflect positive assortative matching across firms, layer by layer: the firm with the better manager has weakly better types in every layer counting from the top. Consider swapping the top managers, z_L and x_K of the two firms. Stability requires that the increase in the wage bill from hiring z_L instead of x_K , $u(z_L) - u(x_K)$, must be at least the gain in output, $\frac{z_L - x_K}{h(1 - x_{K-1})}$, and symmetrically for the other firm:

$$\frac{z_L - x_K}{h(1 - x_{K-1})} \leq u(z_L) - u(x_K) \leq \frac{z_L - x_K}{h(1 - z_{L-1})}.$$

Since $z_L > x_K$, it follows that $z_{L-1} \geq x_{K-1}$. Note that for a manager with second-in-command of type s , $\frac{1}{h(1-s)}$ is the marginal value of the manager's knowledge; that this marginal value is increasing in s (and, more generally, supermodularity of profit in the knowledge of consecutive layers) is responsible for the first half of the accordion property.

On the other hand, inequalities of the form $z_{L-l} \leq x_{K-l+1}$ are more subtle. Consider the base case $x_K \geq z_{L-1}$. This says that the second-in-command of the better firm is no better than the head of the worse firm and follows because the total value of a manager's knowledge decreases with the knowledge of the layer below that manager, i.e. it relies on the *substitutability* between the top two layers of the same firm. For a manager of type a with second-in-command of type s , define the total value of the manager's knowledge to be $V_a(s) = \frac{a-s}{h(1-s)}$.²⁴

²⁴This is the part of the loss in output from firing the manager when the number of production workers is held fixed (the other part arising from the decline in the number of workers). That holding the number of

Now consider reorganizing the two firms by promoting z_{L-1} to the top position in the original firm and letting z_L take over the worse firm, with x_K demoted to second-in-command.²⁵ Stability requires that the payment to z_L must be at least the manager's value to the firm with x_K as second-in-command and at most the manager's value to the original firm: $V_{z_L}(x_K) \leq u(z_L) \leq V_{z_L}(z_{L-1})$. Since V_{z_L} is decreasing, we have $x_K \geq z_{L-1}$.

Thus, whereas the marginal value determines who should hold which position among top managers, the total value determines whether a position is worth filling at all. That both supermodularity and substitutability hold in Garicano and Rossi-Hansberg (2004) is what explains why the accordion property holds in that economy. The result that all active firms must have L or $L - 1$ layers and occupational stratification then follow from the accordion property.

4.2.2 Variation in the Number of Layers

Real firms display more than two numbers of layers. The previous discussion identifies substitutability as key to the accordion property and hence the $(L, L - 1)$ result. One advantage of our framework is that it allows us to explore economies where substitutability does not hold in order to generate larger variation in the number of layers.

Consider a more flexible economy, where for each $1 < L \leq \bar{L}$, Y_L^* is replaced with

$$Y_L^{\text{flex}} = \{(z_1, n_1, \dots, z_L, n_L) \in (Z \times [((1 - \bar{z})/(1 - \underline{z}))^\sigma, (h(1 - \bar{z})^\sigma)^{-1}])^L : n_L = 1, \\ n_1 = (h(1 - z_{L-1})^\sigma)^{-1} \text{ and } n_i = ((1 - z_{i-1})/(1 - z_{L-1}))^\sigma \text{ for each } 1 < i < L\}$$

for $\sigma \geq 1$. Note that when $\sigma = 1$, this coincides with Y_L^* .

For a top manager with knowledge a whose second-in-command has knowledge $s < a$, the value of the top manager's knowledge is $V_a(s) = \frac{a-s}{h(1-s)^\sigma}$. We say that the top two layers are *substitutes* if, for each $a \in Z$, V_a is decreasing on $[\underline{z}, a)$. Note that the top two layers are substitutes when $\sigma = 1$ but not when σ is large.

production workers fixed is the relevant comparison in what follows is because the stability conditions can be normalized per production worker.

²⁵When $K = \bar{L}$, this deviation is not feasible; the proof constructs an alternative deviation for that case.

We now show that the $(L, L - 1)$ result may fail in the flexible economy.²⁶ Let $Z = \{z^l, z^m, z^h\} = \{0.2, 0.6, 0.9\}$ with $\nu(z^l) = 30$, $\nu(z^m) = 2$ and $\nu(z^h) = 1$. Let $h = 0.5$, $\bar{L} = 3$ and $\sigma = 3$. We have $V_{z^h}(z^l) = 2.73 < V_{z^h}(z^m) = 9.38$; hence substitutability of the top two layers fails. In the stable assignment, we have $\text{supp}(\gamma) = \{(z^l, z^m, z^h), (z^l, z^h), (z^l)\}$, i.e. firms with 1, 2, and 3 layers coexist, and the middle manager of the three-layer firm is more knowledgeable than the head of the one-layer firm.

5 Quantitative Application

This section presents two quantitative applications of the knowledge economy model. First, we show how communication costs and the distribution of knowledge affect organizational structure and wages. For the second application, we use the 2008 U.S. wage and establishment-size distributions as quantitative benchmarks for comparing model economies. The general existence and duality results and their computational method apply to both applications. In each application, the maximum number of layers $\bar{L}$ is fixed as a primitive of the economy and bounds feasible organizational depth, while the number of layers adopted by individual firms is determined in equilibrium.

5.1 Organizational Structure

We first use the knowledge economy model to study comparative statics in the endogenous choice of organizational form. We allow firms to have at most three organizational layers, $\bar{L} = 3$. We consider the knowledge economy of Section 4 with the specification that knowledge is distributed according to a Beta distribution, $\text{Beta}(\beta_1, \beta_2)$, on $Z = [\underline{z}, \bar{z}]$, where $\underline{z} = 0.001$ and $\bar{z} = 0.999$. We discretize the knowledge distribution and solve the surplus-maximization problem and its dual described in Section 3.²⁷

Panel A of Figure 1 shows that organizational depth decreases with communication costs

²⁶For simplicity, we consider a three-type economy which violates the assumption that ν has a density. The same point could be made by solving the flexible economy of Section 5 with parameters $h = 0.9$, $\sigma = 4$, $\alpha = 1$, $\bar{L} = 3$, $\beta_1 = 1$, $\beta_2 = 4$ and $Z = [0.001, 0.6]$. Under these parameters, we get firms with 1, 2, and 3 layers, and the middle managers of a three-layer firm are more knowledgeable than the top managers of a two-layer firm. On the other hand, setting $\sigma = 1$, we get self-employment (one-layer) and two-layer firms, with the production workers of the two-layer firms being the least knowledgeable, followed by the self-employed, followed by the managers of the two-layer firms, consistent with stratification and implied by Garicano and Rossi-Hansberg (2004).

²⁷See Section 15 of the supplementary material for computational details.

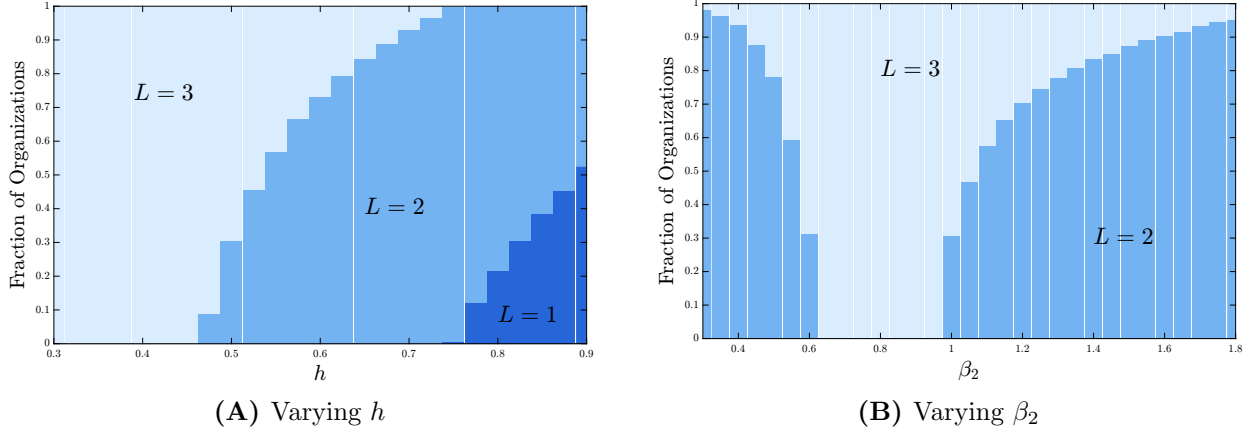


Figure 1: Organizational Choice. The figure reports the fractions of organizations with $L = 1$, $L = 2$, and $L = 3$ as h varies from 0.3 to 0.9 under a uniform knowledge distribution (Panel A) and as β_2 varies holding $h = 0.5$ and $\beta_1 = 1$ (Panel B).

over $h \in [0.3, 0.9]$. At low communication costs, all organizations have three layers. As h rises, $L = 2$ organizations gradually replace $L = 3$ organizations; at the highest communication costs, self-employment appears alongside $L = 2$ organizations. Cheaper communication favors deeper hierarchies: middle managers solve problems left unsolved by workers and pass only the remainder upward, extending the reach of top-manager knowledge.

Panel B shows that organizational structure responds non-monotonically to changes in the distribution of knowledge. We hold $h = 0.5$ and $\beta_1 = 1$, so an increase in β_2 shifts the knowledge distribution toward lower values of z . The fraction of $L = 3$ organizations first rises and then falls as β_2 increases. At each parameter value in both panels, active organizations have either the same number of layers or two adjacent numbers, as characterized in Section 4.

Whether an additional layer is worthwhile depends on the top-manager time saved relative to the middle-manager time required, compared with their relative earnings. Changes in communication costs and the knowledge distribution affect this comparison through equilibrium earnings and the types assigned to different layers, and the above results show that the resulting share of $L = 3$ organizations weakly decreases with h but varies non-monotonically with β_2 .

Figure 2 shows that changes in communication costs and the knowledge distribution also have distinct effects on wage inequality. Panel A shows that changes in h affect wages throughout the distribution. A fall in h raises lower-percentile wages and reduces upper-

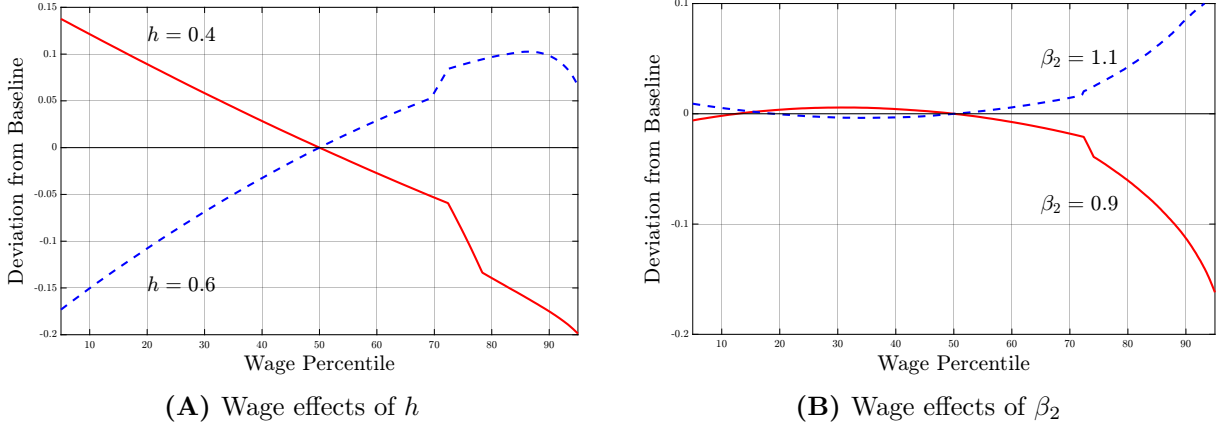


Figure 2: Wage Distribution. The figure reports, at each wage percentile, the difference between log wages relative to the median in the counterfactual economy and in the baseline economy, where $h = 0.5$ and $\beta_1 = \beta_2 = 1$.

percentile wages relative to the median, while an increase in h produces the opposite pattern. By contrast, Panel B shows that changes in β_2 operate primarily through the upper tail. A higher β_2 makes high-knowledge agents scarcer and raises their wages relative to the median, whereas a lower β_2 makes them more abundant and compresses upper-tail wages. Wages below the median change relatively little.

5.2 Wage Inequality and Establishment-Size Distribution

We calibrate alternative knowledge economies to common wage and establishment-size benchmarks to examine the equilibrium patterns each specification can generate. As standard, the goal is to examine which parameter specification yields equilibrium patterns closer to these benchmarks and to evaluate the effect of different specifications on equilibrium outcomes rather than to interpret these calibrated parameters as estimates of their actual values.

For the wage distribution, we use the 5th through 95th percentiles of hourly wages in 2008 constructed by Acemoglu and Autor (2011) from the Current Population Survey (CPS).²⁸ At each percentile, wages are expressed in logs relative to the median. For the establishment-size distribution, we use the Business Dynamics Statistics.²⁹ We also evaluate the top 1% income

²⁸Acemoglu and Autor (2011) use the May/Outgoing Rotation Group CPS samples to construct hourly wages for wage and salary workers aged 16–64 employed during the survey reference week, using CPS sampling weights. They exclude the self-employed and those in military occupations and pool three adjacent years at each date.

²⁹Establishment size is measured by the number of employees. The BDS employer-establishment distribution excludes self-employed individuals without paid employees. Following the same definition, in the model,

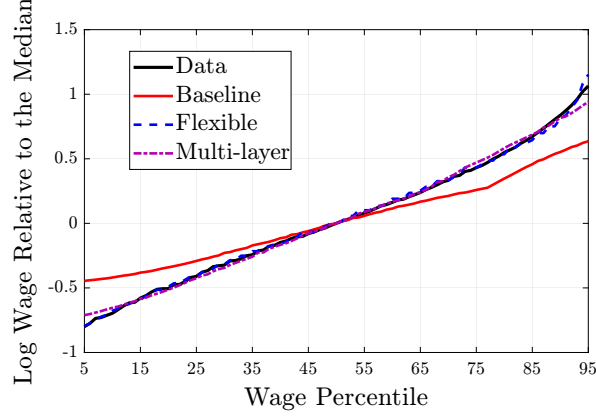


Figure 3: Wage Distributions. The figure compares the wage distributions generated by the baseline model, the flexible model, and the multi-layer model with $\bar{L} = 3$ with the data.

share, although we do not include it among the calibration targets unless otherwise noted.³⁰

Baseline Model. We begin with the standard knowledge-economy model of Garicano and Rossi-Hansberg (2004) with $\bar{L} = 2$, generalized to allow for a flexible Beta distribution of knowledge. We calibrate these parameters by minimizing the maximum of the mean squared deviations for the wage-percentile and establishment-size targets. The calibration yields $(h, \beta_1, \beta_2) = (0.45, 0.58, 0.62)$.

Figure 3 and Table 1 show that the calibrated baseline model generates too little dispersion in both wages and establishment sizes. The baseline technology tightly links wage inequality and establishment-size dispersion. The size of an establishment employing workers of type z_1 is $n_1(z_1) = \frac{1}{h(1-z_1)}$. Let z_w denote the knowledge of the most knowledgeable worker employed in equilibrium. The size of the largest establishment relative to the smallest is then $\frac{n_1(z_w)}{n_1(z)} = \frac{1-z}{1-z_w}$. Substantial establishment-size dispersion therefore requires z_w to be close to the upper bound of the knowledge distribution. However, raising the maximum worker type narrows the range of types assigned as managers and can limit dispersion in managerial earnings.

Flexible Model. We next compare the baseline model with a specification that relaxes its span-of-control and output restrictions. We continue to impose $\bar{L} = 2$. For a manager-worker establishment employment counts workers and middle managers, excluding the top manager.

³⁰Because the model contains only labor earnings, its corresponding statistic is a top 1% earnings share. We nevertheless compare it with the empirical top 1% income share as a benchmark, recognizing that the latter also includes nonlabor income.

Table 1: Establishment-Size Distribution.

Employees	Data	Baseline	Flexible	Multi-layer
1–9	71.56	98.01	70.82	76.29
10–99	25.92	1.99	28.66	23.71
100–999	2.42	0.00	0.52	0.00
1,000+	0.10	0.00	0.00	0.00

Note: The table reports the percentage of employer establishments in each employment-size category.

firm, we specify the span of control and output as $n_1(z_1, z_2) = \frac{1}{h(1-z_1)^\sigma}$ and $F(z_1, z_2) = z_2^\alpha n_1(z_1, z_2)$.³¹ For self-employment, output is $F(z_1) = z_1^\alpha$. The parameter α governs the elasticity of output with respect to knowledge, while σ governs the sensitivity of span of control to worker knowledge.³² The baseline model is nested as the special case $(\alpha, \sigma) = (1, 1)$. These parameters are interpreted relative to our specification of the knowledge distribution and the production function; for example, α is the elasticity with respect to the model’s measure of knowledge z , not a directly measured knowledge.

The calibration yields $(\alpha, \sigma, h, \beta_1, \beta_2) = (0.97, 28.87, 0.23, 1.06, 18.64)$. The calibrated α remains close to the linear baseline, while the large value of σ allows small differences in worker knowledge to generate large differences in establishment size. The large value of β_2 , with β_1 close to one, concentrates the knowledge distribution among relatively low types.³³

Figure 3 shows that it closely reproduces wages between the 5th and 95th percentiles, while Table 1 shows an improved fit within the reported size bins, although the flexible model generates no establishments with 1,000 or more employees. The additional parameters break the tight connection between wage and establishment-size dispersion in the baseline model.

The flexible model generates an untargeted top 1% earnings share of 12.5%, compared with 1.9% in the baseline economy. For reference, the three-year moving average of the top 1% income share reported by Auten and Splinter (2024) is 13.7%. This comparison illustrates how a technology that allows managerial knowledge to operate across organizations of very

³¹As in Section 4, a 2-layer firm $(z_1, n_1(z_1, z_2), z_2, 1)$ is denoted by (z_1, z_2) and a 1-layer firm $(z_1, 1)$ by z_1 .

³²This generalization of the span-of-control function has also been considered by Kapicka and Slavik (2021), who argue that it is necessary to jointly match the wage distribution and the fraction of managers in the data.

³³The model implies that 8.1% of agents are managers, compared with the same 10.9% benchmark for employed persons in management occupations in the 2008 CPS. Source: *Labor Force Statistics from the Current Population Survey: 2008 Annual Averages*, Table 9, the U.S. Bureau of Labor Statistics.

different sizes can generate concentrated top earnings.³⁴

Additional Layers. We next ask whether greater organizational depth can deliver similar gains in fit while retaining the baseline technology. We therefore consider the multi-layer model with $\bar{L} = 3$, retaining $\alpha = \sigma = 1$. The calibration yields $(h, \beta_1, \beta_2) = (0.63, 0.70, 0.78)$.

The multi-layer model improves the fit to both benchmarks (Figure 3 and Table 1). Middle-managers solve some of the problems generated by bottom-layer workers and pass only the remaining problems to the top manager, allowing top-manager knowledge to be applied to a larger workforce. The share of establishments with 10–99 employees rises from 1.99% in the baseline to 23.71%, while the share with 100 or more employees remains zero.

The multi-layer model generates a top 1% earnings share of only 2.5%. When the top 1% share is included among the calibration targets, the recalibrated multi-layer model still yields a share of 3.0%, leaving the three targets jointly unmatched in this exercise.³⁵ An additional layer extends the reach of top-manager knowledge but also allocates earnings to middle managers.

Taken together, the comparisons illustrate how alternative organizational technologies and layer structures translate knowledge into establishment scale and earnings differently. The flexible economy illustrates Rosen’s (1981) idea that organizational scale can magnify differences in managerial talent to generate concentrated top earnings. The multi-layer economy illustrates that organizational scale alone is not enough, in this case because its gains are shared across layers. Taken together, these results distinguish organizational depth from technological flexibility in translating organizational scale into top earnings.

6 Concluding Remarks

This paper develops a unified framework for knowledge economies in which stable outcomes and competitive equilibria are equivalent and characterized by infinite-dimensional linear programs, providing tools for theory and computation. In the canonical multi-layer economy, active firms have at most two adjacent numbers of layers in general, while our numerical

³⁴A further numerical illustration of wage polarization holds the knowledge distribution at its 2008 calibration and fits technology parameters to the 1988 benchmarks, reproducing the broad U-shaped wage change relative to the median between the two dates. Details are in Section 16 of the supplementary material.

³⁵Recalibrating the model with $\bar{L} = 4$ leaves this qualitative conclusion unchanged: the top 1% earnings share remains far below the data, and no four-layer firms are chosen.

exercises show that lower communication costs favor deeper organizations and changes in the knowledge distribution can affect organizational depth non-monotonically. Quantitatively, a flexible version of Garicano and Rossi-Hansberg’s (2004) economy closely reproduces wages between the 5th and 95th percentiles and improves the establishment-size fit by making span of control more sensitive to worker knowledge. As a numerical application of the general framework, this comparison illustrates how changing the span-of-control technology alters equilibrium wages and establishment sizes. It also improves the untargeted top 1% earnings share, whereas an additional managerial layer increases establishment-size dispersion with little effect on top earnings concentration.

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A Knowledge Economies as Large Many-to-One Matching Markets with Occupational Choice

In this section, we show how to represent a knowledge economy as a large many-to-one matching market with occupational choice, as defined in Section 2 of the Supplementary Material.

A.1 Knowledge Economy as a Matching Market

Consider a knowledge economy $E = (Z, \nu, Y, F)$ and recall that a large many-to-one matching market with occupational choice is defined by (i) a set of types, (ii) a type distribution, (iii) a set C of contracts, (iv) feasibility correspondences $P_m : Z \rightrightarrows M(Z \times C)$ and $P_s : Z \rightrightarrows \{1_{(\emptyset, c)} : c \in C\}$ for managers and self-employed, respectively, and (v) preferences $\succ_z$ for each type $z \in Z$, where $\emptyset \notin Z$ is a dummy type used to represent the match of self-employed; we let $Z_\emptyset = Z \cup \{\emptyset\}$ and endow it with a metric such that $\emptyset$ is an isolated point in $Z_\emptyset$. We use the elements of E to define a large many-to-one matching market with occupational choice, which we denote by E_m .

The set of types and the type distribution are as in E , namely Z and ν respectively. Contracts specify transfers from the manager to workers and which layer the worker belongs. Without loss of generality, the transfer can be bounded above by $\bar{t} > \bar{F}/\underline{n}$, where $\bar{F} = \max_{Y \times \{\gamma \in M(Y) : \gamma(Y) \leq \nu(Z)\}} F$. Thus, set $C = D \times T$, with $D = \{0, \dots, \bar{L}\}$ and $T = [0, \bar{t}]$ (the fictitious layer 0 is used to describe the self-employed matches). A manager of type z can choose the types, numbers and wages of workers to hire subject to the firm being feasible. Thus,

$$P_m(z) = \left\{ \sum_{i=1}^k n_i 1_{(z_i, i, t_i)} \in M(Z \times C) : (z_1, n_1, \dots, z_k, n_k, z, 1) \in Y \right\}.$$

For convenience, let $X = \cup_z P_m(z)$ and note that X is compact since Z , C and Y are compact. For the self-employed, set $P_s(z) = \{1_{(\emptyset, 0, 0)}\}$ and let $X_\emptyset = X \cup \{1_{(\emptyset, 0, 0)}\}$. Define (the compact set) $G = \text{graph}(P_m)$, $G_\emptyset = G \cup \text{graph}(P_s)$ and a function $g : G_\emptyset \rightarrow Y$ by setting $g(z, 1_{(\emptyset, 0, 0)}) = (z, 1)$ for each $z \in Z$, and, for each $(z, \sum_{i=1}^k n_i 1_{(z_i, i, t_i)}) \in G$, $g(z, \sum_{i=1}^k n_i 1_{(z_i, i, t_i)}) = (z_1, n_1, \dots, z_k, n_k, z, 1)$.

Lemma 1 *The function g is continuous.*

Proof. First note that g is continuous at $(z, 1_{(\emptyset, 0, 0)})$ since the first coordinate is the identity and $1_{(\emptyset, 0, 0)}$ is an isolated point of $X_\emptyset$.

Next, let $(z, \sum_{i=1}^k n_i 1_{(z_i, i, t_i)}) \in G$ and $\{(z_j, \sum_{i=1}^{k_j} n_{i,j} 1_{(z_{i,j}, i, t_{i,j})})\}_{j=1}^\infty$ be such that, for each $j \in \mathbb{N}$, $(z_j, \sum_{i=1}^{k_j} n_{i,j} 1_{(z_{i,j}, i, t_{i,j})}) \in G$, and $(z_j, \sum_{i=1}^{k_j} n_{i,j} 1_{(z_{i,j}, i, t_{i,j})}) \rightarrow (z, \sum_{i=1}^k n_i 1_{(z_i, i, t_i)})$. For

convenience, let $\delta_j = \sum_{i=1}^{k_j} n_{i,j} 1_{(z_{i,j}, i, t_{i,j})}$ for each $j \in \mathbb{N}$ and $\delta = \sum_{i=1}^k n_i 1_{(z_i, i, t_i)}$.

We have that $k_j = k$ for each j sufficiently large. Indeed, $O = Z \times \{k\} \times T$ is open and, hence, $0 < \delta(O) \leq \liminf_j \delta_j(O)$. Thus, $k_j \geq k$ for all j sufficiently large. If $k_j > k$ for infinitely many j , then since $H = Z \times \{k+1\} \times T$ is closed, $\delta(H) \geq \limsup_j \delta_j(H) \geq \underline{n}$, a contradiction to $\delta(H) = 0$.

Then, $(z_{i,j}, t_{i,j}, n_{i,j}) \rightarrow (z_i, t_i, n_i)$ for each $i \in \{1, \dots, k\}$. Indeed, for each $\varepsilon > 0$, $O = B_\varepsilon(z_i) \times \{i\} \times B_\varepsilon(t_i)$ is open and, hence, $\liminf_j \delta_j(O) \geq \delta(O) \geq \underline{n} > 0$. Hence, there exists $J \in \mathbb{N}$ such that $\delta_j(O) > 0$ and, thus, $(z_{i,j}, t_{i,j}) \in B_\varepsilon(z_i) \times B_\varepsilon(t_i)$ for each $j \geq J$. Since $\delta(O) = n_i$ and $\delta_j(O) = n_{i,j}$ for each $j \geq J$, it follows that $\liminf_j n_{i,j} \geq n_i$. Letting $H = \overline{B_\varepsilon(z_i)} \times \{i\} \times \overline{B_\varepsilon(t_i)}$, it follows that H is closed, $\delta(H) = n_i$ and $\delta_j(H) = n_{i,j}$ for each $j \geq J$; hence, $\limsup_j n_{i,j} = \limsup_j \delta_j(H) \leq \delta(H) = n_i$. It then follows that $n_{i,j} \rightarrow n_i$.

Hence, it follows that $g(z_j, \sum_{i=1}^{k_j} n_{i,j} 1_{(z_{i,j}, i, t_{i,j})}) \rightarrow g(z, \sum_{i=1}^k n_i 1_{(z_i, i, t_i)})$. ■

Note that for each $\mu \in M(G_\emptyset)$, $\gamma = \mu \circ g^{-1}$ is the resulting distribution of firms. We identify measures on $G_\emptyset$ with measures on $Z \times X_\emptyset$ supported on $G_\emptyset$ and use the same notation for both.

Finally, to complete the description of a market, we must specify preferences for each type of individual, which depend on the individual's occupation, match and the externality. If an individual of type z chooses to be a worker, then the match is described by the type of the hiring manager and the worker's contract, thus, by a measure $1_{(\hat{z}, c)}$ with $(\hat{z}, c) \in Z \times C$. The match of a manager of type z is $\delta \in P_m(z)$ and the match of a self-employed individual is $1_{(\emptyset, 0, 0)}$. Hence, set:

$$U_z(a, \delta, \mu) = \begin{cases} t & \text{if } a = w \text{ and } \delta = 1_{(\hat{z}, i, t)} \in M(Z \times C), \\ F(z, 1, \mu \circ g^{-1}) & \text{if } a = s, \\ F(g(z, \delta), \mu \circ g^{-1}) - \sum_{i=1}^k n_i t_i & \text{if } a = m \text{ and } \delta \in P_m(z). \end{cases}$$

Then the utility function U_z represents the preferences of an individual of type z . We identify $1_{(z, c)}$ with (z, c) and often write $U_{z'}(w, z, c, \mu)$ instead of $U_{z'}(w, 1_{(z, c)}, \mu)$ for each $(z, c) \in Z \times C$, $z' \in Z$ and $\mu \in M(G_\emptyset)$.

Note that, for each $(z, \delta) \in G$ and $\mu \in M(G_\emptyset)$, $U_z(m, \delta, \mu) = F(g(z, \delta), \mu \circ g^{-1}) - \int_T t d_T(t)$; moreover, for any $\hat{\delta} \in X$ such that $\hat{\delta}_{Z \times D} = \delta_{Z \times D}$, it follows that $(z, \hat{\delta}) \in G$, $g(z, \delta) = g(z, \hat{\delta})$ and $F(g(z, \delta), \mu \circ g^{-1}) = F(g(z, \hat{\delta}), \mu \circ g^{-1})$.

See Section 7 in the supplementary material for the definition of stable matchings of E_m , which is obtained simply by applying the definition of a stable matching for general large many-to-one matching markets with occupational choice. We then apply Theorem 3.1 in the supplementary material to conclude that E_m has a stable matching.

Lemma 2 E_m has a stable matching.

Proof. It is clear that E_m is rational since preferences are represented by a utility function. It follows from the continuity of F and g , the compactness of Z and N and the closedness of Y that E_m is continuous. We have that E_m is bounded since $\delta(Z \times C) \leq \bar{L}\bar{n}$ for each $\delta \in X$. As we next show, the lower hemicontinuity of N_L for each $2 \leq L \leq \bar{L}$ implies that E_m is rich. Hence, Theorem 3.1 applies.

We establish the richness of E_m using Lemma 3.8 in the supplementary material. We have that condition (β) holds since each $\delta \in X$ has a finite support. It is then sufficient to show condition (α) .

Let $z \in Z$, $\delta \in P_m(z)$ and V_δ be an open neighborhood of δ . Let $J \in \mathbb{N}$ and, for each $j = 1, \dots, J$, $(z_j, n_j, t_j) \in Z \times N \times T$ be such that $\delta = \sum_{j=1}^J n_j 1_{(z_j, j, t_j)}$.

Let $\hat{g} : (Z \times N \times T)^J \rightarrow M(Z \times C)$ be defined by setting, for each $(\hat{z}_1, \hat{n}_1, \hat{t}_1, \dots, \hat{z}_J, \hat{n}_J, \hat{t}_J) \in (Z \times N \times T)^J$, $\hat{g}(\hat{z}_1, \hat{n}_1, \hat{t}_1, \dots, \hat{z}_J, \hat{n}_J, \hat{t}_J) = \sum_{j=1}^J \hat{n}_j 1_{(\hat{z}_j, j, \hat{t}_j)}$. It is straightforward to see that $\hat{g}$ is an homeomorphism between $(Z \times N \times T)^J$ and $\hat{g}((Z \times N \times T)^J)$ and that $\hat{g}(z_1, n_1, t_1, \dots, z_J, n_J, t_J) = \delta$. Thus, let O be an open neighborhood of $(z_1, n_1, t_1, \dots, z_J, n_J, t_J)$ such that $\hat{g}(O) \subseteq V_\delta$.

For each $1 \leq j \leq J$, let $\tilde{V}_{z_j}$, V_{n_j} and V_{t_j} be open neighborhoods of z_j , n_j and t_j , respectively, such that $\prod_{j=1}^J (\tilde{V}_{z_j} \times V_{n_j} \times V_{t_j}) \subseteq O$. Let $O_n = \prod_{j=1}^J V_{n_j}$ and, from the lower hemicontinuity of N_{J+1} , let $\hat{V}_{z_1}, \dots, \hat{V}_{z_J}, V_z$ be open neighborhoods of $z_1, \dots, z_J, z$, respectively, such that $N_{J+1}(\hat{z}_1, \dots, \hat{z}_J, \hat{z}) \cap O_n \neq \emptyset$ for each $(\hat{z}_1, \dots, \hat{z}_J, \hat{z}) \in \hat{V}_{z_1} \times \dots \times \hat{V}_{z_J} \times V_z$.

For each $1 \leq j \leq J$, let $V_{z_j} = \tilde{V}_{z_j} \cap \hat{V}_{z_j}$ and $V_j = \{j\}$. Hence, for each $\hat{z} \in V_z$ and $(\hat{z}_j, j, \hat{t}_j) \in V_{z_j} \times V_j \times V_{t_j}$, let $(\hat{n}_1, \dots, \hat{n}_J) \in N_{J+1}(\hat{z}_1, \dots, \hat{z}_J, \hat{z}) \cap O_n$ and, hence, $(\hat{z}_1, \hat{n}_1, \dots, \hat{z}_J, \hat{n}_J, \hat{z}, 1) \in Y$ by the definition of N_{J+1} . Furthermore, $(\hat{z}_1, \hat{n}_1, \hat{t}_1, \dots, \hat{z}_J, \hat{n}_J, \hat{t}_J) \in O$, hence

$$\sum_{j=1}^J \hat{n}_j 1_{(\hat{z}_j, j, \hat{t}_j)} = \hat{g}(\hat{z}_1, \hat{n}_1, \hat{t}_1, \dots, \hat{z}_J, \hat{n}_J, \hat{t}_J) \in V_\delta.$$

This shows that condition (α) holds and completes the proof that E_m is rich. ■

B Proofs

B.1 Proof of the Measurability of the Function in the Definition of an Assignment

We show that, for each Borel subset B of Z , the function $f : Y \rightarrow \mathbb{R}$ defined by setting, for each $y \in Y$, $f(y) = \sum_{i=1}^{\ell(y)} 1_B(z_i(y)) n_i(y)$ is Borel measurable. Let A be a Borel subset of $\mathbb{R}$. Since $f^{-1}(A) = \cup_{L=1}^{\bar{L}} \{y \in Y_L : f(y) \in A\}$, it suffices to show that its restriction f_L to Y_L is

Borel measurable for each $L \in \{1, \dots, \bar{L}\}$.

Fix $L \in \{1, \dots, \bar{L}\}$, let $g_L : Y_L \times \mathbb{R}^L \rightarrow \mathbb{R}$ be defined by setting, for each $(y, x) \in Y_L \times \mathbb{R}^L$, $g_L(y, x) = \sum_{i=1}^L x_i n_i(y)$ and $k_L : Y_L \rightarrow Y_L \times \mathbb{R}^L$ be defined by setting, for each $y \in Y_L$, $k_L(y) = (y, 1_B(z_1(y)), \dots, 1_B(z_L(y)))$. Then, g_L is continuous and k_L is Borel measurable; for the latter, note that, for each $i \in \{1, \dots, L\}$, the function $y \mapsto 1_B(z_i(y))$ is the composition of the continuous function $y \mapsto z_i(y)$ and the Borel measurable function $z \mapsto 1_B(z)$. Since $f_L = g_L \circ k_L$, it then follows that f_L is Borel measurable.

B.2 Proof of Theorem 1

To establish the existence of stable outcomes, it suffices to show that stable outcomes are equivalent to stable matchings as defined in Section 7 in the supplementary material since we have already established the existence of stable matchings. To this end, we need to show that an equal treatment property holds for stable matchings.

B.2.1 Equal Treatment

Let E be a knowledge economy and μ be a stable matching of E . Let

$$M = \{z \in Z : (z, \delta) \in \text{supp}(\mu) \text{ for some } \delta \in X\}, \quad (\text{A1})$$

$$S = \{z \in Z : (z, \delta) \in \text{supp}(\mu) \text{ for some } \delta \in X_\emptyset \setminus X\} \text{ and} \quad (\text{A2})$$

$$W = \{z \in Z : z \in \text{supp}(\delta_Z) \text{ for some } (\hat{z}, \delta) \in \text{supp}(\mu)\} \quad (\text{A3})$$

denote the set of manager, self-employed and worker types, respectively. Lemma 3 states the equal treatment property.

Lemma 3 *There exists a continuous function $u : Z \rightarrow \mathbb{R}$ such that*

1. $u(z) = U_z(m, \delta, \mu)$ for each $z \in M$ and $\delta \in X$ such that $(z, \delta) \in \text{supp}(\mu)$,
2. $u(z) = U_z(s, \delta, \mu)$ for each $z \in S$ and $\delta \in X_\emptyset \setminus X$ such that $(z, \delta) \in \text{supp}(\mu)$,
3. $u(z) = U_z(w, \hat{z}, c, \mu)$ for each $z \in W$, $(\hat{z}, c) \in Z \times C$ and $\delta \in X$ such that $(\hat{z}, \delta) \in \text{supp}(\mu)$ and $(z, c) \in \text{supp}(\delta)$.

Proof. The following lemma is the equal treatment property for workers.

Lemma 4 *If $z, \hat{z}, z' \in Z$, $\delta, \hat{\delta} \in X$, $t, \hat{t} \in T$ and $d, \hat{d} \in D$ are such that $(z, \delta), (\hat{z}, \hat{\delta}) \in \text{supp}(\mu)$, $(z', d, t) \in \text{supp}(\delta)$ and $(z', \hat{d}, \hat{t}) \in \text{supp}(\hat{\delta})$, then $t = \hat{t}$.*

Proof. Suppose not; then $t \neq \hat{t}$ and we may assume that $t > \hat{t}$. Let $\varepsilon > 0$ be such that $\varepsilon < t - \hat{t}$. Since $(z', d, t) \in \text{supp}(\delta)$ and $\text{supp}(\delta)$ is finite, it follows that $\delta(z', d, t) > 0$. Since $T = [0, \bar{t}]$ and $\hat{t} < \hat{t} + \varepsilon < t$, it follows that $\hat{t} + \varepsilon \in T$.

Let $f : Z \times C \rightarrow Z \times C$ be defined by setting, for each $(\tilde{z}, \tilde{c}) \in Z \times C$,

$$f(\tilde{z}, \tilde{c}) = \begin{cases} (z', d, \hat{t} + \varepsilon) & \text{if } (\tilde{z}, \tilde{c}) = (z', d, t), \\ (\tilde{z}, \tilde{c}) & \text{otherwise,} \end{cases}$$

and $\tilde{\delta} = \delta \circ f^{-1}$. It then follows that $\delta_{Z \times D} = \tilde{\delta}_{Z \times D}$ and, hence, $F(g(z, \delta), \mu \circ g^{-1}) = F(g(z, \tilde{\delta}), \mu \circ g^{-1})$ and $\tilde{\delta} \in P_m(z)$. Moreover, since $(z', d, \hat{t} + \varepsilon) \in T_z^m(\mu)$, it follows that $\text{supp}(\tilde{\delta}) \subseteq \text{supp}(\delta) \cup T_z^m(\mu)$.

We have that

$$\int_{Z \times C} \tilde{t} d\tilde{\delta}(\tilde{z}, \tilde{d}, \tilde{t}) - \int_{Z \times C} \tilde{t} d\delta(\tilde{z}, \tilde{d}, \tilde{t}) = \delta(z', d, t)(t - \hat{t} - \varepsilon) > 0.$$

Hence,

$$\begin{aligned} U_z(m, \tilde{\delta}, \mu) &= F(g(z, \tilde{\delta}), \mu \circ g^{-1}) - \int_{Z \times C} \tilde{t} d\tilde{\delta}(\tilde{z}, \tilde{d}, \tilde{t}) \\ &> F(g(z, \delta), \mu \circ g^{-1}) - \int_{Z \times C} \tilde{t} d\delta(\tilde{z}, \tilde{d}, \tilde{t}) = U_z(m, \delta, \mu). \end{aligned}$$

But this contradicts the stability of μ . ■

Define $t : W \rightarrow T$ by setting, for each $z \in W$, $t(z) = t$, where $t \in T$ is such that $(z, d, t) \in \text{supp}(\delta)$ and $(\hat{z}, \delta) \in \text{supp}(\mu)$ for some $\hat{z} \in Z$, $\delta \in X$ and $d \in D$. Lemma 4 implies that the function t is well-defined. The function t is also bounded since T is compact.

The following lemma asserts the continuity of t and is a simple consequence of the previous lemma.

Lemma 5 *t is continuous.*

Proof. Let $z \in W$ and $\{z_k\}_{k=1}^\infty$ be such that $z_k \rightarrow z$ and $z_k \in W$ for each $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $(\hat{z}_k, \delta_k) \in Z \times X$ be such that $(\hat{z}_k, \delta_k) \in \text{supp}(\mu)$ and $z_k \in \text{supp}(\delta_{k,Z})$. Since $Z \times X$ is compact, we may assume that $\{(\hat{z}_k, \delta_k)\}_{k=1}^\infty$ converges; let $(\hat{z}, \delta) = \lim_k (\hat{z}_k, \delta_k)$.

Note that $\{t(z_k)\}_{k=1}^\infty$ converges to $t(z)$ if and only if every subsequence of $\{t(z_k)\}_{k=1}^\infty$ has a further subsequence converging to $t(z)$. Since $\{t(z_k)\}_{k=1}^\infty$ is bounded, each subsequence of $\{t(z_k)\}_{k=1}^\infty$ has a convergent subsequence; let $\{t(z_j)\}_{j=1}^\infty$ denote the latter subsequence and $t = \lim_j t(z_j)$. Let $\{(\hat{z}_j, \delta_j)\}_{j=1}^\infty$ be the corresponding subsequence of $\{(\hat{z}_k, \delta_k)\}_{k=1}^\infty$. Since $(\hat{z}_j, \delta_j) \in \text{supp}(\mu)$ for each j , $(\hat{z}_j, \delta_j) \rightarrow (\hat{z}, \delta)$ and $\text{supp}(\mu)$ is closed, it follows that

$(\hat{z}, \delta) \in \text{supp}(\mu)$. Since $\text{supp}(\delta)$ is finite, let $\varepsilon > 0$ be such that $\overline{B}_\varepsilon(z) \cap \text{supp}(\delta_Z) \subseteq \{z\}$. Then $\delta_{j,Z} \rightarrow \delta_Z$ and, since $z_j \rightarrow z$,

$$\delta_Z(\{z\}) \geq \delta_Z(\overline{B}_\varepsilon(z)) \geq \limsup_j \delta_{j,Z}(\overline{B}_\varepsilon(z)) \geq \limsup_j \delta_{j,Z}(\{z_j\}) \geq \underline{n} > 0.$$

Thus, $z \in \text{supp}(\delta_Z)$ and $t = t(z)$ by Lemma 4. ■

Lemma 6 *If $(z, \delta) \in \text{supp}(\mu)$, then $t < \bar{t}$ for each $t \in \text{supp}(\delta_T)$.*

Proof. Let $(z, \delta) \in \text{supp}(\mu) \cap G$ and $t \in \text{supp}(\delta_T)$ (in case $(z, \delta) \in \text{supp}(\mu) \cap (G_\emptyset \setminus G)$, $\text{supp}(\delta_T) = \{0\}$). It follows by Lemma 4 that $\delta_T(\{t\}) = \sum_{z' \in \text{supp}(\delta_Z): t(z')=t} \delta_Z(\{z'\}) \geq \underline{n}$. Then

$$F(g(z, \delta), \mu \circ g^{-1}) - \underline{n}t \geq U_z(m, \delta, \mu) \geq U_z(s, 1_{(\emptyset, 0, 0)}, \mu) \geq 0$$

and, hence, $t \leq \frac{F(g(z, \delta), \mu \circ g^{-1})}{\underline{n}} < \bar{t}$. ■

The following is the equal treatment property for managers.

Lemma 7 *If $z \in Z$ and $\delta, \delta' \in X$ are such that $(z, \delta), (z, \delta') \in \text{supp}(\mu)$, then $U_z(m, \delta, \mu) = U_z(m, \delta', \mu)$.*

Proof. Suppose not; then $U_z(m, \delta, \mu) \neq U_z(m, \delta', \mu)$ and we may assume that $U_z(m, \delta, \mu) > U_z(m, \delta', \mu)$. Let $\varepsilon > 0$ be such that $U_z(m, \delta, \mu) - \varepsilon\delta(Z \times C) > U_z(m, \delta', \mu)$ and $\tilde{t} + \varepsilon \in T$ for each $\tilde{t} \in \text{supp}(\delta_T)$, the latter by Lemma 6.

Let $f : Z \times C \rightarrow Z \times D \times \mathbb{R}$ be defined by setting, for each $(\tilde{z}, \tilde{d}, \tilde{t}) \in Z \times C$, $f(\tilde{z}, \tilde{d}, \tilde{t}) = (\tilde{z}, \tilde{d}, \tilde{t} + \varepsilon)$, and $\hat{\delta} = \delta \circ f^{-1}$. We then have that $(\tilde{z}, \tilde{d}, \tilde{t} + \varepsilon) \in T_z^m(\mu)$ for each $(\tilde{z}, \tilde{d}, \tilde{t}) \in \text{supp}(\delta)$ and $\hat{\delta}_{Z \times D} = \delta_{Z \times D}$. Thus, $\hat{\delta} \in P_m(z)$ and

$$\begin{aligned} U_z(m, \hat{\delta}, \mu) &= F(g(z, \hat{\delta}), \mu \circ g^{-1}) - \int_{Z \times C} \tilde{t} d\hat{\delta}(\tilde{z}, \tilde{d}, \tilde{t}) = \\ &= F(g(z, \delta), \mu \circ g^{-1}) - \int_{Z \times C} (\tilde{t} + \varepsilon) d\delta(\tilde{z}, \tilde{d}, \tilde{t}) = U_z(m, \delta, \mu) - \varepsilon\delta(Z \times C) > U_z(m, \delta', \mu), \end{aligned}$$

contradicting the stability of μ . ■

Define $u_m : M \rightarrow \mathbb{R}$ by setting, for each $z \in M$, $u_m(z) = U_z(m, \delta, \mu)$, where $\delta \in X$ is such that $(z, \delta) \in \text{supp}(\mu)$. Lemma 7 implies that the function u_m is well-defined. The following lemma shows that the function $u_m : M \rightarrow \mathbb{R}$ is continuous.

Lemma 8 *u_m is continuous.*

Proof. Let $z \in M$ and $\{z_k\}_{k=1}^\infty$ be such that $z_k \rightarrow z$ and $z_k \in M$ for each $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $\delta_k \in X$ be such that $(z_k, \delta_k) \in \text{supp}(\mu)$. Since $\delta_k \in X$ for each k and X is compact, we may assume that $\{\delta_k\}_{k=1}^\infty$ converges; let $\delta = \lim_k \delta_k$.

Since $\text{supp}(\mu)$ is closed, it follows that $(z, \delta) \in \text{supp}(\mu)$ and, by Lemma 7, $u_m(z) = U_z(m, \delta, \mu)$. Hence, the continuity of $(\tilde{z}, \tilde{\delta}) \mapsto U_{\tilde{z}}(m, \tilde{\delta}, \mu)$ implies that

$$u_m(z_k) = U_{z_k}(m, \delta_k, \mu) \rightarrow U_z(m, \delta, \mu) = u_m(z).$$

■

The following is the equal treatment property for self-employed individuals; it follows immediately from $X_\emptyset \setminus X = \{1_{(\emptyset, 0, 0)}\}$ being singleton.

Lemma 9 *If $z \in Z$ and $\delta, \delta' \in X_\emptyset \setminus X$ are such that $(z, \delta), (z, \delta') \in \text{supp}(\mu)$, then $U_z(s, \delta, \mu) = U_z(s, \delta', \mu)$.*

Define $u_s : S \rightarrow \mathbb{R}_+$ by setting, for each $z \in S$, $u_s(z) = U_z(s, \delta, \mu)$, where $\delta \in X_\emptyset \setminus X$ is such that $(z, \delta) \in \text{supp}(\mu)$. Lemma 9 implies that the function u_s is well-defined; an analogous argument to the one in Lemma 8 implies that $u_s : S \rightarrow \mathbb{R}$ is continuous.

Lemma 10 *$u_s : S \rightarrow \mathbb{R}$ is continuous.*

The following lemma shows that if, for some type z , there are type z individuals with different occupations, then the payoffs from such occupations are the same at z .

Lemma 11 *The following holds: (i) $u_m(z) = t(z)$ if $z \in M \cap W$, (ii) $u_m(z) = u_s(z)$ if $z \in M \cap S$ and (iii) $t(z) = u_s(z)$ if $z \in W \cap S$.*

Proof. The stability of μ implies that $u_m(z) \geq u_s(z)$ if $z \in M \cap S$ and $t(z) \geq u_s(z)$ if $z \in W \cap S$.

Suppose that (a) $z \in M \cap W$ and $u_m(z) > t(z)$ or (b) $z \in M \cap S$ and $u_m(z) > u_s(z)$. Let $\alpha = t(z)$ in case (a), $\alpha = u_s(z)$ in case (b) and $\delta \in X$ be such that $(z, \delta) \in \text{supp}(\mu)$. Then let $\varepsilon > 0$ be such that $U_z(m, \delta, \mu) - \varepsilon \delta(Z \times C) > \alpha$ and $\tilde{t} + \varepsilon \in T$ for each $\tilde{t} \in \text{supp}(\delta_T)$, the latter by Lemma 6.

Let $f : Z \times C \rightarrow Z \times D \times \mathbb{R}$ be defined by setting, for each $(\tilde{z}, \tilde{d}, \tilde{t}) \in Z \times C$, $f(\tilde{z}, \tilde{d}, \tilde{t}) = (\tilde{z}, \tilde{d}, \tilde{t} + \varepsilon)$, and $\hat{\delta} = \delta \circ f^{-1}$. We then have that $(\tilde{z}, \tilde{d}, \tilde{t} + \varepsilon) \in T_z^m(\mu)$ for each $(\tilde{z}, \tilde{d}, \tilde{t}) \in \text{supp}(\delta)$ and $\hat{\delta}_{Z \times D} = \delta_{Z \times D}$. Thus, $\hat{\delta} \in P_m(z)$ and

$$\begin{aligned} U_z(m, \hat{\delta}, \mu) &= F(g(z, \hat{\delta}), \mu \circ g^{-1}) - \int_{Z \times C} \tilde{t} d\hat{\delta}(\tilde{z}, \tilde{d}, \tilde{t}) = \\ &= F(g(z, \delta), \mu \circ g^{-1}) - \int_{Z \times C} (\tilde{t} + \varepsilon) d\delta(\tilde{z}, \tilde{d}, \tilde{t}) = U_z(m, \delta, \mu) - \varepsilon \delta(Z \times C) > \alpha, \end{aligned}$$

contradicting the stability of μ .

It then follows that $u_m(z) \leq t(z)$ if $z \in M \cap W$ and $u_m(z) \leq u_s(z)$ if $z \in M \cap S$; in particular, (ii) follows.

Suppose that (a) $z \in M \cap W$ and $t(z) > u_m(z)$ or (b) $z \in W \cap S$ and $t(z) > u_s(z)$. Let $\alpha = u_m(z)$ in case (a), $\alpha = u_s(z)$ in case (b), $(\hat{z}, \delta) \in \text{supp}(\mu)$ such that $z \in \text{supp}(\delta_Z)$ and $\varepsilon > 0$ be such that $\varepsilon < t(z) - \alpha$. We have that $(z, \tilde{d}, \alpha + \varepsilon) \in T_{\hat{z}}^m(\mu)$ for each $\tilde{d} \in D$ and that $\sum_{d \in \pi_D(\text{supp}(\delta))} \delta(z, d, t(z)) = \delta_Z(z) \geq \underline{n}$. Let $f : Z \times C \rightarrow Z \times D \times \mathbb{R}$ be defined by setting, for each $(\tilde{z}, \tilde{d}, \tilde{t}) \in Z \times C$,

$$f(\tilde{z}, \tilde{d}, \tilde{t}) = \begin{cases} (z, \tilde{d}, \alpha + \varepsilon) & \text{if } \tilde{z} = z, \\ (\tilde{z}, \tilde{d}, \tilde{t}) & \text{otherwise,} \end{cases}$$

and $\hat{\delta} = \delta \circ f^{-1}$. Then $\hat{\delta}_{Z \times D} = \delta_{Z \times D}$, $\text{supp}(\hat{\delta}) \subseteq \text{supp}(\delta) \cup T_{\hat{z}}^m(\mu)$, $\hat{\delta} \in P_m(\hat{z})$ and

$$\begin{aligned} U_{\hat{z}}(m, \hat{\delta}, \mu) &= F(g(\hat{z}, \hat{\delta}), \mu \circ g^{-1}) - \int_{Z \times C} \tilde{t} d\hat{\delta}(\tilde{z}, \tilde{d}, \tilde{t}) = \\ &= F(g(\hat{z}, \delta), \mu \circ g^{-1}) - \int_{Z \times C} \tilde{t} d\delta(\tilde{z}, \tilde{d}, \tilde{t}) + (t(z) - \alpha - \varepsilon)\delta_Z(z) > U_{\hat{z}}(m, \delta, \mu), \end{aligned}$$

contradicting the stability of μ . Thus, $t(z) \leq u_m(z)$ if $z \in M \cap W$ and $t(z) \leq u_s(z)$ if $z \in W \cap S$. Hence, (i) and (iii) follow. ■

The next step is to combine t , u_m and u_s . Before that, we establish that the sets W , M and S are closed, a property that is needed for the argument.

Lemma 12 *W is closed.*

Proof. Let $\{z_k\}_{k=1}^\infty$ be a convergent sequence in W and $z = \lim_k z_k$. For each k , let $(\hat{z}_k, \delta_k) \in Z \times X$ be such that $(\hat{z}_k, \delta_k) \in \text{supp}(\mu)$ and $z_k \in \text{supp}(\delta_{k,Z})$. Since $Z \times X$ is compact, we may assume that $\{(\hat{z}_k, \delta_k)\}_{k=1}^\infty$ converges; let $(\hat{z}, \delta) = \lim_k (\hat{z}_k, \delta_k)$. Since $\text{supp}(\mu)$ is closed, it follows that $(\hat{z}, \delta) \in \text{supp}(\mu)$.

Since $\text{supp}(\delta)$ is finite, let $\varepsilon > 0$ be such that $\overline{B}_\varepsilon(z) \cap \text{supp}(\delta_Z) \subseteq \{z\}$. Then $\delta_{k,Z} \rightarrow \delta_Z$ and

$$\delta_Z(\{z\}) \geq \delta_Z(\overline{B}_\varepsilon(z)) \geq \limsup_k \delta_{k,Z}(\overline{B}_\varepsilon(z)) \geq \limsup_k \delta_{k,Z}(\{z_k\}) \geq \underline{n} > 0.$$

Hence, $z \in \text{supp}(\delta_Z)$ and $z \in W$. ■

Lemma 13 *M is closed.*

Proof. Let $\{z_k\}_{k=1}^\infty$ be a convergent sequence in M and $z = \lim_k z_k$. For each k , let $\delta_k \in X$ be such that $(z_k, \delta_k) \in \text{supp}(\mu)$. Since X is compact, we may assume that $\{\delta_k\}_{k=1}^\infty$

converges; let $\delta = \lim_k \delta_k$. Since $\text{supp}(\mu)$ is closed, it follows that $(z, \delta) \in \text{supp}(\mu)$ and, thus, $z \in M$. ■

Lemma 14 *S is closed.*

Proof. Let $\{z_k\}_{k=1}^\infty$ be a convergent sequence in S and $z = \lim_k z_k$. For each k , $(z_k, 1_{(\emptyset, 0, 0)}) \in \text{supp}(\mu)$ and $(z_k, 1_{(\emptyset, 0, 0)}) \rightarrow (z, 1_{(\emptyset, 0, 0)})$. Since $\text{supp}(\mu)$ is closed, it follows that $(z, 1_{(\emptyset, 0, 0)}) \in \text{supp}(\mu)$ and, thus, $z \in S$. ■

The following lemma shows that M , S and W cover Z .

Lemma 15 $Z = M \cup S \cup W$.

Proof. Let $K = M \cup S \cup W$ and note that we have that $K \subseteq Z$ by definition.

Conversely, note first that K is closed by Lemmas 12, 13 and 14. Moreover, we have that $\text{supp}(\mu) \subseteq \pi_Z(\text{supp}(\mu)) \times X_\emptyset = (M \cup S) \times X_\emptyset \subseteq K \times X_\emptyset$ and, hence,

$$\mu(K \times X_\emptyset) \geq \mu(\text{supp}(\mu)) = \mu(\text{supp}(\mu) \cap (Z \times X_\emptyset)) = \mu(Z \times X_\emptyset).$$

Furthermore, for each $(z, \delta) \in \text{supp}(\mu) \cap (Z \times X)$,

$$\delta(\text{supp}(\delta_Z) \times C) = \delta_Z(\text{supp}(\delta_Z)) = \delta_Z(Z) = \delta(Z \times C).$$

Since $\text{supp}(\delta_Z) \subseteq W$, it follows that

$$\delta(K \times C) \leq \delta(Z \times C) = \delta(\text{supp}(\delta_Z) \times C) \leq \delta(W \times C) \leq \delta(K \times C).$$

Hence,

$$\begin{aligned} \nu(K) &= \mu(K \times X) + \mu(K \times (X_\emptyset \setminus X)) + \int_{Z \times X} \delta(K \times C) d\mu(z, \delta) \\ &= \mu(K \times X_\emptyset) + \int_{(Z \times X) \cap \text{supp}(\mu)} \delta(K \times C) d\mu(z, \delta) \\ &\geq \mu(Z \times X_\emptyset) + \int_{(Z \times X) \cap \text{supp}(\mu)} \delta(Z \times C) d\mu(z, \delta) = \nu(Z). \end{aligned}$$

It then follows by the definition of $\text{supp}(\nu)$ that $Z = \text{supp}(\nu) \subseteq K$. ■

To complete the proof of Lemma 3, define $u : Z \rightarrow \mathbb{R}$ by setting, for each $z \in Z$,

$$u(z) = \begin{cases} u_m(z) & \text{if } z \in M, \\ t(z) & \text{if } z \in W, \\ u_s(z) & \text{if } z \in S. \end{cases}$$

It follows by Lemma 11 that u is well-defined. It follows by Lemmas 4, 7 and 9 that conditions 1–3 in the statement of the lemma hold. We have that u is continuous since M , W and S are closed (by Lemmas 12, 13 and 14), $Z = M \cup W \cup S$ (by Lemma 15) and $u|_M$, $u|_W$ and $u|_S$ are continuous. Indeed, let K be a closed subset of $\mathbb{R}$ and, for each $A \in \{M, S, W\}$, let K_A be a closed subset of Z such that $A \cap K_A = u|_A^{-1}(K)$. Then $u^{-1}(K) = (M \cap K_M) \cup (S \cap K_S) \cup (W \cap K_W)$ is a closed subset of Z . ■

B.2.2 Equivalence between Stable Outcomes and Stable Matchings

Recall that $g : G_\emptyset \rightarrow Y$ is defined by setting $g(z, 1_{(\emptyset, 0, 0)}) = (z, 1)$ for each $z \in Z$, and, for each $(z, \sum_{i=1}^k n_i 1_{(z_i, i, t_i)}) \in G$, $g(z, \sum_{i=1}^k n_i 1_{(z_i, i, t_i)}) = (z_1, n_1, \dots, z_k, n_k, z, 1)$.

In addition, for each earnings function u with $u(Z) \subseteq T$, define $h_u : Y \rightarrow G_\emptyset$ by setting, for each $y = (z_i, n_i)_{i=1}^L \in Y$ with $L \geq 2$, $h_u(y) = (z_L, \sum_{i=1}^{L-1} n_i 1_{(z_i, i, u(z_i))})$, and for each $y = (z, 1) \in Y$, $h_u(y) = (z, 1_{(\emptyset, 0, 0)})$.

Theorem 4 *If μ is a stable matching of E and u is from Lemma 3, then $(\mu \circ g^{-1}, u)$ is a stable outcome. If (γ, v) is a stable outcome of E , then $\gamma \circ h_v^{-1}$ is a stable matching.*

Proof. First, we establish that g and h_v are homeomorphisms.

Lemma 16 *If μ is a stable matching, then g is a homeomorphism between $\text{supp}(\mu)$ and $g(\text{supp}(\mu))$; if (γ, v) is a stable outcome, then h_v is a homeomorphism between $\text{supp}(\gamma)$ and $h_v(\text{supp}(\gamma))$.*

Proof. Let μ be a stable matching and u its earnings function from Lemma 3.

First, g is 1-1 since $g(z, \sum_{i=1}^k n_i 1_{(z_i, i, t_i)}) = g(\hat{z}, \sum_{i=1}^{\hat{k}} \hat{n}_i 1_{(\hat{z}_i, i, \hat{t}_i)})$ implies that $z = \hat{z}$, $k = \hat{k}$, and $(n_i, z_i) = (\hat{n}_i, \hat{z}_i)$ for each $i \in \{1, \dots, k\}$; then $t_i = u(z_i) = u(\hat{z}_i) = \hat{t}_i$ for each $i \in \{1, \dots, k\}$ (since $(z, \sum_{i=1}^k n_i 1_{(z_i, i, t_i)}), (\hat{z}, \sum_{i=1}^{\hat{k}} \hat{n}_i 1_{(\hat{z}_i, i, \hat{t}_i)}) \in \text{supp}(\mu)$) and hence $\sum_{i=1}^k n_i 1_{(z_i, i, t_i)} = \sum_{i=1}^{\hat{k}} \hat{n}_i 1_{(\hat{z}_i, i, \hat{t}_i)}$. In case $g(z, \delta) = g(z', \delta') = (z, 1)$, then $z' = z$ and $\delta' = \delta = 1_{(\emptyset, 0, 0)}$.

The continuity of g follows from Lemma 1. Since $g : \text{supp}(\mu) \rightarrow g(\text{supp}(\mu))$ is 1-1 and continuous, and $\text{supp}(\mu)$ is compact, then g^{-1} is continuous.

Let (γ, v) be stable. Let $y, y' \in \text{supp}(\gamma)$ be such that $h_v(y) = h_v(y')$ and let $(z, \delta) \in G_\emptyset$ be such that $h_v(y) = (z, \delta)$. If $\delta = 1_{(\emptyset, 0, 0)}$, then $y = (z, 1) = y'$. If $\delta \in X$, then, letting $y = (z_i, n_i)_{i=1}^{\ell(y)}$ and $y' = (z'_i, n'_i)_{i=1}^{\ell(y')}$, $\ell(y) = \ell(y') = |\text{supp}(\delta)| + 1$, $z_{\ell(y)} = z'_{\ell(y)}$ and, for each $1 \leq i < \ell(y)$, $n_i = n'_i = \delta_D(\{i\})$ and $z_i = z'_i$ are the unique element of $\text{supp}(\delta_i)$, where $\delta_i \in M(Z)$ is defined by setting, for each $z \in \text{supp}(\delta_Z)$, $\delta_i(z) = \sum_{t \in \text{supp}(\delta_T)} \delta(\{(z, i, t)\})$. Thus, $y = y'$.

The continuity of $h_v : Y \rightarrow G_\emptyset$ is as follows. Let $y \in Y$ and $\{y_k\}_{k=1}^\infty$ be such that $y_k \in Y$ for each $k \in \mathbb{N}$ and $y_k \rightarrow y$; we may assume that $\ell(y_k) = \ell(y)$. Let $L = \ell(y)$, $y = (z_i, n_i)_{i=1}^L$ and $y_k = (z_{k,i}, n_{k,i})_{i=1}^L$ for each $k \in \mathbb{N}$. If $L = 1$, then $h_v(y_k) = (z_{k,1}, 1_{(\emptyset,0,0)}) \rightarrow (z_1, 1_{(\emptyset,0,0)}) = h_v(y)$. If $L > 1$, then $h_v(y_k) = (z_{k,L}, \sum_{i=1}^{L-1} n_{k,i} 1_{(z_{k,i}, i, v(z_{k,i}))}) \rightarrow (z_L, \sum_{i=1}^{L-1} n_i 1_{(z_i, i, v(z_i))}) = h_v(y)$ using the continuity of v . Thus, h_v is continuous.

Since $h_v : \text{supp}(\gamma) \rightarrow h_v(\text{supp}(\gamma))$ is 1-1 and continuous, and $\text{supp}(\gamma)$ is compact, then h_v^{-1} is continuous. ■

Let μ be a stable matching and u be its earnings function from Lemma 3. We will show that $(\mu \circ g^{-1}, u)$ is a stable outcome.

Let $\gamma = \mu \circ g^{-1}$. We first show that γ is an assignment. Let B be a Borel subset of Z ; then

$$\begin{aligned} & \int_Y \sum_{i=1}^{\ell(y)} 1_B(z_i(y)) n_i(y) d\gamma(y) \\ &= \int_Y 1_B(z_{\ell(y)}(y)) d\mu \circ g^{-1}(y) + \int_{Y \setminus Y_1} \sum_{i=1}^{\ell(y)-1} 1_B(z_i(y)) n_i(y) d\mu \circ g^{-1}(y) \\ &= \int_{G_\emptyset} 1_B(z_{\ell(g(z,\delta))}(g(z,\delta))) d\mu(z,\delta) + \int_G \sum_{i=1}^{\ell(g(z,\delta))-1} 1_B(z_i(g(z,\delta))) n_i(g(z,\delta)) d\mu(z,\delta) \\ &= \mu(B \times X_\emptyset) + \int_{Z \times X} \delta(B \times C) d\mu(z,\delta) = \nu(B) \end{aligned}$$

since $g^{-1}(Y \setminus Y_1) = G$, $1_B(z_{\ell(g(z,\delta))}(g(z,\delta))) = 1_B(z)$ for each $(z,\delta) \in G_\emptyset$ and, for each $(z,\delta) \in G$, $\sum_{i=1}^{\ell(g(z,\delta))-1} 1_B(z_i(g(z,\delta))) n_i(g(z,\delta)) = \delta(B \times C)$. To complete the argument that γ is an assignment, note that $\text{supp}(\mu) \subseteq \text{graph}(P_m) \cup \text{graph}(P_s)$ implies $\text{supp}(\gamma) \subseteq Y$.

Let $(z_i, n_i)_{i=1}^L \in \text{supp}(\gamma)$ with $L \geq 2$. Since $\text{supp}(\mu) = g^{-1}(\text{supp}(\gamma))$ by Carmona and Laohakunakorn (2024, Lemma 1), it follows that $(z_L, \sum_{i=1}^{L-1} n_i 1_{(z_i, i, u(z_i))}) \in \text{supp}(\mu)$. Then

$$u(z_L) + \sum_{i=1}^{L-1} n_i u(z_i) = U_{z_L}(m, \sum_{i=1}^{L-1} n_i 1_{(z_i, i, u(z_i))}, \mu) + \sum_{i=1}^{L-1} n_i u(z_i) = F((z_i, n_i)_{i=1}^L, \mu \circ g^{-1}).$$

If $(z, 1) \in \text{supp}(\gamma)$, then $(z, 1_{(\emptyset,0,0)}) \in \text{supp}(\mu)$ and $u(z) = U_z(s, 1_{(\emptyset,0,0)}, \mu) = F(z, 1, \mu \circ g^{-1})$. Thus, condition (i) of a stable outcome is satisfied.

We now show condition (ii). Let $y = (z_i, n_i)_{i=1}^L \in Y$. If $L = 1$, then $u(z_1) \geq F(z_1, 1, \mu \circ g^{-1})$ since μ is stable. If $L \geq 2$ and $u(z_L) + \sum_{i=1}^{L-1} n_i u(z_i) < F(y, \mu \circ g^{-1})$, then letting $\{\varepsilon_i\}_{i=1}^{L-1} > 0$ be such that $F(y, \mu \circ g^{-1}) - \sum_{i=1}^{L-1} n_i (u(z_i) + \varepsilon_i) > u(z_L)$ and $\varepsilon_i < \bar{t} - u(z_i)$ for each i , it follows that $\text{supp}(\sum_{i=1}^{L-1} n_i 1_{(z_i, i, u(z_i) + \varepsilon_i)}) \subseteq T_{z_L}^m(\mu)$ and $U_{z_L}(m, \sum_{i=1}^{L-1} n_i 1_{(z_i, i, u(z_i) + \varepsilon_i)}, \mu) > u(z_L)$, a contradiction to the stability of μ . Thus, condition (ii) is satisfied.

Let (γ, v) be stable. We will show that $\mu = \gamma \circ h_v^{-1}$ is a stable matching. Since $g \circ h_v = \text{id}_Y$, we have $\mu \circ g^{-1} = \gamma$.

We first show that μ is a matching. Let B be a Borel subset of Z ; then

$$\begin{aligned} \mu(B \times X_\emptyset) + \int_{Z \times X} \delta(B \times C) d\mu(z, \delta) &= \\ \gamma \circ h_v^{-1}(B \times X_\emptyset) + \int_{Z \times X} \delta(B \times C) d\gamma \circ h_v^{-1}(z, \delta) &= \\ \int_Y 1_B(z_{\ell(y)}) d\gamma(y) + \int_Y \sum_{i=1}^{\ell(y)-1} 1_B(z_i(y)) n_i(y) d\gamma(y) &= \\ \int_Y \sum_{i=1}^{\ell(y)} 1_B(z_i(y)) n_i(y) d\gamma(y) &= \nu(B). \end{aligned}$$

Furthermore, if $(z, \delta) \in \text{supp}(\mu)$ and $h_v^{-1}(z, \delta) = y$, then $y \in \text{supp}(\gamma)$ since $h_v^{-1}(\text{supp}(\mu)) = \text{supp}(\gamma)$ by Carmona and Laohakunakorn (2024, Lemma 1). Hence, $\text{supp}(\mu) \subseteq \text{graph}(P_m) \cup \text{graph}(P_s)$.

We next show that v is the earnings function of μ . Let $z \in W$. Then $(\hat{z}, \delta) \in \text{supp}(\mu)$ for some $\hat{z} \in Z$, $\delta \in X$ such that $(z, d, t) \in \text{supp}(\delta)$ for some $(d, t) \in C$. Furthermore, $(\hat{z}, \delta) \in h_v(\text{supp}(\gamma))$, and hence $t = v(z)$. Thus, $U_z(w, \hat{z}, (d, t), \mu) = t = v(z)$. Let $z_L \in M$. Then $(z_L, \sum_{i=1}^{L-1} n_i 1_{(z_i, i, v(z_i))}) \in \text{supp}(\mu)$, where $y = (z_i, n_i)_{i=1}^L \in \text{supp}(\gamma)$. Thus, by the stability of (γ, v) , $U_{z_L}(m, \sum_{i=1}^{L-1} n_i 1_{(z_i, i, v(z_i))}, \mu) = F(y, \mu \circ g^{-1}) - \sum_{i=1}^{L-1} n_i v(z_i) = v(z_L)$. Finally, if $z \in S$, then $(z, 1) \in \text{supp}(\gamma)$, and $U_z(s, 1_{(\emptyset, 0, 0)}, \mu) = F((z, 1), \mu \circ g^{-1}) = v(z)$ by the stability of (γ, v) . It follows that v is the earnings function of μ .

We conclude this proof by showing that μ is stable. Let $(z, \delta) \in \text{supp}(\mu)$; then $U_z(m, \delta, \mu) = v(z)$ and, for each $(z', c) \in \text{supp}(\delta)$, $U_{z'}(w, z, c, \mu) = v(z')$ if $\delta \in X$ and $U_z(s, \delta, \mu) = v(z)$ if $\delta \in X_\emptyset \setminus X$. It is clear that $(z, \delta) \in IR(\mu)$ since $v(\hat{z}) \geq F(\hat{z}, 1, \mu \circ g^{-1})$ for each $\hat{z} \in Z$ by the stability of (γ, v) . To see that $(z, \delta) \in S_M(\mu)$, let $\tilde{z} \in \{z\} \cup \text{supp}(\delta_Z)$ if $\delta \in X$ and $\tilde{z} = z$ if $\delta \in X_\emptyset \setminus X$. Note that $\delta' = \sum_{i=1}^k n_i 1_{(z_i, i, t_i)} \in X$ and $\text{supp}(\delta') \subseteq T_{\tilde{z}}^m(\mu) \cup \text{supp}(\delta)$ implies that $\sum_{i=1}^k n_i t_i \geq \sum_{i=1}^k n_i v(z_i)$. Thus, $U_{\tilde{z}}(m, \delta', \mu) \leq F((z_i, n_i)_{i=1}^k, \tilde{z}, 1, \mu \circ g^{-1}) - \sum_{i=1}^k n_i v(z_i) \leq v(\tilde{z})$ and, hence, $(z, \delta) \in S_M(\mu)$. ■

B.3 Proof of Theorem 2

First, note the following implication of feasibility: for every $(\tau, v) \in \Gamma \times C(Z)$:

$$\int_Z v d\nu = \int_Y \sum_{i=1}^{\ell(y)} n_i(y) v(z_i(y)) d\tau(y).$$

Indeed, $\nu = \nu_\tau$ and v is the uniform limit of simple functions; if $f : Z \rightarrow \mathbb{R}$ is simple, i.e. $f = \sum_{j=1}^J a_j 1_{A_j}$ for some $J \in \mathbb{N}$ and, for each $1 \leq j \leq J$, $a_j \in \mathbb{R}$ and A_j a Borel subset of Z , then

$$\begin{aligned} \int_Z f d\nu_\tau &= \sum_{j=1}^J a_j \int_Y \sum_{i=1}^{\ell(y)} n_i(y) 1_{A_j}(z_i(y)) d\tau(y) \\ &= \int_Y \sum_{i=1}^{\ell(y)} n_i(y) \sum_{j=1}^J a_j 1_{A_j}(z_i(y)) d\tau(y) \\ &= \int_Y \sum_{i=1}^{\ell(y)} n_i(y) f(z_i(y)) d\tau(y). \end{aligned}$$

Consider an economy with payoff function F^γ (and no externalities) and let $(\hat{\gamma}, \hat{u})$ be a stable outcome of this economy. Then

$$\int_Z \hat{u} d\nu = \int_Y F^\gamma d\hat{\gamma} \leq \sup_{\tau \in \Gamma} \int_Y F^\gamma d\tau \leq \inf_{v \in U^\gamma} \int_Z v d\nu \leq \int_Z \hat{u} d\nu,$$

where the first equality follows from condition (i) of stable outcomes, the first inequality follows by definition of the supremum, the next inequality follows because for each $(\tau, v) \in \Gamma \times U^\gamma$, $\int_Z v d\nu = \int_Y \sum_{i=1}^{\ell(y)} n_i(y) v(z_i(y)) d\tau(y) \geq \int_Y F^\gamma(y) d\tau(y)$, and the final inequality follows by definition of the infimum. Hence, all the inequalities are in fact equalities and the supremum and infimum are attained by $\hat{\gamma}$ and $\hat{u}$ respectively. This proves the first statement.

Let (γ, u) be a stable outcome of the economy with externalities. Then $\int_Z u d\nu = \int_Y F^\gamma d\gamma \leq \max_{\tau \in \Gamma} \int_Y F^\gamma d\tau \leq \min_{v \in U^\gamma} \int_Z v d\nu \leq \int_Z u d\nu$. Thus, γ solves $\max_{\tau \in \Gamma} \int_Y F^\gamma d\tau$ and u solves $\min_{v \in U^\gamma} \int_Z v d\nu$.

Finally, suppose γ solves $\max_{\tau \in \Gamma} \int_Y F^\gamma d\tau$ and u solves $\min_{v \in U^\gamma} \int_Z v d\nu$. By the first part of the theorem, $\int_Z u d\nu = \int_Y F^\gamma d\gamma$, which implies that condition (i) of stable outcomes holds. Condition (ii) holds since $u \in U^\gamma$; hence (γ, u) is stable.

B.4 Proof of Theorem 3

Necessity. Let $\zeta = \sum_{i=1}^k \zeta_i 1_{y_i}$ and $\tau = \sum_{i=1}^l \tau_i 1_{x_i}$ be such that $\{y_1, \dots, y_k\} = \text{supp}(\zeta) \subseteq \text{supp}(\gamma)$ and $\nu_\zeta = \nu_\tau$. By Theorem 2, for each $i \in \{1, \dots, k\}$, $F^\gamma(y_i) = \sum_{j=1}^{\ell(y_i)} n_j(y_i) u(z_j(y_i))$, where u solves $\min_{v \in U^\gamma} \int_Z v d\nu$. Thus,

$$\sum_{i=1}^k \zeta_i F^\gamma(y_i) = \underbrace{\sum_{i=1}^k \zeta_i \sum_{j=1}^{\ell(y_i)} n_j(y_i) u(z_j(y_i))}_{\int_Z u d\nu_\zeta} = \underbrace{\sum_{i=1}^l \tau_i \sum_{j=1}^{\ell(x_i)} n_j(x_i) u(z_j(x_i))}_{\int_Z u d\nu_\tau} \geq \sum_{i=1}^l \tau_i F^\gamma(x_i),$$

where the first equality follows from Theorem 2, the second follows from $\nu_\zeta = \nu_\tau$ and the inequality follows from $u \in U^\gamma$.

Sufficiency. This follows from the following claim:

Claim 1 *For any $\gamma \in \Gamma$, if $\text{supp}(\gamma)$ is F^γ -monotone, there exists $u \in U^\gamma$ such that:*

$$\text{supp}(\gamma) \subseteq \left\{ y \in Y : \sum_{j=1}^{\ell(y)} n_j(y) u(z_j(y)) = F^\gamma(y) \right\}.$$

Proof. Fix $\gamma \in \Gamma$ and suppose that $S = \text{supp}(\gamma)$ is F^γ -monotone. Let $M = \max_Y F^\gamma$ and $B = M/\underline{n}$. Consider the system

$$\begin{aligned} \sum_{i=1}^{\ell(y)} n_i(y) u(z_i(y)) &\geq F^\gamma(y) \text{ for each } y \in Y \\ \sum_{i=1}^{\ell(y)} n_i(y) u(z_i(y)) &= F^\gamma(y) \text{ for each } y \in S. \end{aligned}$$

Choose finite sets $I \subseteq Y$ and $J \subseteq S$ of inequality and equality constraints, respectively. Let

$$T = \bigcup_{y \in I \cup J} \{z_i(y) : 1 \leq i \leq \ell(y)\}$$

and add $(z, 1)$ to I for every $z \in T$. For $y \in I \cup J$, define its population vector $a_y \in \mathbb{R}^T$ by

$$a_y(z) = \sum_{i=1}^{\ell(y)} n_i(y) \mathbf{1}_{\{z_i(y)=z\}}.$$

The finite system, with unrestricted variables $v \in \mathbb{R}^T$, is

$$\begin{aligned} a_y \cdot v &\geq F^\gamma(y) && \text{for each } y \in I \\ a_s \cdot v &\geq F^\gamma(s) && \text{for each } s \in J \\ -a_s \cdot v &\geq -F^\gamma(s) && \text{for each } s \in J. \end{aligned}$$

By Gale's (1960) theorem of the alternative, the system is infeasible if and only if there exist

nonnegative multipliers $(\lambda_y)_{y \in I}$, $(\beta_s)_{s \in J}$ and $(\eta_s)_{s \in J}$ satisfying

$$\sum_{y \in I} \lambda_y a_y + \sum_{s \in J} \beta_s a_s = \sum_{s \in J} \eta_s a_s, \quad (\text{A4})$$

$$\sum_{y \in I} \lambda_y F^\gamma(y) + \sum_{s \in J} \beta_s F^\gamma(s) > \sum_{s \in J} \eta_s F^\gamma(s). \quad (\text{A5})$$

Define the nonnegative, finitely supported measures

$$\zeta = \sum_{s \in J} \eta_s 1_s, \quad \tau = \sum_{y \in I} \lambda_y 1_y + \sum_{s \in J} \beta_s 1_s.$$

By (A4), $\nu_\zeta = \nu_\tau$; by (A5), $\int F^\gamma d\tau > \int F^\gamma d\zeta$. As $\text{supp}(\zeta) \subseteq J \subseteq S$, this contradicts F^γ -monotonicity. The finite system is therefore feasible, and its self-employment inequalities imply $v_z \geq 0$.

Truncating each v_z at B preserves all constraints. Indeed, an equality implies $n_i(y)v_{z_i(y)} \leq F^\gamma(y) \leq M$, so its coordinates are unchanged. Any truncated coordinate alone contributes at least $\underline{n}B = M \geq F^\gamma(y)$ to an inequality; hence the inequality continues to hold. Extend the truncated vector arbitrarily to a function on Z taking values in $[0, B]$. The product $[0, B]^Z$, equipped with the product topology, is compact by Tychonoff's theorem. For each $y \in Y$,

$$D_y = \left\{ u \in [0, B]^Z : \sum_{i=1}^{\ell(y)} n_i(y)u(z_i(y)) \geq F^\gamma(y) \right\}$$

and

$$E_y = \left\{ u \in [0, B]^Z : \sum_{i=1}^{\ell(y)} n_i(y)u(z_i(y)) = F^\gamma(y) \right\}$$

are closed. The previous paragraph establishes that $\{D_y : y \in Y\} \cup \{E_y : y \in S\}$ satisfies the finite intersection property; hence, by compactness, there exists a function $u \in [0, B]^Z$ satisfying every constraint.

Finally, we show that u is continuous. By Lemma 6.1 in the supplementary material, every type appears in some firm in S . Fix $z^k \rightarrow z$ and choose $y \in S$ and $i \in \{1, \dots, \ell(y)\}$ such that $z_i(y) = z$. If $\ell(y) = 1$, take $y^k = (z^k, 1)$. Otherwise, apply lower hemicontinuity of the span-of-control correspondence $N_{\ell(y)}$ to the type profiles obtained from that of y by replacing only its i th coordinate with z^k . In either case, this gives feasible firms $y^k \rightarrow y$ with

$z_i(y^k) = z^k$ and $z_j(y^k) = z_j(y)$ for $j \neq i$. Consequently,

$$u(z^k) \geq \frac{F^\gamma(y^k) - \sum_{j \neq i} n_j(y^k)u(z_j(y))}{n_i(y^k)} \rightarrow u(z),$$

so u is lower semicontinuous.

Suppose u is not upper semicontinuous. Then there exist $z^k \rightarrow z$ and $\varepsilon > 0$ such that $u(z^k) \geq u(z) + \varepsilon$ for every k . Choose $y^k \in S$ and i_k such that $z_{i_k}(y^k) = z^k$; taking a subsequence if necessary, we may assume that $\ell(y^k) = L$ and $i_k = i$ are constant. Since $S \cap Y_L$ is compact, we may assume that $y^k \rightarrow y \in S \cap Y_L$. In particular, $z_i(y) = z$.

For each $j \neq i$, lower semicontinuity of u implies

$$\liminf_{k \rightarrow \infty} n_j(y^k)u(z_j(y^k)) \geq n_j(y)u(z_j(y)).$$

For i , we instead have

$$\liminf_{k \rightarrow \infty} n_i(y^k)u(z^k) \geq n_i(y)(u(z) + \varepsilon).$$

Using continuity of F^γ and equality at every firm in S ,

$$\begin{aligned} F^\gamma(y) &= \liminf_{k \rightarrow \infty} \sum_{j=1}^L n_j(y^k)u(z_j(y^k)) \\ &\geq n_i(y)(u(z) + \varepsilon) + \sum_{j \neq i} n_j(y)u(z_j(y)) \\ &= F^\gamma(y) + n_i(y)\varepsilon, \end{aligned}$$

a contradiction. Hence u is upper semicontinuous. ■

Let $\text{supp}(\gamma)$ be F^γ -monotone and let u be from the claim. Then $\int_Y F^\gamma d\gamma = \int_Z u d\nu \geq \min_{v \in U^\gamma} \int_Z v d\nu = \max_{\tau \in \Gamma} \int_Y F^\gamma d\tau$, where the last equality follows from Theorem 2.