

Supplementary Material for “Knowledge Economies”

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1 Introduction

This paper contains supplementary material to our paper “Knowledge Economies”. It contains:

Section 2: An extension of the framework of large many-to-one matching markets with occupational choice in Carmona and Laohakunakorn (2024).

Section 3: Existence and limit results for the setting of Section 2.

Section 4: The relationship between the setting of Section 2 with that in Carmona and Laohakunakorn (2024).

Section 5: A characterization of stable outcomes in knowledge economies.

Section 6: The stable outcomes of economies that are the same up to a homeomorphism are homeomorphic. This implies that the knowledge distribution can be normalized to be uniform and that, in Garicano and Rossi-Hansberg (2004), the distribution of problems can be assumed to be uniform.

Section 7: Sufficient conditions for the earnings function to be increasing.

Section 8: A variation of the Garicano and Rossi-Hansberg's (2004) economy where production is $F(z, z', n) = \max\{z, z'\}n$.

Section 9: An example of a Rosen market where all distributions are uniform.

2 Large many-to-one matching markets with occupational choice: An extension

We extend the framework of markets with occupational choice of Carmona and Lao-hakunakorn (2024) by allowing the set of feasible matches of a manager to depend on his type. A matching market with occupational choice (a market, henceforth) is $E = (Z, \nu, C, P_m, P_s, (\succ_z)_{z \in Z})$, where Z is the (nonempty, Polish) set of individual types, ν is the type distribution (a nonzero, finite, Borel measure on Z), C is the (nonempty, Polish) set of contracts, P_m is the correspondence of possible matches of managers, P_s is the correspondence of possible matches of self-employed and \succ_z describes the preferences of an individual of type z , for each $z \in Z$.

A *dummy type* $\emptyset \notin Z$ is used to represent unmatched i.e. self-employed individuals, and we let $Z_\emptyset = Z \cup \{\emptyset\}$, with the assumption that \emptyset is an isolated point in Z_\emptyset .

The correspondence of possible matches of managers is $P_m : Z \rightrightarrows M(Z \times C)$ and we require that managers of type z be matched with $\delta \in P_m(z)$. Similarly, the correspondence of possible matches of self-employed is $P_s : Z \rightrightarrows \{1_{(\emptyset, c)} : c \in C\}$ and we require that self-employed individuals of type z be matched with $\delta \in P_s(z)$. Let $X = \cup_{z \in Z} P_m(z)$ and $X_\emptyset = X \cup \{1_{(\emptyset, c)} : c \in C\}$, the latter being the *set of possible matches of managers and self-employed individuals*.

The *set of occupations* is $A = \{w, s, m\}$, where w stands for worker, s for self-employed and m for manager. The choice set of each individual depends on his occupation: let $X_m = X$, $X_s = \{1_{(\emptyset, c)} : c \in C\}$, $X_w = \{1_{(z, c)} : (z, c) \in Z \times C\}$ and $\Delta = \{(a, \delta) : \delta \in X_a\}$. The set Δ is the *choice set of each individual* as she can choose her occupation and a match feasible for the chosen occupation.

We allow for externalities and, thus, preferences are allowed to depend on the matching. Matchings with occupational choice are elements of $M(Z \times X_\emptyset)$ satisfying certain properties described below. The *preferences* of an individual of type z are

then described by a relation \succ_z defined on $\Delta \times M(Z \times X_\emptyset)$ for each $z \in Z$.

The particular case considered in Carmona and Laohakunakorn (2024) is when there is $X \subseteq M(Z \times C)$ such that $P_m(z) = X$ for each $z \in Z$ and there is a contract correspondence $\mathbb{C} : Z \times Z_\emptyset \rightrightarrows C$. In this case, define $P_m(z) = \{\delta \in X : \{z\} \times \text{supp}(\delta) \subseteq \text{graph}(\mathbb{C})\}$ and $P_s(z) = \{1_{(\emptyset, c)} : c \in \mathbb{C}(z, \emptyset)\}$. See Section 4 for details.

A *matching with occupational choice* (a matching, henceforth) is a Borel measure $\mu \in M(Z \times X_\emptyset)$ such that

1. $\text{supp}(\mu) \subseteq \text{graph}(P_m) \cup \text{graph}(P_s)$, and
2. $\nu_M + \nu_S + \nu_W = \nu$

where, for each Borel subset B of Z , $\nu_M(B) = \mu(B \times X)$, $\nu_S(B) = \mu(B \times (X_\emptyset \setminus X))$ and $\nu_W(B) = \int_{Z \times X} \delta(B \times C) d\mu(z, \delta)$.

Targets at a given matching μ depend on the type z and on the occupational choice a , and are denoted by $T_z^a(\mu)$. For each $z \in Z$, the targets for the prospective self-employed of type z are $T_z^s(\mu) = \{\emptyset\} \times C$.

For each $z \in Z$, the targets of prospective managers of type z are $T_z^m(\mu)$ defined as the set of $(z^*, c) \in Z \times C$ such that there exists

- (a) $(z', \delta', c') \in Z \times X \times C$ such that $(z', \delta') \in \text{supp}(\mu)$, $(z^*, c') \in \text{supp}(\delta')$ and $(w, 1_{(z, c)}, \mu) \succ_{z^*} (w, 1_{(z', c')}, \mu)$, or
- (b) $\delta' \in X_\emptyset \setminus X$ such that $(z^*, \delta') \in \text{supp}(\mu)$ and $(w, 1_{(z, c)}, \mu) \succ_{z^*} (s, \delta', \mu)$, or
- (c) $\delta' \in X$ such that $(z^*, \delta') \in \text{supp}(\mu)$ and $(w, 1_{(z, c)}, \mu) \succ_{z^*} (m, \delta', \mu)$.

For each $z \in Z$, the targets of prospective workers of type z are $T_z^w(\mu)$ defined as the set of $(z^*, c) \in Z \times C$ such that there is $\delta \in P_m(z^*)$ such that $(z, c) \in \text{supp}(\delta)$ and

- (a) $\text{supp}(\delta) \setminus \{(z, c)\} \subseteq T_{z^*}^m(\mu)$ and there is $(z', \delta', c') \in Z \times X \times C$ such that $(z', \delta') \in \text{supp}(\mu)$, $(z^*, c') \in \text{supp}(\delta')$ and $(m, \delta, \mu) \succ_{z^*} (w, 1_{(z', c')}, \mu)$, or
- (b) $\text{supp}(\delta) \setminus \{(z, c)\} \subseteq T_{z^*}^m(\mu)$ and there is $\delta' \in X_\emptyset \setminus X$ such that $(z^*, \delta') \in \text{supp}(\mu)$ and $(m, \delta, \mu) \succ_{z^*} (s, \delta', \mu)$, or

- (c) there is $\delta' \in X$ such that $\text{supp}(\delta) \setminus \{(z, c)\} \subseteq T_{z^*}^m(\mu) \cup \text{supp}(\delta')$, $(z^*, \delta') \in \text{supp}(\mu)$ and $(m, \delta, \mu) \succ_{z^*} (m, \delta', \mu)$.

For each $z \in Z$, let $P_w(z) = X_w$. The stability set $S(\mu)$ of matching μ is the set of $(z, \delta) \in Z \times X_\emptyset$ such that, if $\delta \in X$, then

- (i) there does not exist $(a, \delta') \in A \times P_a(z)$ such that $\text{supp}(\delta') \subseteq T_z^a(\mu) \cup \text{supp}(\delta)$ if $a = m$, $\text{supp}(\delta') \subseteq T_z^a(\mu)$ if $a \neq m$, and $(a, \delta', \mu) \succ_z (m, \delta, \mu)$,
- (ii) for each $(z', c) \in \text{supp}(\delta)$, there does not exist $(a, \delta') \in A \times P_a(z)$ such that $\text{supp}(\delta') \subseteq T_{z'}^a(\mu)$ and $(a, \delta', \mu) \succ_{z'} (w, 1_{(z,c)}, \mu)$,

and, if $\delta \in X_\emptyset \setminus X$, then

- (iii) there does not exist $(a, \delta') \in A \times P_a(z)$ such that $\text{supp}(\delta') \subseteq T_z^a(\mu)$ and $(a, \delta', \mu) \succ_z (s, \delta, \mu)$.

A matching μ is *stable* if $\text{supp}(\mu) \subseteq S(\mu)$.

Theorem 2.1 provides a characterization of stable matchings that is simpler to use. Let $S_M(\mu)$ be the set of $(z, \delta) \in Z \times X_\emptyset$ such that, if $\delta \in X$, then

- (i) there does not exist $\delta' \in P_m(z)$ such that $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ and $(m, \delta', \mu) \succ_z (m, \delta, \mu)$,
- (ii) for each $(z', c) \in \text{supp}(\delta)$, there does not exist $\delta' \in P_m(z)$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu)$ and $(m, \delta', \mu) \succ_{z'} (w, 1_{(z,c)}, \mu)$,

and, if $\delta \in X_\emptyset \setminus X$, then

- (iii) there does not exist $\delta' \in P_m(z)$ such that $\text{supp}(\delta') \subseteq T_z^m(\mu)$ and $(m, \delta', \mu) \succ_z (s, \delta, \mu)$.

Let $IR(\mu)$ be defined as the set of $(z, \delta) \in Z \times X_\emptyset$ such that, if $\delta \in X$, then

- (i) there does not exist $\delta' \in P_s(z)$ such that $(s, \delta', \mu) \succ_z (m, \delta, \mu)$,
- (ii) for each $(z', c) \in \text{supp}(\delta)$, there does not exist $\delta' \in P_s(z')$ such that $(s, \delta', \mu) \succ_{z'} (w, 1_{(z,c)}, \mu)$,

and, if $\delta \in X_\emptyset \setminus X$, then

(iii) there does not exist $\delta' \in P_s(z)$ such that $(s, \delta', \mu) \succ_z (s, \delta, \mu)$.

Theorem 2.1 *A matching μ is stable if and only if $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$.*

2.1 Proof of Theorem 2.1

Note first that $\text{supp}(\mu) \subseteq S(\mu)$ implies that $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$ since $S(\mu) \subseteq S_M(\mu) \cap IR(\mu)$.

Conversely, suppose that $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$. Let $(z, \delta) \in \text{supp}(\mu)$ and assume, in order to reach a contradiction, that $(z, \delta) \notin S(\mu)$. Since $(z, \delta) \in \text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$, it follows that there is $(z^*, c) \in Z \times C$ and $\bar{z} \in Z$ such that $(z^*, c) \in T_{\bar{z}}^w(\mu)$, $\bar{z} = z$ or $(\bar{z}, \bar{c}) \in \text{supp}(\delta)$ for some $\bar{c} \in C$, (1) $(w, 1_{(z^*, c)}, \mu) \succ_{\bar{z}} (m, \delta, \mu)$ if $\bar{z} = z$ and $\delta \in X$, (2) $(w, 1_{(z^*, c)}, \mu) \succ_{\bar{z}} (w, 1_{(z, \bar{c})}, \mu)$ if $(\bar{z}, \bar{c}) \in \text{supp}(\delta)$ and (3) $(w, 1_{(z^*, c)}, \mu) \succ_{\bar{z}} (s, \delta, \mu)$ if $\bar{z} = z$ and $\delta \in X_\emptyset \setminus X$.

We now show that $(\bar{z}, c) \in T_{z^*}^m(\mu)$. Indeed, we have that $(z, \delta) \in \text{supp}(\mu)$. Thus, in case (1), the conclusion follows by condition (c) in the definition of $T_{z^*}^m(\mu)$ since $\bar{z} = z$ and $(w, 1_{(z^*, c)}, \mu) \succ_z (m, \delta, \mu)$; in case (2), the conclusion follows by condition (a) in the definition of $T_{z^*}^m(\mu)$ since $(\bar{z}, \bar{c}) \in \text{supp}(\delta)$ and $(w, 1_{(z^*, c)}, \mu) \succ_{\bar{z}} (w, 1_{(z, \bar{c})}, \mu)$; and, in case (3), the conclusion follows by condition (b) in the definition of $T_{z^*}^m(\mu)$ since $\bar{z} = z$ and $(w, 1_{(z^*, c)}, \mu) \succ_z (s, \delta, \mu)$.

Since $(z^*, c) \in T_{\bar{z}}^w(\mu)$, there is $\tilde{\delta} \in P_m(z^*)$ such that $(\bar{z}, c) \in \text{supp}(\tilde{\delta})$ and (a) or (b) or (c) in the definition of $T_{\bar{z}}^w(\mu)$ holds. In either case, we will show that $\text{supp}(\mu) \subseteq S_M(\mu)$ fails, which is a contradiction to $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$.

Suppose that condition (a) in the definition of $T_{\bar{z}}^w(\mu)$ holds. Then, in addition, $\text{supp}(\tilde{\delta}) \setminus \{(\bar{z}, c)\} \subseteq T_{z^*}^m(\mu)$, and there is $(z', c', \delta') \in Z \times C \times X$ such that $(z', \delta') \in \text{supp}(\mu)$, $(z^*, c') \in \text{supp}(\delta')$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (w, 1_{(z', c')}, \mu)$. Since $(\bar{z}, c) \in T_{z^*}^m(\mu)$, it follows that $(z', \delta') \in \text{supp}(\mu) \setminus S_M(\mu)$ since (ii) of the definition of $S_M(\mu)$ fails. Indeed, $(z', \delta') \in \text{supp}(\mu)$, $(z^*, c') \in \text{supp}(\delta')$, $\text{supp}(\tilde{\delta}) \subseteq T_{z^*}^m(\mu)$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (w, 1_{(z', c')}, \mu)$.

Suppose next that condition (b) in the definition of $T_{\bar{z}}^w(\mu)$ holds. Then, in addition, $\text{supp}(\tilde{\delta}) \setminus \{(\bar{z}, c)\} \subseteq T_{z^*}^m(\mu)$, and there is $\delta' \in X_\emptyset \setminus X$ such that $(z^*, \delta') \in \text{supp}(\mu)$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (s, \delta', \mu)$. Since $(\bar{z}, c) \in T_{z^*}^m(\mu)$, it follows that $(z^*, \delta') \in \text{supp}(\mu) \setminus S_M(\mu)$ since (iii) of the definition of $S_M(\mu)$ fails. Indeed, $(z^*, \delta') \in \text{supp}(\mu)$, $\text{supp}(\tilde{\delta}) \subseteq T_{z^*}^m(\mu)$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (s, \delta', \mu)$.

Finally, suppose that condition (c) in the definition of $T_{\bar{z}}^w(\mu)$ holds. Then, in addition, there is $\delta' \in X_\emptyset \setminus X$ such that $(z^*, \delta') \in \text{supp}(\mu)$, $\text{supp}(\tilde{\delta}) \setminus \{(\bar{z}, c)\} \subseteq T_{z^*}^m(\mu) \cup \text{supp}(\delta')$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (m, \delta', \mu)$. Since $(\bar{z}, c) \in T_{z^*}^m(\mu)$, it follows that $(z^*, \delta') \in \text{supp}(\mu) \setminus S_M(\mu)$ since (i) of the definition of $S_M(\mu)$ fails. Indeed, $(z^*, \delta') \in \text{supp}(\mu)$, $\text{supp}(\tilde{\delta}) \subseteq T_{z^*}^m(\mu) \cup \text{supp}(\delta')$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (m, \delta', \mu)$.

3 Existence and limit results

We establish existence of stable matchings under the following conditions. We say that a market E is *rational* if \succ_z is asymmetric and negative transitive for each $z \in Z$.

We say that a market E is *continuous* if $\{(a, \delta, \mu, a', \delta', \mu', z) \in (\Delta \times M(Z \times X_\emptyset))^2 \times Z : (a, \delta, \mu) \succ_z (a', \delta', \mu')\}$ is open, P_m is upper hemicontinuous with compact values and P_s is continuous with nonempty and compact values.

We say that a market E is *bounded* if there exists $R > 0$ such that $\delta(Z \times C) \leq R$ for each $\delta \in \cup_{z \in Z} P_m(z)$.

Let $\Lambda : \text{graph}(P_m) \times M(Z \times X_\emptyset) \rightrightarrows X$ be defined by setting, for each $(z, \delta, \mu) \in Z \times X \times M(Z \times X_\emptyset)$, $\Lambda(z, \delta, \mu) = \{\delta' \in P_m(z) : \text{supp}(\delta') \subseteq \text{supp}(\delta) \cup T_z^m(\mu)\}$ and let $\Lambda_0 : Z \times M(Z \times X_\emptyset) \rightrightarrows X$ be defined by setting, for each $(z, \mu) \in Z \times M(Z \times X_\emptyset)$, $\Lambda_0(z, \mu) = \{\delta' \in P_m(z) : \text{supp}(\delta') \subseteq T_z^m(\mu)\}$. We say that a market E is *rich* if Λ is lower hemicontinuous for each $(z, \delta, \mu) \in Z \times X \times M(Z \times X_\emptyset)$ such that $(z, \delta) \in \text{supp}(\mu)$ and Λ_0 is lower hemicontinuous for each $(z, \mu) \in Z \times M(Z \times X_\emptyset)$ such that $z \in \text{supp}(\mu_Z)$.

Theorem 3.1 *Every rational, continuous, bounded and rich market has a stable matching.*

The proof of Theorem 3.1 considers first markets where Z , C and $P_m(z)$ are finite for each $z \in Z$ and then uses limit results to extend to more general markets. The same approach is also useful to establish existence of stable matchings in some applications to which Theorem 3.1 does not apply, namely by approximating such market by a sequence of market that satisfy the assumptions of Theorem 3.1 and then use a limit result. The remaining of this section then presents several limits results under general assumptions, which are then summarized by Corollary 3.1 for the use in the proof of Theorem 3.1.

Lemma 3.1 *Let $E = (Z, \nu, C, P_m, P_s, (\succ_z)_{z \in Z})$ be a market and $\{(E_k, \mu_k)\}_{k=1}^\infty$ be a sequence such that, for each $k \in \mathbb{N}$, (i) $E_k = (Z_k, \nu_k, C_k, P_{m,k}, P_{s,k}, (\succ_{z,k})_{z \in Z_k})$ is a market, (ii) μ_k is a stable matching for E_k , (iii) $Z_k \subseteq Z$, (iv) $\text{supp}(\nu_k) \subseteq \text{supp}(\nu)$, (v) $C_k \subseteq C$, (vi) $P_{a,k}(z) \subseteq P_a(z)$ for each $z \in Z_k$ and $a \in \{m, s\}$, and (vii) $\nu_k \rightarrow \nu$. If P_m and P_s are upper hemicontinuous and compact-valued, then $\{\mu_k\}_{k=1}^\infty$ has a convergent subsequence in $\mathcal{M}(Z \times X_\emptyset)$.*

Proof. Since $\mathcal{M}(Z \times X_\emptyset)$ is a separable metrizable space, it suffices to show that $\{\mu_k\}_{k=1}^\infty$ is tight; this follows by Carmona and Laohakunakorn (2024, Lemma 4) together with $\mu_k(Z \times X_\emptyset) \leq \nu_k(Z)$ for each $k \in \mathbb{N}$ and the fact that $\{\nu_k(Z)\}_{k=1}^\infty$ converges (to $\nu(Z)$) and, hence, is bounded.

Let $\varepsilon > 0$. Since $\{\nu_k\}_{k=1}^\infty$ is tight, there exists a compact subset K_Z of Z such that $\nu_k(Z \setminus K_Z) \leq \varepsilon$ for all k . Then $P_m(K_Z)$ and $P_s(K_Z)$ are compact since both P_m and P_s are upper hemicontinuous and compact-valued. Let $K_{X_\emptyset} = P_m(K_Z) \cup P_s(K_Z)$, which is a compact subset of X_\emptyset .

Then, for each $k \in \mathbb{N}$,

$$\begin{aligned} \mu_k((Z \times X_\emptyset) \setminus (K_Z \times K_{X_\emptyset})) &= \mu_k((Z \setminus K_Z) \times X_\emptyset) + \mu_k(K_Z \times (X_\emptyset \setminus K_{X_\emptyset})) \\ &\leq \nu_k(Z \setminus K_Z) \leq \varepsilon, \end{aligned}$$

since $\text{supp}(\mu_k) \cap (K_Z \times (X_\emptyset \setminus K_{X_\emptyset})) = \emptyset$ because of condition 2 of a matching and, hence, $\mu_k(K_Z \times (X_\emptyset \setminus K_{X_\emptyset})) = 0$. ■

Lemma 3.2 *Let $E = (Z, \nu, C, P_m, P_s, (\succ_z)_{z \in Z})$ be a market, $\{(E_k, \mu_k)\}_{k=1}^\infty$ be a sequence such that $\{\mu_k\}_{k=1}^\infty$ converges and, for each $k \in \mathbb{N}$, (i) $E_k = (Z_k, \nu_k, C_k, P_{m,k}, P_{s,k}, (\succ_{z,k})_{z \in Z_k})$ is a market, (ii) μ_k is a stable matching for E_k , (iii) $Z_k \subseteq Z$, (iv) $\text{supp}(\nu_k) \subseteq \text{supp}(\nu)$, (v) $C_k \subseteq C$, (vi) $P_{a,k}(z) \subseteq P_a(z)$ for each $z \in Z_k$ and $a \in \{m, s\}$, and (vii) $\nu_k \rightarrow \nu$, and $\mu = \lim_k \mu_k$. If E is bounded, and P_m and P_s are upper hemicontinuous and closed-valued, then μ is a matching for E .*

Proof. We first consider condition 2 of the definition of a matching. Let $\pi : Z \times X_\emptyset \rightarrow Z$ be the projection of $Z \times X_\emptyset$ onto Z and note that, for each Borel subset B of Z , $\nu_M(B) + \nu_S(B) = \mu(B \times X_\emptyset) = \mu \circ \pi^{-1}(B)$ and $\nu_{M,k}(B) + \nu_{S,k}(B) = \mu_k(B \times X_\emptyset) = \mu_k \circ \pi^{-1}(B)$ for each $k \in \mathbb{N}$. Since π is continuous, $\mu_k \circ \pi^{-1} \rightarrow \mu \circ \pi^{-1}$. Indeed, for each bounded and continuous $f : Z \rightarrow \mathbb{R}$, $\int_Z f d\mu_k \circ \pi^{-1} = \int_{Z \times X_\emptyset} f \circ \pi d\mu_k \rightarrow \int_{Z \times X_\emptyset} f \circ \pi d\mu = \int_Z f d\mu_k \circ \pi^{-1}$ since $f \circ \pi : Z \times X_\emptyset \rightarrow \mathbb{R}$ is bounded and continuous. Hence, since $\mathcal{M}(Z \times X_\emptyset)$ is metrizable, $\nu_M + \nu_S = \mu \circ \pi^{-1} = \lim_k \mu_k \circ \pi^{-1} = \lim_k (\nu_{M,k} + \nu_{S,k})$.

Also, for each Borel subset B of Z , $\nu_W(B) = \int_{Z \times X} \delta(B \times C) d\mu(z, \delta)$ and $\nu_{W,k}(B) = \int_{Z \times X} \delta(B \times C) d\mu_k(z, \delta)$ for each $k \in \mathbb{N}$. We show that $\nu_{W,k} \rightarrow \nu_W$. Let $B \subseteq Z$ be closed and $f : X \rightarrow \mathbb{R}$ be defined by setting, for each $\delta \in X$, $f(\delta) = \delta(B \times C)$. Then f is bounded (since E is bounded) and upper semi-continuous. Hence, by (a suitable adaptation of) Aliprantis and Border (2006, Theorem 15.5), $\limsup_k \nu_{W,k}(B) = \limsup_k \int_{Z \times X} f d\mu_k \leq \int_{Z \times X} f d\mu = \nu_W(B)$ and it follows that $\nu_W = \lim_k \nu_{W,k}$ as claimed. Thus,

$$\nu_M + \nu_S + \nu_W = \lim_k (\nu_{M,k} + \nu_{S,k}) + \lim_k \nu_{W,k} = \lim_k (\nu_{M,k} + \nu_{S,k} + \nu_{W,k}) = \nu.$$

Condition 1 holds because, by Carmona and Podczeck (2009, Lemma 12), for each $(z, \delta) \in \text{supp}(\mu)$, there exists a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding $\{(z_{k_j}, \delta_{k_j})\}_{j=1}^\infty$ such that $(z_{k_j}, \delta_{k_j}) \rightarrow (z, \delta)$ and, for each $j \in \mathbb{N}$, $(z_{k_j}, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$. Hence, $\delta_{k_j} \in P_{m,k}(z_{k_j}) \cup P_{s,k}(z_{k_j}) \subseteq P_m(z_{k_j}) \cup P_s(z_{k_j})$ and, since $P_m \cup P_s$ is upper hemicontinuous and closed-valued and $M(Z \times C)$ is regular, $\delta \in P_m(z) \cup P_s(z)$. ■

Consider a market $E = (Z, \nu, C, P_m, P_s, (\succ_z)_{z \in Z})$. We say that $(\succ_z)_{z \in Z}$ is *continuous* if $\{(a, \delta, \mu, a', \delta', \mu', z) \in (\Delta \times M(Z \times X_\emptyset))^2 \times Z : (a, \delta, \mu) \succ_z (a', \delta', \mu')\}$ is open.

Lemma 3.3 *Let $E = (Z, \nu, C, P_m, P_s, (\succ_z)_{z \in Z})$ be a market, $\{(E_k, \mu_k)\}_{k=1}^\infty$ be a sequence such that $\{\mu_k\}_{k=1}^\infty$ converges and, for each $k \in \mathbb{N}$, (i) $E_k = (Z_k, \nu_k, C_k, P_{m,k}, P_{s,k}, (\succ_{z,k})_{z \in Z_k})$ is a market, (ii) μ_k is a stable matching for E_k , (iii) $Z_k \subseteq Z$, (iv) $\text{supp}(\nu_k) \subseteq \text{supp}(\nu)$, (v) $C_k \subseteq C$, (vi) $P_{a,k}(z) \subseteq P_a(z)$ for each $z \in Z_k$ and $a \in \{m, s\}$, and (vii) $\nu_k \rightarrow \nu$, and $\mu = \lim_k \mu_k$. If $(\succ_z)_{z \in Z}$ is continuous, P_s is lower hemicontinuous and $\succ_{z,k}$ is the restriction of \succ_z to $\Delta_k \times M(Z_k \times X_{\emptyset,k})$ for each $z \in Z_k$, then $\text{supp}(\mu) \subseteq IR(\mu)$.*

Proof. Let $(z, \delta) \in \text{supp}(\mu)$ and suppose that $(z, \delta) \notin IR(\mu)$. Then either (i) there exists $\delta' \in P_s(z)$ such that $(s, \delta', \mu) \succ_z (a(\delta), \delta, \mu)$ where $a(\delta) = m$ if $\delta \in X$ and $a(\delta) = s$ if $\delta \in X_\emptyset \setminus X$, or (ii) there exists $(z', c) \in \text{supp}(\delta)$ and $\delta' \in P_s(z')$ such that $(s, \delta', \mu) \succ_{z'} (w, 1_{(z,c)}, \mu)$.

Consider case (i) first. The continuity of $(\succ_z)_{z \in Z}$ and the lower hemicontinuity of P_s implies that there are open neighborhoods $V_{\delta'}$, V_z , V_δ and V_μ of δ' , z , δ and μ , respectively, such that $(s, \tilde{\delta}, \hat{\mu}) \succ_{\tilde{z}} (a(\delta), \hat{\delta}, \hat{\mu})$ and $P_s(\tilde{z}) \cap V_{\delta'} \neq \emptyset$ for each $\tilde{\delta} \in V_{\delta'}$, $\tilde{z} \in V_z$, $\hat{\delta} \in V_\delta$ and $\hat{\mu} \in V_\mu$. Since $(z, \delta) \in \text{supp}(\mu)$, it follows that $0 < \mu(V_z \times V_\delta) \leq \liminf_k \mu_k(V_z \times V_\delta)$; hence, for each k sufficiently large, $\mu_k(V_z \times V_\delta) > 0$ and $\mu_k \in V_\mu$. This means that, for any such k , there exist $(\hat{z}, \hat{\delta}) \in \text{supp}(\mu_k) \cap (V_z \times V_\delta)$ and $\tilde{\delta} \in P_s(\hat{z}) \cap V_{\delta'}$. But then $(s, \tilde{\delta}, \mu_k) \succ_{\hat{z}} (a(\delta), \hat{\delta}, \mu_k)$ and, hence, $(s, \tilde{\delta}, \mu_k) \succ_{\tilde{z},k} (a(\delta), \hat{\delta}, \mu_k)$, contradicting the individual rationality of μ_k .

Consider next case (ii). The continuity of $(\succ_z)_{z \in Z}$ and the lower hemicontinuity of P_s implies that there are open neighborhoods $V_{\delta'}$, V_c , V_z , $V_{z'}$ and V_μ of δ' , c , z , z' and μ , respectively, such that $(s, \tilde{\delta}, \hat{\mu}) \succ_{\tilde{z}} (w, 1_{(\tilde{z},c)}, \hat{\mu})$ and $P_s(\tilde{z}) \cap V_{\delta'} \neq \emptyset$ for each $\tilde{\delta} \in V_{\delta'}$, $\hat{c} \in V_c$, $\hat{z} \in V_z$, $\tilde{z} \in V_{z'}$ and $\hat{\mu} \in V_\mu$. Since $(z', c) \in \text{supp}(\delta)$, there is an open neighborhood V_δ of δ such that $\text{supp}(\hat{\delta}) \cap (V_{z'} \times V_c) \neq \emptyset$ for each $\hat{\delta} \in V_\delta$. Since $\mu_k \rightarrow \mu$ and $(z, \delta) \in \text{supp}(\mu)$, it follows that $0 < \mu(V_z \times V_\delta) \leq \liminf_k \mu_k(V_z \times V_\delta)$; hence, for all k sufficiently large, $\mu_k(V_z \times V_\delta) > 0$ and $\mu_k \in V_\mu$. This means that, for any such k , there exists $(\hat{z}, \hat{\delta}) \in \text{supp}(\mu_k) \cap (V_z \times V_\delta)$, $(\tilde{z}, \hat{c}) \in \text{supp}(\hat{\delta}) \cap (V_{z'} \times V_c)$ and $\tilde{\delta} \in P_s(\tilde{z}) \cap V_{\delta'}$. But then $(s, \tilde{\delta}, \mu_k) \succ_{\tilde{z}} (w, 1_{(\hat{z},\hat{c})}, \mu_k)$ and, hence, $(s, \tilde{\delta}, \mu_k) \succ_{\tilde{z},k} (w, 1_{(\hat{z},\hat{c})}, \mu_k)$, contradicting the individual rationality of μ_k . ■

Lemma 3.4 Let $E = (Z, \nu, C, P_m, P_s, (\succ_z)_{z \in Z})$ be a market, $\{(E_k, \mu_k)\}_{k=1}^\infty$ be a sequence such that $\{\mu_k\}_{k=1}^\infty$ converges and, for each $k \in \mathbb{N}$, (i) $E_k = (Z_k, \nu_k, C_k, P_{m,k}, P_{s,k}, (\succ_{z,k})_{z \in Z_k})$ is a market, (ii) μ_k is a stable matching for E_k , (iii) $Z_k \subseteq Z$, (iv) $\text{supp}(\nu_k) \subseteq \text{supp}(\nu)$, (v) $C_k \subseteq C$, (vi) $P_{a,k}(z) \subseteq P_a(z)$ for each $z \in Z_k$ and $a \in \{m, s\}$, and (vii) $\nu_k \rightarrow \nu$, and $\mu = \lim_k \mu_k$. If

1. $(\succ_z)_{z \in Z}$ is continuous,
2. $\succ_{z,k}$ is the restriction of \succ_z to $\Delta_k \times M(Z_k \times X_{\emptyset,k})$ for each $z \in Z_k$,
3. for each $(z, \delta, \mu) \in \text{graph}(P_m) \times M(Z \times X_\emptyset)$ such that $(z, \delta) \in \text{supp}(\mu)$, $\delta' \in \Lambda(z, \delta, \mu)$, open neighborhood $V_{\delta'}$ of δ' , subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and sequence $\{(z_{k_j}, \delta_{k_j})\}_{j=1}^\infty$ such that $(z_{k_j}, \delta_{k_j}) \rightarrow (z, \delta)$ and $(z_{k_j}, \delta_{k_j}) \in Z_{k_j} \times X_{k_j}$ for each $j \in \mathbb{N}$, there exists $J \in \mathbb{N}$ such that $P_{m,k_j}(z_{k_j}) \cap \Lambda(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ for each $j \geq J$, and
4. for each $(z, \mu) \in Z \times M(Z \times X_\emptyset)$ such that $z \in \text{supp}(\mu_Z)$, $\delta' \in \Lambda_0(z, \mu)$, open neighborhood $V_{\delta'}$ of δ' , subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and sequence $\{z_{k_j}\}_{j=1}^\infty$ such that $z_{k_j} \rightarrow z$ and $z_{k_j} \in Z_{k_j}$ for each $j \in \mathbb{N}$, there exists $J \in \mathbb{N}$ such that $P_{m,k_j}(z_{k_j}) \cap \Lambda_0(z_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ for each $j \geq J$,

then $\text{supp}(\mu) \subseteq S_M(\mu)$.

Proof. In this proof, to avoid confusion, we write $T_z^m(\mu; E')$ for $T_z^m(\mu)$ in a market E' .

Let $(z, \delta) \in \text{supp}(\mu)$ (hence, $z \in \text{supp}(\mu_Z)$ since $\text{supp}(\mu) \subseteq \text{supp}(\mu_Z) \times X_\emptyset$) and suppose that $(z, \delta) \notin S_M(\mu)$. Then either (i) $\delta \in X$ and there exists $\delta' \in P_m(z)$ such that $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ and $(m, \delta', \mu) \succ_z (m, \delta, \mu)$ or (ii) there exists $(z', c) \in \text{supp}(\delta)$ and $\delta' \in P_m(z')$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu)$ and $(m, \delta', \mu) \succ_{z'} (w, 1_{(z,c)}, \mu)$ or (iii) there exists $\delta' \in P_m(z)$ such that $\text{supp}(\delta') \subseteq T_z^m(\mu)$ and $(m, \delta', \mu) \succ_z (s, \delta, \mu)$.

Consider case (i) first. Since $(\succ_z)_{z \in Z}$ is continuous, let $V_z, V_{\delta'}, V_\delta$ and V_μ be open neighborhoods of z, δ', δ and μ , respectively, such that $(m, \gamma', \bar{\mu}) \succ_{\bar{z}} (m, \gamma, \bar{\mu})$ for each $\bar{z} \in V_z, \gamma' \in V_{\delta'}, \gamma \in V_\delta$ and $\bar{\mu} \in V_\mu$.

By Carmona and Podczeck (2009, Lemma 12), there is a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding sequence $\{(z_{k_j}, \delta_{k_j})\}_{j=1}^\infty$ such that $(z_{k_j}, \delta_{k_j}) \rightarrow (z, \delta)$ and $(z_{k_j}, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$ for each $j \in \mathbb{N}$. Since μ_{k_j} is a matching, it follows that $\delta_{k_j} \in P_{m, k_j}(z_{k_j}) \subseteq P_m(z_{k_j})$ and, hence, $(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \in \text{graph}(P_m) \times M(Z \times X_\emptyset)$ for each $j \in \mathbb{N}$.

Since $\delta' \in \Lambda(z, \delta, \mu)$, let $J \in \mathbb{N}$ be such that $\mu_{k_j} \in V_\mu$, $z_{k_j} \in V_z$, $\delta_{k_j} \in V_\delta$ and $P_{m, k_j}(z_{k_j}) \cap \Lambda(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ for all $j \geq J$ by condition 3. Let $j \geq J$ and let $\delta'_{k_j} \in P_{m, k_j}(z_{k_j}) \cap \Lambda(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \cap V_{\delta'}$. Then $\text{supp}(\delta'_{k_j}) \subseteq T_{z_{k_j}}^m(\mu_{k_j}; E) \cup \text{supp}(\delta_{k_j})$ and $(m, \delta'_{k_j}, \mu_{k_j}) \succ_{z_{k_j}} (m, \delta_{k_j}, \mu_{k_j})$. It then follows that $\text{supp}(\delta'_{k_j}) \subseteq T_{z_{k_j}}^m(\mu_{k_j}; E_{k_j}) \cup \text{supp}(\delta_{k_j})$ and $(m, \delta'_{k_j}, \mu_{k_j}) \succ_{z_{k_j}, k_j} (m, \delta_{k_j}, \mu_{k_j})$. But this contradicts the stability of μ_{k_j} .

Consider next case (ii). Since $(\succ_z)_{z \in Z}$ is continuous, let $V_z, V_{z'}, V_{\delta'}, V_c$ and V_μ be open neighborhoods of z, z', δ', c and μ respectively, such that $(m, \hat{\delta}', \hat{\mu}) \succ_{z'} (w, 1_{(\hat{z}, \hat{c})}, \hat{\mu})$ for each $\hat{z} \in V_z, \hat{z}' \in V_{z'}, \hat{\delta}' \in V_{\delta'}, \hat{c} \in V_c$ and $\hat{\mu} \in V_\mu$. By Carmona and Podczeck (2009, Lemma 12), there is a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding sequence $\{(z_{k_j}, \delta_{k_j}, z'_{k_j}, c_{k_j})\}_{j=1}^\infty$ such that $(z_{k_j}, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$ and $(z'_{k_j}, c_{k_j}) \in \text{supp}(\delta_{k_j})$ for each $j \in \mathbb{N}$ and $(z_{k_j}, \delta_{k_j}, z'_{k_j}, c_{k_j}) \rightarrow (z, \delta, z', c)$.

Since $\delta' \in \Lambda_0(z', \mu)$, let $J \in \mathbb{N}$ be such that $\delta_{k_j} \in V_\delta, z_{k_j} \in V_z, z'_{k_j} \in V_{z'}, c_{k_j} \in V_c, \mu_{k_j} \in V_\mu$ and $P_{m, k_j}(z'_{k_j}) \cap \Lambda_0(z'_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ for all $j \geq J$ by condition 4. Let $j \geq J$ and let $\delta'_{k_j} \in P_{m, k_j}(z'_{k_j}) \cap \Lambda_0(z'_{k_j}, \mu_{k_j}) \cap V_{\delta'}$. Then $\text{supp}(\delta'_{k_j}) \subseteq T_{z'_{k_j}}^m(\mu_{k_j}; E)$ and $(m, \delta'_{k_j}, \mu_{k_j}) \succ_{z'_{k_j}} (w, 1_{(z_{k_j}, c_{k_j})}, \mu_{k_j})$. It then follows that $\text{supp}(\delta'_{k_j}) \subseteq T_{z'_{k_j}}^m(\mu_{k_j}; E_{k_j})$ and $(m, \delta'_{k_j}, \mu_{k_j}) \succ_{z'_{k_j}, k_j} (w, 1_{(z_{k_j}, c_{k_j})}, \mu_{k_j})$. But this contradicts the stability of μ_{k_j} .

Finally, consider case (iii). Since $(\succ_z)_{z \in Z}$ is continuous, let $V_z, V_{\delta'}, V_\delta$ and V_μ be open neighborhoods of z, δ', δ and μ , respectively, such that $(m, \gamma', \bar{\mu}) \succ_{\bar{z}} (s, \gamma, \bar{\mu})$ for each $\bar{z} \in V_z, \gamma' \in V_{\delta'}, \gamma \in V_\delta$ and $\bar{\mu} \in V_\mu$. By Carmona and Podczeck (2009, Lemma 12), there is a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding sequence $\{(z_{k_j}, \delta_{k_j})\}_{j=1}^\infty$ such that $(z_{k_j}, \delta_{k_j}) \rightarrow (z, \delta)$ and $(z_{k_j}, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$ for each $j \in \mathbb{N}$.

Since $\delta' \in \Lambda_0(z, \mu)$, let $J \in \mathbb{N}$ be such that $\mu_{k_j} \in V_\mu, z_{k_j} \in V_z, \delta_{k_j} \in V_\delta$ and $P_{m, k_j}(z_{k_j}) \cap \Lambda_0(z_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ for all $j \geq J$ by condition 4. Let $j \geq J$ and let $\delta'_{k_j} \in P_{m, k_j}(z_{k_j}) \cap \Lambda_0(z_{k_j}, \mu_{k_j}) \cap V_{\delta'}$. Then $\text{supp}(\delta'_{k_j}) \subseteq T_{z_{k_j}}^m(\mu_{k_j}; E)$ and $(m, \delta'_{k_j}, \mu_{k_j}) \succ_{z_{k_j}}$

$(s, \delta_{k_j}, \mu_{k_j})$. It then follows that $\text{supp}(\delta'_{k_j}) \subseteq T_{z_{k_j}}^m(\mu_{k_j}; E_{k_j})$ and $(m, \delta'_{k_j}, \mu_{k_j}) \succ_{z_{k_j}, k_j} (s, \delta_{k_j}, \mu_{k_j})$. But this contradicts the stability of μ_{k_j} . ■

The following corollary, which will be used in the proof of Theorem 3.1, combines Lemmas 3.1–3.4.

Corollary 3.1 *Let $E = (Z, \nu, C, P_m, P_s, (\succ_z)_{z \in Z})$ be a rational, continuous and bounded market and $\{(E_k, \mu_k)\}_{k=1}^\infty$ be a sequence such that, for each $k \in \mathbb{N}$,*

$$E_k = (Z_k, \nu_k, C_k, P_{m,k}, P_{s,k}, (\succ_{z,k})_{z \in Z_k})$$

is a market, μ_k is a stable matching for E_k , $Z_k \subseteq Z$, $\text{supp}(\nu_k) \subseteq \text{supp}(\nu)$, $C_k \subseteq C$, $P_{a,k}(z) \subseteq P_a(z)$ for each $z \in Z_k$ and $a \in \{m, s\}$, and $\nu_k \rightarrow \nu$. Then:

1. $\{\mu_k\}_{k=1}^\infty$ has a convergent subsequence in $\mathcal{M}(Z \times X_\emptyset)$.

Suppose further that $\{\mu_k\}_{k=1}^\infty$ converges and let $\mu = \lim_k \mu_k$. Then:

2. μ is a matching for E .

Suppose further that $\succ_{z,k}$ is the restriction of \succ_z to $\Delta_k \times M(Z_k \times X_{\emptyset,k})$ for each $z \in Z_k$. Then:

3. $\text{supp}(\mu) \subseteq IR(\mu)$.

4. $\text{supp}(\mu) \subseteq S_M(\mu)$ if

(a) *for each $(z, \delta, \mu) \in \text{graph}(P_m) \times M(Z \times X_\emptyset)$ such that $(z, \delta) \in \text{supp}(\mu)$, $\delta' \in \Lambda(z, \delta, \mu)$, open neighborhood $V_{\delta'}$ of δ' , subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and sequence $\{(z_{k_j}, \delta_{k_j})\}_{j=1}^\infty$ such that $(z_{k_j}, \delta_{k_j}) \rightarrow (z, \delta)$ and $(z_{k_j}, \delta_{k_j}) \in Z_{k_j} \times X_{k_j}$ for each $j \in \mathbb{N}$, there exists $J \in \mathbb{N}$ such that $P_{m,k_j}(z_{k_j}) \cap \Lambda(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ for each $j \geq J$, and*

(b) *for each $(z, \mu) \in Z \times M(Z \times X_\emptyset)$ such that $z \in \text{supp}(\mu_Z)$, $\delta' \in \Lambda_0(z, \mu)$, open neighborhood $V_{\delta'}$ of δ' , subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and sequence $\{z_{k_j}\}_{j=1}^\infty$ such that $z_{k_j} \rightarrow z$ and $z_{k_j} \in Z_{k_j}$ for each $j \in \mathbb{N}$, there exists $J \in \mathbb{N}$ such that $P_{m,k_j}(z_{k_j}) \cap \Lambda_0(z_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ for each $j \geq J$.*

3.1 Proof of Theorem 3.1

The first step in the proof of our existence result consists in the following lemma, which considers the special case where Z , C and $P_m(z)$ are finite for each $z \in Z$.

Lemma 3.5 *If E is a rational and continuous market such that Z , C and $P_m(z)$ are finite for each $z \in Z$, then E has a stable matching.*

Proof. Let $X = \cup_{z \in Z} P_m(z)$. Then X is finite and so are Z_\emptyset , X_\emptyset and Δ . Define $\bar{\tau} \in \mathbb{R}^{Z \times \Delta}$ by setting, for each $(z, a, \delta) \in Z \times \Delta$,

$$\bar{\tau}(z, a, \delta) = \begin{cases} 0 & \text{if } \delta \notin P_a(z) \text{ and } a \neq w, \\ \nu(z) & \text{otherwise.} \end{cases}$$

Define

$$\mathcal{T} = \left\{ \tau \in \mathbb{R}_+^{Z \times \Delta} : \tau(z, a, \delta) \leq \bar{\tau}(z, a, \delta) \text{ and } \sum_{(a, \delta) \in \Delta} \tau(z, a, \delta) \leq \nu(z) \right. \\ \left. \text{for each } (z, a, \delta) \in Z \times \Delta \right\} \text{ and}$$

$$\mathcal{K} = \left\{ \kappa \in \mathbb{R}_+^{Z \times Z \times C} : \kappa(z, z', c) \leq \nu(z') \text{ for each } (z, z', c) \in Z \times Z \times C \right\}.$$

Note that \mathcal{T} and \mathcal{K} are nonempty, convex, and compact subsets of Euclidean spaces.

Let $u : Z \times \Delta \times \mathcal{M}(Z \times X_\emptyset) \rightarrow \mathbb{R}$ be a continuous utility function that represents preferences. We normalize $u \geq 1$. For each $n \in \mathbb{N}$, let $u_n = u^n$. Since $x \mapsto x^n$ is strictly increasing on $[1, \infty)$, u_n and u represent the same preferences.

Define $d : \mathcal{T} \rightarrow \mathbb{R}_+^{Z \times X_\emptyset}$ by setting, for each $\tau \in \mathcal{T}$ and $(z, \delta) \in Z \times X_\emptyset$,

$$d(\tau)(z, \delta) = \begin{cases} \tau(z, m, \delta) & \text{if } \delta \in X, \\ \tau(z, s, \delta) & \text{if } \delta \in X_\emptyset \setminus X. \end{cases}$$

The function d is continuous. We abuse notation and, for each $(z, a, \delta, \tau) \in Z \times \Delta \times \mathcal{T}$, write $u(z, a, \delta, \tau)$ for $u(z, a, \delta, d(\tau))$ and analogously for u_n . We also write $(a, \delta, \tau) \succ_z (a', \delta', \tau)$ for $(a, \delta, d(\tau)) \succ_z (a', \delta', d(\tau))$, where $(a', \delta') \in \Delta$.

For each $n \in \mathbb{N}$, let $D_n : \mathcal{T} \times \mathcal{K} \rightrightarrows \mathcal{T}$ be defined by setting, for each $(\mu, \kappa) \in \mathcal{T} \times \mathcal{K}$,

$$D_n(\mu, \kappa) = \left\{ \tau \in \mathcal{T} : \tau \in \arg \max_{\tau' \in \mathcal{T}} \sum_{z \in Z, (a, \delta) \in \Delta} u_n(z, a, \delta, \mu) \tau'(z, a, \delta) \right.$$

$$\text{subject to } \sum_{(a, \delta) \in \Delta} \tau'(z, a, \delta) = \nu(z) \text{ for all } z \in Z,$$

$$\sum_{\delta \in X} \tau'(z, m, \delta) \delta(z', c) \leq \kappa(z, z', c) \text{ for all } (z, z', c) \in Z \times Z \times C, \text{ and}$$

$$\tau'(z, w, 1_{(z', c)}) \leq \sum_{\delta \in X} \mu(z', m, \delta) \delta(z, c) \text{ for all } (z, z', c) \in Z \times Z \times C \left. \right\}.$$

Claim 1 D_n is upper hemicontinuous with nonempty, compact and convex values.

Proof. It follows by the linearity of the objective function together with the convexity of the constraint set that D_n has convex values. It follows from Berge's maximum theorem that D_n is upper hemicontinuous with nonempty and compact values. To see this, first note that the objective function is continuous and that the constraint set, denoted by $\Phi(\mu, \kappa)$, is contained in the compact set \mathcal{T} . It is clear that Φ is upper hemicontinuous with compact values. To see that Φ has nonempty values, define $\bar{\tau} \in \mathcal{T}$ as follows. For each $z \in Z$, let $c_z \in C$ be such that $1_{(\emptyset, c_z)} \in P_s(z)$, $\bar{\tau}(z, s, 1_{(\emptyset, c_z)}) = \nu(z)$ and $\bar{\tau}(z, a, \delta) = 0$ for each $(a, \delta) \in \Delta \setminus \{(s, 1_{(\emptyset, c_z)})\}$. We then have that $\bar{\tau} \in \Phi(\mu, \kappa)$ for each $(\mu, \kappa) \in \mathcal{T} \times \mathcal{K}$. Finally, to see that Φ is lower hemicontinuous, let $(\mu, \kappa) \in \mathcal{T} \times \mathcal{K}$, $O \subseteq \mathcal{T}$ be an open set such that $\Phi(\mu, \kappa) \cap O \neq \emptyset$, and $\tau \in \Phi(\mu, \kappa) \cap O$. Let $\hat{\tau} = \lambda\tau + (1-\lambda)\bar{\tau} \in O$ for some $\lambda \in (0, 1)$. Note that for each $z \in Z$, $\sum_{(a, \delta) \in \Delta} \hat{\tau}(z, a, \delta) = \nu(z)$, $\sum_{\delta \in X} \hat{\tau}(z, m, \delta) \delta(z', c) < \kappa(z, z', c)$ for each $(z, z', c) \in Z \times Z \times C$ such that $\kappa(z, z', c) > 0$ and $\hat{\tau}(z, w, 1_{(z', c)}) < \sum_{\delta \in X} \mu(z', m, \delta) \delta(z, c)$ for each $(z, z', c) \in Z \times Z \times C$ such that $\sum_{\delta \in X} \mu(z', m, \delta) \delta(z, c) > 0$. Thus, there is an open neighborhood V of (μ, κ) such that $\hat{\tau} \in \Phi(\mu', \kappa') \cap O$ for each $(\mu', \kappa') \in V$. ■

Claim 2 If $(\mu, \kappa) \in \mathcal{T} \times \mathcal{K}$, $\tau \in D_n(\mu, \kappa)$ and $(z, a, \delta') \in Z \times \Delta$ is such that $\tau(z, a, \delta') > 0$, then $\tau(z, w, 1_{(\hat{z}, \hat{c})}) = \sum_{\delta \in X} \mu(\hat{z}, m, \delta) \delta(z, \hat{c})$ for each $(\hat{z}, \hat{c}) \in Z \times C$ such that $(w, 1_{(\hat{z}, \hat{c})}, \mu) \succ_z (a, \delta', \mu)$.

Proof. If not, then $\tau(z, w, 1_{(\hat{z}, \hat{c})}) < \sum_{\delta \in X} \mu(\hat{z}, m, \delta) \delta(z, \hat{c})$ for some $(\hat{z}, \hat{c}) \in Z \times C$ such that $(w, 1_{(\hat{z}, \hat{c})}, \mu) \succ_z (a, \delta', \mu)$. Then $\sum_{\delta \in X} \mu(\hat{z}, m, \delta) \delta(z, \hat{c}) > 0$. Thus, increase

$\tau(z, w, 1_{(\hat{z}, \hat{c})})$ and decrease $\tau(z, a, \delta')$ by the same amount $\varepsilon \in (0, \tau(z, a, \delta'))$ such that $\tau(z, w, 1_{(\hat{z}, \hat{c})}) + \varepsilon < \sum_{\delta \in X} \mu(\hat{z}, m, \delta) \delta(z, \hat{c})$. This increases the objective function in $D_n(\mu, \kappa)$ while satisfying the constraints, thus contradicting $\tau \in D_n(\mu, \kappa)$. ■

For each $\mu \in \mathcal{T}$ and $(z, z', c) \in Z \times Z \times C$, let

$$W(z, z', c, \mu) = \{(a, \delta) \in \Delta : u(z', w, 1_{(z, c)}, \mu) > u(z', a, \delta, \mu)\}.$$

Let $g : \mathcal{T} \rightarrow \mathcal{K}$ be defined by setting, for each $\mu \in \mathcal{T}$ and $(z, z', c) \in Z \times Z \times C$,

$$g(\mu)(z, z', c) = \sum_{(a, \delta) \in W(z, z', c, \mu)} \mu(z', a, \delta)$$

To see that $g(\mu) \in \mathcal{K}$, note that since $\mu \in \mathcal{T}$, $0 \leq g(\mu)(z, z', c) \leq \nu(z')$.

The function g may fail to be continuous and, thus, we will consider a continuous approximation to it. For each $n \in \mathbb{N}$ and $(z, z', c) \in Z \times Z \times C$, let $\alpha_{n, (z, z', c)} : \Delta \times \mathcal{T} \rightarrow [0, 1]$ be defined by setting, for each $(a, \delta, \mu) \in \Delta \times \mathcal{T}$,

$$\alpha_{n, (z, z', c)}(a, \delta, \mu) = n \max \left\{ 0, \min \left\{ u(z', w, 1_{(z, c)}, \mu) - u(z', a, \delta, \mu), \frac{1}{n} \right\} \right\}.$$

Let $g_n : \mathcal{T} \rightarrow \mathcal{K}$ be defined by setting, for each $\mu \in \mathcal{T}$ and $(z, z', c) \in Z \times Z \times C$,

$$g_n(\mu)(z, z', c) = \sum_{(a, \delta) \in \Delta} \alpha_{n, (z, z', c)}(a, \delta, \mu) \mu(z', a, \delta).$$

We have that g_n is continuous since $\alpha_{n, (z, z', c)}$ is continuous for each $(z, z', c) \in Z \times Z \times C$. Note that

$$\alpha_{n, (z, z', c)}(a, \delta, \mu) \in \begin{cases} \{0\} & \text{if } u(z', a, \delta, \mu) \geq u(z', w, 1_{(z, c)}, \mu), \\ (0, 1) & \text{if } u(z', w, 1_{(z, c)}, \mu) - \frac{1}{n} < u(z', a, \delta, \mu) < u(z', w, 1_{(z, c)}, \mu), \\ \{1\} & \text{if } u(z', a, \delta, \mu) \leq u(z', w, 1_{(z, c)}, \mu) - \frac{1}{n}. \end{cases}$$

Hence, it follows that

$$g_n(\mu)(z, z', c) \leq g(\mu)(z, z', c) \tag{3.1}$$

for each $\mu \in \mathcal{T}$ and $(z, z', c) \in Z \times Z \times C$ since

$$g(\mu)(z, z', c) = \sum_{(a, \delta) \in \Delta} \alpha_{(z, z', c)}(a, \delta, \mu) \mu(z', a, \delta)$$

with

$$\alpha_{(z,z',c)}(a, \delta, \mu) = \begin{cases} 1 & \text{if } u(z', a, \delta, \mu) < u(z', w, 1_{(z,c)}, \mu), \\ 0 & \text{otherwise.} \end{cases}$$

To see that $g_n(\mu) \in \mathcal{K}$, note that $0 \leq g_n(\mu) \leq g(\mu)$.

Let $f_n : \mathcal{T} \rightarrow \mathcal{K}$ be defined by setting, for each $\mu \in \mathcal{T}$ and $(z, z', c) \in Z \times Z \times C$,

$$f_n(\mu)(z, z', c) = \mu(z', w, 1_{(z,c)}) + \frac{1}{n}g_n(\mu)(z, z', c).$$

To see that $f_n(\mu) \in \mathcal{K}$, note that

$$0 \leq \mu(z', w, 1_{(z,c)}) + \frac{1}{n}g_n(\mu)(z, z', c) \leq \mu(z', w, 1_{(z,c)}) + g(\mu)(z, z', c) \leq \nu(z').$$

We have that f_n is continuous since g_n is continuous.

Let $\Psi_n : \mathcal{T} \times \mathcal{K} \rightrightarrows \mathcal{T} \times \mathcal{K}$ be defined by setting, for each $(\mu, \kappa) \in \mathcal{T} \times \mathcal{K}$,

$$\Psi_n(\mu, \kappa) = D_n(\mu, \kappa) \times \{f_n(\mu)\}.$$

It follows from the continuity of f_n and from Claim 1 that Ψ_n is upper hemicontinuous with nonempty, compact and convex values. Hence, by the Kakutani fixed point theorem, let (μ_n, κ_n) be a fixed point of Ψ_n . Thus, $\mu_n \in D_n(\mu_n, \kappa_n)$ and $\kappa_n = f_n(\mu_n)$.

Since $\mathcal{T} \times \mathcal{K}$ is compact, taking a subsequence if necessary, we may assume that $\{(\mu_n, \kappa_n)\}_{n=1}^\infty$ converges; let $(\mu, \kappa) = \lim_{n \rightarrow \infty} (\mu_n, \kappa_n)$. For each n , we have $\kappa_n = f_n(\mu_n)$, and so

$$\kappa(z, z', c) = \lim_{n \rightarrow \infty} f_n(\mu_n)(z, z', c) = \mu(z', w, 1_{(z,c)}) \quad (3.2)$$

for each $(z, z', c) \in Z \times Z \times C$. Let

$$\mu^* = d(\mu)$$

and $\mu_n^* = d(\mu_n)$ for each $n \in \mathbb{N}$.

For each $z \in Z$ and $n \in \mathbb{N}$, it follows from $\mu_n \in D_n(\mu_n, \kappa_n)$ that

$$\sum_{(z',c) \in Z \times C} \mu_n(z, w, 1_{(z',c)}) \leq \sum_{(z',c) \in Z \times C} \sum_{\delta \in X} \mu_n(z', m, \delta) \delta(z, c) \leq \sum_{(z',c) \in Z \times C} \kappa_n(z', z, c).$$

By (3.2), $\lim_n \sum_{(z',c) \in Z \times C} \kappa_n(z', z, c) = \sum_{(z',c) \in Z \times C} \mu(z, w, 1_{(z',c)})$ and, hence,

$$\sum_{(z',c) \in Z \times C} \mu(z, w, 1_{(z',c)}) = \sum_{(z',c) \in Z \times C} \sum_{\delta \in X} \mu(z', m, \delta) \delta(z, c) \text{ for each } z \in Z.$$

Thus, for each $z \in Z$,

$$\sum_{(z',\delta) \in Z \times X} \mu^*(z', \delta) \delta_Z(z) = \sum_{(z',c) \in Z \times C} \sum_{\delta \in X} \mu(z', m, \delta) \delta(z, c) = \sum_{(z',c) \in Z \times C} \mu(z, w, 1_{(z',c)}). \quad (3.3)$$

Claim 3 μ^* is a matching.

Proof. Condition 1 follows because if $(z, \delta) \in Z \times X_\emptyset$ is such that $\mu^*(z, \delta) > 0$, then $\mu(z, a, \delta) > 0$ for some $a \neq w$ and hence $(z, \delta) \in \text{graph}(P_m) \cup \text{graph}(P_s)$ since $\mu \in \mathcal{T}$.

Condition 2 holds since, for each $z \in Z$, $\mu = \lim_n \mu_n$ and $\mu_n \in D_n(\mu_n, \kappa_n)$ for each $n \in \mathbb{N}$ imply that

$$\begin{aligned} \nu(z) &= \sum_{\delta \in X} \mu(z, m, \delta) + \sum_{\delta \in X_\emptyset \setminus X} \mu(z, s, \delta) + \sum_{(z',c) \in Z \times C} \mu(z, w, 1_{(z',c)}) \\ &= \sum_{\delta \in X} \mu^*(z, \delta) + \sum_{\delta \in X_\emptyset \setminus X} \mu^*(z, \delta) + \sum_{(z',\delta) \in Z \times X} \mu^*(z', \delta) \delta_Z(z), \end{aligned}$$

the last equality following by (3.3). ■

Claim 4 If $(z, z', c, \delta) \in Z \times Z \times C \times X$ is such that $(z, \delta) \in \text{supp}(\mu^*)$ and $(z', c) \in \text{supp}(\delta)$, then $\mu(z', w, 1_{(z',c)}) > 0$.

Proof. We have that $\sum_{\delta' \in X} \mu(z, m, \delta') \delta'(z', c) \geq \mu(z, m, \delta) \delta(z', c) > 0$ and, thus, $\kappa(z, z', c) > 0$ since $\mu_n \in D_n(\mu_n, \kappa_n)$ for each $n \in \mathbb{N}$. Hence, (3.2) implies that $\mu(z', w, 1_{(z',c)}) > 0$. ■

Claim 5 $\text{supp}(\mu^*) \subseteq IR(\mu^*)$.

Proof. Suppose not; then there exists $(z^*, \delta^*) \in \text{supp}(\mu^*) \setminus IR(\mu^*)$. We claim that there exists $z \in Z$, $(a, \delta) \in \Delta$ and $1_{(\emptyset, c')} \in P_s(z)$ such that

1. $\mu(z, a, \delta) > 0$ and

2. $(s, 1_{(\emptyset, c')}, \mu^*) \succ_z (a, \delta, \mu^*)$.

Indeed, in cases (i) and (iii) of the definition of $IR(\mu^*)$, let $(z, \delta) = (z^*, \delta^*)$ in both cases and $a = m$ in case (i) and $a = s$ in case (iii). In case (ii) of the definition of $IR(\mu^*)$, we have that $\delta^* \in X$ and there exist $(z', c) \in \text{supp}(\delta^*)$ and $c' \in C$ such that $1_{(\emptyset, c')} \in P_s(z')$ such that $(s, 1_{(\emptyset, c')}, \mu^*) \succ_{z'} (w, 1_{(z^*, c)}, \mu^*)$. Claim 4 implies that $\mu(z', w, 1_{(z^*, c)}) > 0$; hence, let $z = z'$, $a = w$ and $\delta = 1_{(z^*, c)}$.

We then have that $\mu_n(z, a, \delta) > 0$ and $(s, 1_{(\emptyset, c')}, \mu_n^*) \succ_z (a, \delta, \mu_n^*)$ for n sufficiently large. Then decrease $\mu_n(z, a, \delta)$ and increase $\mu_n(z, s, 1_{(\emptyset, c)})$ by the same amount $\varepsilon \in (0, \mu_n(z, a, \delta))$ to increase the objective function in $D_n(\mu_n, \kappa_n)$ while satisfying the constraints. But this is a contradiction to $\mu_n \in D_n(\mu_n, \kappa_n)$. ■

Claim 6 *If $(z, z', c) \in Z \times Z \times C$ is such that $(z', c) \in T_z^m(\mu^*)$, then there is $N_{z, z', c} \in \mathbb{N}$ such that $\sum_{\delta \in X} \mu_n(z, m, \delta) \delta(z', c) < \kappa_n(z, z', c)$ for each $n \geq N_{z, z', c}$.*

Proof. Let $(z', c) \in T_z^m(\mu^*)$. In case (a) of the definition of $T_z^m(\mu^*)$, there exists $(\tilde{z}, \tilde{c}, \tilde{\delta}) \in Z \times C \times X$ such that $(\tilde{z}, \tilde{\delta}) \in \text{supp}(\mu^*)$, $(z', \tilde{c}) \in \text{supp}(\tilde{\delta})$ and $(w, 1_{(z, c)}, \mu^*) \succ_{z'} (w, 1_{(\tilde{z}, \tilde{c})}, \mu^*)$. Hence, $\mu(z', w, 1_{(\tilde{z}, \tilde{c})}) > 0$ by Claim 4.

In cases (b) and (c) of the definition of $T_z^m(\mu^*)$, there exists $(a, \delta') \in \Delta$ such that $a \neq w$, $(z', \delta') \in \text{supp}(\mu^*)$ and $(w, 1_{(z, c)}, \mu^*) \succ_{z'} (a, \delta', \mu^*)$. Thus, letting $a = w$ and $\delta' = 1_{(\tilde{z}, \tilde{c})}$ in case (a), it follows that, in all cases, there exists $(a, \delta') \in \Delta$ such that $(z', a, \delta') \in \text{supp}(\mu)$ and $(w, 1_{(z, c)}, \mu^*) \succ_{z'} (a, \delta', \mu^*)$.

Let $N_{z, z', c} \in \mathbb{N}$ be such that $\mu_n(z', a, \delta') > 0$ and $(w, 1_{(z, c)}, \mu_n^*) \succ_{z'} (a, \delta', \mu_n^*)$ for each $n \geq N_{z, z', c}$. Thus, for each $n \geq N_{z, z', c}$,

$$\mu_n(z', w, 1_{(z, c)}) = \sum_{\delta \in X} \mu_n(z, m, \delta) \delta(z', c)$$

by Claim 2 and, since $\alpha_{n, (z, z', c)}(a, \delta', \mu_n) > 0$,

$$\begin{aligned} \kappa_n(z, z', c) &= \mu_n(z', w, 1_{(z, c)}) + \frac{1}{n} g_n(\mu_n)(z, z', c) \\ &\geq \mu_n(z', w, 1_{(z, c)}) + \frac{1}{n} \alpha_{n, (z, z', c)}(a, \delta', \mu_n) \mu_n(z', a, \delta') \\ &> \mu_n(z', w, 1_{(z, c)}) = \sum_{\delta \in X} \mu_n(z, m, \delta) \delta(z', c). \end{aligned}$$

■

Claim 7 $\text{supp}(\mu^*) \subseteq S_M(\mu^*)$.

Proof. Suppose not; then there exists $(z^*, \delta^*) \in \text{supp}(\mu^*) \setminus S_M(\mu^*)$. We claim that there exists $z \in Z$, $(a, \delta) \in \Delta$ and $\delta' \in P_m(z)$ such that

1. $\mu(z, a, \delta) > 0$,
2. $\text{supp}(\delta') \subseteq T_z^m(\mu^*) \cup \text{supp}(\delta)$ if $a = m$ and $\text{supp}(\delta') \subseteq T_z^m(\mu^*)$ if $a \neq m$, and
3. $(m, \delta', \mu^*) \succ_z (a, \delta, \mu^*)$.

Indeed, in cases (i) and (iii) of the definition of $S_M(\mu^*)$, let $(z, \delta) = (z^*, \delta^*)$ in both cases and $a = m$ in case (i) and $a = s$ in case (iii). In case (ii) of the definition of $S_M(\mu^*)$, we have that $\delta^* \in X$ and there exist $(z', c) \in \text{supp}(\delta^*)$ and $\delta' \in P_m(z')$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu^*)$ and $(m, \delta', \mu^*) \succ_{z'} (w, 1_{(z^*, c)}, \mu^*)$. Claim 4 implies that $\mu(z', w, 1_{(z^*, c)}) > 0$; hence, let $z = z'$, $a = w$ and $\delta = 1_{(z^*, c)}$.

Let $\theta = 1$ if $\text{supp}(\delta) \cap \text{supp}(\delta') = \emptyset$; otherwise, let $(\bar{z}, \bar{c}) \in \text{supp}(\delta) \cap \text{supp}(\delta')$ be such that $\delta(\bar{z}, \bar{c})/\delta'(\bar{z}, \bar{c}) \leq \delta(z, c)/\delta'(z, c)$ for each $(z, c) \in \text{supp}(\delta) \cap \text{supp}(\delta')$ and define

$$\theta = \min \left\{ 1, \frac{\delta(\bar{z}, \bar{c})}{\delta'(\bar{z}, \bar{c})} \right\}.$$

Let $k \in \mathbb{N}$ be such that $k\theta > 1$.

There is $N \in \mathbb{N}$ such that, for each $n \geq N$,

- (i) $\text{supp}(\mu) \subseteq \text{supp}(\mu_n)$,
- (ii) $\sum_{\delta \in X} \mu_n(z, m, \hat{\delta}) \hat{\delta}(z', c) < \kappa_n(z, z', c)$ for each $(z', c) \in T_z^m(\mu^*)$, and
- (iii) $u_n(z, m, \delta', \mu_n) \geq k u_n(z, a, \delta, \mu_n)$.

Indeed, (i) is clear since Z and Δ are finite. As for (ii), take $N \geq \max_{(z', c) \in Z \times C} N_{z, z', c}$ where, for each $(z', c) \in Z \times C$, $N_{z, z', c}$ is given by Claim 6. Finally, for (iii), we have that $u(z, m, \delta', \mu)/u(z, a, \delta, \mu) > 1$ and $u(z, m, \delta', \mu_n)/u(z, a, \delta, \mu_n) \geq \beta$ for some $\beta > 1$ for all n sufficiently large. Hence,

$$\frac{u_n(z, m, \delta', \mu_n)}{u_n(z, a, \delta, \mu_n)} = \left(\frac{u(z, m, \delta', \mu_n)}{u(z, a, \delta, \mu_n)} \right)^n \geq \beta^n > k$$

for all n sufficiently large.

Fix $n \geq N$ and let $c^* \in C$ be such that $1_{(\emptyset, c^*)} \in P_s(z)$. For each $\varepsilon > 0$, define π_ε by setting, for each $(\hat{z}, \hat{a}, \hat{\delta}) \in Z \times \Delta$,

$$\pi_\varepsilon(\hat{z}, \hat{a}, \hat{\delta}) = \begin{cases} \mu_n(z, a, \delta) - \varepsilon & \text{if } \hat{z} = z, \hat{a} = a \text{ and } \hat{\delta} = \delta, \\ \mu_n(z, m, \delta') + \theta\varepsilon & \text{if } \hat{z} = z, \hat{a} = m \text{ and } \hat{\delta} = \delta', \\ \mu_n(z, s, 1_{(\emptyset, c^*)}) + (1 - \theta)\varepsilon & \text{if } \hat{z} = z, \hat{a} = s \text{ and } \hat{\delta} = 1_{(\emptyset, c^*)}, \\ \mu_n(\hat{z}, \hat{a}, \hat{\delta}) & \text{otherwise.} \end{cases}$$

By (i), $\mu_n(z, a, \delta) > 0$. We have that, for each $\varepsilon \in (0, \mu_n(z, a, \delta))$, $\pi_\varepsilon(\hat{z}, \hat{a}, \hat{\delta}) \geq 0$ for each $(\hat{z}, \hat{a}, \hat{\delta}) \in Z \times \Delta$, $\pi_\varepsilon(\hat{z}, w, 1_{(z', c)}) \leq \mu_n(\hat{z}, w, 1_{(z', c)}) \leq \sum_{\hat{\delta} \in X} \mu_n(z', m, \hat{\delta}) \hat{\delta}(\hat{z}, c)$ for each $(\hat{z}, z', c) \in Z \times Z \times C$ and

$$\sum_{(\hat{a}, \hat{\delta}) \in \Delta} \pi_\varepsilon(\hat{z}, \hat{a}, \hat{\delta}) = \sum_{(\hat{a}, \hat{\delta}) \in \Delta} \mu_n(\hat{z}, \hat{a}, \hat{\delta}) = \nu(\hat{z})$$

for each $\hat{z} \in Z$. In particular, $\pi_\varepsilon \in \mathcal{T}$.

We also have that, for some $\varepsilon \in (0, \mu_n(z, a, \delta))$,

$$\sum_{\hat{\delta} \in X} \pi_\varepsilon(\hat{z}, m, \hat{\delta}) \hat{\delta}(z', c) \leq \kappa_n(\hat{z}, z', c) \text{ for all } (\hat{z}, z', c) \in Z \times Z \times C. \quad (3.4)$$

First, note that it is enough to consider $\hat{z} = z$ and that, for each $(z', c) \in Z \times C$,

$$\sum_{\hat{\delta} \in X} \pi_\varepsilon(z, m, \hat{\delta}) \hat{\delta}(z', c) = \sum_{\hat{\delta} \in X} \mu_n(z, m, \hat{\delta}) \hat{\delta}(z', c) + \varepsilon(-\delta(z', c)1_m(a) + \theta\delta'(z', c)).$$

Thus, (3.4) holds if $(z', c) \notin \text{supp}(\delta')$.

If $a = m$ and $(z', c) \in \text{supp}(\delta') \cap \text{supp}(\delta)$, the definition of (\bar{z}, \bar{c}) implies that:

$$\sum_{\hat{\delta} \in X} \pi_\varepsilon(z, m, \hat{\delta}) \hat{\delta}(z', c) \leq \sum_{\hat{\delta} \in X} \mu_n(z, m, \hat{\delta}) \hat{\delta}(z', c) \leq \kappa_n(z, z', c).$$

If $a \neq m$ and $(z', c) \in \text{supp}(\delta')$ or if $a = m$ and $(z', c) \in \text{supp}(\delta') \setminus \text{supp}(\delta)$, then $(z', c) \in T_z^m(\mu^*)$ and, thus, $\sum_{\hat{\delta} \in X} \mu_n(z, m, \hat{\delta}) \hat{\delta}(z', c) < \kappa_n(z, z', c)$ by (ii). Hence, there is $\varepsilon(z', c) > 0$ such that

$$\sum_{\hat{\delta} \in X} \pi_\varepsilon(z, m, \hat{\delta}) \hat{\delta}(z', c) = \sum_{\hat{\delta} \in X} \mu_n(z, m, \hat{\delta}) \hat{\delta}(z', c) + \varepsilon\theta\delta'(z', c) < \kappa_n(z, z', c)$$

for each $0 < \varepsilon < \varepsilon(z', c)$. Thus, letting $B = \text{supp}(\delta')$ if $a \neq m$, $B = \text{supp}(\delta') \setminus \text{supp}(\delta)$ if $a = m$ and $0 < \varepsilon < \min_{(z', c) \in B} \varepsilon(z', c)$, we have that $\sum_{\hat{\delta} \in X} \pi_\varepsilon(\hat{z}, m, \hat{\delta}) \hat{\delta}(z', c) \leq \kappa_n(\hat{z}, z', c)$ for each $(\hat{z}, z', c) \in Z \times Z \times C$.

Finally, note that

$$\sum_{\hat{z} \in Z, (\hat{a}, \hat{\delta}) \in \Delta} u_n(\hat{z}, \hat{a}, \hat{\delta}, \mu_n) \pi_\varepsilon(\hat{z}, \hat{a}, \hat{\delta}) > \sum_{\hat{z} \in Z, (\hat{a}, \hat{\delta}) \in \Delta} u_n(\hat{z}, \hat{a}, \hat{\delta}, \mu_n) \mu_n(\hat{z}, \hat{a}, \hat{\delta})$$

since $u_n(z, m, \delta', \mu_n) \geq k u_n(z, a, \delta, \mu_n)$ by (iii) since $n \geq N$, $u_n(z, s, 1_{(\emptyset, c^*)}, \mu_n) \geq 1$ and, hence,

$$\sum_{\hat{z} \in Z, (\hat{\delta}, \hat{a}) \in \Delta} u_n(\hat{z}, \hat{a}, \hat{\delta}, \mu_n) \left(\pi_\varepsilon(\hat{z}, \hat{a}, \hat{\delta}) - \mu_n(\hat{z}, \hat{a}, \hat{\delta}) \right) \geq u_n(z, a, \delta, \mu_n) \varepsilon (-1 + k\theta) > 0.$$

In conclusion $\mu_n \notin D_n(\mu_n, \kappa_n)$, a contradiction. ■

It follows from Claims 3, 5 and 7 that μ^* is a stable matching. ■

In the remainder of the proof, we extend Lemma 3.5 using the following limit result.

The second step in the proof of our existence result consists in the following lemma, which considers the special case where Z is finite and, for some $R > 0$, $P_m(z) = M_R(Z \times C)$ for all $z \in Z$.

Lemma 3.6 *If E is a rational and continuous market such that Z is finite and, for some $R > 0$, $P_m(z) = M_R(Z \times C)$ for all $z \in Z$, then E has a stable matching.*

Proof. Let $\{c_n\}_{n=1}^\infty$ be a dense subset of C . For each $k \in \mathbb{N}$, define $C_k = \{c_n : n \leq k\}$.

In addition, enumerate $\mathbb{Q} = \{q_1, q_2, \dots\}$ and, for each $k \in \mathbb{N}$, let, for each $z \in Z$, $P_{m,k}(z)$ be the set of $\delta \in M_R(Z \times C)$ such that $\text{supp}(\delta)$ is a subset of $Z \times C_k$ and, for each $(z', c) \in Z \times C_k$, $\delta(z', c) \in \{q_n : n \leq k\}$, and let $P_{s,k}(z)$ be the set of $\delta \in P_s(z)$ such that $\text{supp}(\delta)$ is a subset of $\{\emptyset\} \times C_k$. Let $X_k = \cup_{z \in Z} P_{m,k}(z)$, $X_{\emptyset,k} = X_k \cup \{1_{(\emptyset, c)} : c \in C_k\}$, $X_{m,k} = X_k$, $X_{s,k} = \{1_{(\emptyset, c)} : c \in C_k\}$, $X_{w,k} = \{1_{(z,c)} : (z,c) \in Z \times C_k\}$ and $\Delta_k = \{(a, \delta) : \delta \in X_{a,k}\}$.

For each $k \in \mathbb{N}$, let $E_k = (Z, \nu, C_k, P_{m,k}, P_{s,k}, (\succ_z)_{z \in Z})$ be a market where \succ_z is restricted to $\Delta_k \times M(Z \times X_{\emptyset,k})$ for each $z \in Z$. Let $\mu_k \in M(Z \times X_{\emptyset,k})$ be a stable

matching in E_k , which exists by Lemma 3.5 since Z , C_k and $P_{m,k}(z)$ are finite for each $z \in Z$.

It follows by part 1 of Corollary 3.1 that we may assume that $\{\mu_k\}_{k=1}^\infty$ converges; let $\mu = \lim_k \mu_k$. It then follows by parts 2 and 3 of Corollary 3.1 that μ is a matching and that $\text{supp}(\mu) \subseteq IR(\mu)$.

The following claim will be used to show that condition (a) of part 4 of Corollary 3.1 holds.

Claim 8 *Let $(\tilde{z}, \tilde{c}) \in T_z^m(\mu)$ and $V_{\tilde{c}}$ be an open neighborhood of \tilde{c} . Then, for all k sufficiently large, there exists $c_k \in C_k$ such that $(\tilde{z}, c_k) \in T_z^m(\mu_k) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$.*

Proof. Let $(\tilde{z}, \tilde{c}) \in T_z^m(\mu)$ and $V_{\tilde{c}}$ be an open neighborhood of \tilde{c} . Then either (i) there exists $(\hat{z}, \hat{\delta}, \hat{c})$ such that $(\hat{z}, \hat{\delta}) \in \text{supp}(\mu)$, $(\tilde{z}, \hat{c}) \in \text{supp}(\hat{\delta})$ and $(w, 1_{(z, \hat{c})}, \mu) \succ_{\tilde{z}} (w, 1_{(\hat{z}, \hat{c})}, \mu)$, or (ii) there exists $\tilde{\delta} \in X_\emptyset$ such that $(\tilde{z}, \tilde{\delta}) \in \text{supp}(\mu)$ and $(w, 1_{(z, \tilde{c})}, \mu) \succ_{\tilde{z}} (a(\tilde{\delta}), \tilde{\delta}, \mu)$, where $a(\tilde{\delta}) = s$ if $\tilde{\delta} \in X_\emptyset \setminus X$ and $a(\tilde{\delta}) = m$ if $\tilde{\delta} \in X$.

Consider case (i) first. Let $O_{\tilde{c}}, O_{\hat{c}}, O_{\hat{\delta}}$ and O_μ be open neighborhoods of $\tilde{c}, \hat{c}, \hat{\delta}$ and μ , respectively, such that $(w, 1_{(z, \tilde{c}'), \mu'}) \succ_{\tilde{z}} (w, 1_{(\hat{z}, \tilde{c}'), \mu'})$ and $\text{supp}(\hat{\delta}') \cap (\{\tilde{z}\} \times O_{\hat{c}}) \neq \emptyset$ for each $\tilde{c}' \in O_{\tilde{c}}, \tilde{c}' \in O_{\hat{c}}, \hat{\delta}' \in O_{\hat{\delta}}$ and $\mu' \in O_\mu$. Since $0 < \mu(\{\tilde{z}\} \times O_{\hat{\delta}}) \leq \liminf_k \mu(\{\tilde{z}\} \times O_{\hat{\delta}})$, it follows that, for each k sufficiently large, there is $\hat{\delta}_k \in O_{\hat{\delta}}$ such that $(\hat{z}, \hat{\delta}_k) \in \text{supp}(\mu_k)$ and, for some $\hat{c}_k \in O_{\hat{c}}, (\tilde{z}, \hat{c}_k) \in \text{supp}(\hat{\delta}_k)$. In addition, $\mu_k \in O_\mu$ and there exists $c_k \in C_k \cap O_{\tilde{c}} \cap V_{\tilde{c}}$ since, respectively, $\mu_k \rightarrow \mu$ and C_k increases to a dense subset of C . Then $(w, 1_{(z, c_k), \mu_k}) \succ_{\tilde{z}} (w, 1_{(\hat{z}, \hat{c}_k), \mu_k})$ and, hence, $(\tilde{z}, c_k) \in T_z^m(\mu_k) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$ for all k sufficiently large.

Consider next case (ii). Let $O_{\tilde{c}}, O_{\tilde{\delta}}$ and O_μ be open neighborhoods of $\tilde{c}, \tilde{\delta}$ and μ , respectively, such that $(w, 1_{(z, \tilde{c}'), \mu'}) \succ_{\tilde{z}} (a(\tilde{\delta}), \tilde{\delta}', \mu')$ for each $\tilde{c}' \in O_{\tilde{c}}, \tilde{\delta}' \in O_{\tilde{\delta}}$ and $\mu' \in O_\mu$. Since $0 < \mu(\{\tilde{z}\} \times O_{\tilde{\delta}}) \leq \liminf_k \mu_k(\{\tilde{z}\} \times O_{\tilde{\delta}})$, it follows that, for each k sufficiently large there is $\tilde{\delta}_k \in O_{\tilde{\delta}}$ such that $(\tilde{z}, \tilde{\delta}_k) \in \text{supp}(\mu_k)$. In addition, $\mu_k \in O_\mu$ and there exists $c_k \in C_k \cap O_{\tilde{c}} \cap V_{\tilde{c}}$ since, respectively, $\mu_k \rightarrow \mu$ and C_k increases to a dense subset of C . Then $(w, 1_{(z, c_k), \mu_k}) \succ_{\tilde{z}} (a(\tilde{\delta}), \tilde{\delta}_k, \mu_k)$ and, hence, $(\tilde{z}, c_k) \in T_z^m(\mu_k) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$ for all k sufficiently large. ■

We now show that condition (a) of part 4 of Corollary 3.1 holds. Let $(z, \delta, \mu) \in$

graph(P_m) \times $M(Z \times X_\emptyset)$, $\delta' \in \Lambda(z, \delta, \mu)$, $V_{\delta'}$ be an open neighborhood of δ' and $\{(z_{k_j}, \delta_{k_j}, \mu_{k_j})\}_{j=1}^\infty$ be a sequence such that $(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \rightarrow (z, \delta, \mu)$ and $(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \in Z_{k_j} \times X_{k_j} \times M(Z_{k_j} \times X_{\emptyset, k_j})$ for each $j \in \mathbb{N}$.

In particular, $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ and we may assume that $\delta'(Z \times C) < R$ and $\text{supp}(\delta')$ is finite, i.e. $\delta' = \sum_{(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')} a(\tilde{z}, \tilde{c}) 1_{(\tilde{z}, \tilde{c})}$ for some $a = (a(\tilde{z}, \tilde{c}))_{(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')}$. Let V_a be an open neighborhood of a and, for each $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')$, $V_{(\tilde{z}, \tilde{c})}$ be an open neighborhood of (\tilde{z}, \tilde{c}) be such that

$$\sum_{(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')} \hat{a}(\tilde{z}, \tilde{c}) 1_{(z(\tilde{z}, \tilde{c}), c(\tilde{z}, \tilde{c}))} \in V_{\delta'}$$

whenever $(z(\tilde{z}, \tilde{c}), c(\tilde{z}, \tilde{c})) \in V_{(\tilde{z}, \tilde{c})}$ for each $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')$ and $\hat{a} \in V_a$. Let $\hat{a} = (\hat{a}(\tilde{z}, \tilde{c}))_{(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')} \in \mathbb{Q}_+^{|\text{supp}(\delta')|} \cap V_a$ and $V_{\tilde{c}}$ be an open neighborhood of \tilde{c} such that $\{\tilde{z}\} \times V_{\tilde{c}} \subseteq V_{(\tilde{z}, \tilde{c})}$.

For each $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \cap T_z^m(\mu)$, and for each k sufficiently large, let $c_k(\tilde{z}, \tilde{c}) \in C_k$ be such that $(\tilde{z}, c_k(\tilde{z}, \tilde{c})) \in T_z^m(\mu_k) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$, which exists by Claim 8.

If $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \setminus T_z^m(\mu)$, then $\delta \in X$, $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta)$ and $0 < \delta(\{\tilde{z}\} \times V_{\tilde{c}}) \leq \liminf_j \delta_{k_j}(\{\tilde{z}\} \times V_{\tilde{c}})$. Hence, for each j sufficiently large, let $c_{k_j}(\tilde{z}, \tilde{c}) \in V_{\tilde{c}}$ be such that $(\tilde{z}, c_{k_j}(\tilde{z}, \tilde{c})) \in \text{supp}(\delta_{k_j})$.

Let $J' \in \mathbb{N}$ be such that, for each $j \geq J'$, $(\tilde{z}, c_{k_j}(\tilde{z}, \tilde{c})) \in T_z^m(\mu_{k_j}) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$ if $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \cap T_z^m(\mu)$ and $(\tilde{z}, c_{k_j}(\tilde{z}, \tilde{c})) \in \text{supp}(\delta_{k_j}) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$ if $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \setminus T_z^m(\mu)$. Thus, letting $\delta'_{k_j} = \sum_{(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')} \hat{a}(\tilde{z}, \tilde{c}) 1_{(\tilde{z}, c_{k_j}(\tilde{z}, \tilde{c}))}$ for each $j \geq J'$, we have that $\delta'_{k_j} \in V_{\delta'}$ and $\text{supp}(\delta'_{k_j}) \subseteq T_z^m(\mu_{k_j}) \cup \text{supp}(\delta_{k_j})$. Since $\{\hat{a}(\tilde{z}, \tilde{c}) : (\tilde{z}, \tilde{c}) \in \text{supp}(\delta')\}$ is finite, it follows that there is $J > J'$ such that $\delta'_{k_j} \in X_{k_j}$ for each $j \geq J$.

An analogous argument shows that condition (b) of part 4 of Corollary 3.1 also holds. Hence, it follows that $\text{supp}(\mu) \subseteq S_M(\mu)$. This together with the fact that μ is a matching and $\text{supp}(\mu) \subseteq IR(\mu)$ shows that μ is stable. ■

The next step of the proof of Theorem 3.1 extends Lemma 3.6 by requiring only that E be rich.

Lemma 3.7 *If E is a rational, continuous, bounded and rich market such that Z is finite, then E has a stable matching.*

Proof. It follows by Debreu (1964, Proposition 3) and by the finiteness of Z that there exists a continuous function $u : Z \times \Delta \times M(Z \times X_\emptyset) \rightarrow [1, 2]$ such that $(a, \delta, \mu) \mapsto u(z, a, \delta, \mu)$ represents \succeq_z for each $z \in Z$, using the fact that $[1, 2]$ and the extended reals are homeomorphic.

Let $R > 0$ be such that $X \subseteq M_R(Z \times C)$, $\Delta^* = (\{m\} \times M_R(Z \times C)) \cup (\{w\} \times X_w) \cup (\{s\} \times X_s)$ and $X^* = M_R(Z \times C) \cup \{1_{(\emptyset, c)} : c \in C\}$. By Tietze Extension Theorem, let $U : Z \times \Delta^* \times M(Z \times X^*) \rightarrow [1, 2]$ be a continuous extension of u .

Let ρ be a metric on $M_R(Z \times C)$. For each $k \in \mathbb{N}$, let

$$G_k = \{(z, \delta) \in Z \times M_R(Z \times C) : \rho(\delta, P_m(z)) \geq k^{-1}\}.$$

Let, by Urysohn's Lemma, $g_k : Z \times M_R(Z \times C) \rightarrow [0, 1]$ be a continuous function such that $g_k^{-1}(1) = \text{graph}(P_m)$ and $g_k^{-1}(0) = G_k$. Then define $U_k : Z \times \Delta^* \times M(Z \times X^*) \rightarrow \mathbb{R}$ by setting, for each $(z, a, \delta, \mu) \in Z \times \Delta^* \times M(Z \times X^*)$,

$$U_k(z, a, \delta, \mu) = \begin{cases} g_k(a, \delta)U(z, a, \delta, \mu) & \text{if } a = m, \\ U(z, a, \delta, \mu) & \text{otherwise.} \end{cases}$$

Let $P_{m,k}(z) = M_R(Z \times C)$ for each $z \in Z$.

Consider the market $E_k = (Z, \nu, C, P_{m,k}, P_s, U_k)$, i.e. E_k is equal to E except that $P_m(z)$ is replaced with $M_R(Z \times C)$ for each $z \in Z$ and u with U_k . Since E_k is rational and continuous with Z finite and $P_{m,k}(z) = M_R(Z \times C)$ for each $z \in Z$, then E_k has a stable matching μ_k by Lemma 3.6.

Let $E^* = (Z, \nu, C, P_m^*, P_s, U)$ where $P_m^*(z) = M_R(Z \times C)$ for each $z \in Z$. To avoid confusion, we write $IR(\mu; E')$ for $IR(\mu)$ and $S_M(\mu; E')$ for $S_M(\mu)$ whenever μ is a matching of a market E' . It follows by part 1 of Corollary 3.1 that we may assume that $\{\mu_k\}_{k=1}^\infty$ converges; let $\mu = \lim_k \mu_k$. It then follows by part 2 of Corollary 3.1 that μ is a matching of E^* .

The proof of part 3 of Corollary 3.1 implies that $\text{supp}(\mu) \subseteq IR(\mu; E^*)$ since the requirement that $\succ_{z,k}$ is the restriction of \succ_z to $\Delta_k \times M(Z_k \times X_{\emptyset,k})$ for each $z \in Z_k$ can be replaced with the following condition: $(s, \delta, \hat{\mu}) \succ_{z,k} (a, \delta', \hat{\mu})$ for each $k \in \mathbb{N}$, $z \in Z_k$, $\delta \in X_{s,k}$, $(a, \delta') \in \Delta_k$ and $\hat{\mu} \in M(Z_k \times X_{\emptyset,k})$ such that $(s, \delta, \hat{\mu}) \succ_z (a, \delta', \hat{\mu})$.

This condition holds because $U_k(z, a, \delta', \hat{\mu}) \leq U(z, a, \delta', \hat{\mu})$ and $U_k(z, s, 1_{(\emptyset, \hat{c})}, \hat{\mu}) = U(z, s, 1_{(\emptyset, \hat{c})}, \hat{\mu})$ for each $k \in \mathbb{N}$, $z \in Z$, $(a, \delta') \in \Delta^*$, $\hat{c} \in C$ and $\hat{\mu} \in M(Z \times X^*)$.

We have that $\text{supp}(\mu) \subseteq \text{graph}(P_m) \cup \text{graph}(P_s)$. Indeed, let $k \in \mathbb{N}$ and $(z, \delta) \in \text{supp}(\mu_k)$. If $\delta \notin M(Z \times C)$, then $\delta \in P_s(z)$ since μ_k is a matching of E_k . If $\delta \in M(Z \times C)$ and $\rho(\delta, P_m(z)) \geq k^{-1}$, then let $\delta' \in P_s(z)$ to obtain that $\text{supp}(\delta') \subseteq T_z^s(\mu_k)$ and $U_k(z, s, \delta', \mu) = U(z, s, \delta', \mu) > 0 = U_k(z, m, \delta, \mu)$, the latter since $U(z, s, \delta', \mu) \in [1, 2]$ and $g_k(m, \delta) = 0$. But this contradicts the stability of μ_k . Hence, it follows that $\rho(\delta, P_m(z)) < k^{-1}$.

Thus, for each $k \in \mathbb{N}$,

$$\text{supp}(\mu_k) \subseteq (\{(z, \delta) \in Z \times M_R(Z \times C) : \rho(\delta, P_m(z)) \leq k^{-1}\}) \cup \text{graph}(P_s).$$

Hence, $\text{supp}(\mu) \subseteq \text{graph}(P_m) \cup \text{graph}(P_s)$ as claimed.

It then follows that μ is a matching of E and that $\text{supp}(\mu) \subseteq IR(\mu; E)$ since $IR(\mu; E^*) \cap (Z \times X_\emptyset) \subseteq IR(\mu; E)$. Claim 9, which is analogous to part 4 of Corollary 3.1, shows that $\text{supp}(\mu) \subseteq S_M(\mu; E)$.

Claim 9 $\text{supp}(\mu) \subseteq S_M(\mu; E)$.

Proof. Let $(z, \delta) \in \text{supp}(\mu)$ and suppose that $(z, \delta) \notin S_M(\mu; E)$. Then either (i) there exists $\delta' \in P_m(z)$ such that $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ and $U(z, m, \delta', \mu) > U(z, a(\delta), \delta, \mu)$, where $a(\delta) = m$ if $\delta \in X$ and $a(\delta) = s$ if $\delta \in X_\emptyset \setminus X$,¹ or (ii) there exists $(z', c) \in \text{supp}(\delta)$ and $\delta' \in P_m(z')$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu)$ and $U(z', m, \delta', \mu) > U(z', w, 1_{(z', c)}, \mu)$.

Consider case (i) first. Let $V_{\delta'}$, V_δ and V_μ be open neighborhoods of δ' , δ and μ , respectively, such that $U(z, m, \gamma', \bar{\mu}) > U(z, a(\delta), \gamma, \bar{\mu})$ for each $\gamma' \in V_{\delta'}$, $\gamma \in V_\delta$ and $\bar{\mu} \in V_\mu$. Let, by the richness of E , \tilde{V}_δ and \tilde{V}_μ be open neighborhoods of δ and μ , respectively, such that $\Lambda_{a(\delta)}(z, \gamma, \bar{\mu}) \cap V_{\delta'} \neq \emptyset$ for each $(\gamma, \bar{\mu}) \in \tilde{V}_\delta \times \tilde{V}_\mu$, where $\Lambda_m(z, \gamma, \bar{\mu}) = \Lambda(z, \gamma, \bar{\mu})$ and $\Lambda_s(z, \gamma, \bar{\mu}) = \Lambda_\emptyset(z, \bar{\mu})$.

¹Note that when $\delta \in X_\emptyset \setminus X$ and $\delta' \in X$, $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ if and only if $\text{supp}(\delta') \subseteq T_z^m(\mu)$.

By Carmona and Podczeck (2009, Lemma 12), there is a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding sequence $\{\delta_{k_j}\}_{j=1}^\infty$ such that $\delta_{k_j} \rightarrow \delta$ and $(z, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$ for each $j \in \mathbb{N}$.

Let $J \in \mathbb{N}$ be such that $\mu_{k_j} \in V_\mu \cap \tilde{V}_\mu$ and $\delta_{k_j} \in V_\delta \cap \tilde{V}_\delta$ for all $j \geq J$ and, for each $j \geq J$, let $\delta'_{k_j} \in \Lambda(z, \delta_{k_j}, \mu_{k_j}) \cap V_{\delta'}$. Then, for each $j \geq J$,

$$U_{k_j}(z, m, \delta'_{k_j}, \mu_{k_j}) = U(z, m, \delta'_{k_j}, \mu_{k_j}) > U(z, a(\delta), \delta_{k_j}, \mu_{k_j}) \geq U_{k_j}(z, a(\delta), \delta_{k_j}, \mu_{k_j})$$

since $\delta'_{k_j} \in P_m(z)$ by the definition of $\Lambda_{a(\delta)}$, and $\text{supp}(\delta'_{k_j}) \subseteq T_z^m(\mu_{k_j}) \cup \text{supp}(\delta_{k_j})$. But this contradicts the stability of μ_{k_j} .

Now assume there exists $(z', c) \in \text{supp}(\delta)$ and $\delta' \in P_m(z')$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu)$ and $U(z', m, \delta', \mu) > U(z', w, 1_{(z,c)}, \mu)$. Let $V_{\delta'}$, V_c and V_μ be open neighborhoods of δ' , c and μ respectively, such that $U(z', m, \hat{\delta}', \hat{\mu}) > U(z', w, 1_{(z,\hat{c})}, \hat{\mu})$ for each $\hat{\delta}' \in V_{\delta'}$, $\hat{c} \in V_c$ and $\hat{\mu} \in V_\mu$. Let, by the richness of E , \tilde{V}_μ be an open neighborhood of μ such that $\Lambda_0(z', \hat{\mu}) \cap V_{\delta'} \neq \emptyset$ for each $\hat{\mu} \in \tilde{V}_\mu$.

By Carmona and Podczeck (2009, Lemma 12), there is a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding sequence $\{(\delta_{k_j}, c_{k_j})\}_{j=1}^\infty$ such that $(\delta_{k_j}, c_{k_j}) \rightarrow (\delta, c)$, $(z, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$ and $(z', c_{k_j}) \in \text{supp}(\delta_{k_j})$ for each $j \in \mathbb{N}$.

Let $J \in \mathbb{N}$ be such that $\delta_{k_j} \in V_\delta$, $c_{k_j} \in V_c$ and $\mu_{k_j} \in V_\mu \cap \tilde{V}_\mu$ for all $j \geq J$ and, for each $j \geq J$, let $\delta'_{k_j} \in \Lambda_0(z', \mu_{k_j}) \cap V_{\delta'}$. Then, for each $j \geq J$, $U_{k_j}(z', m, \delta'_{k_j}, \mu_{k_j}) = U(z', m, \delta'_{k_j}, \mu_{k_j}) > U(z', w, 1_{(z,c_{k_j})}, \mu_{k_j}) \geq U_{k_j}(z', w, 1_{(z,c_{k_j})}, \mu_{k_j})$ since $\delta'_{k_j} \in P_m(z')$ by the definition of Λ_0 , and $\text{supp}(\delta'_{k_j}) \subseteq T_{z'}^m(\mu_{k_j})$. But this contradicts the stability of μ_{k_j} . ■

It follows by $\text{supp}(\mu) \subseteq IR(\mu; E)$ and by Claim 9 that $\text{supp}(\mu) \subseteq S_M(\mu; E) \cap IR(\mu; E)$. Thus, μ is stable. ■

We now complete the proof of our existence result.

Proof of Theorem 3.1. Let $\{\nu_k\}_{k=1}^\infty$ be such that $\nu_k \rightarrow \nu$ and $\text{supp}(\nu_k)$ is a finite subset of Z for each $k \in \mathbb{N}$. Define $Z_k = \text{supp}(\nu_k)$, $Z_{\emptyset,k} = Z_k \cup \{\emptyset\}$, and for each $z \in Z_k$, $P_{m,k}(z) = P_m(z) \cap M(Z_k \times C)$ and $P_{s,k}(z) = P_s(z)$. Let $X_k = \cup_{z \in Z_k} P_{m,k}(z)$, $X_{\emptyset,k} = X_k \cup \{1_{(\emptyset,c)} : c \in C\}$, $X_{m,k} = X_k$, $X_{s,k} = \{1_{(\emptyset,c)} : c \in C\}$, $X_{w,k} = \{1_{(z,c)} : (z,c) \in Z_k \times C\}$ and $\Delta_k = \{(a, \delta) : \delta \in X_{a,k}\}$ for each $k \in \mathbb{N}$. Note

that $P_{m,k}$ is upper hemicontinuous (since its domain is finite) and compact valued (since P_m is compact valued and $M(Z_k \times C)$ is closed) for each $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$, let $\tilde{E}_k = (Z_k, \nu_k, C, P_{m,k}, P_{s,k}, (\succ_z)_{z \in Z})$ be a market where \succ_z is restricted to $\Delta_k \times M(Z_k \times X_{\emptyset,k})$ for each $z \in Z$. Furthermore, let E_k be exactly as \tilde{E}_k , except with P_m in place of $P_{m,k}$, P_s in place of $P_{s,k}$ and Z in place of Z_k ; more precisely, $E_k = (Z, \nu_k, C, P_m, P_s, (\succ_z)_{z \in Z})$.

Claim 10 *For each $k \in \mathbb{N}$, if μ is a stable matching of \tilde{E}_k , then μ is a stable matching of E_k .*

Proof. In this proof, to avoid confusion, we write $IR(\mu; E)$ for $IR(\mu)$ and $S_M(\mu; E)$ for $S_M(\mu)$ whenever μ is a matching of a market E .

Let $k \in \mathbb{N}$ and μ be a stable matching of \tilde{E}_k . Clearly, μ is a matching of E_k and $\text{supp}(\mu) \subseteq IR(\mu; E_k)$. We show that $\text{supp}(\mu) \subseteq S_M(\mu; E_k)$. Suppose not; then let $(z, \delta) \in \text{supp}(\mu) \setminus S_M(\mu; E_k)$.

First suppose that there exists $\delta' \in P_m(z)$ such that $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ and $(m, \delta', \mu) \succ_z (a(\delta), \delta, \mu)$, where $a(\delta) = m$ if $\delta \in X$ and $a(\delta) = s$ if $\delta \in X_\emptyset \setminus X$. We claim that $\delta' \in P_{m,k}(z)$, i.e. that $\text{supp}(\delta') \subseteq Z_k \times C$, from which we obtain a contradiction to the stability of μ in \tilde{E}_k .

Note that $\text{supp}(\bar{\delta}) \subseteq Z_k \times C$ whenever $\bar{\delta} \in X$ and $(\bar{z}, \bar{\delta}) \in \text{supp}(\mu)$ for some $\bar{z} \in Z_k$ since μ is stable in \tilde{E}_k . Thus, it follows that $\text{supp}(\delta') \cap \text{supp}(\delta) \subseteq Z_k \times C$ since if $\text{supp}(\delta') \cap \text{supp}(\delta) \neq \emptyset$, then $\delta \in X$. We also have that $\text{supp}(\delta') \cap T_z^m(\mu) \subseteq Z_k \times C$. Indeed, if $(z', c) \in T_z^m(\mu)$, then either $z' \in \text{supp}(\mu_Z) \subseteq Z_k$ or $(z', \bar{c}) \in \text{supp}(\bar{\delta})$ and $(\bar{z}, \bar{\delta}) \in \text{supp}(\mu)$ for some $\bar{c} \in C$, $\bar{z} \in Z_k$ and $\bar{\delta} \in X$; hence, $z' \in Z_k$. Thus, $\text{supp}(\delta') = (\text{supp}(\delta') \cap \text{supp}(\delta)) \cup (\text{supp}(\delta') \cap T_z^m(\mu)) \subseteq Z_k \times C$ as desired.

Consider next the case where there exists $(z', c) \in \text{supp}(\delta)$ and $\delta' \in P_m(z')$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu)$ and $(m, \delta', \mu) \succ_{z'} (w, 1_{(z,c)}, \mu)$. As above, we obtain a contradiction to the stability of μ in \tilde{E}_k by showing that $\delta' \in P_{m,k}(z')$. To establish this claim, it suffices to show that $T_{z'}^m(\mu) \subseteq Z_k \times C$. If $(\tilde{z}, \tilde{c}) \in T_{z'}^m(\mu)$, then either $\tilde{z} \in \text{supp}(\mu_Z) \subseteq Z_k$ or $(\tilde{z}, \tilde{c}) \in \text{supp}(\bar{\delta})$ and $(\bar{z}, \bar{\delta}) \in \text{supp}(\mu)$ for some $\tilde{c} \in C$, $\tilde{z} \in Z_k$ and $\bar{\delta} \in X$; hence, $(\tilde{z}, \tilde{c}) \in Z_k \times C$ as required. ■

For each $k \in \mathbb{N}$, let $\mu_k \in M(Z \times X_{\emptyset,k})$ be a stable matching in E_k , which exists by Lemma 3.7 (since Z_k is finite and \tilde{E}_k satisfies its assumptions) and Claim 10.²

It follows by part 1 of Corollary 3.1 that we may assume that $\{\mu_k\}_{k=1}^\infty$ converges; let $\mu = \lim_k \mu_k$. It then follows by parts 2 – 4 of Corollary 3.1 that μ is a matching and that $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$. Hence, μ is stable. ■

3.2 Sufficient condition for richness

It is often easy to establish the richness of a market by showing that the following two conditions hold for each $z \in Z$:

- (α) For each $\delta \in P_m(z)$ such that $\delta = \sum_{j=1}^J a_j 1_{(z_j, c_j)}$ for some $J \in \mathbb{N}$, $a_j \in \mathbb{R}_{++}$ and $(z_j, c_j) \in Z \times C$ for each $j = 1, \dots, J$ and each open neighborhood V_δ of δ , there exist open neighborhoods V_z of z and $V_{(z_j, c_j)}$ of (z_j, c_j) for each $j = 1, \dots, J$ such that, whenever $\hat{z} \in V_z$ and $(\hat{z}_j, \hat{c}_j) \in V_{(z_j, c_j)}$ for each $j = 1, \dots, J$, there exists $\hat{a} = (\hat{a}_1, \dots, \hat{a}_J) \in \mathbb{R}_+^J$ such that $\sum_{j=1}^J \hat{a}_j 1_{(\hat{z}_j, \hat{c}_j)} \in V_\delta \cap P_m(\hat{z})$.
- (β) For each $\delta \in P_m(z)$ and open neighborhood V_δ of δ , there exists $\hat{\delta} \in V_\delta \cap P_m(z)$ such that $\text{supp}(\hat{\delta})$ is a finite subset of $\text{supp}(\delta)$.

Lemma 3.8 *Let E be a market such that $\{(a, \delta, \mu, a', \delta', \mu', z) \in (\Delta \times M(Z \times X_\emptyset))^2 \times Z : (a, \delta, \mu) \succ_z (a', \delta', \mu')\}$ is open. If (α) and (β) hold, then E is rich.*

We use the following lemma in the proof of Lemma 3.8.

²It is clear that \tilde{E}_k is rational, continuous and bounded. It can be shown that \tilde{E}_k is rich along the lines of Claim 10: Let $\Lambda_k : Z_k \times X_k \times \mathcal{M}(Z_k \times X_{\emptyset,k}) \rightrightarrows X_k$ be defined by setting, for each $(z, \delta, \mu) \in Z_k \times X_k \times \mathcal{M}(Z_k \times X_{\emptyset,k})$, $\Lambda_k(z, \delta, \mu) = \{\delta' \in X_k : \text{supp}(\delta') \subseteq \text{supp}(\delta) \cup T_z^m(\mu; \tilde{E}_k)\}$. Let $(z, \delta, \mu) \in Z_k \times X_k \times \mathcal{M}(Z_k \times X_{\emptyset,k})$ and O be open and such that $(z, \delta) \in \text{supp}(\mu)$ and $\Lambda_k(z, \delta, \mu) \cap O \neq \emptyset$. Since $T_z(\mu; \tilde{E}_k) \subseteq T_z(\mu; E)$, it follows that $\Lambda(z, \delta, \mu) \cap O \neq \emptyset$. Hence, there is an open neighborhood V of (z, δ, μ) such that, for each $(\tilde{z}, \tilde{\delta}, \tilde{\mu}) \in V$, $\Lambda(\tilde{z}, \tilde{\delta}, \tilde{\mu}) \cap O \neq \emptyset$. Thus, for each $(\tilde{z}, \tilde{\delta}, \tilde{\mu}) \in V \cap (Z_k \times X_k \times \mathcal{M}(Z_k \times X_{\emptyset,k}))$, let $\delta' \in \Lambda(\tilde{z}, \tilde{\delta}, \tilde{\mu}) \cap O$. We have that $\text{supp}(\delta') \subseteq Z_k \times C$ and that $T_{\tilde{z}}^m(\tilde{\mu}; E) \subseteq (Z_k \times C) \cap T_{\tilde{z}}^m(\tilde{\mu}; \tilde{E}_k)$. Hence, $\delta' \in X_k$ and $\text{supp}(\delta') \subseteq \text{supp}(\tilde{\delta}) \cup T_{\tilde{z}}^m(\tilde{\mu}; \tilde{E}_k)$ and it follows that $\delta' \in \Lambda_k(\tilde{z}, \tilde{\delta}, \tilde{\mu}) \cap O$. Thus, Λ_k is lower hemicontinuous and an analogous argument shows that $\Lambda_{0,k}$ is also lower hemicontinuous.

Lemma 3.9 *Let $z \in Z$, $\mu \in M(Z \times X_\emptyset)$, $\{(z_k, \mu_k)\}_{k=1}^\infty$ be such that $(z_k, \mu_k) \rightarrow (z, \mu)$, $(\tilde{z}, \tilde{c}) \in T_z^m(\mu)$, and $V_{\tilde{z}}$ be an open neighborhood of \tilde{z} . Then, for all k sufficiently large, there exists $\tilde{z}_k \in V_{\tilde{z}}$ such that $(\tilde{z}_k, \tilde{c}) \in T_{z_k}^m(\mu_k)$.*

Proof. Let $(\tilde{z}, \tilde{c}) \in T_z^m(\mu)$. Then (i) there exists $(\hat{z}, \hat{\delta}, \hat{c})$ such that $(\hat{z}, \hat{\delta}) \in \text{supp}(\mu)$, $(\tilde{z}, \tilde{c}) \in \text{supp}(\hat{\delta})$ and $(w, 1_{(z, \tilde{c})}, \mu) \succ_{\tilde{z}} (w, 1_{(\hat{z}, \hat{c})}, \mu)$, or (ii) there exists $\tilde{\delta} \in X_\emptyset$ such that $(\tilde{z}, \tilde{\delta}) \in \text{supp}(\mu)$ and $(w, 1_{(z, \tilde{c})}, \mu) \succ_{\tilde{z}} (a(\tilde{\delta}), \tilde{\delta}, \mu)$, where $a(\tilde{\delta}) = s$ if $\tilde{\delta} \in X_\emptyset \setminus X$ and $a(\tilde{\delta}) = m$ if $\tilde{\delta} \in X$.

Consider case (i) first. Let $O_z, O_{\tilde{z}}, O_{\hat{z}}, O_{\hat{c}}, O_{\hat{\delta}}$ and O_μ be open neighborhoods of $z, \tilde{z}, \hat{z}, \hat{c}, \hat{\delta}$ and μ , respectively, such that $(w, 1_{(z', \tilde{c})}, \mu') \succ_{\tilde{z}'} (w, 1_{(\hat{z}', \hat{c}')}, \mu')$ and $\text{supp}(\hat{\delta}') \cap ((O_{\tilde{z}} \cap V_{\tilde{z}}) \times O_{\hat{c}}) \neq \emptyset$ for each $z' \in O_z, \tilde{z}' \in O_{\tilde{z}}, \hat{z}' \in O_{\hat{z}}, \hat{c}' \in O_{\hat{c}}, \hat{\delta}' \in O_{\hat{\delta}}$ and $\mu' \in O_\mu$. We have that $0 < \mu(O_{\hat{z}} \times O_{\hat{\delta}}) \leq \liminf_k \mu_k(O_{\hat{z}} \times O_{\hat{\delta}})$ since $\mu_k \rightarrow \mu$. Thus, it follows from $z_k \rightarrow z$ and $\mu_k \rightarrow \mu$ that, for each k sufficiently large, $\mu_k \in O_\mu, z_k \in O_z$ and there is $(\hat{z}_k, \hat{\delta}_k) \in \text{supp}(\mu_k) \cap (O_{\hat{z}} \times O_{\hat{\delta}})$ and $(\tilde{z}_k, \hat{c}_k) \in \text{supp}(\hat{\delta}_k) \cap ((O_{\tilde{z}} \cap V_{\tilde{z}}) \times O_{\hat{c}})$. Then $(w, 1_{(z_k, \tilde{c})}, \mu_k) \succ_{\tilde{z}_k} (w, 1_{(\hat{z}_k, \hat{c}_k)}, \mu_k)$ and, hence, $(\tilde{z}_k, \tilde{c}) \in T_{z_k}^m(\mu_k)$.

Consider case (ii) next. Let $O_z, O_{\tilde{z}}, O_{\tilde{\delta}}$ and O_μ be open neighborhoods of $z, \tilde{z}, \tilde{\delta}$ and μ , respectively, such that $(w, 1_{(z', \tilde{c})}, \mu') \succ_{\tilde{z}'} (a(\tilde{\delta}), \tilde{\delta}', \mu')$ for each $z' \in O_z, \tilde{z}' \in O_{\tilde{z}}, \tilde{\delta}' \in O_{\tilde{\delta}}$ and $\mu' \in O_\mu$. We have that $0 < \mu((O_{\tilde{z}} \cap V_{\tilde{z}}) \times O_{\tilde{\delta}}) \leq \liminf_k \mu_k((O_{\tilde{z}} \cap V_{\tilde{z}}) \times O_{\tilde{\delta}})$ since $\mu_k \rightarrow \mu$; hence, for each k sufficiently large, there is $(\tilde{z}_k, \tilde{\delta}_k) \in \text{supp}(\mu_k) \cap ((O_{\tilde{z}} \cap V_{\tilde{z}}) \times O_{\tilde{\delta}})$. Since $\mu_k \rightarrow \mu$ and $z_k \rightarrow z$, it follows that, for each k sufficiently large, $\mu_k \in O_\mu$ and $z_k \in O_z$. Then $(w, 1_{(z_k, \tilde{c})}, \mu_k) \succ_{\tilde{z}_k} (a(\tilde{\delta}), \tilde{\delta}_k, \mu_k)$ and, hence, $(\tilde{z}_k, \tilde{c}) \in T_{z_k}^m(\mu_k)$. ■

We can now turn to the proof of the lemma.

Proof of Lemma 3.8. Let $(z, \delta, \mu) \in \text{graph}(P_m) \times M(Z \times X_\emptyset)$ and $V \subseteq X$ be open and such that $\Lambda(z, \delta, \mu) \cap V \neq \emptyset$. Let $\delta' \in \Lambda(z, \delta, \mu) \cap V$, i.e. V is an open neighborhood of δ' , $\delta' \in P_m(z)$ and $\text{supp}(\delta') \subseteq \text{supp}(\delta) \cup T_z^m(\mu)$.

We may assume that $\text{supp}(\delta')$ is finite by condition (β) since $\delta' \in P_m(z)$. Let V_z and $V_{(\tilde{z}, \tilde{c})}$ for each $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')$ be given by condition (α) . Let $V_{\tilde{z}}$ and $V_{\tilde{c}}$ be open neighborhoods of \tilde{z} and \tilde{c} such that $V_{\tilde{z}} \times V_{\tilde{c}} \subseteq V_{(\tilde{z}, \tilde{c})}$. Let O_δ be an open neighborhood of δ such that $\text{supp}(\tilde{\delta}) \cap (V_{\tilde{z}} \times V_{\tilde{c}}) \neq \emptyset$ for each $\tilde{\delta} \in O_\delta$.

We claim that there is an open neighborhood of (z, δ, μ) such that, for each

(z^*, δ^*, μ^*) in it, $\Lambda(z^*, \delta^*, \mu^*) \cap V \neq \emptyset$. Suppose not; then there is a sequence $\{(z_k, \delta_k, \mu_k)\}_{k=1}^\infty$ such that $(z_k, \delta_k, \mu_k) \rightarrow (z, \delta, \mu)$ and

$$\Lambda(z_k, \delta_k, \mu_k) \cap V = \emptyset \text{ for each } k \in \mathbb{N}. \quad (3.5)$$

For each $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \cap T_z^m(\mu)$ and k sufficiently large, let $z_k(\tilde{z}, \tilde{c}) \in V_{\tilde{z}}$ and $c_k(\tilde{z}, \tilde{c}) = \tilde{c} \in V_{\tilde{c}}$ be such that $(z_k(\tilde{z}, \tilde{c}), c_k(\tilde{z}, \tilde{c})) \in T_{z_k}^m(\mu_k) \cap V_{(\tilde{z}, \tilde{c})}$, which exists by Lemma 3.9.

For each $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \cap \text{supp}(\delta)$ and k sufficiently large, let $z_k(\tilde{z}, \tilde{c}) \in V_{\tilde{z}}$ and $c_k(\tilde{z}, \tilde{c}) \in V_{\tilde{c}}$ be such that $(z_k(\tilde{z}, \tilde{c}), c_k(\tilde{z}, \tilde{c})) \in \text{supp}(\delta_k) \cap V_{(\tilde{z}, \tilde{c})}$.

Let $K \in \mathbb{N}$ be such that, for each $k \geq K$, $z_k \in V_z$, $(z_k(\tilde{z}, \tilde{c}), c_k(\tilde{z}, \tilde{c})) \in T_{z_k}^m(\mu_k) \cap V_{(\tilde{z}, \tilde{c})}$ if $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \cap T_z^m(\mu)$ and $(z_k(\tilde{z}, \tilde{c}), c_k(\tilde{z}, \tilde{c})) \in \text{supp}(\delta_k) \cap V_{(\tilde{z}, \tilde{c})}$ if $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \setminus T_z^m(\mu)$. Condition (α) gives $(a_k(\tilde{z}, \tilde{c}))_{(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')} \in \mathbb{R}_+^{|\text{supp}(\delta')|}$ such that, setting $\delta'_k = \sum_{(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')} a_k(\tilde{z}, \tilde{c}) 1_{(z_k(\tilde{z}, \tilde{c}), c_k(\tilde{z}, \tilde{c}))}$ for each $k \geq K$, we have that $\delta'_k \in V \cap P_m(z_k)$ and $\text{supp}(\delta'_k) \subseteq T_{z_k}^m(\mu_k) \cup \text{supp}(\delta_k)$. But this contradicts (3.5). Thus, it follows that Λ is lower hemicontinuous.

The proof that Λ_0 is lower hemicontinuous is analogous. ■

4 Relationship with Carmona and Laohakunakorn (2024)

A market in Carmona and Laohakunakorn (2024) is $E = (Z, \nu, C, \mathbb{C}, X, (\succ_z)_{z \in Z})$, where Z , ν , C and $(\succ_z)_{z \in Z}$ are as in this section, X is a subset of $M(Z \times C)$ and $\mathbb{C} : Z \times Z_\emptyset \rightrightarrows C$ is the contract correspondence. We define a market \hat{E} as in this section by setting, for each $z \in Z$, $P_m(z) = \{\delta \in X : \{z\} \times \text{supp}(\delta) \subseteq \text{graph}(\mathbb{C})\}$ and $P_s(z) = \{1_{(\emptyset, c)} : c \in \mathbb{C}(z, \emptyset)\}$.

Note that $\cup_{z \in Z} P_m(z) \subseteq X$ and, hence, a matching in \hat{E} is a measure on $M(Z \times X_\emptyset)$. We have that $\mu \in M(Z \times X_\emptyset)$ is a matching in E if and only if it is a matching in \hat{E} . Indeed, for each $(z, \delta) \in \text{supp}(\mu)$, $\delta \in P_m(z) \cup P_s(z)$ holds if and only if $\{z\} \times \text{supp}(\delta) \subseteq \text{graph}(\mathbb{C})$.

We also have that a matching μ is stable in E if and only if it is stable in \hat{E} . To see this, note that $IR(\mu)$ and $\hat{I}R(\mu)$ differ only because the requirement that $\text{supp}(\delta') \subseteq T_{z^*}^s(\mu) = \{\emptyset\} \times \mathbb{C}(z^*, \emptyset)$ in $IR(\mu)$ is replaced with $\delta' \in P_s(z^*)$ in \hat{E} , where $z^* \in \{z, z'\}$. Since $\delta' \in P_s(z^*)$ is equivalent to $\text{supp}(\delta') \subseteq \{\emptyset\} \times \mathbb{C}(z^*, \emptyset)$, it follows that $IR(\mu) = \hat{I}R(\mu)$.

Similarly, $S_M(\mu) \cap \text{supp}(\mu)$ and $\hat{S}_M(\mu) \cap \text{supp}(\mu)$ differ only because the requirements that (i) $\delta' \in X$ and $\text{supp}(\delta') \subseteq T_{z^*}^m(\mu)$ where $z^* \in \{z, z'\}$, and (ii) $\delta' \in X$ and $\text{supp}(\delta') \subseteq \text{supp}(\delta) \cup T_z^m(\mu)$ in $S_M(\mu)$, where $(z, \delta) \in \text{supp}(\mu)$, are replaced with (a) $\delta' \in P_m(z^*)$ and $\text{supp}(\delta') \subseteq \hat{T}_{z^*}^m(\mu)$ and (b) $\delta' \in P_m(z^*)$ and $\text{supp}(\delta') \subseteq \text{supp}(\delta) \cup \hat{T}_{z^*}^m(\mu)$ in $S_M(\mu)$ in \hat{E} . Note that (i) is equivalent to $\delta' \in X$, $\text{supp}(\delta') \subseteq \hat{T}_{z^*}^m(\mu)$ and $\{z\} \times \text{supp}(\delta') \subseteq \text{graph}(\mathbb{C})$ and, thus, to (a). Regarding (ii), since μ is a matching and $(z, \delta) \in \text{supp}(\mu)$, it follows $\{z\} \times \text{supp}(\delta) \subseteq \text{graph}(\mathbb{C})$. Hence, (ii) is equivalent to $\delta' \in X$, $\text{supp}(\delta') \subseteq \text{supp}(\delta) \cup \hat{T}_{z^*}^m(\mu)$ and $\{z\} \times \text{supp}(\delta') \subseteq \text{graph}(\mathbb{C})$ and, thus, to (b).

We also claim that the existence result in Carmona and Laohakunakorn (2024) follows from the one established here in the case where Z and C are compact. To this end, suppose that E is rational, continuous, bounded and rich; we will show that \hat{E} satisfies the properties of Theorem 3.1. Since $\hat{X} = \cup_{z \in Z} P_m(z) \subseteq X$, it follows that \hat{E} is bounded and that the set $\{(a, \delta, \mu, a', \delta', \mu', z) \in (\Delta \times M(Z \times \hat{X}_\emptyset))^2 \times Z : (a, \delta, \mu) \succ_z (a', \delta', \mu')\}$ is open.

We have that $\hat{\Lambda}(z, \delta, \mu) = \Lambda(z, \delta, \mu)$ for each $(z, \delta, \mu) \in \text{graph}(P_m) \times M(Z \times \hat{X}_\emptyset)$ and $\hat{\Lambda}_0(z, \mu) = \Lambda_0(z, \mu)$ for each $(z, \mu) \in Z \times M(Z \times \hat{X}_\emptyset)$ by what was noted above. Hence, \hat{E} is rich.

We next show that P_s is continuous with nonempty and compact values. Note that $g : C \rightarrow \{1_{(\emptyset, c)} : c \in C\}$ defined by setting, for each $c \in C$, $g(c) = 1_{(\emptyset, c)}$ is continuous and that $P_s = g \circ \mathbb{C}$. Hence, P_s is continuous with nonempty and compact values since \mathbb{C} has the same properties.

We conclude by establishing that P_m is compact-valued and upper hemicontinuous. Note that, for each $z \in Z$,

$$P_m(z) = X \cap \{\delta \in M(Z \times C) : c \in \mathbb{C}(z, z') \text{ for each } (z', c) \in \text{supp}(\delta)\}$$

and that X is compact since it is a closed subset of $M_R(Z \times C)$. If $(z, \delta) \in Z \times M(Z \times \delta)$ and $\{(z_k, \delta_k)\}_{k=1}^\infty$ are such that $(z_k, \delta_k) \rightarrow (z, \delta)$ and $\delta_k \in P_m(z_k)$ for each $k \in \mathbb{N}$, then, for each $(z', c) \in \text{supp}(\delta)$, there is a subsequence $\{z_{k_j}, \delta_{k_j}\}_{j=1}^\infty$ of $\{(z_k, \delta_k)\}_{k=1}^\infty$ and $\{(z'_{k_j}, c_{k_j})\}_{j=1}^\infty$ such that $(z'_{k_j}, c_{k_j}) \rightarrow (z', c)$ and $(z'_{k_j}, c_{k_j}) \in \text{supp}(\delta_{k_j})$ for each $j \in \mathbb{N}$. Then $c_{k_j} \in \mathbb{C}(z_{k_j}, z'_{k_j})$ since $\delta_{k_j} \in P_m(z_{k_j})$ for each $j \in \mathbb{N}$ and, hence, $c \in \mathbb{C}(z, z')$ since \mathbb{C} is upper hemicontinuous. It then follows that P_m is closed and has a compact range, thus, P_m is compact-valued and upper hemicontinuous.

5 A characterization of stable outcomes in knowledge economies

We provide a proof of the characterization of stable outcomes in knowledge economies stated in Section 2.2. We start with the following lemma.

Lemma 5.1 *If (γ, v) is an outcome in an economy E and $z \in Z$, then there exists $(z^*, n^*) \in Z_\emptyset \times N$ such that $(z, z^*, n^*) \in \text{supp}(\gamma)$ or $(z^*, z, n^*) \in \text{supp}(\gamma)$.*

Proof. Let (γ, v) be an outcome in an economy E and assume that the conclusion of the lemma fails. Then $(\{z\} \times Z_\emptyset \times N) \cap \text{supp}(\gamma) = \emptyset = (Z \times \{z\} \times N) \cap \text{supp}(\gamma)$. Since $Z \times Z_\emptyset \times N$ is a metric space, hence normal, and $\{z\} \times Z_\emptyset \times N$, $Z \times \{z\} \times N$ and $\text{supp}(\gamma)$ are closed, there exist open sets O_1 , O_2 , V_1 and V_2 such that $\{z\} \times Z_\emptyset \times N \subseteq O_1$, $\text{supp}(\gamma) \subseteq V_1$, $O_1 \cap V_1 = \emptyset$, $Z \times \{z\} \times N \subseteq O_2$, $\text{supp}(\gamma) \subseteq V_2$ and $O_2 \cap V_2 = \emptyset$.

We claim that there is an open neighborhoods O_z of z and an open set V such that $\{z\} \times Z_\emptyset \times N \subseteq O_z \times Z_\emptyset \times N$, $Z \times \{z\} \times N \subseteq Z \times O_z \times N$, $\text{supp}(\gamma) \subseteq V$, $(O_z \times Z_\emptyset \times N) \cap V = \emptyset$ and $(Z \times O_z \times N) \cap V = \emptyset$.

To establish the above claim, first note that letting $O = O_1 \cup O_2$ and $V = V_1 \cap V_2$, it follows that O and V are open, $\{z\} \times Z_\emptyset \times N \subseteq O$, $Z \times \{z\} \times N \subseteq O$, $\text{supp}(\gamma) \subseteq V$ and $O \cap V = \emptyset$.

Since O is open, for each $y \in Z_\emptyset \times N$, there are open sets $O_y \subseteq Z$ and $\hat{O}_y \subseteq Z_\emptyset \times N$ such that $(z, y) \in O_y \times \hat{O}_y \subseteq O$. Since $Z_\emptyset \times N$ is compact, there is a finite set $\{y_1, \dots, y_k\}$ such that $Z_\emptyset \times N = \cup_{i=1}^k \hat{O}_{y_i}$. Let $\hat{O}_z = \cap_{i=1}^k O_{y_i}$. Then $\{z\} \times Z_\emptyset \times N \subseteq$

$\hat{O}_z \times Z_\emptyset \times N$ since $z \in O_{y_i}$ for each $1 \leq i \leq k$ and $(\hat{O}_z \times Z_\emptyset \times N) \cap V = \emptyset$ since if $(\hat{z}, y) \in \hat{O}_z \times (Z_\emptyset \times N)$, then $y \in \hat{O}_{y_i}$ for some $1 \leq i \leq k$ and, hence, $(\hat{z}, y) \in O_{y_i} \times \hat{O}_{y_i} \subseteq O \subseteq V^c$.

The above argument shows that there is an open neighborhood \hat{O}_z of z such that $\{z\} \times Z_\emptyset \times N \subseteq \hat{O}_z \times Z_\emptyset \times N$ and $(\hat{O}_z \times Z_\emptyset \times N) \cap V = \emptyset$. An analogous argument shows that there is an open neighborhood \tilde{O}_z of z such that $Z \times \{z\} \times N \subseteq Z \times \tilde{O}_z \times N$ and $(Z \times \tilde{O}_z \times N) \cap V = \emptyset$. Then let $O_z = \hat{O}_z \cap \tilde{O}_z$ to establish the above claim.

It then follows that $(O_z \times Z_\emptyset \times N) \cap \text{supp}(\gamma) = \emptyset = (Z \times O_z \times N) \cap \text{supp}(\gamma)$ and $\nu(O_z) > 0$, the latter since $\text{supp}(\nu) = Z$. Since γ is an assignment,

$$0 < \nu(O_z) = \gamma(O_z \times Z_\emptyset \times N) + \int_{Z \times O_z \times N} \tilde{n} d\gamma(\hat{z}, \tilde{z}, \tilde{n}) = 0,$$

a contradiction. This contradiction establishes the lemma. ■

With Lemma 5.1, it is then easy to establish the characterization of stable outcomes in Section 2.2.

Theorem 5.1 *An outcome (γ, v) in an economy E is stable if and only if*

$$v(z) + nv(z') = F(z, z', n) \text{ for each } (z, z', n) \in \text{supp}(\gamma), \text{ and}$$

$$v(z) = \max \left\{ \max_{\tilde{z} \in Z, \tilde{n} \in \bar{N}(z, \tilde{z})} (F(z, \tilde{z}, \tilde{n}) - \tilde{n}v(\tilde{z})), U_s(z), \max_{\hat{z} \in Z, \hat{n} \in \bar{N}(\hat{z}, z)} \frac{F(\hat{z}, z, \hat{n}) - v(\hat{z})}{\hat{n}} \right\}$$

for each $z \in Z$.

Proof. (Sufficiency) Let (γ, v) be an outcome in an economy E and assume that it satisfies the two condition in the statement. The first is the same as (i) in the definition of a stable outcome, hence it suffices to establish (ii). Let $(z, z', n) \in \text{graph}(\bar{N})$, hence $z' \in Z$ and $n \in \bar{N}(z, z')$. It then follows by the second condition in the statement that $v(z) \geq F(z, z', n) - nv(z)$. Thus, (ii) holds.

(Necessity) Let (γ, v) be a stable outcome in an economy E . Then (i) implies that the first condition in the statement holds.

To show that the first condition in the statement holds, let $z \in Z$. Let, by Lemma 5.1, $(z^*, n^*) \in Z_\emptyset \times N$ be such that $(z, z^*, n^*) \in \text{supp}(\gamma)$ or $(z^*, z, n^*) \in \text{supp}(\gamma)$.

Consider first the case where $(z, z^*, n^*) \in \text{supp}(\gamma)$. For each $\tilde{z} \in Z$ and $\tilde{n} \in \bar{N}(z, \tilde{z})$, (ii) implies that $v(z) \geq F(z, \tilde{z}, \tilde{n}) - \tilde{n}v(\tilde{z})$; thus, using (i),

$$v(z) \geq \max_{\tilde{z} \in Z, \tilde{n} \in \bar{N}(z, \tilde{z})} (F(z, \tilde{z}, \tilde{n}) - \tilde{n}v(\tilde{z})) \geq F(z, z^*, n^*) - n^*v(z^*) = v(z).$$

Hence, to establish the second condition in the statement in the case where $(z, z^*, n^*) \in \text{supp}(\gamma)$, it suffices to show that $v(z) \geq U_s(z)$ and $v(z) \geq \max_{\hat{z} \in Z, \hat{n} \in \bar{N}(\hat{z}, z)} \frac{F(\hat{z}, z, \hat{n}) - v(\hat{z})}{\hat{n}}$. The former follows by (ii) because $(z, \emptyset, 1) \in \text{graph}(\bar{N})$ and, hence, $v(z) = v(z) + v(\emptyset) \geq F(z, \emptyset, 1) = U_s(z)$. The latter follows because, for each $\hat{z} \in Z$ and $\hat{n} \in \bar{N}(\hat{z}, z)$, (ii) implies that $v(\hat{z}) + nv(z) \geq F(\hat{z}, z, \hat{n})$, i.e. $v(z) \geq \frac{F(\hat{z}, z, \hat{n}) - v(\hat{z})}{\hat{n}}$.

The argument to establish the second condition in the statement when $(z^*, z, n^*) \in \text{supp}(\gamma)$ is analogous to the one above. For each $\hat{z} \in Z$ and $\hat{n} \in \bar{N}(\hat{z}, z)$, (ii) implies that $v(\hat{z}) + nv(z) \geq F(\hat{z}, z, \hat{n})$. Thus, using (i),

$$v(z) \geq \max_{\hat{z} \in Z, \hat{n} \in \bar{N}(\hat{z}, z)} \frac{F(\hat{z}, z, \hat{n}) - v(\hat{z})}{\hat{n}} \geq \frac{F(z^*, z, n^*) - v(z^*)}{n^*} = v(z).$$

Hence, to establish the second condition in the statement, it suffices to show that $v(z) \geq U_s(z)$ and $v(z) \geq \max_{\tilde{z} \in Z, \tilde{n} \in \bar{N}(z, \tilde{z})} (F(z, \tilde{z}, \tilde{n}) - \tilde{n}v(\tilde{z}))$. The former follows because $(z, \emptyset, 1) \in \text{graph}(\bar{N})$ and, hence, $v(z) = v(z) + v(\emptyset) \geq F(z, \emptyset, 1) = U_s(z)$. The latter follows because, for each $\tilde{z} \in Z$ and $\tilde{n} \in \bar{N}(z, \tilde{z})$, (ii) implies that $v(z) + \tilde{n}v(\tilde{z}) \geq F(z, \tilde{z}, \tilde{n})$, i.e. $v(z) \geq F(z, \tilde{z}, \tilde{n}) - \tilde{n}v(\tilde{z})$. ■

6 Equivalent knowledge economies

The definition of knowledge economies is sufficiently general to allow for some redundancy, namely if the cumulative distribution function of the knowledge distribution is continuous and strictly increasing (e.g. if ν has a continuous and strictly positive density), then we may assume that the knowledge distribution is uniform on the unit interval by adjusting the other elements of the knowledge economy.

The formal argument is as follows. Let $E = (Z, \nu, \underline{n}, \bar{n}, F)$ be a knowledge economy, Y be a compact metric space and $\psi : Z \rightarrow Y$ be a homeomorphism. Extend ψ to Z_\emptyset by setting $\psi(\emptyset) = \psi^{-1}(\emptyset) = \emptyset$ and define $\tilde{\psi} : Z \times Z_\emptyset \rightarrow Y \times Y_\emptyset$

by setting, for each $(z, z') \in Z \times Z_\emptyset$, $\tilde{\psi}(z, z') = (\psi(z), \psi(z'))$. Then note that $N = [0, \max_{(z, z') \in Z \times Z_\emptyset} \bar{n}(z, z')] = [0, \max_{(y, y') \in Y \times Y_\emptyset} \bar{n}(\psi^{-1}(y), \psi^{-1}(y'))]$. Define $\hat{\psi} : Z \times Z_\emptyset \times N \rightarrow Y \times Y_\emptyset \times N$ by setting, for each $(z, z', n) \in Z \times Z_\emptyset \times N$, $\hat{\psi}(z, z', n) = (\psi(z), \psi(z'), n)$. Then consider the knowledge economy

$$E_\psi = (Y, \nu \circ \psi^{-1}, \underline{n} \circ \tilde{\psi}^{-1}, \bar{n} \circ \tilde{\psi}^{-1}, F \circ \hat{\psi}^{-1}).$$

In the concrete case where ν has a continuous and strictly positive density on $Z = [\underline{z}, \bar{z}]$, where $\underline{z}, \bar{z} \in \mathbb{R}$ and $\underline{z} < \bar{z}$, let $Y = [0, \nu(Z)]$ and $\psi(z) = \nu([\underline{z}, z])$ for each $z \in Z$, i.e. ψ is the cumulative distribution function of ν and is continuous, strictly increasing and satisfies $\psi(Z) = Y$. Thus, ψ is a homeomorphism between Z and Y . The knowledge economy E_ψ has a uniform knowledge distribution since, for each $y, y' \in Y$ with $y < y'$,

$$\begin{aligned} \nu \circ \psi^{-1}([y, y']) &= \nu(\psi^{-1}([y, y'])) = \nu([\psi^{-1}(y), \psi^{-1}(y')]) \\ &= \nu([\underline{z}, \psi^{-1}(y')]) - \nu([\underline{z}, \psi^{-1}(y)]) = y' - y. \end{aligned}$$

The knowledge economies E and E_ψ have the same stable outcomes up to homeomorphisms.

Theorem 6.1 *An outcome (γ, v) is stable in E if and only if $(\gamma \circ \hat{\psi}^{-1}, v \circ \psi^{-1})$ is a stable outcome in E_ψ .*

Proof. (Necessity) Let (γ, v) be a stable outcome of E . For each Borel B of Y , we have that

$$\begin{aligned} \gamma \circ \hat{\psi}^{-1}(B \times Y_\emptyset \times N) &+ \int_{Y \times B \times N} nd\gamma \circ \hat{\psi}^{-1} \\ &= \gamma(\psi^{-1}(B) \times Z_\emptyset \times N) + \int_{Z \times \psi^{-1}(B) \times N} nd\gamma \\ &= \nu(\psi^{-1}(B)) = \nu \circ \psi^{-1}(B). \end{aligned}$$

Since $\hat{\psi}$ is an homeomorphism, it follows that $\text{supp}(\gamma) = \hat{\psi}^{-1}(\text{supp}(\gamma \circ \hat{\psi}^{-1}))$. Hence, if $(y, y', n) \in \text{supp}(\gamma \circ \hat{\psi}^{-1})$, then $(\psi^{-1}(y), \psi^{-1}(y'), n) = \hat{\psi}^{-1}(y, y', n) \in \text{supp}(\gamma)$ and

$$\underline{n} \circ \tilde{\psi}^{-1}(y, y') = \underline{n}(\psi^{-1}(y), \psi^{-1}(y')) \leq n \leq \bar{n}(\psi^{-1}(y), \psi^{-1}(y')) = \bar{n} \circ \tilde{\psi}^{-1}(y, y').$$

Thus, $\gamma \circ \hat{\psi}^{-1}$ is an assignment in E_ψ .

For each $(y, y', n) \in \text{supp}(\gamma \circ \hat{\psi}^{-1})$, then, as above, $(\psi^{-1}(y), \psi^{-1}(y'), n) = \hat{\psi}^{-1}(y, y', n) \in \text{supp}(\gamma)$ and

$$\begin{aligned} v \circ \psi^{-1}(y) + nv \circ \psi^{-1}(y') &= v(\psi^{-1}(y)) + nv(\psi^{-1}(y')) \\ &= F(\psi^{-1}(y), \psi^{-1}(y'), n) = F \circ \hat{\psi}^{-1}(y, y', n). \end{aligned}$$

Furthermore, for each $(y, y', n) \in Y \times Y_\emptyset \times N$ such that $\underline{n} \circ \tilde{\psi}^{-1}(y, y') \leq n \leq \bar{n} \circ \tilde{\psi}^{-1}(y, y')$, it follows that

$$\underline{n}(\psi^{-1}(y), \psi^{-1}(y')) = \underline{n} \circ \tilde{\psi}^{-1}(y, y') \leq n \leq \bar{n} \circ \tilde{\psi}^{-1}(y, y') = \bar{n}(\psi^{-1}(y), \psi^{-1}(y'))$$

and, hence,

$$\begin{aligned} v \circ \psi^{-1}(y) + nv \circ \psi^{-1}(y') &= v(\psi^{-1}(y)) + nv(\psi^{-1}(y')) \\ &\geq F(\psi^{-1}(y), \psi^{-1}(y'), n) = F \circ \hat{\psi}^{-1}(y, y', n). \end{aligned}$$

Thus, $(\gamma \circ \hat{\psi}^{-1}, v \circ \psi^{-1})$ is a stable outcome of E_ψ .

(Sufficiency) Let $\lambda : Y \rightarrow Z$ be the inverse of ψ and note that $\hat{\lambda} = (\lambda, \lambda, \text{id})$ and $\tilde{\lambda} = (\lambda, \lambda)$ are the inverses of $\hat{\psi}$ and $\tilde{\psi}$ respectively, where id is the identity in N .

Thus, the economy

$$E_\lambda = (Z, (\nu \circ \psi^{-1}) \circ \lambda^{-1}, (\underline{n} \circ \tilde{\psi}^{-1}) \circ \tilde{\lambda}^{-1}, (\bar{n} \circ \tilde{\psi}^{-1}) \circ \tilde{\lambda}^{-1}, (F \circ \hat{\psi}^{-1}) \circ \hat{\lambda}^{-1})$$

equals E . Since λ is a homeomorphism, the necessity part above implies that if $(\gamma \circ \hat{\psi}^{-1}, v \circ \psi^{-1})$ is a stable outcome in E_ψ , then $\gamma = (\gamma \circ \hat{\psi}^{-1}) \circ \hat{\lambda}^{-1}$ and $v = (v \circ \psi^{-1}) \circ \lambda^{-1}$ form a stable outcome in $E = E_\lambda$. ■

We give two illustrations of Theorem 6.1 in the context of Garicano and Rossi-Hansberg (2004). The first one was already described in Section 2.3 and consists in transforming the economy so that it has a uniform knowledge distribution. Let E be defined by $Z = [0, \bar{z}]$ with $0 < \bar{z} < 1$, ν such that it has a continuous and strictly positive density, and $\nu(Z) = 1$, $\underline{n}(z, z') = \bar{n}(z, z') = \frac{1}{h(1-z')}$ with $0 < h < 1$, $U_s(z) = z$ and $F(z, z', n) = zn$ for each $(z, z', n) \in Z \times Z \times N$. Let $Y = [0, 1]$ and $\psi : Z \rightarrow Y$ be defined by setting, for each $z \in Z$, $\psi(z) = \nu([0, z])$. Then

ψ is a homeomorphism between Z and Y ; letting G denote the inverse of ψ , it follows that $E_\psi = ([0, 1], \text{unif}, \underline{n}_\psi, \bar{n}_\psi, F_\psi)$ where unif is the uniform distribution on $[0, 1]$, $\underline{n}_\psi(y, y') = \bar{n}_\psi(y, y') = \frac{1}{h(1-G(y'))}$ and $F_\psi(y, y', n) = G(y)n$ for each $(z, z', n) \in Z \times Z \times N$ with $F_\psi(y, \emptyset, n) = G(y)$. Theorem 6.1 implies that the function $(\gamma, v) \mapsto (\gamma \circ \hat{\psi}^{-1}, v \circ \psi^{-1})$ is a homeomorphism between the stable pairs of E and E_ψ . Note that the stable pairs of E_ψ are described in terms of the percentiles of ν , e.g. if $y = \frac{k}{100}$ with $k \in \{0, \dots, 100\}$, $v \circ \psi^{-1}(y)$ is the earnings of the k th percentile of the distribution ν .

Another application of Theorem 6.1 shows that assuming that the distribution of problems is uniform in Garicano and Rossi-Hansberg (2004) is without loss of generality, in the following sense. Let $Z = [0, \bar{z}]$ with $\bar{z} > 0$ and $G : Z \rightarrow \mathbb{R}_+$ be the cumulative distribution function of a continuous and decreasing density $g : Z \rightarrow \mathbb{R}_+$. Then define an economy E by letting $\underline{n}(z, z') = \bar{n}(z, z') = \frac{1}{h(1-G(z'))}$ and $F(z, z', n) = G(z)n$ for each $(z, z', n) \in Z \times Z \times N$ and $F(z, \emptyset, n) = G(z)$. The interpretation is that each worker draws a problem $x \in \mathbb{R}_+$ from g and production equals 1 if the problem is solved, which happens if and only if $x \leq z$, and zero otherwise.

Let $Y = [0, G(\bar{z})]$ and $\psi : Z \rightarrow Y$ be defined by setting, for each $z \in Z$, $\psi(z) = G(z)$. Then ψ is a homeomorphism between Z and Y . It then follows that $E_\psi = ([0, G(\bar{z})], \nu \circ G^{-1}, \underline{n}_\psi, \bar{n}_\psi, F_\psi)$ where $\underline{n}_\psi(y, y') = \bar{n}_\psi(y, y') = \frac{1}{h(1-y')}$ and $F_\psi(y, y', n) = yn$ for each $(z, z', n) \in Z \times Z \times N$ with $F_\psi(y, \emptyset, n) = y$. Theorem 6.1 implies that the function $(\gamma, v) \mapsto (\gamma \circ \hat{\psi}^{-1}, v \circ \psi^{-1})$ is a homeomorphism between the stable pairs of E and E_ψ . Note that the stable pairs of E_ψ are described in terms of the distribution G , i.e. $v \circ \psi^{-1}(y)$ is the earnings of the individuals who can solve a fraction y of problems.

7 Increasing earnings

This section provides sufficient conditions for the earnings function to be increasing under condition B. These conditions require the production function and the upper bound on firm size to be increasing. In addition, they require that either the lower

bound on firm size is decreasing (e.g. $\underline{n} \equiv 0$) or part (i) of condition C holds, i.e. production is linear in n and every type can get strictly positive earnings.

Theorem 7.1 *Let E be an economy satisfying condition B and (γ, u) be a stable outcome. Then u is increasing if (i) U_s is increasing, (ii) F is increasing in z and in z' , (iii) \bar{n} is increasing in z and z' and (iv) either (a) \underline{n} is decreasing in z and z' or (b) part (i) of condition C holds.*

Proof. Let (γ, u) be stable in E , $z, \hat{z} \in Z$ be such that $z > \hat{z}$ and suppose that $u(z) < u(\hat{z})$. Let a be \hat{z} 's occupation; we consider the three possible cases regarding a .

If $a = m$, then let $z' \in Z$ and $n \in \mathbb{R}_+$ be such that $(\hat{z}, z', n) \in \text{supp}(\gamma)$. Then

$$u(z) < u(\hat{z}) = F(\hat{z}, z', n) - nu(z') \leq F(z, z', n) - nu(z').$$

When \underline{n} is decreasing in z and z' , it follows that $\underline{n}(z, z') \leq \underline{n}(\hat{z}, z') \leq n \leq \bar{n}(\hat{z}, z') \leq \bar{n}(z, z')$ and this together with $u(z) + nu(z') < F(z, z', n)$ contradicts the stability of (γ, u) . If part (b) of condition (iv) hold, then $n = \bar{n}(\hat{z}, z') \leq \bar{n}(z, z')$ and $F(z, z', n) - nu(z') \leq F(z, z', \bar{n}(z, z')) - \bar{n}(z, z')u(z')$. Indeed, $0 \leq u(z) < (F(z, z', 1) - u(z'))n$ implies that $F(z, z', 1) - u(z') > 0$ and hence that $F(z, z', n) - nu(z') = (F(z, z', 1) - u(z'))n \leq (F(z, z', 1) - u(z'))\bar{n}(z, z') = F(z, z', \bar{n}(z, z')) - \bar{n}(z, z')u(z')$. Thus, it follows that $(z, z', \bar{n}(z, z')) \in \text{graph}(\bar{N})$ and $u(z) + \bar{n}(z, z')u(z') < F(z, z', \bar{n}(z, z'))$, a contradiction to the stability of (γ, u) .

If $a = s$, then $u(z) < u(\hat{z}) = U_s(\hat{z}) \leq U_s(z)$ i.e. $u(z) + u(\emptyset) < F(z, \emptyset, 1)$, a contradiction to the stability of (γ, u) since $(z, \emptyset, 1) \in \text{graph}(\bar{N})$.

If $a = w$, then let $\tilde{z} \in Z$ and $n \in \mathbb{R}_+$ be such that $(\tilde{z}, \hat{z}, n) \in \text{supp}(\gamma)$. Lemma 2 implies that $n > 0$. Hence,

$$u(\tilde{z}) = F(\tilde{z}, \hat{z}, n) - nu(\hat{z}) < F(\tilde{z}, \hat{z}, n) - nu(z) \leq F(\tilde{z}, z, n) - nu(z).$$

When \underline{n} is decreasing in z and z' , it follows that $\underline{n}(\tilde{z}, z) \leq \underline{n}(\tilde{z}, \hat{z}) \leq n \leq \bar{n}(\tilde{z}, \hat{z}) \leq \bar{n}(\tilde{z}, z)$ and this together with $u(\tilde{z}) + nu(z) < F(\tilde{z}, z, n)$ contradicts the stability of (γ, u) . If part (b) of condition (iv) hold, then $n = \bar{n}(\tilde{z}, \hat{z}) \leq \bar{n}(\tilde{z}, z)$ and $F(\tilde{z}, z, n) -$

$nu(z) \leq F(\tilde{z}, z, \bar{n}(\tilde{z}, z)) - \bar{n}(\tilde{z}, z)u(z)$. Indeed, $0 \leq u(\tilde{z}) < (F(\tilde{z}, z, 1) - u(z))n$ implies that $F(\tilde{z}, z, 1) - u(z) > 0$ and hence that $F(\tilde{z}, z, n) - nu(z) = (F(\tilde{z}, z, 1) - u(z))n \leq (F(\tilde{z}, z, 1) - u(z))\bar{n}(\tilde{z}, z) = F(\tilde{z}, z, \bar{n}(\tilde{z}, z)) - \bar{n}(\tilde{z}, z)u(z)$. Thus, it follows that $(\tilde{z}, z, \bar{n}(\tilde{z}, z)) \in \text{graph}(\bar{N})$ and $u(\tilde{z}) + \bar{n}(\tilde{z}, z)u(z) < F(\tilde{z}, z, \bar{n}(\tilde{z}, z))$, a contradiction to the stability of (γ, u) . ■

8 A variation of the Garicano and Rossi-Hansberg's (2004) economy

We show that changing the definition of the Garicano and Rossi-Hansberg's (2004) economy in Section 2.3.2 by letting the production function be

$$F(z, z', n) = \max\{z, z'\}n,$$

yields an economy with the same stable outcomes.

Let E_{grh}^* be as in Section 2.3.2, i.e. $Z = [0, \bar{z}]$ with $0 < \bar{z} < 1$, $\underline{n}(z, z') = \bar{n}(z, z') = \frac{1}{h(1-z')}$ with $0 < h < 1$, $F(z, z', n) = zn$ and $U_s(z) = z$. Let E_{grh} be exactly as E_{grh}^* but for $F(z, z', n) = \max\{z, z'\}n$.

It follows by Corollary 3 that E_{grh}^* has a unique stable outcome. Hence, to show that E_{grh} has the same stable outcomes as E_{grh}^* , it suffices to establish the following result.

Lemma 8.1 *If (γ, u) is a stable outcome of E_{grh} , then (γ, u) is a stable outcome of E_{grh}^* .*

Proof. Let (γ, u) be a stable outcome of E_{grh} . We first show that, for each $(z, z', n) \in \text{supp}(\gamma)$, $\max\{z, z'\} = z$. Let $(z, z', n) \in \text{supp}(\gamma)$. Then $u(z) \geq z$ and $u(z') \geq z'$ since $(z, \emptyset, 1), (z', \emptyset, 1) \in \text{graph}(\bar{N})$. If $z > 0$, then $u(z) = (\max\{z, z'\} - u(z'))n \geq z$ implies that $\max\{z, z'\} > u(z') \geq z'$ and, hence, $\max\{z, z'\} = z$. If $z = 0$, we claim that $z' = 0$ and, hence, $\max\{z, z'\} = z$ as well. Indeed, consider $z = 0$ and assume that $z' > 0$. Then $\max\{z, z'\} = z'$, $u(z) = (z' - u(z'))n \geq z = 0$ and $u(z') \geq z'$

imply that $u(z') = z'$. Since $(z', z, \bar{n}(z', z)) \in \text{graph}(\bar{N})$ and $\bar{n}(z', z) > 1$, it follows that

$$z' = u(z') + \bar{n}(z', z)u(z) \geq F(z', z, \bar{n}(z', z)) = \max\{z', z\}\bar{n}(z', z) = z'\bar{n}(z', z) > z',$$

a contradiction.

It then follows that (γ, u) is a stable outcome of E_{grh}^* . Indeed, condition (i) holds since, for each $(z, z', n) \in \text{supp}(\gamma)$, $u(z) + nu(z') = \max\{z, z'\}n = zn$ by what has been shown in the previous paragraph. Condition (ii) holds since, for each $(z, z', n) \in \text{graph}(\bar{N})$, $u(z) + nu(z') \geq \max\{z, z'\}n \geq zn$. ■

9 An example of a Rosen market

We consider the example of $E_{r,\alpha}$ in which $\theta \equiv 1$ and $F(z) = z$ for each $z \in Z$. In this case, for each $w > 0$ and $z \in Z$,

$$\begin{aligned} n(z, w) &= \left(\frac{1-\alpha}{w}\right)^{\frac{1}{\alpha}} z^{\frac{1+\alpha}{\alpha}}, \text{ and} \\ wn(z, w) &= (1-\alpha)z^{1+\alpha}n(z, w)^{1-\alpha}. \end{aligned}$$

This implies that

$$F(z_1)z_1^\alpha n(z_1, w)^{1-\alpha} - wn(z_1, w) = \frac{wn(z_1, w)}{1-\alpha} - wn_1(z_1, w) = \frac{\alpha}{1-\alpha}wn(z_1, w).$$

It then follows by condition 4 in Theorem 11 that

$$\frac{\alpha}{1-\alpha}wn(z_1, w) = w \Leftrightarrow n(z_1, w) = \frac{1-\alpha}{\alpha} \Leftrightarrow w = \alpha^\alpha(1-\alpha)^{1-\alpha}z_1^{1+\alpha}.$$

Recall from the proof of Theorem 11 that $\nu([0, z_1]) = \int_{[z_1, \bar{z}]} n(z, w)d\nu(z)$. Since $\nu([0, z_1]) = z_1$ and

$$\int_{[z_1, \bar{z}]} n(z, w)d\nu(z) = \left(\frac{1-\alpha}{w}\right)^{\frac{1}{\alpha}} \int_{z_1}^{\bar{z}} z^{\frac{1+\alpha}{\alpha}} dz = \left(\frac{1-\alpha}{w}\right)^{\frac{1}{\alpha}} \frac{\alpha}{1+2\alpha} \left(\bar{z}^{\frac{1+2\alpha}{\alpha}} - z_1^{\frac{1+2\alpha}{\alpha}}\right),$$

it follows that

$$z_1 = \left(\frac{1-\alpha}{w}\right)^{\frac{1}{\alpha}} \frac{\alpha}{1+2\alpha} \left(\bar{z}^{\frac{1+2\alpha}{\alpha}} - z_1^{\frac{1+2\alpha}{\alpha}}\right). \quad (9.1)$$

By substituting $w = \alpha^\alpha(1 - \alpha)^{1-\alpha}z_1^{1+\alpha}$ in (9.1) and then solving it, we obtain that

$$z_1 = \left(\frac{1 - \alpha}{2 + \alpha} \right)^{\frac{\alpha}{1+2\alpha}} \bar{z}.$$

The payment u for each individual is

$$u(z) = \begin{cases} w & \text{if } z < z_1, \\ \alpha \left(\frac{1-\alpha}{w} \right)^{\frac{1-\alpha}{\alpha}} z^{\frac{1+\alpha}{\alpha}} & \text{if } z \geq z_1. \end{cases}$$

The expression for $u(z)$ when $z \geq z_1$ is obtained by noting that, if $z \geq z_1$,

$$u(z) = z^{1+\alpha}n(z, w)^{1-\alpha} - wn(z, w) = \alpha z^{1+\alpha}n(z, w)^{1-\alpha}.$$

It follows that, as $\alpha \rightarrow 0$, $z_1 \rightarrow \bar{z}$ and $w \rightarrow \bar{z}$. Furthermore, by substituting $w = \alpha^\alpha(1 - \alpha)^{1-\alpha}z_1^{1+\alpha}$ in $u(\bar{z})$, it follows that

$$u(\bar{z}) = \alpha^\alpha(1 - \alpha)^{1-\alpha} \left(\frac{\bar{z}}{z_1} \right)^{\frac{1+\alpha}{\alpha}} z_1^{1+\alpha}.$$

Using $z_1 = \left(\frac{1-\alpha}{2+\alpha} \right)^{\frac{\alpha}{1+2\alpha}} \bar{z}$ to substitute for $\frac{\bar{z}}{z_1}$, it follows that

$$u(\bar{z}) = \alpha^\alpha(1 - \alpha)^{1-\alpha} \left(\frac{2 + \alpha}{1 - \alpha} \right)^{\frac{1+\alpha}{1+2\alpha}} z_1^{1+\alpha}.$$

Hence, $u(\bar{z}) \rightarrow 2\bar{z}$ as $\alpha \rightarrow 0$.

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