

Supplementary Material for “Knowledge Economies”

Guilherme Carmona* Krittanai Laohakunakorn[†]
University of Surrey University of Surrey

Hitoshi Tsujiyama[‡]
University of Surrey

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*Address: University of Surrey, School of Economics, Guildford, GU2 7XH, UK; email: g.carmona@surrey.ac.uk.

[†]Address: University of Surrey, School of Economics, Guildford, GU2 7XH, UK; email: k.laohakunakorn@surrey.ac.uk.

[‡]Address: University of Surrey, School of Economics, Guildford, GU2 7XH, UK; email: h.tsujiyama@surrey.ac.uk.

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1 Introduction

This paper contains supplementary material to our paper “Knowledge Economies”.
It contains:

Section 2: An extension of the framework of large many-to-one matching markets with occupational choice in Carmona and Laohakunakorn (2024).

Section 3: Existence and limit results for the setting of Section 2.

Section 4: The relationship between the setting of Section 2 with that in Carmona and Laohakunakorn (2024).

Section 5: Several models in the literature are special cases of our framework of knowledge economies.

Section 6: A characterization of stable outcomes in knowledge economies.

Section 7: Definition of a stable matching in knowledge economies.

Section 8: The equivalence between competitive equilibria and stable outcomes in knowledge economies.

Section 9: Stable outcomes in the multi-layer extension of the Garicano and Rossi-Hansberg (2004) economy.

Section 10: Technical difficulties in Chen and Suen (2019).

Section 11: The stable outcomes of economies that are the same up to an homeomorphism are homeomorphic. This implies that the knowledge distribution can be normalized to be uniform and that, in Garicano and Rossi-Hansberg (2004), the distribution of problems can be assumed to be uniform.

Section 12: A variation of the Garicano and Rossi-Hansberg (2004) economy with multi-layers.

Section 13: Some details on the topology of $\cup_{L=1}^{\bar{L}}(Z \times N)^L$.

Section 14: A sufficient condition to reformulate the surplus maximization problem as a network-flow program.

Section 15: The numerical solution and calibration procedures.

Section 16: The calibration and counterfactual exercises for U.S. wage polarization between 1988 and 2008.

2 Large many-to-one matching markets with occupational choice: An extension

We extend the framework of markets with occupational choice of Carmona and Lao-hakunakorn (2024) by allowing the set of feasible matches of a manager to depend on the manager's type. A matching market with occupational choice (a market, henceforth) is $E = (Z, \nu, C, P_m, P_s, (\succ_z)_{z \in Z})$, where Z is the (nonempty, Polish) set of individual types, ν is the type distribution (a nonzero, finite, Borel measure on Z), C is the (nonempty, Polish) set of contracts, P_m is the correspondence of possible matches of managers, P_s is the correspondence of possible matches of self-employed and $\succ_z$ describes the preferences of an individual of type z , for each $z \in Z$.

A *dummy type* $\emptyset \notin Z$ is used to represent unmatched i.e. self-employed individuals, and we let $Z_\emptyset = Z \cup \{\emptyset\}$, with the assumption that $\emptyset$ is an isolated point in $Z_\emptyset$.

The correspondence of possible matches of managers is $P_m : Z \rightrightarrows M(Z \times C)$ and we require that managers of type z be matched with $\delta \in P_m(z)$. Similarly, the correspondence of possible matches of self-employed is $P_s : Z \rightrightarrows \{1_{(\emptyset, c)} : c \in C\}$ and we require that self-employed individuals of type z be matched with $\delta \in P_s(z)$. Let $X = \cup_{z \in Z} P_m(z)$ and $X_\emptyset = X \cup \{1_{(\emptyset, c)} : c \in C\}$, the latter being the *set of possible matches of managers and self-employed individuals*.

The *set of occupations* is $A = \{w, s, m\}$, where w stands for worker, s for self-employed and m for manager. The choice set of each individual depends on occupation: let $X_m = X$, $X_s = \{1_{(\emptyset, c)} : c \in C\}$, $X_w = \{1_{(z, c)} : (z, c) \in Z \times C\}$ and $\Delta = \{(a, \delta) : \delta \in X_a\}$. The set Δ is the *choice set of each individual* because an individual can choose an occupation and a match feasible for that occupation.

We allow for externalities and, thus, preferences are allowed to depend on the matching. Matchings with occupational choice are elements of $M(Z \times X_\emptyset)$ satisfying certain properties described below. The *preferences* of an individual of type z are then described by a relation $\succ_z$ defined on $\Delta \times M(Z \times X_\emptyset)$ for each $z \in Z$.

The particular case considered in Carmona and Laohakunakorn (2024) starts with a common set $X \subseteq M(Z \times C)$ of candidate managerial matches and a contract correspondence $\mathbb{C} : Z \times Z_\emptyset \rightrightarrows C$. In this case, define $P_m(z) = \{\delta \in X : \{z\} \times \text{supp}(\delta) \subseteq \text{graph}(\mathbb{C})\}$ and $P_s(z) = \{1_{(\emptyset, c)} : c \in \mathbb{C}(z, \emptyset)\}$. See Section 4 for details.

A *matching with occupational choice* (a matching, henceforth) is a Borel measure $\mu \in M(Z \times X_\emptyset)$ such that

1. $\text{supp}(\mu) \subseteq \text{graph}(P_m) \cup \text{graph}(P_s)$, and
2. $\nu_M + \nu_S + \nu_W = \nu$

where, for each Borel subset B of Z , $\nu_M(B) = \mu(B \times X)$, $\nu_S(B) = \mu(B \times (X_\emptyset \setminus X))$ and $\nu_W(B) = \int_{Z \times X} \delta(B \times C) d\mu(z, \delta)$.

Targets at a given matching μ depend on the type z and on the occupational choice a , and are denoted by $T_z^a(\mu)$. For each $z \in Z$, the targets for the prospective self-employed of type z are $T_z^s(\mu) = \{\emptyset\} \times C$.

For each $z \in Z$, the targets of prospective managers of type z are $T_z^m(\mu)$ defined as the set of $(z^*, c) \in Z \times C$ such that there exists

- (a) $(z', \delta', c') \in Z \times X \times C$ such that $(z', \delta') \in \text{supp}(\mu)$, $(z^*, c') \in \text{supp}(\delta')$ and $(w, 1_{(z,c)}, \mu) \succ_{z^*} (w, 1_{(z',c')}, \mu)$, or
- (b) $\delta' \in X_\emptyset \setminus X$ such that $(z^*, \delta') \in \text{supp}(\mu)$ and $(w, 1_{(z,c)}, \mu) \succ_{z^*} (s, \delta', \mu)$, or
- (c) $\delta' \in X$ such that $(z^*, \delta') \in \text{supp}(\mu)$ and $(w, 1_{(z,c)}, \mu) \succ_{z^*} (m, \delta', \mu)$.

For each $z \in Z$, the targets of prospective workers of type z are $T_z^w(\mu)$ defined as the set of $(z^*, c) \in Z \times C$ such that there is $\delta \in P_m(z^*)$ such that $(z, c) \in \text{supp}(\delta)$ and

- (a) $\text{supp}(\delta) \setminus \{(z, c)\} \subseteq T_{z^*}^m(\mu)$ and there is $(z', \delta', c') \in Z \times X \times C$ such that $(z', \delta') \in \text{supp}(\mu)$, $(z^*, c') \in \text{supp}(\delta')$ and $(m, \delta, \mu) \succ_{z^*} (w, 1_{(z',c')}, \mu)$, or
- (b) $\text{supp}(\delta) \setminus \{(z, c)\} \subseteq T_{z^*}^m(\mu)$ and there is $\delta' \in X_\emptyset \setminus X$ such that $(z^*, \delta') \in \text{supp}(\mu)$ and $(m, \delta, \mu) \succ_{z^*} (s, \delta', \mu)$, or
- (c) there is $\delta' \in X$ such that $\text{supp}(\delta) \setminus \{(z, c)\} \subseteq T_{z^*}^m(\mu) \cup \text{supp}(\delta')$, $(z^*, \delta') \in \text{supp}(\mu)$ and $(m, \delta, \mu) \succ_{z^*} (m, \delta', \mu)$.

For each $z \in Z$, let $P_w(z) = X_w$. The stability set $S(\mu)$ of matching μ is the set of $(z, \delta) \in Z \times X_\emptyset$ such that, if $\delta \in X$, then

- (i) there does not exist $(a, \delta') \in \Delta$ with $\delta' \in P_a(z)$ such that $\text{supp}(\delta') \subseteq T_z^a(\mu) \cup \text{supp}(\delta)$ if $a = m$, $\text{supp}(\delta') \subseteq T_z^a(\mu)$ if $a \neq m$, and $(a, \delta', \mu) \succ_z (m, \delta, \mu)$.
- (ii) for each $(z', c) \in \text{supp}(\delta)$, there does not exist $(a, \delta') \in \Delta$ with $\delta' \in P_a(z')$ such that $\text{supp}(\delta') \subseteq T_{z'}^a(\mu)$ and $(a, \delta', \mu) \succ_{z'} (w, 1_{(z,c)}, \mu)$.

and, if $\delta \in X_\emptyset \setminus X$, then

- (iii) there does not exist $(a, \delta') \in \Delta$ with $\delta' \in P_a(z)$ such that $\text{supp}(\delta') \subseteq T_z^a(\mu)$ and $(a, \delta', \mu) \succ_z (s, \delta, \mu)$.

A matching μ is *stable* if $\text{supp}(\mu) \subseteq S(\mu)$.

Theorem 2.1 provides a characterization of stable matchings that is simpler to use. Let $S_M(\mu)$ be the set of $(z, \delta) \in Z \times X_\emptyset$ such that, if $\delta \in X$, then

- (i) there does not exist $\delta' \in P_m(z)$ such that $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ and $(m, \delta', \mu) \succ_z (m, \delta, \mu)$,
- (ii) for each $(z', c) \in \text{supp}(\delta)$, there does not exist $\delta' \in P_m(z')$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu)$ and $(m, \delta', \mu) \succ_{z'} (w, 1_{(z', c)}, \mu)$.

and, if $\delta \in X_\emptyset \setminus X$, then

- (iii) there does not exist $\delta' \in P_m(z)$ such that $\text{supp}(\delta') \subseteq T_z^m(\mu)$ and $(m, \delta', \mu) \succ_z (s, \delta, \mu)$.

Let $IR(\mu)$ be defined as the set of $(z, \delta) \in Z \times X_\emptyset$ such that, if $\delta \in X$, then

- (i) there does not exist $\delta' \in P_s(z)$ such that $(s, \delta', \mu) \succ_z (m, \delta, \mu)$,
- (ii) for each $(z', c) \in \text{supp}(\delta)$, there does not exist $\delta' \in P_s(z')$ such that $(s, \delta', \mu) \succ_{z'} (w, 1_{(z', c)}, \mu)$,

and, if $\delta \in X_\emptyset \setminus X$, then

- (iii) there does not exist $\delta' \in P_s(z)$ such that $(s, \delta', \mu) \succ_z (s, \delta, \mu)$.

Theorem 2.1 *A matching μ is stable if and only if $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$.*

2.1 Proof of Theorem 2.1

Note first that $\text{supp}(\mu) \subseteq S(\mu)$ implies that $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$ since $S(\mu) \subseteq S_M(\mu) \cap IR(\mu)$.

Conversely, suppose that $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$. Let $(z, \delta) \in \text{supp}(\mu)$ and assume, in order to reach a contradiction, that $(z, \delta) \notin S(\mu)$. Since $(z, \delta) \in \text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$, it follows that there is $(z^*, c) \in Z \times C$ and $\bar{z} \in Z$ such that $(z^*, c) \in T_{\bar{z}}^w(\mu)$, $\bar{z} = z$ or $(\bar{z}, \bar{c}) \in \text{supp}(\delta)$ for some $\bar{c} \in C$, (1) $(w, 1_{(z^*, c)}, \mu) \succ_{\bar{z}} (m, \delta, \mu)$ if $\bar{z} = z$ and $\delta \in X$, (2) $(w, 1_{(z^*, c)}, \mu) \succ_{\bar{z}} (w, 1_{(z, \bar{c})}, \mu)$ if $(\bar{z}, \bar{c}) \in \text{supp}(\delta)$ and (3) $(w, 1_{(z^*, c)}, \mu) \succ_{\bar{z}} (s, \delta, \mu)$ if $\bar{z} = z$ and $\delta \in X_\emptyset \setminus X$.

We now show that $(\bar{z}, c) \in T_{z^*}^m(\mu)$. Indeed, we have that $(z, \delta) \in \text{supp}(\mu)$. Thus, in case (1), the conclusion follows by condition (c) in the definition of $T_{z^*}^m(\mu)$ since

$\bar{z} = z$ and $(w, 1_{(z^*, c)}, \mu) \succ_z (m, \delta, \mu)$; in case (2), the conclusion follows by condition (a) in the definition of $T_{z^*}^m(\mu)$ since $(\bar{z}, \bar{c}) \in \text{supp}(\delta)$ and $(w, 1_{(z^*, c)}, \mu) \succ_{\bar{z}} (w, 1_{(z, \bar{c})}, \mu)$; and, in case (3), the conclusion follows by condition (b) in the definition of $T_{z^*}^m(\mu)$ since $\bar{z} = z$ and $(w, 1_{(z^*, c)}, \mu) \succ_z (s, \delta, \mu)$.

Since $(z^*, c) \in T_{\bar{z}}^w(\mu)$, there is $\tilde{\delta} \in P_m(z^*)$ such that $(\bar{z}, c) \in \text{supp}(\tilde{\delta})$ and (a) or (b) or (c) in the definition of $T_{\bar{z}}^w(\mu)$ holds. In either case, we will show that $\text{supp}(\mu) \subseteq S_M(\mu)$ fails, which is a contradiction to $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$.

Suppose that condition (a) in the definition of $T_{\bar{z}}^w(\mu)$ holds. Then, in addition, $\text{supp}(\tilde{\delta}) \setminus \{(\bar{z}, c)\} \subseteq T_{z^*}^m(\mu)$, and there is $(z', c', \delta') \in Z \times C \times X$ such that $(z', \delta') \in \text{supp}(\mu)$, $(z^*, c') \in \text{supp}(\delta')$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (w, 1_{(z', c')}, \mu)$. Since $(\bar{z}, c) \in T_{z^*}^m(\mu)$, it follows that $(z', \delta') \in \text{supp}(\mu) \setminus S_M(\mu)$ since (ii) of the definition of $S_M(\mu)$ fails. Indeed, $(z', \delta') \in \text{supp}(\mu)$, $(z^*, c') \in \text{supp}(\delta')$, $\text{supp}(\tilde{\delta}) \subseteq T_{z^*}^m(\mu)$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (w, 1_{(z', c')}, \mu)$.

Suppose next that condition (b) in the definition of $T_{\bar{z}}^w(\mu)$ holds. Then, in addition, $\text{supp}(\tilde{\delta}) \setminus \{(\bar{z}, c)\} \subseteq T_{z^*}^m(\mu)$, and there is $\delta' \in X_\emptyset \setminus X$ such that $(z^*, \delta') \in \text{supp}(\mu)$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (s, \delta', \mu)$. Since $(\bar{z}, c) \in T_{z^*}^m(\mu)$, it follows that $(z^*, \delta') \in \text{supp}(\mu) \setminus S_M(\mu)$ since (iii) of the definition of $S_M(\mu)$ fails. Indeed, $(z^*, \delta') \in \text{supp}(\mu)$, $\text{supp}(\tilde{\delta}) \subseteq T_{z^*}^m(\mu)$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (s, \delta', \mu)$.

Finally, suppose that condition (c) in the definition of $T_{\bar{z}}^w(\mu)$ holds. Then, in addition, there is $\delta' \in X$ such that $(z^*, \delta') \in \text{supp}(\mu)$, $\text{supp}(\tilde{\delta}) \setminus \{(\bar{z}, c)\} \subseteq T_{z^*}^m(\mu) \cup \text{supp}(\delta')$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (m, \delta', \mu)$. Since $(\bar{z}, c) \in T_{z^*}^m(\mu)$, it follows that $(z^*, \delta') \in \text{supp}(\mu) \setminus S_M(\mu)$ since (i) of the definition of $S_M(\mu)$ fails. Indeed, $(z^*, \delta') \in \text{supp}(\mu)$, $\text{supp}(\tilde{\delta}) \subseteq T_{z^*}^m(\mu) \cup \text{supp}(\delta')$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (m, \delta', \mu)$.

3 Existence and limit results

We establish existence of stable matchings under the following conditions. We say that a market E is *rational* if $\succ_z$ is asymmetric and negatively transitive for each $z \in Z$.

We say that a market E is *continuous* if $\{(a, \delta, \mu, a', \delta', \mu', z) \in (\Delta \times M(Z \times X_\emptyset))^2 \times$

$Z : (a, \delta, \mu) \succ_z (a', \delta', \mu')\}$ is open, C is compact, P_m is upper hemicontinuous with compact values and P_s is continuous with nonempty and compact values.

We say that a market E is *bounded* if there exists $R > 0$ such that $\delta(Z \times C) \leq R$ for each $\delta \in \cup_{z \in Z} P_m(z)$.

Let E be bounded and $R > 0$ be such that $\delta(Z \times C) \leq R$ for each $\delta \in \cup_{z \in Z} P_m(z)$. Let $X^* = M_R(Z \times C) \cup \{1_{(\emptyset, c)} : c \in C\}$, $\Lambda : Z \times M_R(Z \times C) \times M(Z \times X^*) \rightrightarrows X$ be defined by setting, for each $(z, \delta, \mu) \in Z \times M_R(Z \times C) \times M(Z \times X^*)$,

$$\Lambda(z, \delta, \mu) = \{\delta' \in P_m(z) : \text{supp}(\delta') \subseteq \text{supp}(\delta) \cup T_z^m(\mu)\}$$

and $\Lambda_0 : Z \times M(Z \times X^*) \rightrightarrows X$ be defined by setting, for each $(z, \mu) \in Z \times M(Z \times X^*)$,

$$\Lambda_0(z, \mu) = \{\delta' \in P_m(z) : \text{supp}(\delta') \subseteq T_z^m(\mu)\}.$$

We say that a market E is *rich* if Λ is lower hemicontinuous for each $(z, \delta, \mu) \in Z \times X \times M(Z \times X_\emptyset)$ such that $(z, \delta) \in \text{supp}(\mu) \cap \text{graph}(P_m)$ and Λ_0 is lower hemicontinuous for each $(z, \mu) \in Z \times M(Z \times X_\emptyset)$.

Theorem 3.1 *Every rational, continuous, bounded and rich market has a stable matching.*

The proof of Theorem 3.1 considers first markets where Z , C and $P_m(z)$ are finite for each $z \in Z$ and then uses limit results to extend to more general markets. The same approach is also useful to establish existence of stable matchings in some applications to which Theorem 3.1 does not apply, namely by approximating such market by a sequence of markets that satisfy the assumptions of Theorem 3.1 and then use a limit result. The remainder of this section then presents several limit results under general assumptions, which are then summarized by Corollary 3.1 for the use in the proof of Theorem 3.1.

Lemma 3.1 *Let $E = (Z, \nu, C, P_m, P_s, (\succ_z)_{z \in Z})$ be a market and $\{(E_k, \mu_k)\}_{k=1}^\infty$ be a sequence such that, for each $k \in \mathbb{N}$, (i) $E_k = (Z_k, \nu_k, C_k, P_{m,k}, P_{s,k}, (\succ_{z,k})_{z \in Z_k})$ is a market, (ii) μ_k is a stable matching for E_k , (iii) $Z_k \subseteq Z$, (iv) $\text{supp}(\nu_k) \subseteq \text{supp}(\nu)$, (v)*

$C_k \subseteq C$, (vi) $P_{a,k}(z) \subseteq P_a(z)$ for each $z \in Z_k$ and $a \in \{m, s\}$, and (vii) $\nu_k \rightarrow \nu$. If P_m and P_s are upper hemicontinuous and compact-valued, then $\{\mu_k\}_{k=1}^\infty$ has a convergent subsequence in $M(Z \times X_\emptyset)$.

Proof. Since $M(Z \times X_\emptyset)$ is a separable metrizable space, it suffices to show that $\{\mu_k\}_{k=1}^\infty$ is tight; this follows by Carmona and Laohakunakorn (2024, Lemma 4) together with $\mu_k(Z \times X_\emptyset) \leq \nu_k(Z)$ for each $k \in \mathbb{N}$ and the fact that $\{\nu_k(Z)\}_{k=1}^\infty$ converges (to $\nu(Z)$) and, hence, is bounded.

Let $\varepsilon > 0$. Since $\{\nu_k\}_{k=1}^\infty$ is tight, there exists a compact subset K_Z of Z such that $\nu_k(Z \setminus K_Z) \leq \varepsilon$ for all k . Then $P_m(K_Z)$ and $P_s(K_Z)$ are compact since both P_m and P_s are upper hemicontinuous and compact-valued. Let $K_{X_\emptyset} = P_m(K_Z) \cup P_s(K_Z)$, which is a compact subset of $X_\emptyset$.

Then, for each $k \in \mathbb{N}$,

$$\begin{aligned} \mu_k((Z \times X_\emptyset) \setminus (K_Z \times K_{X_\emptyset})) &= \mu_k((Z \setminus K_Z) \times X_\emptyset) + \mu_k(K_Z \times (X_\emptyset \setminus K_{X_\emptyset})) \\ &\leq \nu_k(Z \setminus K_Z) \leq \varepsilon, \end{aligned}$$

since $\text{supp}(\mu_k) \cap (K_Z \times (X_\emptyset \setminus K_{X_\emptyset})) = \emptyset$ because of condition 1 of a matching and, hence, $\mu_k(K_Z \times (X_\emptyset \setminus K_{X_\emptyset})) = 0$. ■

Lemma 3.2 *Let $E = (Z, \nu, C, P_m, P_s, (\succ_z)_{z \in Z})$ be a market, $\{(E_k, \mu_k)\}_{k=1}^\infty$ be a sequence such that $\{\mu_k\}_{k=1}^\infty$ converges and, for each $k \in \mathbb{N}$, (i) $E_k = (Z_k, \nu_k, C_k, P_{m,k}, P_{s,k}, (\succ_{z,k})_{z \in Z_k})$ is a market, (ii) μ_k is a stable matching for E_k , (iii) $Z_k \subseteq Z$, (iv) $\text{supp}(\nu_k) \subseteq \text{supp}(\nu)$, (v) $C_k \subseteq C$, (vi) $P_{a,k}(z) \subseteq P_a(z)$ for each $z \in Z_k$ and $a \in \{m, s\}$, and (vii) $\nu_k \rightarrow \nu$, and $\mu = \lim_k \mu_k$. If E is bounded, and P_m and P_s are upper hemicontinuous and closed-valued, then μ is a matching for E .*

Proof. We first consider condition 2 of the definition of a matching. Let $\pi : Z \times X_\emptyset \rightarrow Z$ be the projection of $Z \times X_\emptyset$ onto Z and note that, for each Borel subset B of Z , $\nu_M(B) + \nu_S(B) = \mu(B \times X_\emptyset) = \mu \circ \pi^{-1}(B)$ and $\nu_{M,k}(B) + \nu_{S,k}(B) = \mu_k(B \times X_\emptyset) = \mu_k \circ \pi^{-1}(B)$ for each $k \in \mathbb{N}$. Since π is continuous, $\mu_k \circ \pi^{-1} \rightarrow \mu \circ \pi^{-1}$. Indeed, for each bounded and continuous $f : Z \rightarrow \mathbb{R}$, $\int_Z f d\mu_k \circ \pi^{-1} = \int_{Z \times X_\emptyset} f \circ \pi d\mu_k \rightarrow \int_{Z \times X_\emptyset} f \circ \pi d\mu$.

$\pi d\mu = \int_Z f d\mu \circ \pi^{-1}$ since $f \circ \pi : Z \times X_\emptyset \rightarrow \mathbb{R}$ is bounded and continuous. Hence, since $M(Z \times X_\emptyset)$ is metrizable, $\nu_M + \nu_S = \mu \circ \pi^{-1} = \lim_k \mu_k \circ \pi^{-1} = \lim_k (\nu_{M,k} + \nu_{S,k})$.

Also, for each Borel subset B of Z , $\nu_W(B) = \int_{Z \times X} \delta(B \times C) d\mu(z, \delta)$ and $\nu_{W,k}(B) = \int_{Z \times X} \delta(B \times C) d\mu_k(z, \delta)$ for each $k \in \mathbb{N}$. We show that $\nu_{W,k} \rightarrow \nu_W$. Let $B \subseteq Z$ be closed and $f_B : X \rightarrow \mathbb{R}$ be defined by setting, for each $\delta \in X$, $f_B(\delta) = \delta(B \times C)$. Then f_B is bounded (since E is bounded) and upper semi-continuous. Hence, by (a suitable adaptation of) Aliprantis and Border (2006, Theorem 15.5), $\limsup_k \nu_{W,k}(B) = \limsup_k \int_{Z \times X} f_B d\mu_k \leq \int_{Z \times X} f_B d\mu = \nu_W(B)$. Furthermore, since f_Z is continuous and bounded, it follows that $\nu_{W,k}(Z) \rightarrow \nu_W(Z)$. Hence, $\nu_W = \lim_k \nu_{W,k}$ as claimed. Thus,

$$\nu_M + \nu_S + \nu_W = \lim_k (\nu_{M,k} + \nu_{S,k}) + \lim_k \nu_{W,k} = \lim_k (\nu_{M,k} + \nu_{S,k} + \nu_{W,k}) = \nu.$$

Condition 1 holds because, by Carmona and Podczeck (2009, Lemma 12), for each $(z, \delta) \in \text{supp}(\mu)$, there exists a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding $\{(z_{k_j}, \delta_{k_j})\}_{j=1}^\infty$ such that $(z_{k_j}, \delta_{k_j}) \rightarrow (z, \delta)$ and, for each $j \in \mathbb{N}$, $(z_{k_j}, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$. Hence, $\delta_{k_j} \in P_{m,k_j}(z_{k_j}) \cup P_{s,k_j}(z_{k_j}) \subseteq P_m(z_{k_j}) \cup P_s(z_{k_j})$ and, since $P_m \cup P_s$ is upper hemicontinuous and closed-valued and $M(Z \times C)$ is regular, $\delta \in P_m(z) \cup P_s(z)$. ■

Consider a market $E = (Z, \nu, C, P_m, P_s, (\succ_z)_{z \in Z})$. We say that $(\succ_z)_{z \in Z}$ is *continuous* if $\{(a, \delta, \mu, a', \delta', \mu', z) \in (\Delta \times M(Z \times X_\emptyset))^2 \times Z : (a, \delta, \mu) \succ_z (a', \delta', \mu')\}$ is open.

Lemma 3.3 *Let $E = (Z, \nu, C, P_m, P_s, (\succ_z)_{z \in Z})$ be a market, $\{(E_k, \mu_k)\}_{k=1}^\infty$ be a sequence such that $\{\mu_k\}_{k=1}^\infty$ converges and, for each $k \in \mathbb{N}$, (i) $E_k = (Z_k, \nu_k, C_k, P_{m,k}, P_{s,k}, (\succ_{z,k})_{z \in Z_k})$ is a market, (ii) μ_k is a stable matching for E_k , (iii) $Z_k \subseteq Z$, (iv) $\text{supp}(\nu_k) \subseteq \text{supp}(\nu)$, (v) $C_k \subseteq C$, (vi) $P_{a,k}(z) \subseteq P_a(z)$ for each $z \in Z_k$ and $a \in \{m, s\}$, and (vii) $\nu_k \rightarrow \nu$, and $\mu = \lim_k \mu_k$. If*

1. $(\succ_z)_{z \in Z}$ is continuous,
2. $\succ_{z,k}$ is the restriction of $\succ_z$ to $\Delta_k \times M(Z_k \times X_{\emptyset,k})$ for each $z \in Z_k$, and
3. for each $z \in Z$, $\delta' \in P_s(z)$, open neighborhood $V_{\delta'}$ of δ' , subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of

$\{\mu_k\}_{k=1}^\infty$ and sequence $\{z_{k_j}\}_{j=1}^\infty$ such that $z_{k_j} \rightarrow z$ and $z_{k_j} \in Z_{k_j}$ for each $j \in \mathbb{N}$,
there exists $J \in \mathbb{N}$ such that $P_{s,k_j}(z_{k_j}) \cap V_{\delta'} \neq \emptyset$ for each $j \geq J$,

then $\text{supp}(\mu) \subseteq IR(\mu)$.

Proof. Let $(z, \delta) \in \text{supp}(\mu)$ and suppose that $(z, \delta) \notin IR(\mu)$. Then either (i) there exists $\delta' \in P_s(z)$ such that $(s, \delta', \mu) \succ_z (a(\delta), \delta, \mu)$ where $a(\delta) = m$ if $\delta \in X$ and $a(\delta) = s$ if $\delta \in X_\emptyset \setminus X$, or (ii) there exists $(z', c) \in \text{supp}(\delta)$ and $\delta' \in P_s(z')$ such that $(s, \delta', \mu) \succ_{z'} (w, 1_{(z,c)}, \mu)$.

Consider case (i) first. The continuity of $(\succ_z)_{z \in Z}$ implies that there are open neighborhoods $V_{\delta'}$, V_z , V_δ and V_μ of δ' , z , δ and μ , respectively, such that $(s, \tilde{\delta}, \hat{\mu}) \succ_{\tilde{z}} (a(\delta), \hat{\delta}, \hat{\mu})$ for each $\tilde{\delta} \in V_{\delta'}$, $\tilde{z} \in V_z$, $\hat{\delta} \in V_\delta$ and $\hat{\mu} \in V_\mu$.

By Carmona and Podczeck (2009, Lemma 12), there is a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding sequence $\{(z_{k_j}, \delta_{k_j})\}_{j=1}^\infty$ such that $(z_{k_j}, \delta_{k_j}) \rightarrow (z, \delta)$ and $(z_{k_j}, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$ for each $j \in \mathbb{N}$. Thus, for each j sufficiently large, $\mu_{k_j} \in V_\mu$, $(z_{k_j}, \delta_{k_j}) \in \text{supp}(\mu_{k_j}) \cap (V_z \times V_\delta)$ and there exists $\tilde{\delta} \in P_{s,k_j}(z_{k_j}) \cap V_{\delta'}$. But then $(s, \tilde{\delta}, \mu_{k_j}) \succ_{z_{k_j}} (a(\delta), \delta_{k_j}, \mu_{k_j})$ and, hence, $(s, \tilde{\delta}, \mu_{k_j}) \succ_{z_{k_j}, k_j} (a(\delta), \delta_{k_j}, \mu_{k_j})$, contradicting the individual rationality of μ_{k_j} .

Consider next case (ii). The continuity of $(\succ_z)_{z \in Z}$ implies that there are open neighborhoods $V_{\delta'}$, V_c , V_z , $V_{z'}$ and V_μ of δ' , c , z , z' and μ , respectively, such that $(s, \tilde{\delta}, \hat{\mu}) \succ_{\tilde{z}} (w, 1_{(\tilde{z}, \hat{c})}, \hat{\mu})$ for each $\tilde{\delta} \in V_{\delta'}$, $\hat{c} \in V_c$, $\tilde{z} \in V_z$, $\tilde{z} \in V_{z'}$ and $\hat{\mu} \in V_\mu$.

By Carmona and Podczeck (2009, Lemma 12), there is a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding sequence $\{(z_{k_j}, \delta_{k_j}, z'_{k_j}, c_{k_j})\}_{j=1}^\infty$ such that $(z_{k_j}, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$ and $(z'_{k_j}, c_{k_j}) \in \text{supp}(\delta_{k_j})$ for each $j \in \mathbb{N}$ and $(z_{k_j}, \delta_{k_j}, z'_{k_j}, c_{k_j}) \rightarrow (z, \delta, z', c)$. Thus, for each j sufficiently large, $\mu_{k_j} \in V_\mu$, $(z_{k_j}, \delta_{k_j}) \in \text{supp}(\mu_{k_j}) \cap (V_z \times V_\delta)$, $(z'_{k_j}, c_{k_j}) \in \text{supp}(\delta_{k_j}) \cap (V_{z'} \times V_c)$ and there exists $\tilde{\delta} \in P_{s,k_j}(z'_{k_j}) \cap V_{\delta'}$. But then $(s, \tilde{\delta}, \mu_{k_j}) \succ_{z'_{k_j}} (w, 1_{(z_{k_j}, c_{k_j})}, \mu_{k_j})$ and, hence, $(s, \tilde{\delta}, \mu_{k_j}) \succ_{z'_{k_j}, k_j} (w, 1_{(z_{k_j}, c_{k_j})}, \mu_{k_j})$, contradicting the individual rationality of μ_{k_j} . ■

Lemma 3.4 Let $E = (Z, \nu, C, P_m, P_s, (\succ_z)_{z \in Z})$ be a market, $\{(E_k, \mu_k)\}_{k=1}^\infty$ be a sequence such that $\{\mu_k\}_{k=1}^\infty$ converges and, for each $k \in \mathbb{N}$, (i) $E_k = (Z_k, \nu_k, C_k, P_{m,k}, P_{s,k}, (\succ_{z,k})_{z \in Z_k})$ is a market, (ii) μ_k is a stable matching for E_k , (iii) $Z_k \subseteq Z$, (iv)

$\text{supp}(\nu_k) \subseteq \text{supp}(\nu)$, (v) $C_k \subseteq C$, (vi) $P_{a,k}(z) \subseteq P_a(z)$ for each $z \in Z_k$ and $a \in \{m, s\}$, and (vii) $\nu_k \rightarrow \nu$, and $\mu = \lim_k \mu_k$. If

1. $(\succ_z)_{z \in Z}$ is continuous,
2. $\succ_{z,k}$ is the restriction of $\succ_z$ to $\Delta_k \times M(Z_k \times X_{\emptyset,k})$ for each $z \in Z_k$,
3. for each $(z, \delta, \mu) \in \text{graph}(P_m) \times M(Z \times X_{\emptyset})$ such that $(z, \delta) \in \text{supp}(\mu)$, $\delta' \in \Lambda(z, \delta, \mu)$, open neighborhood $V_{\delta'}$ of δ' , subsequence $\{\mu_{k_j}\}_{j=1}^{\infty}$ of $\{\mu_k\}_{k=1}^{\infty}$ and sequence $\{(z_{k_j}, \delta_{k_j})\}_{j=1}^{\infty}$ such that $(z_{k_j}, \delta_{k_j}) \rightarrow (z, \delta)$ and $(z_{k_j}, \delta_{k_j}) \in Z_{k_j} \times X_{k_j}$ for each $j \in \mathbb{N}$, there exists $J \in \mathbb{N}$ such that $P_{m,k_j}(z_{k_j}) \cap \Lambda(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ for each $j \geq J$, and
4. for each $(z, \mu) \in Z \times M(Z \times X_{\emptyset})$, $\delta' \in \Lambda_0(z, \mu)$, open neighborhood $V_{\delta'}$ of δ' , subsequence $\{\mu_{k_j}\}_{j=1}^{\infty}$ of $\{\mu_k\}_{k=1}^{\infty}$ and sequence $\{z_{k_j}\}_{j=1}^{\infty}$ such that $z_{k_j} \rightarrow z$ and $z_{k_j} \in Z_{k_j}$ for each $j \in \mathbb{N}$, there exists $J \in \mathbb{N}$ such that $P_{m,k_j}(z_{k_j}) \cap \Lambda_0(z_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ for each $j \geq J$,

then $\text{supp}(\mu) \subseteq S_M(\mu)$.

Proof. In this proof, to avoid confusion, we write $T_z^m(\mu; E')$ for $T_z^m(\mu)$ in a market E' .

Let $(z, \delta) \in \text{supp}(\mu)$ and suppose that $(z, \delta) \notin S_M(\mu)$. Then either (i) $\delta \in X$ and there exists $\delta' \in P_m(z)$ such that $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ and $(m, \delta', \mu) \succ_z (m, \delta, \mu)$ or (ii) there exists $(z', c) \in \text{supp}(\delta)$ and $\delta' \in P_m(z')$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu)$ and $(m, \delta', \mu) \succ_{z'} (w, 1_{(z,c)}, \mu)$ or (iii) there exists $\delta' \in P_m(z)$ such that $\text{supp}(\delta') \subseteq T_z^m(\mu)$ and $(m, \delta', \mu) \succ_z (s, \delta, \mu)$.

Consider case (i) first. Since $(\succ_z)_{z \in Z}$ is continuous, let V_z , $V_{\delta'}$, V_{δ} and V_{μ} be open neighborhoods of z , δ' , δ and μ , respectively, such that $(m, \gamma', \bar{\mu}) \succ_{\bar{z}} (m, \gamma, \bar{\mu})$ for each $\bar{z} \in V_z$, $\gamma' \in V_{\delta'}$, $\gamma \in V_{\delta}$ and $\bar{\mu} \in V_{\mu}$.

By Carmona and Podczeck (2009, Lemma 12), there is a subsequence $\{\mu_{k_j}\}_{j=1}^{\infty}$ of $\{\mu_k\}_{k=1}^{\infty}$ and corresponding sequence $\{(z_{k_j}, \delta_{k_j})\}_{j=1}^{\infty}$ such that $(z_{k_j}, \delta_{k_j}) \rightarrow (z, \delta)$ and $(z_{k_j}, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$ for each $j \in \mathbb{N}$. Since μ_{k_j} is a matching, it follows that

$\delta_{k_j} \in P_{m,k_j}(z_{k_j}) \subseteq P_m(z_{k_j})$ and, hence, $(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \in \text{graph}(P_m) \times M(Z \times X_\emptyset)$ for each $j \in \mathbb{N}$.

Since $\delta' \in \Lambda(z, \delta, \mu)$, let $J \in \mathbb{N}$ be such that $\mu_{k_j} \in V_\mu$, $z_{k_j} \in V_z$, $\delta_{k_j} \in V_\delta$ and $P_{m,k_j}(z_{k_j}) \cap \Lambda(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ for all $j \geq J$ by condition 3. Let $j \geq J$ and let $\delta'_{k_j} \in P_{m,k_j}(z_{k_j}) \cap \Lambda(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \cap V_{\delta'}$. Then $\text{supp}(\delta'_{k_j}) \subseteq T_{z_{k_j}}^m(\mu_{k_j}; E) \cup \text{supp}(\delta_{k_j})$ and $(m, \delta'_{k_j}, \mu_{k_j}) \succ_{z_{k_j}} (m, \delta_{k_j}, \mu_{k_j})$. It then follows that $\text{supp}(\delta'_{k_j}) \subseteq T_{z_{k_j}}^m(\mu_{k_j}; E_{k_j}) \cup \text{supp}(\delta_{k_j})$ and $(m, \delta'_{k_j}, \mu_{k_j}) \succ_{z_{k_j}, k_j} (m, \delta_{k_j}, \mu_{k_j})$. But this contradicts the stability of μ_{k_j} .

Consider next case (ii). Since $(\succ_z)_{z \in Z}$ is continuous, let V_z , $V_{z'}$, $V_{\delta'}$, V_c and V_μ be open neighborhoods of z , z' , δ' , c and μ respectively, such that $(m, \hat{\delta}', \hat{\mu}) \succ_{\hat{z}'} (w, 1_{(\hat{z}, \hat{c})}, \hat{\mu})$ for each $\hat{z} \in V_z$, $\hat{z}' \in V_{z'}$, $\hat{\delta}' \in V_{\delta'}$, $\hat{c} \in V_c$ and $\hat{\mu} \in V_\mu$. By Carmona and Podczeck (2009, Lemma 12), there is a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding sequence $\{(z_{k_j}, \delta_{k_j}, z'_{k_j}, c_{k_j})\}_{j=1}^\infty$ such that $(z_{k_j}, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$ and $(z'_{k_j}, c_{k_j}) \in \text{supp}(\delta_{k_j})$ for each $j \in \mathbb{N}$ and $(z_{k_j}, \delta_{k_j}, z'_{k_j}, c_{k_j}) \rightarrow (z, \delta, z', c)$.

Since $\delta' \in \Lambda_0(z', \mu)$, let $J \in \mathbb{N}$ be such that $z_{k_j} \in V_z$, $z'_{k_j} \in V_{z'}$, $c_{k_j} \in V_c$, $\mu_{k_j} \in V_\mu$ and $P_{m,k_j}(z'_{k_j}) \cap \Lambda_0(z'_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ for all $j \geq J$ by condition 4. Let $j \geq J$ and let $\delta'_{k_j} \in P_{m,k_j}(z'_{k_j}) \cap \Lambda_0(z'_{k_j}, \mu_{k_j}) \cap V_{\delta'}$. Then $\text{supp}(\delta'_{k_j}) \subseteq T_{z'_{k_j}}^m(\mu_{k_j}; E)$ and $(m, \delta'_{k_j}, \mu_{k_j}) \succ_{z'_{k_j}} (w, 1_{(z_{k_j}, c_{k_j})}, \mu_{k_j})$. It then follows that $\text{supp}(\delta'_{k_j}) \subseteq T_{z'_{k_j}}^m(\mu_{k_j}; E_{k_j})$ and $(m, \delta'_{k_j}, \mu_{k_j}) \succ_{z'_{k_j}, k_j} (w, 1_{(z_{k_j}, c_{k_j})}, \mu_{k_j})$. But this contradicts the stability of μ_{k_j} .

Finally, consider case (iii). Since $(\succ_z)_{z \in Z}$ is continuous, let V_z , $V_{\delta'}$, V_δ and V_μ be open neighborhoods of z , δ' , δ and μ , respectively, such that $(m, \gamma', \bar{\mu}) \succ_{\bar{z}} (s, \gamma, \bar{\mu})$ for each $\bar{z} \in V_z$, $\gamma' \in V_{\delta'}$, $\gamma \in V_\delta$ and $\bar{\mu} \in V_\mu$. By Carmona and Podczeck (2009, Lemma 12), there is a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding sequence $\{(z_{k_j}, \delta_{k_j})\}_{j=1}^\infty$ such that $(z_{k_j}, \delta_{k_j}) \rightarrow (z, \delta)$ and $(z_{k_j}, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$ for each $j \in \mathbb{N}$.

Since $\delta' \in \Lambda_0(z, \mu)$, let $J \in \mathbb{N}$ be such that $\mu_{k_j} \in V_\mu$, $z_{k_j} \in V_z$, $\delta_{k_j} \in V_\delta$ and $P_{m,k_j}(z_{k_j}) \cap \Lambda_0(z_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ for all $j \geq J$ by condition 4. Let $j \geq J$ and let $\delta'_{k_j} \in P_{m,k_j}(z_{k_j}) \cap \Lambda_0(z_{k_j}, \mu_{k_j}) \cap V_{\delta'}$. Then $\text{supp}(\delta'_{k_j}) \subseteq T_{z_{k_j}}^m(\mu_{k_j}; E)$ and $(m, \delta'_{k_j}, \mu_{k_j}) \succ_{z_{k_j}} (s, \delta_{k_j}, \mu_{k_j})$. It then follows that $\text{supp}(\delta'_{k_j}) \subseteq T_{z_{k_j}}^m(\mu_{k_j}; E_{k_j})$ and $(m, \delta'_{k_j}, \mu_{k_j}) \succ_{z_{k_j}, k_j} (s, \delta_{k_j}, \mu_{k_j})$. But this contradicts the stability of μ_{k_j} . ■

The following corollary, which will be used in the proof of Theorem 3.1, combines

Lemmas 3.1–3.4.

Corollary 3.1 *Let $E = (Z, \nu, C, P_m, P_s, (\succ_z)_{z \in Z})$ be a rational, continuous and bounded market and $\{(E_k, \mu_k)\}_{k=1}^\infty$ be a sequence such that, for each $k \in \mathbb{N}$,*

$$E_k = (Z_k, \nu_k, C_k, P_{m,k}, P_{s,k}, (\succ_{z,k})_{z \in Z_k})$$

is a market, μ_k is a stable matching for E_k , $Z_k \subseteq Z$, $\text{supp}(\nu_k) \subseteq \text{supp}(\nu)$, $C_k \subseteq C$, $P_{a,k}(z) \subseteq P_a(z)$ for each $z \in Z_k$ and $a \in \{m, s\}$, and $\nu_k \rightarrow \nu$. Then:

1. $\{\mu_k\}_{k=1}^\infty$ has a convergent subsequence in $M(Z \times X_\emptyset)$.

Suppose further that $\{\mu_k\}_{k=1}^\infty$ converges and let $\mu = \lim_k \mu_k$. Then:

2. μ is a matching for E .

Suppose further that $\succ_{z,k}$ is the restriction of $\succ_z$ to $\Delta_k \times M(Z_k \times X_{\emptyset,k})$ for each $z \in Z_k$. Then:

3. $\text{supp}(\mu) \subseteq IR(\mu)$ if

(a) *for each $z \in Z$, $\delta \in P_s(z)$, open neighborhood V_δ of δ , subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and sequence $\{z_{k_j}\}_{j=1}^\infty$ such that $z_{k_j} \rightarrow z$ and $z_{k_j} \in Z_{k_j}$ for each $j \in \mathbb{N}$, there exists $J \in \mathbb{N}$ such that $P_{s,k_j}(z_{k_j}) \cap V_\delta \neq \emptyset$ for each $j \geq J$,*

4. $\text{supp}(\mu) \subseteq S_M(\mu)$ if

(a) *for each $(z, \delta, \mu) \in \text{graph}(P_m) \times M(Z \times X_\emptyset)$ such that $(z, \delta) \in \text{supp}(\mu)$, $\delta' \in \Lambda(z, \delta, \mu)$, open neighborhood $V_{\delta'}$ of δ' , subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and sequence $\{(z_{k_j}, \delta_{k_j})\}_{j=1}^\infty$ such that $(z_{k_j}, \delta_{k_j}) \rightarrow (z, \delta)$ and $(z_{k_j}, \delta_{k_j}) \in Z_{k_j} \times X_{k_j}$ for each $j \in \mathbb{N}$, there exists $J \in \mathbb{N}$ such that $P_{m,k_j}(z_{k_j}) \cap \Lambda(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ for each $j \geq J$, and*

(b) *for each $(z, \mu) \in Z \times M(Z \times X_\emptyset)$, $\delta' \in \Lambda_0(z, \mu)$, open neighborhood $V_{\delta'}$ of δ' , subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and sequence $\{z_{k_j}\}_{j=1}^\infty$ such that $z_{k_j} \rightarrow z$ and $z_{k_j} \in Z_{k_j}$ for each $j \in \mathbb{N}$, there exists $J \in \mathbb{N}$ such that $P_{m,k_j}(z_{k_j}) \cap \Lambda_0(z_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ for each $j \geq J$.*

3.1 Proof of Theorem 3.1

The first step in the proof of our existence result consists in the following lemma, which considers the special case where Z , C and $P_m(z)$ are finite for each $z \in Z$.

Lemma 3.5 *If E is a rational and continuous market such that Z , C and $P_m(z)$ are finite for each $z \in Z$, then E has a stable matching.*

Proof. Let $X = \cup_{z \in Z} P_m(z)$. Then X is finite and so are $Z_\emptyset$, $X_\emptyset$ and Δ . Define $\bar{\tau} \in \mathbb{R}^{Z \times \Delta}$ by setting, for each $(z, a, \delta) \in Z \times \Delta$,

$$\bar{\tau}(z, a, \delta) = \begin{cases} 0 & \text{if } \delta \notin P_a(z) \text{ and } a \neq w, \\ \nu(z) & \text{otherwise.} \end{cases}$$

Define

$$\mathcal{T} = \left\{ \tau \in \mathbb{R}_+^{Z \times \Delta} : \tau(z, a, \delta) \leq \bar{\tau}(z, a, \delta) \text{ and } \sum_{(a, \delta) \in \Delta} \tau(z, a, \delta) \leq \nu(z) \right.$$

for each $(z, a, \delta) \in Z \times \Delta \}$ and

$$\mathcal{K} = \left\{ \kappa \in \mathbb{R}_+^{Z \times Z \times C} : \kappa(z, z', c) \leq \nu(z') \text{ for each } (z, z', c) \in Z \times Z \times C \right\}.$$

Note that $\mathcal{T}$ and $\mathcal{K}$ are nonempty, convex, and compact subsets of Euclidean spaces.

Let $u : Z \times \Delta \times M(Z \times X_\emptyset) \rightarrow \mathbb{R}$ be a continuous utility function that represents preferences. We normalize $u \geq 1$. For each $n \in \mathbb{N}$, let $u_n = u^n$. Since $x \mapsto x^n$ is strictly increasing on $[1, \infty)$, u_n and u represent the same preferences.

Define $d : \mathcal{T} \rightarrow \mathbb{R}_+^{Z \times X_\emptyset}$ by setting, for each $\tau \in \mathcal{T}$ and $(z, \delta) \in Z \times X_\emptyset$,

$$d(\tau)(z, \delta) = \begin{cases} \tau(z, m, \delta) & \text{if } \delta \in X, \\ \tau(z, s, \delta) & \text{if } \delta \in X_\emptyset \setminus X. \end{cases}$$

The function d is continuous. We abuse notation and, for each $(z, a, \delta, \tau) \in Z \times \Delta \times \mathcal{T}$, write $u(z, a, \delta, \tau)$ for $u(z, a, \delta, d(\tau))$ and analogously for u_n . We also write $(a, \delta, \tau) \succ_z (a', \delta', \tau)$ for $(a, \delta, d(\tau)) \succ_z (a', \delta', d(\tau))$, where $(a', \delta') \in \Delta$.

For each $n \in \mathbb{N}$, let $D_n : \mathcal{T} \times \mathcal{K} \rightrightarrows \mathcal{T}$ be defined by setting, for each $(\mu, \kappa) \in \mathcal{T} \times \mathcal{K}$,

$$D_n(\mu, \kappa) = \left\{ \tau \in \mathcal{T} : \tau \in \arg \max_{\tau' \in \mathcal{T}} \sum_{z \in Z, (a, \delta) \in \Delta} u_n(z, a, \delta, \mu) \tau'(z, a, \delta) \right.$$

$$\text{subject to } \sum_{(a, \delta) \in \Delta} \tau'(z, a, \delta) = \nu(z) \text{ for all } z \in Z,$$

$$\sum_{\delta \in X} \tau'(z, m, \delta) \delta(z', c) \leq \kappa(z, z', c) \text{ for all } (z, z', c) \in Z \times Z \times C, \text{ and}$$

$$\tau'(z, w, 1_{(z', c)}) \leq \sum_{\delta \in X} \mu(z', m, \delta) \delta(z, c) \text{ for all } (z, z', c) \in Z \times Z \times C \}.$$

Claim 1 D_n is upper hemicontinuous with nonempty, compact and convex values.

Proof. It follows by the linearity of the objective function together with the convexity of the constraint set that D_n has convex values. It follows from Berge's maximum theorem that D_n is upper hemicontinuous with nonempty and compact values. To see this, first note that the objective function is continuous and that the constraint set, denoted by $\Phi(\mu, \kappa)$, is contained in the compact set $\mathcal{T}$. It is clear that Φ is upper hemicontinuous with compact values. To see that Φ has nonempty values, define $\bar{\tau} \in \mathcal{T}$ as follows. For each $z \in Z$, let $c_z \in C$ be such that $1_{(\emptyset, c_z)} \in P_s(z)$, $\bar{\tau}(z, s, 1_{(\emptyset, c_z)}) = \nu(z)$ and $\bar{\tau}(z, a, \delta) = 0$ for each $(a, \delta) \in \Delta \setminus \{(s, 1_{(\emptyset, c_z)})\}$. We then have that $\bar{\tau} \in \Phi(\mu, \kappa)$ for each $(\mu, \kappa) \in \mathcal{T} \times \mathcal{K}$. Finally, to see that Φ is lower hemicontinuous, let $(\mu, \kappa) \in \mathcal{T} \times \mathcal{K}$, $O \subseteq \mathcal{T}$ be an open set such that $\Phi(\mu, \kappa) \cap O \neq \emptyset$, and $\tau \in \Phi(\mu, \kappa) \cap O$. Let $\hat{\tau} = \lambda\tau + (1-\lambda)\bar{\tau} \in O$ for some $\lambda \in (0, 1)$. Note that for each $z \in Z$, $\sum_{(a, \delta) \in \Delta} \hat{\tau}(z, a, \delta) = \nu(z)$, $\sum_{\delta \in X} \hat{\tau}(z, m, \delta) \delta(z', c) < \kappa(z, z', c)$ for each $(z, z', c) \in Z \times Z \times C$ such that $\kappa(z, z', c) > 0$ and $\hat{\tau}(z, w, 1_{(z', c)}) < \sum_{\delta \in X} \mu(z', m, \delta) \delta(z, c)$ for each $(z, z', c) \in Z \times Z \times C$ such that $\sum_{\delta \in X} \mu(z', m, \delta) \delta(z, c) > 0$. Thus, there is an open neighborhood V of (μ, κ) such that $\hat{\tau} \in \Phi(\mu', \kappa') \cap O$ for each $(\mu', \kappa') \in V$. ■

Claim 2 If $(\mu, \kappa) \in \mathcal{T} \times \mathcal{K}$, $\tau \in D_n(\mu, \kappa)$ and $(z, a, \delta') \in Z \times \Delta$ is such that $\tau(z, a, \delta') > 0$, then $\tau(z, w, 1_{(\hat{z}, \hat{c})}) = \sum_{\delta \in X} \mu(\hat{z}, m, \delta) \delta(z, \hat{c})$ for each $(\hat{z}, \hat{c}) \in Z \times C$ such that $(w, 1_{(\hat{z}, \hat{c})}, \mu) \succ_z (a, \delta', \mu)$.

Proof. If not, then $\tau(z, w, 1_{(\hat{z}, \hat{c})}) < \sum_{\delta \in X} \mu(\hat{z}, m, \delta) \delta(z, \hat{c})$ for some $(\hat{z}, \hat{c}) \in Z \times C$ such that $(w, 1_{(\hat{z}, \hat{c})}, \mu) \succ_z (a, \delta', \mu)$. Then $\sum_{\delta \in X} \mu(\hat{z}, m, \delta) \delta(z, \hat{c}) > 0$. Thus, increase

$\tau(z, w, 1_{(\hat{z}, \hat{c})})$ and decrease $\tau(z, a, \delta')$ by the same amount $\varepsilon \in (0, \tau(z, a, \delta'))$ such that $\tau(z, w, 1_{(\hat{z}, \hat{c})}) + \varepsilon < \sum_{\delta \in X} \mu(\hat{z}, m, \delta) \delta(z, \hat{c})$. This increases the objective function in $D_n(\mu, \kappa)$ while satisfying the constraints, thus contradicting $\tau \in D_n(\mu, \kappa)$. ■

For each $\mu \in \mathcal{T}$ and $(z, z', c) \in Z \times Z \times C$, let

$$W(z, z', c, \mu) = \{(a, \delta) \in \Delta : u(z', w, 1_{(z, c)}, \mu) > u(z', a, \delta, \mu)\}.$$

Let $g : \mathcal{T} \rightarrow \mathcal{K}$ be defined by setting, for each $\mu \in \mathcal{T}$ and $(z, z', c) \in Z \times Z \times C$,

$$g(\mu)(z, z', c) = \sum_{(a, \delta) \in W(z, z', c, \mu)} \mu(z', a, \delta)$$

To see that $g(\mu) \in \mathcal{K}$, note that since $\mu \in \mathcal{T}$, $0 \leq g(\mu)(z, z', c) \leq \nu(z')$.

The function g may fail to be continuous and, thus, we will consider a continuous approximation to it. For each $n \in \mathbb{N}$ and $(z, z', c) \in Z \times Z \times C$, let $\alpha_{n, (z, z', c)} : \Delta \times \mathcal{T} \rightarrow [0, 1]$ be defined by setting, for each $(a, \delta, \mu) \in \Delta \times \mathcal{T}$,

$$\alpha_{n, (z, z', c)}(a, \delta, \mu) = n \max \left\{ 0, \min \left\{ u(z', w, 1_{(z, c)}, \mu) - u(z', a, \delta, \mu), \frac{1}{n} \right\} \right\}.$$

Let $g_n : \mathcal{T} \rightarrow \mathcal{K}$ be defined by setting, for each $\mu \in \mathcal{T}$ and $(z, z', c) \in Z \times Z \times C$,

$$g_n(\mu)(z, z', c) = \sum_{(a, \delta) \in \Delta} \alpha_{n, (z, z', c)}(a, \delta, \mu) \mu(z', a, \delta).$$

We have that g_n is continuous since $\alpha_{n, (z, z', c)}$ is continuous for each $(z, z', c) \in Z \times Z \times C$. Note that

$$\alpha_{n, (z, z', c)}(a, \delta, \mu) \in \begin{cases} \{0\} & \text{if } u(z', a, \delta, \mu) \geq u(z', w, 1_{(z, c)}, \mu), \\ (0, 1) & \text{if } u(z', w, 1_{(z, c)}, \mu) - \frac{1}{n} < u(z', a, \delta, \mu) < u(z', w, 1_{(z, c)}, \mu), \\ \{1\} & \text{if } u(z', a, \delta, \mu) \leq u(z', w, 1_{(z, c)}, \mu) - \frac{1}{n}. \end{cases}$$

Hence, it follows that

$$g_n(\mu)(z, z', c) \leq g(\mu)(z, z', c) \tag{3.1}$$

for each $\mu \in \mathcal{T}$ and $(z, z', c) \in Z \times Z \times C$ since

$$g(\mu)(z, z', c) = \sum_{(a, \delta) \in \Delta} \alpha_{n, (z, z', c)}(a, \delta, \mu) \mu(z', a, \delta)$$

with

$$\alpha_{(z,z',c)}(a, \delta, \mu) = \begin{cases} 1 & \text{if } u(z', a, \delta, \mu) < u(z', w, 1_{(z,c)}, \mu), \\ 0 & \text{otherwise.} \end{cases}$$

To see that $g_n(\mu) \in \mathcal{K}$, note that $0 \leq g_n(\mu) \leq g(\mu)$.

Let $f_n : \mathcal{T} \rightarrow \mathcal{K}$ be defined by setting, for each $\mu \in \mathcal{T}$ and $(z, z', c) \in Z \times Z \times C$,

$$f_n(\mu)(z, z', c) = \mu(z', w, 1_{(z,c)}) + \frac{1}{n}g_n(\mu)(z, z', c).$$

To see that $f_n(\mu) \in \mathcal{K}$, note that

$$0 \leq \mu(z', w, 1_{(z,c)}) + \frac{1}{n}g_n(\mu)(z, z', c) \leq \mu(z', w, 1_{(z,c)}) + g(\mu)(z, z', c) \leq \nu(z').$$

We have that f_n is continuous since g_n is continuous.

Let $\Psi_n : \mathcal{T} \times \mathcal{K} \rightrightarrows \mathcal{T} \times \mathcal{K}$ be defined by setting, for each $(\mu, \kappa) \in \mathcal{T} \times \mathcal{K}$,

$$\Psi_n(\mu, \kappa) = D_n(\mu, \kappa) \times \{f_n(\mu)\}.$$

It follows from the continuity of f_n and from Claim 1 that Ψ_n is upper hemicontinuous with nonempty, compact and convex values. Hence, by the Kakutani fixed point theorem, let (μ_n, κ_n) be a fixed point of Ψ_n . Thus, $\mu_n \in D_n(\mu_n, \kappa_n)$ and $\kappa_n = f_n(\mu_n)$.

Since $\mathcal{T} \times \mathcal{K}$ is compact, taking a subsequence if necessary, we may assume that $\{(\mu_n, \kappa_n)\}_{n=1}^\infty$ converges; let $(\mu, \kappa) = \lim_{n \rightarrow \infty} (\mu_n, \kappa_n)$. For each n , we have $\kappa_n = f_n(\mu_n)$, and so

$$\kappa(z, z', c) = \lim_{n \rightarrow \infty} f_n(\mu_n)(z, z', c) = \mu(z', w, 1_{(z,c)}) \quad (3.2)$$

for each $(z, z', c) \in Z \times Z \times C$. Let

$$\mu^* = d(\mu)$$

and $\mu_n^* = d(\mu_n)$ for each $n \in \mathbb{N}$.

For each $z \in Z$ and $n \in \mathbb{N}$, it follows from $\mu_n \in D_n(\mu_n, \kappa_n)$ that

$$\sum_{(z',c) \in Z \times C} \mu_n(z, w, 1_{(z',c)}) \leq \sum_{(z',c) \in Z \times C} \sum_{\delta \in X} \mu_n(z', m, \delta) \delta(z, c) \leq \sum_{(z',c) \in Z \times C} \kappa_n(z', z, c).$$

By (3.2), $\lim_n \sum_{(z',c) \in Z \times C} \kappa_n(z', z, c) = \sum_{(z',c) \in Z \times C} \mu(z, w, 1_{(z',c)})$ and, hence,

$$\sum_{(z',c) \in Z \times C} \mu(z, w, 1_{(z',c)}) = \sum_{(z',c) \in Z \times C} \sum_{\delta \in X} \mu(z', m, \delta) \delta(z, c) \text{ for each } z \in Z.$$

Thus, for each $z \in Z$,

$$\sum_{(z',\delta) \in Z \times X} \mu^*(z', \delta) \delta_Z(z) = \sum_{(z',c) \in Z \times C} \sum_{\delta \in X} \mu(z', m, \delta) \delta(z, c) = \sum_{(z',c) \in Z \times C} \mu(z, w, 1_{(z',c)}). \quad (3.3)$$

Claim 3 μ^* is a matching.

Proof. Condition 1 follows because if $(z, \delta) \in Z \times X_\emptyset$ is such that $\mu^*(z, \delta) > 0$, then $\mu(z, a, \delta) > 0$ for some $a \neq w$ and hence $(z, \delta) \in \text{graph}(P_m) \cup \text{graph}(P_s)$ since $\mu \in \mathcal{T}$.

Condition 2 holds since, for each $z \in Z$, $\mu = \lim_n \mu_n$ and $\mu_n \in D_n(\mu_n, \kappa_n)$ for each $n \in \mathbb{N}$ imply that

$$\begin{aligned} \nu(z) &= \sum_{\delta \in X} \mu(z, m, \delta) + \sum_{\delta \in X_\emptyset \setminus X} \mu(z, s, \delta) + \sum_{(z',c) \in Z \times C} \mu(z, w, 1_{(z',c)}) \\ &= \sum_{\delta \in X} \mu^*(z, \delta) + \sum_{\delta \in X_\emptyset \setminus X} \mu^*(z, \delta) + \sum_{(z',\delta) \in Z \times X} \mu^*(z', \delta) \delta_Z(z), \end{aligned}$$

the last equality following by (3.3). ■

Claim 4 If $(z, z', c, \delta) \in Z \times Z \times C \times X$ is such that $(z, \delta) \in \text{supp}(\mu^*)$ and $(z', c) \in \text{supp}(\delta)$, then $\mu(z', w, 1_{(z',c)}) > 0$.

Proof. We have that $\sum_{\delta' \in X} \mu(z, m, \delta') \delta'(z', c) \geq \mu(z, m, \delta) \delta(z', c) > 0$ and, thus, $\kappa(z, z', c) > 0$ since $\mu_n \in D_n(\mu_n, \kappa_n)$ for each $n \in \mathbb{N}$. Hence, (3.2) implies that $\mu(z', w, 1_{(z',c)}) > 0$. ■

Claim 5 $\text{supp}(\mu^*) \subseteq IR(\mu^*)$.

Proof. Suppose not; then there exists $(z^*, \delta^*) \in \text{supp}(\mu^*) \setminus IR(\mu^*)$. We claim that there exists $z \in Z$, $(a, \delta) \in \Delta$ and $1_{(\emptyset, c')} \in P_s(z)$ such that

1. $\mu(z, a, \delta) > 0$ and

2. $(s, 1_{(\emptyset, c')}, \mu^*) \succ_z (a, \delta, \mu^*)$.

Indeed, in cases (i) and (iii) of the definition of $IR(\mu^*)$, let $(z, \delta) = (z^*, \delta^*)$ in both cases and $a = m$ in case (i) and $a = s$ in case (iii). In case (ii) of the definition of $IR(\mu^*)$, we have that $\delta^* \in X$ and there exist $(z', c) \in \text{supp}(\delta^*)$ and $c' \in C$ such that $1_{(\emptyset, c')} \in P_s(z')$ such that $(s, 1_{(\emptyset, c')}, \mu^*) \succ_{z'} (w, 1_{(z^*, c)}, \mu^*)$. Claim 4 implies that $\mu(z', w, 1_{(z^*, c)}) > 0$; hence, let $z = z'$, $a = w$ and $\delta = 1_{(z^*, c)}$.

We then have that $\mu_n(z, a, \delta) > 0$ and $(s, 1_{(\emptyset, c')}, \mu_n^*) \succ_z (a, \delta, \mu_n^*)$ for n sufficiently large. Then decrease $\mu_n(z, a, \delta)$ and increase $\mu_n(z, s, 1_{(\emptyset, c')})$ by the same amount $\varepsilon \in (0, \mu_n(z, a, \delta))$ to increase the objective function in $D_n(\mu_n, \kappa_n)$ while satisfying the constraints. But this is a contradiction to $\mu_n \in D_n(\mu_n, \kappa_n)$. ■

Claim 6 *If $(z, z', c) \in Z \times Z \times C$ is such that $(z', c) \in T_z^m(\mu^*)$, then there is $N_{z, z', c} \in \mathbb{N}$ such that $\sum_{\delta \in X} \mu_n(z, m, \delta) \delta(z', c) < \kappa_n(z, z', c)$ for each $n \geq N_{z, z', c}$.*

Proof. Let $(z', c) \in T_z^m(\mu^*)$. In case (a) of the definition of $T_z^m(\mu^*)$, there exists $(\tilde{z}, \tilde{c}, \tilde{\delta}) \in Z \times C \times X$ such that $(\tilde{z}, \tilde{\delta}) \in \text{supp}(\mu^*)$, $(z', \tilde{c}) \in \text{supp}(\tilde{\delta})$ and $(w, 1_{(z, c)}, \mu^*) \succ_{z'} (w, 1_{(\tilde{z}, \tilde{c})}, \mu^*)$. Hence, $\mu(z', w, 1_{(\tilde{z}, \tilde{c})}) > 0$ by Claim 4.

In cases (b) and (c) of the definition of $T_z^m(\mu^*)$, there exists $(a, \delta') \in \Delta$ such that $a \neq w$, $(z', \delta') \in \text{supp}(\mu^*)$ and $(w, 1_{(z, c)}, \mu^*) \succ_{z'} (a, \delta', \mu^*)$. Thus, letting $a = w$ and $\delta' = 1_{(\tilde{z}, \tilde{c})}$ in case (a), it follows that, in all cases, there exists $(a, \delta') \in \Delta$ such that $(z', a, \delta') \in \text{supp}(\mu)$ and $(w, 1_{(z, c)}, \mu^*) \succ_{z'} (a, \delta', \mu^*)$.

Let $N_{z, z', c} \in \mathbb{N}$ be such that $\mu_n(z', a, \delta') > 0$ and $(w, 1_{(z, c)}, \mu_n^*) \succ_{z'} (a, \delta', \mu_n^*)$ for each $n \geq N_{z, z', c}$. Thus, for each $n \geq N_{z, z', c}$,

$$\mu_n(z', w, 1_{(z, c)}) = \sum_{\delta \in X} \mu_n(z, m, \delta) \delta(z', c)$$

by Claim 2 and, since $\alpha_{n, (z, z', c)}(a, \delta', \mu_n) > 0$,

$$\begin{aligned} \kappa_n(z, z', c) &= \mu_n(z', w, 1_{(z, c)}) + \frac{1}{n} g_n(\mu_n)(z, z', c) \\ &\geq \mu_n(z', w, 1_{(z, c)}) + \frac{1}{n} \alpha_{n, (z, z', c)}(a, \delta', \mu_n) \mu_n(z', a, \delta') \\ &> \mu_n(z', w, 1_{(z, c)}) = \sum_{\delta \in X} \mu_n(z, m, \delta) \delta(z', c). \end{aligned}$$

■

Claim 7 $\text{supp}(\mu^*) \subseteq S_M(\mu^*)$.

Proof. Suppose not; then there exists $(z^*, \delta^*) \in \text{supp}(\mu^*) \setminus S_M(\mu^*)$. We claim that there exists $z \in Z$, $(a, \delta) \in \Delta$ and $\delta' \in P_m(z)$ such that

1. $\mu(z, a, \delta) > 0$,
2. $\text{supp}(\delta') \subseteq T_z^m(\mu^*) \cup \text{supp}(\delta)$ if $a = m$ and $\text{supp}(\delta') \subseteq T_z^m(\mu^*)$ if $a \neq m$, and
3. $(m, \delta', \mu^*) \succ_z (a, \delta, \mu^*)$.

Indeed, in cases (i) and (iii) of the definition of $S_M(\mu^*)$, let $(z, \delta) = (z^*, \delta^*)$ in both cases and $a = m$ in case (i) and $a = s$ in case (iii). In case (ii) of the definition of $S_M(\mu^*)$, we have that $\delta^* \in X$ and there exist $(z', c) \in \text{supp}(\delta^*)$ and $\delta' \in P_m(z')$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu^*)$ and $(m, \delta', \mu^*) \succ_{z'} (w, 1_{(z^*, c)}, \mu^*)$. Claim 4 implies that $\mu(z', w, 1_{(z^*, c)}) > 0$; hence, let $z = z'$, $a = w$ and $\delta = 1_{(z^*, c)}$.

Let $\theta = 1$ if $\text{supp}(\delta) \cap \text{supp}(\delta') = \emptyset$; otherwise, let $(\bar{z}, \bar{c}) \in \text{supp}(\delta) \cap \text{supp}(\delta')$ be such that $\delta(\bar{z}, \bar{c})/\delta'(\bar{z}, \bar{c}) \leq \delta(z, c)/\delta'(z, c)$ for each $(z, c) \in \text{supp}(\delta) \cap \text{supp}(\delta')$ and define

$$\theta = \min \left\{ 1, \frac{\delta(\bar{z}, \bar{c})}{\delta'(\bar{z}, \bar{c})} \right\}.$$

Let $k \in \mathbb{N}$ be such that $k\theta > 1$.

There is $N \in \mathbb{N}$ such that, for each $n \geq N$,

- (i) $\text{supp}(\mu) \subseteq \text{supp}(\mu_n)$,
- (ii) $\sum_{\hat{\delta} \in X} \mu_n(z, m, \hat{\delta}) \hat{\delta}(z', c) < \kappa_n(z, z', c)$ for each $(z', c) \in T_z^m(\mu^*)$, and
- (iii) $u_n(z, m, \delta', \mu_n) \geq k u_n(z, a, \delta, \mu_n)$.

Indeed, (i) is clear since Z and Δ are finite. As for (ii), take $N \geq \max_{(z', c) \in T_z^m(\mu^*)} N_{z, z', c}$ where, for each $(z', c) \in T_z^m(\mu^*)$, $N_{z, z', c}$ is given by Claim 6. Finally, for (iii), we have that $u(z, m, \delta', \mu)/u(z, a, \delta, \mu) > 1$ and $u(z, m, \delta', \mu_n)/u(z, a, \delta, \mu_n) \geq \beta$ for some $\beta > 1$ for all n sufficiently large. Hence,

$$\frac{u_n(z, m, \delta', \mu_n)}{u_n(z, a, \delta, \mu_n)} = \left(\frac{u(z, m, \delta', \mu_n)}{u(z, a, \delta, \mu_n)} \right)^n \geq \beta^n > k$$

for all n sufficiently large.

Fix $n \geq N$ and let $c^* \in C$ be such that $1_{(\emptyset, c^*)} \in P_s(z)$. For each $\varepsilon > 0$, define π_ε by setting

$$\pi_\varepsilon = \mu_n - \varepsilon 1_{(z, a, \delta)} + \theta \varepsilon 1_{(z, m, \delta')} + (1 - \theta) \varepsilon 1_{(z, s, 1_{(\emptyset, c^*)})}.$$

By (i), $\mu_n(z, a, \delta) > 0$. We have that, for each $\varepsilon \in (0, \mu_n(z, a, \delta))$, $\pi_\varepsilon(\hat{z}, \hat{a}, \hat{\delta}) \geq 0$ for each $(\hat{z}, \hat{a}, \hat{\delta}) \in Z \times \Delta$, $\pi_\varepsilon(\hat{z}, w, 1_{(z', c)}) \leq \mu_n(\hat{z}, w, 1_{(z', c)}) \leq \sum_{\hat{\delta} \in X} \mu_n(z', m, \hat{\delta}) \hat{\delta}(\hat{z}, c)$ for each $(\hat{z}, z', c) \in Z \times Z \times C$ and

$$\sum_{(\hat{a}, \hat{\delta}) \in \Delta} \pi_\varepsilon(\hat{z}, \hat{a}, \hat{\delta}) = \sum_{(\hat{a}, \hat{\delta}) \in \Delta} \mu_n(\hat{z}, \hat{a}, \hat{\delta}) = \nu(\hat{z})$$

for each $\hat{z} \in Z$. In particular, $\pi_\varepsilon \in \mathcal{T}$.

We also have that, for some $\varepsilon \in (0, \mu_n(z, a, \delta))$,

$$\sum_{\hat{\delta} \in X} \pi_\varepsilon(\hat{z}, m, \hat{\delta}) \hat{\delta}(z', c) \leq \kappa_n(\hat{z}, z', c) \text{ for all } (\hat{z}, z', c) \in Z \times Z \times C. \quad (3.4)$$

First, note that it is enough to consider $\hat{z} = z$ and that, for each $(z', c) \in Z \times C$,

$$\sum_{\hat{\delta} \in X} \pi_\varepsilon(z, m, \hat{\delta}) \hat{\delta}(z', c) = \sum_{\hat{\delta} \in X} \mu_n(z, m, \hat{\delta}) \hat{\delta}(z', c) + \varepsilon (-\delta(z', c) 1_m(a) + \theta \delta'(z', c)).$$

Thus, (3.4) holds if $(z', c) \notin \text{supp}(\delta')$.

If $a = m$ and $(z', c) \in \text{supp}(\delta') \cap \text{supp}(\delta)$, the definition of $(\bar{z}, \bar{c})$ implies that:

$$\sum_{\hat{\delta} \in X} \pi_\varepsilon(z, m, \hat{\delta}) \hat{\delta}(z', c) \leq \sum_{\hat{\delta} \in X} \mu_n(z, m, \hat{\delta}) \hat{\delta}(z', c) \leq \kappa_n(z, z', c).$$

If $a \neq m$ and $(z', c) \in \text{supp}(\delta')$ or if $a = m$ and $(z', c) \in \text{supp}(\delta') \setminus \text{supp}(\delta)$, then $(z', c) \in T_z^m(\mu^*)$ and, thus, $\sum_{\hat{\delta} \in X} \mu_n(z, m, \hat{\delta}) \hat{\delta}(z', c) < \kappa_n(z, z', c)$ by (ii). Hence, there is $\varepsilon(z', c) > 0$ such that

$$\sum_{\hat{\delta} \in X} \pi_\varepsilon(z, m, \hat{\delta}) \hat{\delta}(z', c) = \sum_{\hat{\delta} \in X} \mu_n(z, m, \hat{\delta}) \hat{\delta}(z', c) + \varepsilon \theta \delta'(z', c) < \kappa_n(z, z', c)$$

for each $0 < \varepsilon < \varepsilon(z', c)$. Thus, letting $B = \text{supp}(\delta')$ if $a \neq m$, $B = \text{supp}(\delta') \setminus \text{supp}(\delta)$ if $a = m$ and $0 < \varepsilon < \min_{(z', c) \in B} \varepsilon(z', c)$, we have that $\sum_{\hat{\delta} \in X} \pi_\varepsilon(\hat{z}, m, \hat{\delta}) \hat{\delta}(z', c) \leq \kappa_n(\hat{z}, z', c)$ for each $(\hat{z}, z', c) \in Z \times Z \times C$.

Finally, note that

$$\sum_{\hat{z} \in Z, (\hat{a}, \hat{\delta}) \in \Delta} u_n(\hat{z}, \hat{a}, \hat{\delta}, \mu_n) \pi_\varepsilon(\hat{z}, \hat{a}, \hat{\delta}) > \sum_{\hat{z} \in Z, (\hat{a}, \hat{\delta}) \in \Delta} u_n(\hat{z}, \hat{a}, \hat{\delta}, \mu_n) \mu_n(\hat{z}, \hat{a}, \hat{\delta})$$

since $u_n(z, m, \delta', \mu_n) \geq k u_n(z, a, \delta, \mu_n)$ by (iii) since $n \geq N$, $u_n(z, s, 1_{(\emptyset, c^*)}, \mu_n) \geq 1$ and, hence,

$$\sum_{\hat{z} \in Z, (\hat{\delta}, \hat{a}) \in \Delta} u_n(\hat{z}, \hat{a}, \hat{\delta}, \mu_n) \left(\pi_\varepsilon(\hat{z}, \hat{a}, \hat{\delta}) - \mu_n(\hat{z}, \hat{a}, \hat{\delta}) \right) \geq u_n(z, a, \delta, \mu_n) \varepsilon (-1 + k\theta) > 0.$$

In conclusion $\mu_n \notin D_n(\mu_n, \kappa_n)$, a contradiction. ■

It follows from Claims 3, 5 and 7 that μ^* is a stable matching. ■

In the remainder of the proof, we extend Lemma 3.5 first to a finite type space with a general contract space, then to type-dependent feasible matches, and finally to a general type space.

The second step in the proof of our existence result consists in the following lemma, which considers the special case where Z is finite and, for some $R > 0$, $P_m(z) = M_R(Z \times C)$ for all $z \in Z$.

Lemma 3.6 *If E is a rational and continuous market such that Z is finite and, for some $R > 0$, $P_m(z) = M_R(Z \times C)$ for all $z \in Z$, then E has a stable matching.*

Proof. For each $z \in Z$, let $\{c_{z,n}\}_{n=1}^\infty$ be a dense subset of $\{c \in C : 1_{(\emptyset, c)} \in P_s(z)\}$. For each $k \in \mathbb{N}$ and $z \in Z$, define $P_{s,k}(z) = \{1_{(\emptyset, c_{z,n})} : n \leq k\}$. Let $\{c_n\}_{n=1}^\infty$ be a dense subset of C and $C_k = \{c_n : n \leq k\} \cup (\cup_{z \in Z} \{c_{z,n} : n \leq k\})$.

In addition, enumerate $\mathbb{Q} = \{q_1, q_2, \dots\}$ and, for each $k \in \mathbb{N}$, let, for each $z \in Z$, $P_{m,k}(z)$ be the set of $\delta \in M_R(Z \times C)$ such that $\text{supp}(\delta)$ is a subset of $Z \times C_k$ and, for each $(z', c) \in Z \times C_k$, $\delta(z', c) \in \{q_n : n \leq k\}$. Let $X_k = \cup_{z \in Z} P_{m,k}(z)$, $X_{\emptyset,k} = X_k \cup \{1_{(\emptyset, c)} : c \in C_k\}$, $X_{m,k} = X_k$, $X_{s,k} = \{1_{(\emptyset, c)} : c \in C_k\}$, $X_{w,k} = \{1_{(z,c)} : (z, c) \in Z \times C_k\}$ and $\Delta_k = \{(a, \delta) : \delta \in X_{a,k}\}$.

For each $k \in \mathbb{N}$, let $E_k = (Z, \nu, C_k, P_{m,k}, P_{s,k}, (\succ_z)_{z \in Z})$ be a market where $\succ_z$ is restricted to $\Delta_k \times M(Z \times X_{\emptyset,k})$ for each $z \in Z$. Let $\mu_k \in M(Z \times X_{\emptyset,k})$ be a stable matching in E_k , which exists by Lemma 3.5 since Z , C_k and $P_{m,k}(z)$ are finite for each $z \in Z$.

It follows by part 1 of Corollary 3.1 that we may assume that $\{\mu_k\}_{k=1}^\infty$ converges; let $\mu = \lim_k \mu_k$. It then follows by parts 2 and 3 of Corollary 3.1 that μ is a matching and that $\text{supp}(\mu) \subseteq IR(\mu)$. For the latter, note that, for each $z \in Z$, $\delta \in P_s(z)$, open neighborhood V_δ of δ , subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and sequence $\{z_{k_j}\}_{j=1}^\infty$ such that $z_{k_j} \rightarrow z$ and $z_{k_j} \in Z_{k_j}$ for each $j \in \mathbb{N}$, there exists $J_1 \in \mathbb{N}$ such that $z_{k_j} = z$ for each $j \geq J_1$. Let $c \in C$ be such that $\delta = 1_{(\emptyset, c)}$ and $c_{z, n^*} \in \{c_{z, n}\}_{n=1}^\infty$ be such that $1_{(\emptyset, c_{z, n^*})} \in V_\delta$. Then there exists $J_2 \in \mathbb{N}$ such that $1_{(\emptyset, c_{z, n^*})} \in P_{s, k_j}(z)$ for each $j \geq J_2$. Thus, $P_{s, k_j}(z_{k_j}) \cap V_\delta \neq \emptyset$ for each $j \geq J = \max\{J_1, J_2\}$.

The following claim will be used to show that condition (a) of part 4 of Corollary 3.1 holds.

Claim 8 *Let $(\tilde{z}, \tilde{c}) \in T_z^m(\mu)$ and $V_{\tilde{c}}$ be an open neighborhood of $\tilde{c}$. Then, for all k sufficiently large, there exists $c_k \in C_k$ such that $(\tilde{z}, c_k) \in T_z^m(\mu_k) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$.*

Proof. Let $(\tilde{z}, \tilde{c}) \in T_z^m(\mu)$ and $V_{\tilde{c}}$ be an open neighborhood of $\tilde{c}$. Then either (i) there exists $(\hat{z}, \hat{\delta}, \hat{c})$ such that $(\hat{z}, \hat{\delta}) \in \text{supp}(\mu)$, $(\tilde{z}, \hat{c}) \in \text{supp}(\hat{\delta})$ and $(w, 1_{(z, \hat{c})}, \mu) \succ_{\tilde{z}} (w, 1_{(\hat{z}, \hat{c})}, \mu)$, or (ii) there exists $\tilde{\delta} \in X_\emptyset$ such that $(\tilde{z}, \tilde{\delta}) \in \text{supp}(\mu)$ and $(w, 1_{(z, \tilde{c})}, \mu) \succ_{\tilde{z}} (a(\tilde{\delta}), \tilde{\delta}, \mu)$, where $a(\tilde{\delta}) = s$ if $\tilde{\delta} \in X_\emptyset \setminus X$ and $a(\tilde{\delta}) = m$ if $\tilde{\delta} \in X$.

Consider case (i) first. Let $O_{\tilde{c}}$, $O_{\hat{c}}$, $O_{\hat{\delta}}$ and O_μ be open neighborhoods of $\tilde{c}$, $\hat{c}$, $\hat{\delta}$ and μ , respectively, such that $(w, 1_{(z, \tilde{c}')} , \mu') \succ_{\tilde{z}} (w, 1_{(\hat{z}, \tilde{c}')} , \mu')$ and $\text{supp}(\hat{\delta}') \cap (\{\tilde{z}\} \times O_{\tilde{c}}) \neq \emptyset$ for each $\tilde{c}' \in O_{\tilde{c}}$, $\tilde{c}' \in O_{\hat{c}}$, $\hat{\delta}' \in O_{\hat{\delta}}$ and $\mu' \in O_\mu$. Since $0 < \mu(\{\hat{z}\} \times O_{\hat{\delta}}) \leq \liminf_k \mu_k(\{\hat{z}\} \times O_{\hat{\delta}})$, it follows that, for each k sufficiently large, there is $\hat{\delta}_k \in O_{\hat{\delta}}$ such that $(\hat{z}, \hat{\delta}_k) \in \text{supp}(\mu_k)$ and, for some $\hat{c}_k \in O_{\hat{c}}$, $(\tilde{z}, \hat{c}_k) \in \text{supp}(\hat{\delta}_k)$. In addition, $\mu_k \in O_\mu$ and there exists $c_k \in C_k \cap O_{\tilde{c}} \cap V_{\tilde{c}}$ since, respectively, $\mu_k \rightarrow \mu$ and C_k increases to a dense subset of C . Then $(w, 1_{(z, c_k)}, \mu_k) \succ_{\tilde{z}} (w, 1_{(\hat{z}, \hat{c}_k)}, \mu_k)$ and, hence, $(\tilde{z}, c_k) \in T_z^m(\mu_k) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$ for all k sufficiently large.

Consider next case (ii). Let $O_{\tilde{c}}$, $O_{\tilde{\delta}}$ and O_μ be open neighborhoods of $\tilde{c}$, $\tilde{\delta}$ and μ , respectively, such that $(w, 1_{(z, \tilde{c}')} , \mu') \succ_{\tilde{z}} (a(\tilde{\delta}), \tilde{\delta}', \mu')$ for each $\tilde{c}' \in O_{\tilde{c}}$, $\tilde{\delta}' \in O_{\tilde{\delta}}$ and $\mu' \in O_\mu$. Since $0 < \mu(\{\tilde{z}\} \times O_{\tilde{\delta}}) \leq \liminf_k \mu_k(\{\tilde{z}\} \times O_{\tilde{\delta}})$, it follows that, for each k sufficiently large, there is $\tilde{\delta}_k \in O_{\tilde{\delta}}$ such that $(\tilde{z}, \tilde{\delta}_k) \in \text{supp}(\mu_k)$. In addition, $\mu_k \in O_\mu$ and there exists $c_k \in C_k \cap O_{\tilde{c}} \cap V_{\tilde{c}}$ since, respectively, $\mu_k \rightarrow \mu$ and C_k

increases to a dense subset of C . Then $(w, 1_{(z, c_k)}, \mu_k) \succ_{\tilde{z}} (a(\tilde{\delta}), \tilde{\delta}_k, \mu_k)$ and, hence, $(\tilde{z}, c_k) \in T_z^m(\mu_k) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$ for all k sufficiently large. ■

We now show that condition (a) of part 4 of Corollary 3.1 holds. Let $(z, \delta, \mu) \in \text{graph}(P_m) \times M(Z \times X_\emptyset)$, $\delta' \in \Lambda(z, \delta, \mu)$, $V_{\delta'}$ be an open neighborhood of δ' and $\{(z_{k_j}, \delta_{k_j}, \mu_{k_j})\}_{j=1}^\infty$ be a sequence such that $(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \rightarrow (z, \delta, \mu)$ and $(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \in Z_{k_j} \times X_{k_j} \times M(Z_{k_j} \times X_{\emptyset, k_j})$ for each $j \in \mathbb{N}$.

In particular, $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ and we may assume that $\delta'(Z \times C) < R$ and $\text{supp}(\delta')$ is finite, i.e. $\delta' = \sum_{(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')} a(\tilde{z}, \tilde{c}) 1_{(\tilde{z}, \tilde{c})}$ for some $a = (a(\tilde{z}, \tilde{c}))_{(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')}$. Let V_a be an open neighborhood of a and, for each $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')$, $V_{(\tilde{z}, \tilde{c})}$ be an open neighborhood of $(\tilde{z}, \tilde{c})$ be such that

$$\sum_{(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')} \hat{a}(\tilde{z}, \tilde{c}) 1_{(z(\tilde{z}, \tilde{c}), c(\tilde{z}, \tilde{c}))} \in V_{\delta'}$$

whenever $(z(\tilde{z}, \tilde{c}), c(\tilde{z}, \tilde{c})) \in V_{(\tilde{z}, \tilde{c})}$ for each $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')$ and $\hat{a} \in V_a$. Let $\hat{a} = (\hat{a}(\tilde{z}, \tilde{c}))_{(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')} \in \mathbb{Q}_+^{|\text{supp}(\delta')|} \cap V_a$ and $V_{\tilde{c}}$ be an open neighborhood of $\tilde{c}$ such that $\{\tilde{z}\} \times V_{\tilde{c}} \subseteq V_{(\tilde{z}, \tilde{c})}$ for each $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')$.

For each $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \cap T_z^m(\mu)$, and for each k sufficiently large, let $c_k(\tilde{z}, \tilde{c}) \in C_k$ be such that $(\tilde{z}, c_k(\tilde{z}, \tilde{c})) \in T_z^m(\mu_k) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$, which exists by Claim 8.

If $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \setminus T_z^m(\mu)$, then $\delta \in X$, $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta)$ and $0 < \delta(\{\tilde{z}\} \times V_{\tilde{c}}) \leq \liminf_j \delta_{k_j}(\{\tilde{z}\} \times V_{\tilde{c}})$. Hence, for each j sufficiently large, let $c_{k_j}(\tilde{z}, \tilde{c}) \in V_{\tilde{c}}$ be such that $(\tilde{z}, c_{k_j}(\tilde{z}, \tilde{c})) \in \text{supp}(\delta_{k_j})$.

Let $J' \in \mathbb{N}$ be such that, for each $j \geq J'$, $(\tilde{z}, c_{k_j}(\tilde{z}, \tilde{c})) \in T_z^m(\mu_{k_j}) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$ if $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \cap T_z^m(\mu)$ and $(\tilde{z}, c_{k_j}(\tilde{z}, \tilde{c})) \in \text{supp}(\delta_{k_j}) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$ if $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \setminus T_z^m(\mu)$. Thus, letting $\delta'_{k_j} = \sum_{(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')} \hat{a}(\tilde{z}, \tilde{c}) 1_{(\tilde{z}, c_{k_j}(\tilde{z}, \tilde{c}))}$ for each $j \geq J'$, we have that $\delta'_{k_j} \in V_{\delta'}$ and $\text{supp}(\delta'_{k_j}) \subseteq T_z^m(\mu_{k_j}) \cup \text{supp}(\delta_{k_j})$. Since $\{\hat{a}(\tilde{z}, \tilde{c}) : (\tilde{z}, \tilde{c}) \in \text{supp}(\delta')\}$ is finite, it follows that there is $J > J'$ such that $\delta'_{k_j} \in P_{m, k_j}(z)$ for each $j \geq J$.

An analogous argument shows that condition (b) of part 4 of Corollary 3.1 also holds. Hence, it follows that $\text{supp}(\mu) \subseteq S_M(\mu)$. This together with the fact that μ is a matching and $\text{supp}(\mu) \subseteq IR(\mu)$ shows that μ is stable. ■

The next step of the proof of Theorem 3.1 extends Lemma 3.6 by requiring only that E be rich.

Lemma 3.7 *If E is a rational, continuous, bounded and rich market such that Z is finite, then E has a stable matching.*

Proof. It follows by Debreu (1964, Proposition 3) and by the finiteness of Z that there exists a continuous function $u : Z \times \Delta \times M(Z \times X_\emptyset) \rightarrow [1, 2]$ such that $(a, \delta, \mu) \mapsto u(z, a, \delta, \mu)$ represents $\succeq_z$ for each $z \in Z$, using the fact that $[1, 2]$ and the extended reals are homeomorphic.

Let $R > 0$ be such that $X \subseteq M_R(Z \times C)$, $\Delta^* = (\{m\} \times M_R(Z \times C)) \cup (\{w\} \times X_w) \cup (\{s\} \times X_s)$ and $X^* = M_R(Z \times C) \cup \{1_{(\emptyset, c)} : c \in C\}$. By Tietze Extension Theorem, let $U : Z \times \Delta^* \times M(Z \times X^*) \rightarrow [1, 2]$ be a continuous extension of u .

Let ρ be a metric on $M_R(Z \times C)$. For each $k \in \mathbb{N}$, let

$$G_k = \{(z, \delta) \in Z \times M_R(Z \times C) : \rho(\delta, P_m(z)) \geq k^{-1}\}.$$

Let, by Urysohn's Lemma, $g_k : Z \times M_R(Z \times C) \rightarrow [0, 1]$ be a continuous function such that $g_k^{-1}(1) = \text{graph}(P_m)$ and $g_k^{-1}(0) = G_k$. Then define $U_k : Z \times \Delta^* \times M(Z \times X^*) \rightarrow \mathbb{R}$ by setting, for each $(z, a, \delta, \mu) \in Z \times \Delta^* \times M(Z \times X^*)$,

$$U_k(z, a, \delta, \mu) = \begin{cases} g_k(z, \delta)U(z, a, \delta, \mu) & \text{if } a = m, \\ U(z, a, \delta, \mu) & \text{otherwise.} \end{cases}$$

Let $P_{m,k}(z) = M_R(Z \times C)$ for each $z \in Z$.

Consider the market $E_k = (Z, \nu, C, P_{m,k}, P_s, U_k)$, i.e. E_k is equal to E except that $P_m(z)$ is replaced with $M_R(Z \times C)$ for each $z \in Z$ and u with U_k . Since E_k is rational and continuous with Z finite and $P_{m,k}(z) = M_R(Z \times C)$ for each $z \in Z$, then E_k has a stable matching μ_k by Lemma 3.6.

Let $E^* = (Z, \nu, C, P_m^*, P_s, U)$ where $P_m^*(z) = M_R(Z \times C)$ for each $z \in Z$. To avoid confusion, we write $IR(\mu; E')$ for $IR(\mu)$ and $S_M(\mu; E')$ for $S_M(\mu)$ whenever μ is a matching of a market E' . It follows by part 1 of Corollary 3.1 that we may assume that $\{\mu_k\}_{k=1}^\infty$ converges; let $\mu = \lim_k \mu_k$. It then follows by part 2 of Corollary 3.1 that μ is a matching of E^* .

Note that condition (a) of part 3 of Corollary 3.1 holds since Z is finite and $P_{s,k} = P_s$. The proof of part 3 of Corollary 3.1 implies that $\text{supp}(\mu) \subseteq IR(\mu; E^*)$

since the requirement that $\succ_{z,k}$ is the restriction of $\succ_z$ to $\Delta_k \times M(Z_k \times X_{\emptyset,k})$ for each $z \in Z_k$ can be replaced with the following condition: $(s, \delta, \hat{\mu}) \succ_{z,k} (a, \delta', \hat{\mu})$ for each $k \in \mathbb{N}$, $z \in Z_k$, $\delta \in X_{s,k}$, $(a, \delta') \in \Delta_k$ and $\hat{\mu} \in M(Z_k \times X_{\emptyset,k})$ such that $(s, \delta, \hat{\mu}) \succ_z (a, \delta', \hat{\mu})$. This condition holds because $U_k(z, a, \delta', \hat{\mu}) \leq U(z, a, \delta', \hat{\mu})$ and $U_k(z, s, 1_{(\emptyset, \hat{c})}, \hat{\mu}) = U(z, s, 1_{(\emptyset, \hat{c})}, \hat{\mu})$ for each $k \in \mathbb{N}$, $z \in Z$, $(a, \delta') \in \Delta^*$, $\hat{c} \in C$ and $\hat{\mu} \in M(Z \times X^*)$.

We have that $\text{supp}(\mu) \subseteq \text{graph}(P_m) \cup \text{graph}(P_s)$. Indeed, let $k \in \mathbb{N}$ and $(z, \delta) \in \text{supp}(\mu_k)$. If $\delta \notin M(Z \times C)$, then $\delta \in P_s(z)$ since μ_k is a matching of E_k . If $\delta \in M(Z \times C)$ and $\rho(\delta, P_m(z)) \geq k^{-1}$, then let $\delta' \in P_s(z)$ to obtain that $\text{supp}(\delta') \subseteq T_z^s(\mu_k)$ and $U_k(z, s, \delta', \mu_k) = U(z, s, \delta', \mu_k) > 0 = U_k(z, m, \delta, \mu_k)$, the latter since $U(z, s, \delta', \mu_k) \in [1, 2]$ and $g_k(z, \delta) = 0$. But this contradicts the stability of μ_k . Hence, it follows that $\rho(\delta, P_m(z)) < k^{-1}$.

Thus, for each $k \in \mathbb{N}$,

$$\text{supp}(\mu_k) \subseteq (\{(z, \delta) \in Z \times M_R(Z \times C) : \rho(\delta, P_m(z)) \leq k^{-1}\}) \cup \text{graph}(P_s).$$

Hence, $\text{supp}(\mu) \subseteq \text{graph}(P_m) \cup \text{graph}(P_s)$ as claimed.

It then follows that μ is a matching of E and that $\text{supp}(\mu) \subseteq IR(\mu; E)$ since $IR(\mu; E^*) \cap (Z \times X_\emptyset) \subseteq IR(\mu; E)$. Claim 9, which is analogous to part 4 of Corollary 3.1, shows that $\text{supp}(\mu) \subseteq S_M(\mu; E)$.

Claim 9 $\text{supp}(\mu) \subseteq S_M(\mu; E)$.

Proof. Let $(z, \delta) \in \text{supp}(\mu)$ and suppose that $(z, \delta) \notin S_M(\mu; E)$. Then either (i) there exists $\delta' \in P_m(z)$ such that $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ and $U(z, m, \delta', \mu) > U(z, a(\delta), \delta, \mu)$, where $a(\delta) = m$ if $\delta \in X$ and $a(\delta) = s$ if $\delta \in X_\emptyset \setminus X$,¹ or (ii) there exists $(z', c) \in \text{supp}(\delta)$ and $\delta' \in P_m(z')$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu)$ and $U(z', m, \delta', \mu) > U(z', w, 1_{(z,c)}, \mu)$.

Consider case (i) first. Let $V_{\delta'}$, V_δ and V_μ be open neighborhoods of δ' , δ and μ , respectively, such that $U(z, m, \gamma', \bar{\mu}) > U(z, a(\delta), \gamma, \bar{\mu})$ for each $\gamma' \in V_{\delta'}$, $\gamma \in V_\delta$

¹Note that when $\delta \in X_\emptyset \setminus X$ and $\delta' \in X$, $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ if and only if $\text{supp}(\delta') \subseteq T_z^m(\mu)$.

and $\bar{\mu} \in V_\mu$. Let, by the richness of E , $\tilde{V}_\delta$ and $\tilde{V}_\mu$ be open neighborhoods of δ and μ , respectively, such that $\Lambda_{a(\delta)}(z, \gamma, \bar{\mu}) \cap V_{\delta'} \neq \emptyset$ for each $(\gamma, \bar{\mu}) \in \tilde{V}_\delta \times \tilde{V}_\mu$, where $\Lambda_m(z, \gamma, \bar{\mu}) = \Lambda(z, \gamma, \bar{\mu})$ and $\Lambda_s(z, \gamma, \bar{\mu}) = \Lambda_0(z, \bar{\mu})$.

By Carmona and Podczeck (2009, Lemma 12), there is a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding sequence $\{\delta_{k_j}\}_{j=1}^\infty$ such that $\delta_{k_j} \rightarrow \delta$ and $(z, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$ for each $j \in \mathbb{N}$.

Let $J \in \mathbb{N}$ be such that $\mu_{k_j} \in V_\mu \cap \tilde{V}_\mu$ and $\delta_{k_j} \in V_\delta \cap \tilde{V}_\delta$ for all $j \geq J$ and, for each $j \geq J$, let $\delta'_{k_j} \in \Lambda_{a(\delta)}(z, \delta_{k_j}, \mu_{k_j}) \cap V_{\delta'}$. Then, for each $j \geq J$,

$$U_{k_j}(z, m, \delta'_{k_j}, \mu_{k_j}) = U(z, m, \delta'_{k_j}, \mu_{k_j}) > U(z, a(\delta), \delta_{k_j}, \mu_{k_j}) \geq U_{k_j}(z, a(\delta), \delta_{k_j}, \mu_{k_j})$$

since $\delta'_{k_j} \in P_m(z)$ by the definition of $\Lambda_{a(\delta)}$, and $\text{supp}(\delta'_{k_j}) \subseteq T_z^m(\mu_{k_j}) \cup \text{supp}(\delta_{k_j})$. But this contradicts the stability of μ_{k_j} .

Consider case (ii) next, i.e. there exists $(z', c) \in \text{supp}(\delta)$ and $\delta' \in P_m(z')$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu)$ and $U(z', m, \delta', \mu) > U(z', w, 1_{(z,c)}, \mu)$. Let $V_{\delta'}$, V_c and V_μ be open neighborhoods of δ' , c and μ respectively, such that $U(z', m, \hat{\delta}', \hat{\mu}) > U(z', w, 1_{(z,\hat{c})}, \hat{\mu})$ for each $\hat{\delta}' \in V_{\delta'}$, $\hat{c} \in V_c$ and $\hat{\mu} \in V_\mu$. Let, by the richness of E , $\tilde{V}_\mu$ be an open neighborhood of μ such that $\Lambda_0(z', \hat{\mu}) \cap V_{\delta'} \neq \emptyset$ for each $\hat{\mu} \in \tilde{V}_\mu$.

By Carmona and Podczeck (2009, Lemma 12), there is a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding sequence $\{(\delta_{k_j}, c_{k_j})\}_{j=1}^\infty$ such that $(\delta_{k_j}, c_{k_j}) \rightarrow (\delta, c)$, $(z, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$ and $(z', c_{k_j}) \in \text{supp}(\delta_{k_j})$ for each $j \in \mathbb{N}$.

Let $J \in \mathbb{N}$ be such that $c_{k_j} \in V_c$ and $\mu_{k_j} \in V_\mu \cap \tilde{V}_\mu$ for all $j \geq J$ and, for each $j \geq J$, let $\delta'_{k_j} \in \Lambda_0(z', \mu_{k_j}) \cap V_{\delta'}$. Then, for each $j \geq J$, $U_{k_j}(z', m, \delta'_{k_j}, \mu_{k_j}) = U(z', m, \delta'_{k_j}, \mu_{k_j}) > U(z', w, 1_{(z,c_{k_j})}, \mu_{k_j}) \geq U_{k_j}(z', w, 1_{(z,c_{k_j})}, \mu_{k_j})$ since $\delta'_{k_j} \in P_m(z')$ by the definition of Λ_0 , and $\text{supp}(\delta'_{k_j}) \subseteq T_{z'}^m(\mu_{k_j})$. But this contradicts the stability of μ_{k_j} . ■

It follows by $\text{supp}(\mu) \subseteq IR(\mu; E)$ and by Claim 9 that $\text{supp}(\mu) \subseteq S_M(\mu; E) \cap IR(\mu; E)$. Thus, μ is stable. ■

We now complete the proof of our existence result.

Proof of Theorem 3.1. Let $\{\nu_k\}_{k=1}^\infty$ be such that $\nu_k \rightarrow \nu$ and $\text{supp}(\nu_k)$ is a finite subset of Z for each $k \in \mathbb{N}$. Define $Z_k = \text{supp}(\nu_k)$, $Z_{\emptyset,k} = Z_k \cup \{\emptyset\}$,

and for each $z \in Z_k$, $P_{m,k}(z) = P_m(z) \cap M(Z_k \times C)$ and $P_{s,k}(z) = P_s(z)$. Let $X_k = \cup_{z \in Z_k} P_{m,k}(z)$, $X_{\emptyset,k} = X_k \cup \{1_{(\emptyset,c)} : c \in C\}$, $X_{m,k} = X_k$, $X_{s,k} = \{1_{(\emptyset,c)} : c \in C\}$, $X_{w,k} = \{1_{(z,c)} : (z,c) \in Z_k \times C\}$ and $\Delta_k = \{(a,\delta) : \delta \in X_{a,k}\}$ for each $k \in \mathbb{N}$. Note that $P_{m,k}$ is upper hemicontinuous (since its domain is finite) and compact valued (since P_m is compact valued and $M(Z_k \times C)$ is closed) for each $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$, let $\tilde{E}_k = (Z_k, \nu_k, C, P_{m,k}, P_{s,k}, (\succ_z)_{z \in Z})$ be a market where $\succ_z$ is restricted to $\Delta_k \times M(Z_k \times X_{\emptyset,k})$ for each $z \in Z$. Furthermore, let E_k be exactly as $\tilde{E}_k$, except with P_m in place of $P_{m,k}$, P_s in place of $P_{s,k}$ and Z in place of Z_k ; more precisely, $E_k = (Z, \nu_k, C, P_m, P_s, (\succ_z)_{z \in Z})$.

Claim 10 *For each $k \in \mathbb{N}$, if μ is a stable matching of $\tilde{E}_k$, then μ is a stable matching of E_k .*

Proof. In this proof, to avoid confusion, we write $IR(\mu; E)$ for $IR(\mu)$ and $S_M(\mu; E)$ for $S_M(\mu)$ whenever μ is a matching of a market E .

Let $k \in \mathbb{N}$ and μ be a stable matching of $\tilde{E}_k$. Clearly, μ is a matching of E_k and $\text{supp}(\mu) \subseteq IR(\mu; E_k)$. We show that $\text{supp}(\mu) \subseteq S_M(\mu; E_k)$. Suppose not; then let $(z, \delta) \in \text{supp}(\mu) \setminus S_M(\mu; E_k)$.

First suppose that there exists $\delta' \in P_m(z)$ such that $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ and $(m, \delta', \mu) \succ_z (a(\delta), \delta, \mu)$, where $a(\delta) = m$ if $\delta \in X$ and $a(\delta) = s$ if $\delta \in X_\emptyset \setminus X$. We claim that $\delta' \in P_{m,k}(z)$, i.e. that $\text{supp}(\delta') \subseteq Z_k \times C$, from which we obtain a contradiction to the stability of μ in $\tilde{E}_k$.

Note that $\text{supp}(\bar{\delta}) \subseteq Z_k \times C$ whenever $\bar{\delta} \in X$ and $(\bar{z}, \bar{\delta}) \in \text{supp}(\mu)$ for some $\bar{z} \in Z_k$ since μ is stable in $\tilde{E}_k$. Thus, it follows that $\text{supp}(\delta') \cap \text{supp}(\delta) \subseteq Z_k \times C$ since if $\text{supp}(\delta') \cap \text{supp}(\delta) \neq \emptyset$, then $\delta \in X$. We also have that $\text{supp}(\delta') \cap T_z^m(\mu) \subseteq Z_k \times C$. Indeed, if $(z', c) \in T_z^m(\mu)$, then either $z' \in \text{supp}(\mu_Z) \subseteq Z_k$ or $(z', \bar{c}) \in \text{supp}(\bar{\delta})$ and $(\bar{z}, \bar{\delta}) \in \text{supp}(\mu)$ for some $\bar{c} \in C$, $\bar{z} \in Z_k$ and $\bar{\delta} \in X$; hence, $z' \in Z_k$. Thus, $\text{supp}(\delta') = (\text{supp}(\delta') \cap \text{supp}(\delta)) \cup (\text{supp}(\delta') \cap T_z^m(\mu)) \subseteq Z_k \times C$ as desired.

Consider next the case where there exists $(z', c) \in \text{supp}(\delta)$ and $\delta' \in P_m(z')$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu)$ and $(m, \delta', \mu) \succ_{z'} (w, 1_{(z,c)}, \mu)$. As above, we obtain a contradiction to the stability of μ in $\tilde{E}_k$ by showing that $\delta' \in P_{m,k}(z')$. To establish

this claim, it suffices to show that $T_{z'}^m(\mu) \subseteq Z_k \times C$. If $(\tilde{z}, \tilde{c}) \in T_{z'}^m(\mu)$, then either $\tilde{z} \in \text{supp}(\mu_Z) \subseteq Z_k$ or $(\tilde{z}, \tilde{c}) \in \text{supp}(\bar{\delta})$ and $(\bar{z}, \bar{\delta}) \in \text{supp}(\mu)$ for some $\bar{c} \in C$, $\bar{z} \in Z_k$ and $\bar{\delta} \in X$; hence, $(\tilde{z}, \tilde{c}) \in Z_k \times C$ as required. ■

For each $k \in \mathbb{N}$, let $\mu_k \in M(Z \times X_{\emptyset, k})$ be a stable matching in E_k , which exists by Lemma 3.7 (since Z_k is finite and $\tilde{E}_k$ satisfies its assumptions) and Claim 10.²

It follows by part 1 of Corollary 3.1 that we may assume that $\{\mu_k\}_{k=1}^\infty$ converges; let $\mu = \lim_k \mu_k$. It then follows by parts 2 – 4 of Corollary 3.1 that μ is a matching and that $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$. Hence, μ is stable. ■

3.2 Sufficient condition for richness

It is often easy to establish the richness of a market by showing that the following two conditions hold for each $z \in Z$:

- (α) For each $\delta \in P_m(z)$ such that $\delta = \sum_{j=1}^J a_j 1_{(z_j, c_j)}$ for some $J \in \mathbb{N}_0$, $a_j \in \mathbb{R}_{++}$ and $(z_j, c_j) \in Z \times C$ for each $j = 1, \dots, J$ and each open neighborhood V_δ of δ , there exist open neighborhoods V_z of z and $V_{(z_j, c_j)}$ of (z_j, c_j) for each $j = 1, \dots, J$ such that, whenever $\hat{z} \in V_z$ and $(\hat{z}_j, \hat{c}_j) \in V_{(z_j, c_j)}$ for each $j = 1, \dots, J$, there exists $\hat{a} = (\hat{a}_1, \dots, \hat{a}_J) \in \mathbb{R}_+^J$ such that $\sum_{j=1}^J \hat{a}_j 1_{(\hat{z}_j, \hat{c}_j)} \in V_\delta \cap P_m(\hat{z})$.³
- (β) For each $\delta \in P_m(z)$ and open neighborhood V_δ of δ , there exists $\hat{\delta} \in V_\delta \cap P_m(z)$ such that $\text{supp}(\hat{\delta})$ is a finite subset of $\text{supp}(\delta)$.

²It is clear that $\tilde{E}_k$ is rational, continuous and bounded. It can be shown that $\tilde{E}_k$ is rich along the lines of Claim 10: Let $X_k^* = M_R(Z_k \times C) \cup \{1_{(\emptyset, c)} : c \in C\}$ and $\Lambda_k : Z_k \times M_R(Z_k \times C) \times M(Z_k \times X_k^*) \rightrightarrows X_k$ be defined by setting, for each $(z, \delta, \mu) \in Z_k \times M_R(Z_k \times C) \times M(Z_k \times X_k^*)$, $\Lambda_k(z, \delta, \mu) = \{\delta' \in P_{m,k}(z) : \text{supp}(\delta') \subseteq \text{supp}(\delta) \cup T_z^m(\mu; \tilde{E}_k)\}$. Let $(z, \delta, \mu) \in Z_k \times X_k \times M(Z_k \times X_{\emptyset, k})$ and O be open and such that $(z, \delta) \in \text{supp}(\mu) \cap \text{graph}(P_{m,k})$ and $\Lambda_k(z, \delta, \mu) \cap O \neq \emptyset$. Since $T_z(\mu; \tilde{E}_k) \subseteq T_z(\mu; E)$, it follows that $\Lambda(z, \delta, \mu) \cap O \neq \emptyset$. Hence, there is an open neighborhood V of (z, δ, μ) such that, for each $(\tilde{z}, \tilde{\delta}, \tilde{\mu}) \in V$, $\Lambda(\tilde{z}, \tilde{\delta}, \tilde{\mu}) \cap O \neq \emptyset$. Thus, for each $(\tilde{z}, \tilde{\delta}, \tilde{\mu}) \in V \cap (Z_k \times M_R(Z_k \times C) \times M(Z_k \times X_k^*))$, let $\delta' \in \Lambda(\tilde{z}, \tilde{\delta}, \tilde{\mu}) \cap O$. We have that $\text{supp}(\tilde{\delta}) \subseteq Z_k \times C$ and that $T_{\tilde{z}}^m(\tilde{\mu}; E) \subseteq (Z_k \times C) \cap T_{\tilde{z}}^m(\tilde{\mu}; \tilde{E}_k)$. Hence, $\delta' \in P_{m,k}(\tilde{z})$ and $\text{supp}(\delta') \subseteq \text{supp}(\tilde{\delta}) \cup T_{\tilde{z}}^m(\tilde{\mu}; \tilde{E}_k)$ and it follows that $\delta' \in \Lambda_k(\tilde{z}, \tilde{\delta}, \tilde{\mu}) \cap O$. Thus, Λ_k is lower hemicontinuous and an analogous argument shows that $\Lambda_{0,k}$ is also lower hemicontinuous.

³The case $J = 0$ means that $\delta = 0$ and (α) requires that $0 \in P_m(\hat{z})$ for $\hat{z} \in V_z$ when $0 \in P_m(z)$.

Lemma 3.8 *Let E be a market such that $\{(a, \delta, \mu, a', \delta', \mu', z) \in (\Delta \times M(Z \times X_\emptyset))^2 \times Z : (a, \delta, \mu) \succ_z (a', \delta', \mu')\}$ is open. If (α) and (β) hold, then E is rich.*

We use the following lemma in the proof of Lemma 3.8.

Lemma 3.9 *Let $z \in Z$, $\mu \in M(Z \times X_\emptyset)$, $\{(z_k, \mu_k)\}_{k=1}^\infty$ be such that $(z_k, \mu_k) \rightarrow (z, \mu)$, $(\tilde{z}, \tilde{c}) \in T_z^m(\mu)$, and $V_{\tilde{z}}$ be an open neighborhood of $\tilde{z}$. Then, for all k sufficiently large, there exists $\tilde{z}_k \in V_{\tilde{z}}$ such that $(\tilde{z}_k, \tilde{c}) \in T_{z_k}^m(\mu_k)$.*

Proof. Let $(\tilde{z}, \tilde{c}) \in T_z^m(\mu)$. Then (i) there exists $(\hat{z}, \hat{\delta}, \hat{c})$ such that $(\hat{z}, \hat{\delta}) \in \text{supp}(\mu)$, $(\tilde{z}, \hat{c}) \in \text{supp}(\hat{\delta})$ and $(w, 1_{(z, \tilde{c})}, \mu) \succ_{\tilde{z}} (w, 1_{(\hat{z}, \hat{c})}, \mu)$, or (ii) there exists $\tilde{\delta} \in X_\emptyset$ such that $(\tilde{z}, \tilde{\delta}) \in \text{supp}(\mu)$ and $(w, 1_{(z, \tilde{c})}, \mu) \succ_{\tilde{z}} (a(\tilde{\delta}), \tilde{\delta}, \mu)$, where $a(\tilde{\delta}) = s$ if $\tilde{\delta} \in X_\emptyset \setminus X$ and $a(\tilde{\delta}) = m$ if $\tilde{\delta} \in X$.

Consider case (i) first. Let $O_z, O_{\tilde{z}}, O_{\hat{z}}, O_{\hat{c}}, O_{\hat{\delta}}$ and O_μ be open neighborhoods of $z, \tilde{z}, \hat{z}, \hat{c}, \hat{\delta}$ and μ , respectively, such that $(w, 1_{(z', \tilde{c})}, \mu') \succ_{\tilde{z}'} (w, 1_{(\hat{z}', \hat{c})}, \mu')$ and $\text{supp}(\hat{\delta}') \cap ((O_{\tilde{z}} \cap V_{\tilde{z}}) \times O_{\hat{c}}) \neq \emptyset$ for each $z' \in O_z, \tilde{z}' \in O_{\tilde{z}}, \hat{z}' \in O_{\hat{z}}, \hat{c}' \in O_{\hat{c}}, \hat{\delta}' \in O_{\hat{\delta}}$ and $\mu' \in O_\mu$. We have that $0 < \mu(O_{\hat{z}} \times O_{\hat{\delta}}) \leq \liminf_k \mu_k(O_{\hat{z}} \times O_{\hat{\delta}})$ since $\mu_k \rightarrow \mu$. Thus, it follows from $z_k \rightarrow z$ and $\mu_k \rightarrow \mu$ that, for each k sufficiently large, $\mu_k \in O_\mu, z_k \in O_z$ and there is $(\hat{z}_k, \hat{\delta}_k) \in \text{supp}(\mu_k) \cap (O_{\hat{z}} \times O_{\hat{\delta}})$ and $(\tilde{z}_k, \hat{c}_k) \in \text{supp}(\hat{\delta}_k) \cap ((O_{\tilde{z}} \cap V_{\tilde{z}}) \times O_{\hat{c}})$. Then $(w, 1_{(z_k, \tilde{c})}, \mu_k) \succ_{\tilde{z}_k} (w, 1_{(\hat{z}_k, \hat{c}_k)}, \mu_k)$ and, hence, $(\tilde{z}_k, \tilde{c}) \in T_{z_k}^m(\mu_k)$.

Consider case (ii) next. Let $O_z, O_{\tilde{z}}, O_{\tilde{\delta}}$ and O_μ be open neighborhoods of $z, \tilde{z}, \tilde{\delta}$ and μ , respectively, such that $(w, 1_{(z', \tilde{c})}, \mu') \succ_{\tilde{z}} (a(\tilde{\delta}), \tilde{\delta}', \mu')$ for each $z' \in O_z, \tilde{z}' \in O_{\tilde{z}}, \tilde{\delta}' \in O_{\tilde{\delta}}$ and $\mu' \in O_\mu$. We have that $0 < \mu((O_{\tilde{z}} \cap V_{\tilde{z}}) \times O_{\tilde{\delta}}) \leq \liminf_k \mu_k((O_{\tilde{z}} \cap V_{\tilde{z}}) \times O_{\tilde{\delta}})$ since $\mu_k \rightarrow \mu$; hence, for each k sufficiently large, there is $(\tilde{z}_k, \tilde{\delta}_k) \in \text{supp}(\mu_k) \cap ((O_{\tilde{z}} \cap V_{\tilde{z}}) \times O_{\tilde{\delta}})$. Since $\mu_k \rightarrow \mu$ and $z_k \rightarrow z$, it follows that, for each k sufficiently large, $\mu_k \in O_\mu$ and $z_k \in O_z$. Then $(w, 1_{(z_k, \tilde{c})}, \mu_k) \succ_{\tilde{z}_k} (a(\tilde{\delta}), \tilde{\delta}_k, \mu_k)$ and, hence, $(\tilde{z}_k, \tilde{c}) \in T_{z_k}^m(\mu_k)$. ■

We can now turn to the proof of the lemma.

Proof of Lemma 3.8. Let $(z, \delta, \mu) \in \text{graph}(P_m) \times M(Z \times X_\emptyset)$ and $V \subseteq X$ be open and such that $\Lambda(z, \delta, \mu) \cap V \neq \emptyset$. Let $\delta' \in \Lambda(z, \delta, \mu) \cap V$, i.e. V is an open neighborhood of δ' , $\delta' \in P_m(z)$ and $\text{supp}(\delta') \subseteq \text{supp}(\delta) \cup T_z^m(\mu)$.

We may assume that $\text{supp}(\delta')$ is finite by condition (β) since $\delta' \in P_m(z)$. Let V_z and $V_{(\tilde{z}, \tilde{c})}$ for each $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')$ be given by condition (α) . Let $V_{\tilde{z}}$ and $V_{\tilde{c}}$ be open neighborhoods of $\tilde{z}$ and $\tilde{c}$ such that $V_{\tilde{z}} \times V_{\tilde{c}} \subseteq V_{(\tilde{z}, \tilde{c})}$. Let O_δ be an open neighborhood of δ such that $\text{supp}(\tilde{\delta}) \cap (V_{\tilde{z}} \times V_{\tilde{c}}) \neq \emptyset$ for each $\tilde{\delta} \in O_\delta$ and each $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \setminus T_z^m(\mu)$.

We claim that there is an open neighborhood of (z, δ, μ) such that, for each (z^*, δ^*, μ^*) in it, $\Lambda(z^*, \delta^*, \mu^*) \cap V \neq \emptyset$. Suppose not; then there is a sequence $\{(z_k, \delta_k, \mu_k)\}_{k=1}^\infty$ such that $(z_k, \delta_k, \mu_k) \rightarrow (z, \delta, \mu)$ and

$$\Lambda(z_k, \delta_k, \mu_k) \cap V = \emptyset \text{ for each } k \in \mathbb{N}. \quad (3.5)$$

For each $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \cap T_z^m(\mu)$ and k sufficiently large, let $z_k(\tilde{z}, \tilde{c}) \in V_{\tilde{z}}$ and $c_k(\tilde{z}, \tilde{c}) = \tilde{c} \in V_{\tilde{c}}$ be such that $(z_k(\tilde{z}, \tilde{c}), c_k(\tilde{z}, \tilde{c})) \in T_{z_k}^m(\mu_k) \cap V_{(\tilde{z}, \tilde{c})}$, which exists by Lemma 3.9.

For each $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \cap \text{supp}(\delta)$ and k sufficiently large, let $z_k(\tilde{z}, \tilde{c}) \in V_{\tilde{z}}$ and $c_k(\tilde{z}, \tilde{c}) \in V_{\tilde{c}}$ be such that $(z_k(\tilde{z}, \tilde{c}), c_k(\tilde{z}, \tilde{c})) \in \text{supp}(\delta_k) \cap V_{(\tilde{z}, \tilde{c})}$.

Let $K \in \mathbb{N}$ be such that, for each $k \geq K$, $z_k \in V_z$, $(z_k(\tilde{z}, \tilde{c}), c_k(\tilde{z}, \tilde{c})) \in T_{z_k}^m(\mu_k) \cap V_{(\tilde{z}, \tilde{c})}$ if $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \cap T_z^m(\mu)$ and $(z_k(\tilde{z}, \tilde{c}), c_k(\tilde{z}, \tilde{c})) \in \text{supp}(\delta_k) \cap V_{(\tilde{z}, \tilde{c})}$ if $(\tilde{z}, \tilde{c}) \in \text{supp}(\delta') \setminus T_z^m(\mu)$. Condition (α) gives $(a_k(\tilde{z}, \tilde{c}))_{(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')} \in \mathbb{R}_+^{|\text{supp}(\delta')|}$ such that, setting $\delta'_k = \sum_{(\tilde{z}, \tilde{c}) \in \text{supp}(\delta')} a_k(\tilde{z}, \tilde{c}) 1_{(z_k(\tilde{z}, \tilde{c}), c_k(\tilde{z}, \tilde{c}))}$ for each $k \geq K$, we have that $\delta'_k \in V \cap P_m(z_k)$ and $\text{supp}(\delta'_k) \subseteq T_{z_k}^m(\mu_k) \cup \text{supp}(\delta_k)$. But this contradicts (3.5). Thus, it follows that Λ is lower hemicontinuous.

The proof that Λ_0 is lower hemicontinuous is analogous. ■

4 Relationship with Carmona and Laohakunakorn (2024)

A market in Carmona and Laohakunakorn (2024) is $E = (Z, \nu, C, \mathbb{C}, X, (\succ_z)_{z \in Z})$, where Z , ν , C and $(\succ_z)_{z \in Z}$ are as in this section, X is a subset of $M(Z \times C)$ and $\mathbb{C} : Z \times Z_\emptyset \rightrightarrows C$ is the contract correspondence. We define a market $\hat{E}$ as in this section by setting, for each $z \in Z$, $P_m(z) = \{\delta \in X : \{z\} \times \text{supp}(\delta) \subseteq \text{graph}(\mathbb{C})\}$ and $P_s(z) = \{1_{(\emptyset, c)} : c \in \mathbb{C}(z, \emptyset)\}$.

Note that $\cup_{z \in Z} P_m(z) \subseteq X$ and, hence, a matching in $\hat{E}$ is an element of $M(Z \times X_\emptyset)$. We have that $\mu \in M(Z \times X_\emptyset)$ is a matching in E if and only if it is a matching in $\hat{E}$. Indeed, for each $(z, \delta) \in \text{supp}(\mu)$, $\delta \in P_m(z) \cup P_s(z)$ holds if and only if $\{z\} \times \text{supp}(\delta) \subseteq \text{graph}(\mathbb{C})$.

We also have that a matching μ is stable in E if and only if it is stable in $\hat{E}$. To see this, note that $IR(\mu)$ and $\hat{I}R(\mu)$ differ only because the requirement that $\text{supp}(\delta') \subseteq T_{z^*}^s(\mu) = \{\emptyset\} \times \mathbb{C}(z^*, \emptyset)$ in $IR(\mu)$ is replaced with $\delta' \in P_s(z^*)$ in $\hat{E}$, where $z^* \in \{z, z'\}$. Since $\delta' \in P_s(z^*)$ is equivalent to $\text{supp}(\delta') \subseteq \{\emptyset\} \times \mathbb{C}(z^*, \emptyset)$, it follows that $IR(\mu) = \hat{I}R(\mu)$.

Similarly, $S_M(\mu) \cap \text{supp}(\mu)$ and $\hat{S}_M(\mu) \cap \text{supp}(\mu)$ differ only because the requirements that (i) $\delta' \in X$ and $\text{supp}(\delta') \subseteq T_{z^*}^m(\mu)$ where $z^* \in \{z, z'\}$, and (ii) $\delta' \in X$ and $\text{supp}(\delta') \subseteq \text{supp}(\delta) \cup T_z^m(\mu)$ in $S_M(\mu)$, where $(z, \delta) \in \text{supp}(\mu)$, are replaced with (a) $\delta' \in P_m(z^*)$ and $\text{supp}(\delta') \subseteq \hat{T}_{z^*}^m(\mu)$ and (b) $\delta' \in P_m(z^*)$ and $\text{supp}(\delta') \subseteq \text{supp}(\delta) \cup \hat{T}_{z^*}^m(\mu)$ in $\hat{S}_M(\mu)$ in $\hat{E}$. Note that (i) is equivalent to $\delta' \in X$, $\text{supp}(\delta') \subseteq \hat{T}_{z^*}^m(\mu)$ and $\{z\} \times \text{supp}(\delta') \subseteq \text{graph}(\mathbb{C})$ and, thus, to (a). Regarding (ii), since μ is a matching and $(z, \delta) \in \text{supp}(\mu)$, it follows $\{z\} \times \text{supp}(\delta) \subseteq \text{graph}(\mathbb{C})$. Hence, (ii) is equivalent to $\delta' \in X$, $\text{supp}(\delta') \subseteq \text{supp}(\delta) \cup \hat{T}_{z^*}^m(\mu)$ and $\{z\} \times \text{supp}(\delta') \subseteq \text{graph}(\mathbb{C})$ and, thus, to (b).

We also claim that the existence result in Carmona and Laohakunakorn (2024) follows from the one established here in the case where Z and C are compact. To this end, suppose that E is rational, continuous, bounded and rich; we will show that $\hat{E}$ satisfies the properties of Theorem 3.1. Since $\hat{X} = \cup_{z \in Z} P_m(z) \subseteq X$, it follows that $\hat{E}$ is bounded and that the set $\{(a, \delta, \mu, a', \delta', \mu', z) \in (\Delta \times M(Z \times \hat{X}_\emptyset))^2 \times Z : (a, \delta, \mu) \succ_z (a', \delta', \mu')\}$ is open.

We have that $\hat{\Lambda}(z, \delta, \mu) = \Lambda(z, \delta, \mu)$ for each $(z, \delta, \mu) \in Z \times M_R(Z \times C) \times M(Z \times X^*)$ and $\hat{\Lambda}_0(z, \mu) = \Lambda_0(z, \mu)$ for each $(z, \mu) \in Z \times M(Z \times \hat{X}_\emptyset)$ by what was noted above. Hence, $\hat{E}$ is rich.

We next show that P_s is continuous with nonempty and compact values. Note that $g : C \rightarrow \{1_{(\emptyset, c)} : c \in C\}$ defined by setting, for each $c \in C$, $g(c) = 1_{(\emptyset, c)}$ is continuous and that $P_s = g \circ \mathbb{C}$. Hence, P_s is continuous with nonempty and compact

values since $\mathbb{C}$ has the same properties.

We conclude by establishing that P_m is compact-valued and upper hemicontinuous. Note that, for each $z \in Z$,

$$P_m(z) = X \cap \{\delta \in M(Z \times C) : c \in \mathbb{C}(z, z') \text{ for each } (z', c) \in \text{supp}(\delta)\}$$

and that X is compact since it is a closed subset of $M_R(Z \times C)$. If $(z, \delta) \in Z \times M(Z \times C)$ and $\{(z_k, \delta_k)\}_{k=1}^\infty$ are such that $(z_k, \delta_k) \rightarrow (z, \delta)$ and $\delta_k \in P_m(z_k)$ for each $k \in \mathbb{N}$, then, for each $(z', c) \in \text{supp}(\delta)$, there is a subsequence $\{z_{k_j}, \delta_{k_j}\}_{j=1}^\infty$ of $\{(z_k, \delta_k)\}_{k=1}^\infty$ and $\{(z'_{k_j}, c_{k_j})\}_{j=1}^\infty$ such that $(z'_{k_j}, c_{k_j}) \rightarrow (z', c)$ and $(z'_{k_j}, c_{k_j}) \in \text{supp}(\delta_{k_j})$ for each $j \in \mathbb{N}$. Then $c_{k_j} \in \mathbb{C}(z_{k_j}, z'_{k_j})$ since $\delta_{k_j} \in P_m(z_{k_j})$ for each $j \in \mathbb{N}$ and, hence, $c \in \mathbb{C}(z, z')$ since $\mathbb{C}$ is upper hemicontinuous. It then follows that P_m is closed and has a compact range, thus, P_m is compact-valued and upper hemicontinuous.

5 Special Cases

Several models in the literature are special cases of our framework of knowledge economies.

5.1 Rosen (1982)

A truncated version of Rosen's (1982) setting is equivalent to an economy where $Z = [\underline{z}, \bar{z}]$, with $0 \leq \underline{z} < \bar{z} < \infty$, $\bar{L} = 2$ so that there are only firms with 1 or 2 layers, managers in the latter firms can hire any number of workers in $N = [\underline{n}, \bar{n}]$ and production equals $g(r(z))f(r(z), nq(z'))$ for each $z, z' \in Z$, where, in particular, f, g, q and r are continuous, $q(z) > 0$ and $r(z) > 0$ for each $z \in Z$, $g(r) > 0$ for each $r > 0$, and $f(z, 0) = f(0, z') = 0$ and $f(z, z') > 0$ for each $z, z' > 0$. The interpretation is that, for each $z \in Z$, $r(z)$ is the individual's ability as a manager and $q(z)$ is the individual's ability as a worker; hence, production depends on the ability of the manager (through the functions g and f) and the aggregate ability $nq(z')$ of the workers. Self-employed individuals i.e. managers of layer 1 firms produce nothing.

Thus, set

$$Y = (Z \times \{1\}) \cup (Z \times N \times Z \times \{1\}),$$

$$F(z, 1) = 0, \text{ for each } z \in Z \text{ and}$$

$$F(z', n, z, 1) = g(r(z))f(r(z), nq(z')) \text{ for each } z', z \in Z \text{ and } n \in N.$$

5.2 Garicano and Rossi-Hansberg (2004)

Garicano and Rossi-Hansberg's (2004) setting corresponds to $Z = [0, \bar{z}]$ with $0 < \bar{z} < 1$, $\bar{L} = 2$,

$$Y = (Z \times \{1\}) \cup \left\{ (z', n, z, 1) : z', z \in Z \text{ and } n = \frac{1}{h(1 - z')} \right\},$$

with $0 < h < 1$ so a manager in a two-layered firm hires more workers the higher is worker knowledge,

$$F(z, 1) = z, \text{ for each } z \in Z \text{ and}$$

$$F(z', n, z, 1) = zn \text{ for each } z', z \in Z \text{ and } n = (h(1 - z'))^{-1}$$

so that production is the product of the manager knowledge and the number of workers (with the convention that in firms with one layer, the manager is also a worker).⁴

A variation of the above specification is obtained by requiring only $\bar{z} > 0$ and introducing a function $G : Z \rightarrow \mathbb{R}_+$ which is the cumulative distribution function of a continuous and strictly positive density and satisfies $G(\bar{z}) < 1$. In this case,

$$Y = (Z \times \{1\}) \cup \left\{ (z', n, z, 1) : z', z \in Z \text{ and } n = \frac{1}{h(1 - G(z'))} \right\},$$

$$F(z, 1) = G(z) \text{ for each } z \in Z \text{ and}$$

$$F(z', n, z, 1) = G(z)n \text{ for each } z', z \in Z \text{ and } n = (h(1 - G(z')))^{-1}.$$

This formalization is only apparently more general. Indeed, if E denotes the economy just defined and E_u is such that $Z_u = G(Z)$, $\nu_u = \nu \circ G^{-1}$, $Y_u = (Z_u \times \{1\}) \cup$

⁴Garicano and Rossi-Hansberg (2004) allow $\bar{z} = 1$; our results can be extended to this case.

$\{(z', n, z, 1) : z', z \in Z_u \text{ and } n = (h(1 - z'))^{-1}\}$, $F_u(z, 1) = z$ and $F_u(z', n, z, 1) = zn$ for each $z', z \in Z$ and $n = (h(1 - z'))^{-1}$, then the stable outcomes of E_u are in a one-to-one relationship with those of E .⁵

The above result is implied by Theorem 11.1 which also implies that the knowledge distribution can be normalized to be uniform. Indeed, if E is the knowledge economy defined in the first paragraph of this section and ν has a continuous and strictly positive density, then let $X = [0, \nu(Z)]$ and $\psi(z) = \nu([0, z])$ for each $z \in Z$, i.e. ψ is the cumulative distribution function of ν and is continuous, strictly increasing and satisfies $\psi(Z) = X$. Then the stable outcomes of the economy E_ψ are in a one-to-one relationship with those of E , where E_ψ has knowledge set X , a uniform knowledge distribution on X , $Y_\psi = (X \times \{1\}) \cup \{(x', n, x, 1) : x', x \in X \text{ and } n = (h(1 - \psi^{-1}(x')))^{-1}\}$, $F_\psi(x, 1) = \psi^{-1}(x)$ and $F_\psi(x', n, x, 1) = \psi^{-1}(x)n$.

5.3 Garicano and Rossi-Hansberg (2006)

In the setting of Garicano and Rossi-Hansberg (2006), individuals are characterized by their cognitive ability. The set of possible abilities is $[0, 1]$ and its distribution is ν with $\text{supp}(\nu) = [0, 1]$.

Each individual may acquire knowledge on how to solve problems. The set of all possible problems is $\mathbb{R}_+$ and problems are drawn according to a probability measure G with a continuous density. An individual's ability affects the cost of acquiring knowledge: if an individual with ability $z \in [0, 1]$ learns how to solve problems in $[0, k]$, with $0 \leq k \leq \bar{k}$ and $\bar{k} \in \mathbb{R}_{++}$ such that $G(\bar{k}) < 1$, then the cost is $e(z)k$, where $e(z) = \alpha - \beta z$ with $\alpha, \beta > 0$ and $\alpha \geq \beta$.

Individuals in layer 1 of a firm spend all their time in production. Problems that they cannot solve are asked to layer 2; those that layer 2 cannot solve are then passed to layer 3 and so on. If there are L layer in total, then those in layer L attempt to solve only those problems that layers $1, \dots, L - 1$ cannot solve. If the knowledge of

⁵This result provides a justification for the claim in footnote 4 of Fuchs, Garicano, and Rayo (2015).

those in layer $1 \leq i \leq L$ is k_i , then the number n_i of individuals in layer i satisfies:

$$n_1 h(1 - G(k_{i-1})) = n_i \text{ if } 1 < i < L \text{ and} \quad (5.1)$$

$$n_1 h(1 - G(k_{L-1})) = 1. \quad (5.2)$$

Equation (5.1) is the time constraint of layer $i \in \{2, \dots, L-1\}$ as it attempts to solve the problems drawn by those in layer 1 that layers $1, \dots, i-1$ have not been able to solve. Equation (5.2) is the time constraint of layer L recalling that $n_L = 1$.

Production is then

$$G(k_L) n_1 - \sum_{i=1}^L e(z_i) k_i n_i.{}^6$$

The representation of this setting as a knowledge economy has the following two features: the output function at some firm is the value function of the problem of maximizing that firm's output by choosing knowledge optimally and it need not be nonnegative since it is more properly interpreted as output net of learning costs.⁷ Thus, we represent this setting by letting $Z = [0, 1]$, $N = [1 - G(\bar{k}), (h(1 - G(\bar{k})))^{-1}]$, $Y = \cup_{L=1}^{\bar{L}} Y_L$, $Y_1 = Z \times \{1\}$, $F(z, 1) = \max_{k \in [0, \bar{k}]} G(k) - e(z)k$ for each $z \in Z$,

$$Y_L = \{(z_1, n_1, \dots, z_L, n_L) \in (Z \times N)^L : n_L = 1 \text{ and there exists } (k_1, \dots, k_{L-1}) \in [0, \bar{k}]^{L-1}$$

$$\text{such that } n_1 = (h(1 - G(k_{L-1})))^{-1} \text{ and}$$

$$n_i = (1 - G(k_{i-1}))(1 - G(k_{L-1}))^{-1} \text{ for each } i = 2, \dots, L-1\},$$

⁶These equations have the intended interpretation when k_{i-1} is replaced with $\max\{k_1, \dots, k_{i-1}\}$ in (5.1) and (5.2), and $G(k_L)$ with $G(\max\{k_1, \dots, k_L\})$ in this equation. The two economies have the same stable outcomes.

⁷Here is the formal claim that allows us to dispense with the nonnegativity constraint on output. Let $E = (Z, \nu, Y, F)$ be a knowledge economy, $B \in \mathbb{R}_+$ be such that $|F(y, \gamma)| \leq B$ for each $(y, \gamma) \in Y \times \Gamma$ (such B exists since F is continuous and $Y \times \Gamma$ is compact) and $E_B = (Z, \nu, Y, F_B)$ is equal to E except for its output function, which is such that $F_B(y, \gamma) = \max\{F(y, \gamma) + B \sum_{i=1}^{\ell(y)} n_i(y), 0\}$ for each $(y, \gamma) \in Y \times M(Y)$; note that F_B is continuous and nonnegative. It is straightforward to see that (γ, u) is a stable outcome of E if and only if $(\gamma, u + B)$ is a stable outcome of E_B .

and setting, for each $L \in \{2, \dots, \bar{L}\}$ and $(z_1, n_1, \dots, z_L, n_L) \in Y_L$,

$$\begin{aligned}
F(z_1, n_1, \dots, z_L, n_L) &= \max_{(k_1, \dots, k_L) \in [0, \bar{k}]^L} G(k_L) n_1 - \sum_{i=1}^L e(z_i) k_i n_i \\
&\text{subject to} \\
n_1 &= \frac{1}{h(1 - G(k_{L-1}))} \\
n_i &= \frac{1 - G(k_{i-1})}{1 - G(k_{L-1})} \text{ for each } i = 2, \dots, L-1, \text{ and} \\
n_L &= 1.
\end{aligned}$$

5.4 Multi-layer version of Garicano and Rossi-Hansberg (2004)

A version of Garicano and Rossi-Hansberg (2004) allowing for firms with more than two layers is obtained by changing the set Y of firms and the production function F from Section 5.2 as follows. Let $Z = [\underline{z}, \bar{z}]$, $0 < \underline{z} < \bar{z} < 1$, $N = [(1 - \bar{z})(1 - \underline{z})^{-1}, (h(1 - \bar{z}))^{-1}]$, $Y = \cup_{L=1}^{\bar{L}} Y_L$ with $Y_1 = Z \times \{1\}$ and, for each $L \geq 2$,

$$\begin{aligned}
Y_L &= \left\{ (z_1, n_1, \dots, z_L, n_L) \in (Z \times N)^L : n_L = 1, n_1 = \frac{1}{h(1 - z_{L-1})} \text{ and} \right. \\
&\quad \left. n_i = \frac{1 - z_{i-1}}{1 - z_{L-1}} \text{ for each } i = 2, \dots, L-1 \right\}.
\end{aligned}$$

Then, for each $L \in \{1, \dots, \bar{L}\}$ and $(z_1, n_1, \dots, z_L, n_L) \in Y_L$, let $F(z, 1) = z$ if $L = 1$ and $F(z_1, n_1, \dots, z_L, n_L) = z_L n_1$ if $L \geq 2$,

An alternative specification of multi-layer version of Garicano and Rossi-Hansberg's (2004) setting that corresponds more closely to the description in Garicano (2000) is obtained by letting, for each $L \geq 2$,

$$\begin{aligned}
Y_L &= \left\{ (z_1, n_1, \dots, z_L, n_L) \in (Z \times N)^L : n_1 = \frac{1}{h(1 - \max\{z_1, \dots, z_{L-1}\})} \right. \\
&\quad \left. n_L = 1, \text{ and } n_i = \frac{1 - \max\{z_1, \dots, z_{i-1}\}}{1 - \max\{z_1, \dots, z_{L-1}\}} \text{ for each } i = 2, \dots, L-1 \right\} \text{ and} \\
F(z_1, n_1, \dots, z_L, n_L) &= \max\{z_1, \dots, z_L\} n_1 \text{ for each } (z_1, n_1, \dots, z_L, n_L) \in Y_L.
\end{aligned}$$

As we show in Section 12, these two economies have the same stable outcomes.

5.5 Ide and Talamàs (2025)

Ide and Talamàs (2025) introduce artificial intelligence (AI henceforth) to the setting of Garicano and Rossi-Hansberg (2004). Their formalization of AI can be added to our setting by treating it as another type of knowledge as follows. Let $\bar{z} \in (0, 1)$ and interpret $z \in [0, \bar{z}]$ as the fraction of problems that someone can solve. Then let $z_{AI} \in [0, \bar{z}]$ be the fraction of problems AI can solve and define $Z = ([0, \bar{z}] \times \{0\}) \cup \{(z_{AI}, 1)\}$; thus, the first coordinate of $z \in Z$ indicates the fraction of problems someone can solve and the second whether that someone is a person or AI. Letting $p : Z \rightarrow [0, \bar{z}]$ denote the projection of Z onto $[0, \bar{z}]$, then let

$$Y = (Z \times \{1\}) \cup \left\{ (z', n, z, 1) : z', z \in Z \text{ and } n = \frac{1}{h(1 - p(z'))} \right\},$$

with $0 < h < 1$ so a manager in a two-layered firm hires more workers the higher is worker knowledge,

$$F(z, 1) = p(z), \text{ for each } z \in Z \text{ and}$$

$$F(z', n, z, 1) = p(z)n \text{ for each } z', z \in Z \text{ and } n = (h(1 - p(z')))^{-1}$$

The knowledge distribution ν on Z can be described by a distribution G on $[0, \bar{z}]$ and a number α , which is the measure of AI, i.e. of $\{(z_{AI}, 1)\}$. Since whether knowledge is provided by a person or by AI does not matter for production and firm size, it follows that this economy can also be described by the Garicano and Rossi-Hansberg's (2004) economy of Section 5.2 with a knowledge distribution $\hat{\nu}$ such that, for each Borel subset B of $[0, \bar{z}]$, $\hat{\nu}(B) = G(B)$ if $z_{AI} \notin B$ and $\hat{\nu}(B) = G(B) + \alpha$ otherwise.

Ide and Talamàs (2025) also consider the case where AI can only take a managerial role. This case can be represented in our setting by letting

$$F(z', n, z, 1) = \begin{cases} p(z)n & \text{if } z' \in [0, \bar{z}] \times \{0\}, \\ 0 & \text{otherwise,} \end{cases}$$

and analogously for $F(z, 1)$.

5.6 Mak and Siow (2025)

The setting in Mak and Siow (2025) corresponds to the following case: Let Z be a compact subset of $\mathbb{R}_+^2$; the interpretation of $z = (z_1, z_2) \in Z$ is that z_1 is the individual's ability as a manager and z_2 as a worker. Matching is one-to-one, hence $Y = (Z \times \{1\}) \cup (Z \times \{1\})^2$. Production is $F(z', 1, z, 1) = R(p_1(z), p_2(z'))$, where $R: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is continuous and p_i is the projection of $\mathbb{R}_+^2$ onto its i th coordinate for each $i = 1, 2$; hence, production is a continuous function of the manager's and the worker's abilities, and self-employed individuals produce nothing, $F(z, 1) = 0$.⁸

This setting can be thought of as a one-to-one matching version of Rosen (1982) when for some $\underline{z}, \bar{z} \in \mathbb{R}_+$ with $\underline{z} < \bar{z}$, and continuous functions f, g, q and r , $Z = G([\underline{z}, \bar{z}])$ where $G = (r, q)$, and $R(r(z), q(z')) = g(r(z))f(r(z), q(z'))$ for each $z, z' \in [\underline{z}, \bar{z}]$.

5.7 Gabaix and Landier (2008) and Gola (2021)

Gabaix and Landier (2008) consider a two-sided one-to-one matching model, which can be represented as a knowledge economy as follows. Let M and W be disjoint compact subsets of $\mathbb{R}_+$, with the interpretation that M is the set of managers and W is the set of workers, fixed a priori. Let $Z = M \cup W$, $\bar{L} = 2$ and $Y = (Z \times \{1\}) \cup (W \times \{1\} \times M \times \{1\})$ so that in any firm a manager must take the managerial role and a worker be the workforce. Then, let $F(z, 1) = 0$ and $F(z', 1, z, 1) = f(z, z')$, where f is continuous and strictly positive.

Gola (2021) also considers a two-sided one-to-one matching model that can be written in our setting in a similar way to the one above. The main differences are that (i) W is a subset of $\mathbb{R}^L$ for some $L \in \mathbb{N}$, where $z \in W$ describes a worker's skills, and (ii) M is a subset of $\mathbb{R} \times \{0, 1\}$, with the first coordinate describing a manager's productivity and the second indicating in which of two sectors the manager can operate.

⁸Mak and Siow (2025) do not require Z to be compact; it seems possible to extend our results to this case.

6 A characterization of stable outcomes in knowledge economies

We provide a proof of the characterization of stable outcomes in knowledge economies stated in Section 2.2 of the paper. We start with the following lemma which shows that each knowledge type is part of a firm in the support of γ . It is a consequence of the full support assumption on ν since every type is present in the population and an assignment accounts for every one.

Lemma 6.1 *If (γ, v) is an outcome in an economy E and $z \in Z$, then there exists $y \in \text{supp}(\gamma)$ and $i \in \{1, \dots, \ell(y)\}$ such that $z_i(y) = z$.*

Proof. Suppose there is $z \in Z$ such that, for each $L \in \{1, \dots, \bar{L}\}$ and each $i \in \{1, \dots, L\}$, $z \notin z_i(\text{supp}(\gamma) \cap Y_L)$, i.e. $z \notin A = \bigcup_{L=1}^{\bar{L}} \bigcup_{i=1}^L z_i(\text{supp}(\gamma) \cap Y_L)$. Note that A is compact, $z \in O_z = Z \setminus A$, O_z is open, $1_{O_z}(z_i(y)) = 0$ for each $y \in \text{supp}(\gamma) \cap Y_L$ and, hence,

$$\sum_{L=1}^{\bar{L}} \int_{Y_L} \sum_{i=1}^L 1_{O_z}(z_i(y)) n_i(y) d\gamma(y) = 0.$$

We have that $\nu(O_z) > 0$ since $\text{supp}(\nu) = Z$. Since γ is an assignment, it follows that

$$0 < \nu(O_z) = \sum_{L=1}^{\bar{L}} \int_{Y_L} \sum_{i=1}^L 1_{O_z}(z_i(y)) n_i(y) d\gamma(y) = 0,$$

a contradiction. This contradiction establishes the lemma. ■

With Lemma 6.1, it is then easy to establish the characterization of stable outcomes in Section 2.2 of the paper.

Theorem 6.1 *An outcome (γ, v) in an economy E is stable if and only if*

$$\sum_{i=1}^{\ell(y)} n_i(y) v(z_i(y)) = F(y, \gamma) \text{ for each } y \in \text{supp}(\gamma), \text{ and}$$

$$v(z) = \max_{y \in Y, i \in \{1, \dots, \ell(y)\}: z_i(y)=z} \frac{F(y, \gamma) - \sum_{j \in \{1, \dots, \ell(y)\} \setminus \{i\}} n_j(y) v(z_j(y))}{n_i(y)} \text{ for each } z \in Z.$$

Proof. (Sufficiency) Let (γ, v) be an outcome in an economy E and assume that it satisfies the two conditions in the statement. The first is the same as (i) in the definition of a stable outcome, hence it suffices to establish (ii). Let $y \in Y$. It then follows by the second condition in the statement that $v(z_{\ell(y)}(y)) \geq F(y, \gamma) - \sum_{i=1}^{\ell(y)-1} n_i(y)v(z_i(y))$. Thus, (ii) holds.

(Necessity) Let (γ, v) be a stable outcome in an economy E . Then (i) implies that the first condition in the statement holds.

To show that the second condition in the statement holds, let $z \in Z$. Let, by Lemma 6.1, $\tilde{y} \in \text{supp}(\gamma)$ and $l \in \{1, \dots, \ell(\tilde{y})\}$ be such that $z_l(\tilde{y}) = z$. For each $y \in Y$ and $i \in \{1, \dots, \ell(y)\}$ such that $z_i(y) = z$, (ii) implies that

$$v(z) \geq \frac{F(y, \gamma) - \sum_{j \in \{1, \dots, \ell(y)\} \setminus \{i\}} n_j(y)v(z_j(y))}{n_i(y)};$$

thus, using (i),

$$\begin{aligned} v(z) &\geq \max_{y \in Y, i \in \{1, \dots, \ell(y)\} : z_i(y) = z} \frac{F(y, \gamma) - \sum_{j \in \{1, \dots, \ell(y)\} \setminus \{i\}} n_j(y)v(z_j(y))}{n_i(y)} \\ &\geq \frac{F(\tilde{y}, \gamma) - \sum_{j \in \{1, \dots, \ell(\tilde{y})\} \setminus \{l\}} n_j(\tilde{y})v(z_j(\tilde{y}))}{n_l(\tilde{y})} = v(z). \end{aligned}$$

Hence,

$$v(z) = \max_{y \in Y, i \in \{1, \dots, \ell(y)\} : z_i(y) = z} \frac{F(y, \gamma) - \sum_{j \in \{1, \dots, \ell(y)\} \setminus \{i\}} n_j(y)v(z_j(y))}{n_i(y)}.$$

■

7 Stable Matchings in Knowledge Economies

We provide in this section a formal definition of stable matchings for a knowledge economy. This is done by applying the definition of a stable matching for general large many-to-one matching markets with occupational choice to E_m defined in Section A.1 of the paper.

Recall that a matching in an economy E is a Borel measure μ on $Z \times X_\emptyset$ such that $\text{supp}(\mu) \subseteq G_\emptyset$ and $\nu_M + \nu_S + \nu_W = \nu$.⁹ Roughly, a matching is stable if no

⁹For each Borel subset B of Z , $\nu_M(B) = \mu(B \times X)$, $\nu_S(B) = \mu(B \times (X_\emptyset \setminus X))$ and $\nu_W(B) = \int_{Z \times X} \delta(B \times C) d\mu(z, \delta)$.

individual can increase utility by changing occupation, matching with target partners, or both. An individual's targets are those who would be better off matching with that individual than remaining in their current matches.

Given a matching μ , a type $z \in Z$ and an occupation $a \in \{w, m, s\}$, individuals of type z can target certain types and contracts (z^*, c) when choosing occupation a provided that someone of type z^* is better off with someone of type z with occupation a and with contract c than in the current match. The set of such (z^*, c) is denoted by $T_z^a(\mu)$.

The targets for the prospective self-employed are the type-contract pairs that are feasible when someone is unmatched: For each $z \in Z$, let $T_z^s(\mu) = \{(\emptyset, 0, 0)\}$.

The targets of prospective managers are as follows. Someone of type z^* can be targeted at wage t by anyone of type z who chooses to be a manager when t is bigger than the payment that the type- z^* individual obtains in the current match. If such a type- z^* individual is a worker, then

- (a) there exists $(z', \delta', i', t') \in Z \times X \times C$ such that $(z', \delta') \in \text{supp}(\mu)$, $(z^*, i', t') \in \text{supp}(\delta')$ and $t > t'$,

i.e. z' is the type of the manager who hires the worker, t' is the worker's wage and, hence, $t > t'$ for the worker to prefer to be matched with a type z at wage t . If such z^* individual is self-employed, then

- (b) there exists $\delta' \in X_\emptyset \setminus X$ such that $(z^*, \delta') \in \text{supp}(\mu)$ and $t > U_{z^*}(s, \delta', \mu)$

since $(z^*, \delta') \in \text{supp}(\mu)$ and $\delta' \in X_\emptyset \setminus X$ indicate that the type- z^* individual is self-employed and, hence, $t > U_{z^*}(s, \delta', \mu)$ for that individual to prefer to be matched with a type z at wage t . Finally, when such a type- z^* individual is a manager,

- (c) there exists $\delta' \in X$ such that $(z^*, \delta') \in \text{supp}(\mu)$ and $t > U_{z^*}(m, \delta', \mu)$.

Thus, for each $z \in Z$, let $T_z^m(\mu)$ be the set of $(z^*, i, t) \in Z \times C$ such that (a), (b) or (c) hold.¹⁰

¹⁰It follows by Theorem 2.1 in Section 2 that the targets $T_z^w(\mu)$ of prospective workers are not needed for the definition of a stable matching.

Matches are pairs (z, δ) and consist, in particular, of a manager of type z and workers described by δ if $\delta \in X$ or a self-employed of type z if $\delta \in X_\emptyset \setminus X$. Stability requires that none of these individuals can increase utility by changing occupation and matching with occupation-specific targets. We decompose this requirement by final occupation and note that it is enough to consider the case where such final occupation is m or s .¹¹

The set $S_M(\mu)$ below describes matches (z, δ) such that no one in it gains by changing their occupation-match pair by matching with their targets while becoming a manager (from being a worker as in condition (ii) below or self-employed as in condition (iii)) or while remaining a manager (as in condition (i) below). Formally, let $S_M(\mu)$ be the set of $(z, \delta) \in G_\emptyset$ such that, if $\delta \in X$, then

- (i) there does not exist $\delta' \in P_m(z)$ such that $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ and $U_z(m, \delta', \mu) > U_z(m, \delta, \mu)$,
- (ii) for each $(z', i, t) \in \text{supp}(\delta)$, there does not exist $\delta' \in P_m(z')$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu)$ and $U_{z'}(m, \delta', \mu) > t$,

and, if $\delta \in X_\emptyset \setminus X$, then

- (iii) there does not exist $\delta' \in P_m(z)$ such that $\text{supp}(\delta') \subseteq T_z^m(\mu)$ and $U_z(m, \delta', \mu) > U_z(s, \delta, \mu)$.

The set $IR(\mu)$ describes matches (z, δ) such that no one in it gains by becoming self-employed. Formally, let $IR(\mu)$ be the set of $(z, \delta) \in G_\emptyset$ such that, if $\delta \in X$, then

- (i) $U_z(m, \delta, \mu) \geq U_z(s, 1_{(\emptyset, 0, 0)}, \mu)$ and
- (ii) $t \geq U_{z'}(s, 1_{(\emptyset, 0, 0)}, \mu)$ for each $(z', i, t) \in \text{supp}(\delta)$.

A matching μ is *stable* if $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$.

¹¹This follows by Theorem 2.1 in Section 2.

8 Competitive Equilibrium

Our definition of competitive equilibrium for a knowledge economy is obtained by representing the economy in a general equilibrium setting and then applying the corresponding notion of competitive equilibrium. The general equilibrium setting we consider is analogous to the classical one in Debreu (1959) with the following differences. First, in contrast to the finitely many individuals in Debreu (1959), knowledge economies are populated by non-atomic individuals described by a distribution, as in Hildenbrand (1974). Also in contrast to both Debreu (1959) and Hildenbrand (1974), a knowledge economy features infinitely many commodities since the set of knowledge levels (i.e. the type space Z) is allowed to be infinite and, correspondingly, there are infinitely many types of labor, i.e. labor is a differentiated commodity, as in Jones (1984).

We represent a knowledge economy in a general equilibrium setting as follows. The commodity space is $L = \mathbb{R} \times \text{ca}(Z)$, representing allocations of the consumption good and labor, where $\text{ca}(Z)$ denotes the space of countably additive set functions on Z . An element $(x, \psi) \in L$ is interpreted as follows: x is the amount of the consumption good, used as output if $x > 0$ and as input if $x < 0$; ψ is a countably additive set function defined on the Borel subsets of Z , with $\psi(B)$ denoting the number of individuals with knowledge in the Borel subset B of Z , used as input if $\psi(B) < 0$ and as output if $\psi(B) > 0$.

Prices are in $L^* = \mathbb{R} \times C(Z)$. We normalize the price of the consumption good to one, and the prices of labor are non-negative across knowledge types. Let $L_+^* = \{(p, c) \in \mathbb{R} \times C(Z) : p = 1 \text{ and } c(z) \geq 0 \text{ for each } z \in Z\}$. The value of $\phi = (x, \psi) \in L$ at price $q = (p, c) \in L^*$ is $q(\phi) = px + \int_Z c d\psi$.

Consumers are characterized by their knowledge level $z \in Z$ and are described implicitly by the knowledge distribution ν . The consumption set is $\Omega = \mathbb{R}_+ \times M(Z)$, and preferences are represented by the utility function $u : \Omega \rightarrow \mathbb{R}$ defined by $u(x, \psi) = x$, for each $(x, \psi) \in \Omega$; individuals value only the consumption good. Endowments depend on the knowledge level z since an individual with knowledge z is endowed

with one unit of labor of type z : for each $z \in Z$, let $e(z) = (0, 1_z)$ be the endowment of those with knowledge z . Thus, the value of the endowment at price $(p, c) \in L^*$ is $c(z)$. We write $e_x(z) = 0$ and $e_\psi(z) = 1_z$ for each $z \in Z$.

The description of production follows Hornstein and Prescott (1993) in that only the aggregate production set is specified in such a way that the formation of firms, i.e. the distribution of firms, is part of the description of production. The aggregate production set is indexed by $\gamma^* \in M(Y)$, which describes the production externalities in the economy, and is

$$\Phi_{\gamma^*} = \{(x, \psi) \in L : \text{there exists } \gamma \in M(Y) \text{ such that } x \leq \int_Y F(y, \gamma^*) d\gamma(y) \\ \text{and } \psi(B) \leq -\nu_\gamma(B) \text{ for each Borel } B \subseteq Z\}.$$

Production therefore requires the formation of firms described by the distribution γ . A commodity (x, ψ) is feasible if consumption does not exceed total output and if, for each Borel set of types $B \subseteq Z$, the labor $\nu_\gamma(B)$ used as managerial or worker input does not exceed the amount $-\psi(B)$ available to those firms. For each $\phi = (x, \psi) \in L$, let

$$\Phi_{\gamma^*}(\phi) = \{\gamma \in M(Y) : x \leq \int_Y F(y, \gamma^*) d\gamma(y) \\ \text{and } \psi(B) \leq -\nu_\gamma(B) \text{ for each Borel } B \subseteq Z\}$$

be the set of distributions of firms that can support ϕ .

A competitive equilibrium consists of a measure τ over $Z \times \Omega$ describing the consumers' choices, an externality γ^* , a production vector ϕ^* , and a price q such that (i) the marginal of τ on Z is equal to the distribution ν of individuals in the economy, (ii) almost every consumer maximizes utility subject to the budget constraint, (iii) production maximizes profit, (iv) the production vector is supported by the firm distribution γ^* , and (v) aggregate consumption does not exceed the sum of aggregate endowments and production. Formally, a *competitive equilibrium* is a measure $\tau \in M(Z \times \Omega)$, an externality $\gamma^* \in M(Y)$, a production vector $\phi^* \in \Phi_{\gamma^*}$ and a price $q \in L_+^*$ such that

- (i) $\tau_Z = \nu$,
- (ii) $\tau(\{(z, x, \psi) \in Z \times \Omega : (x, \psi) \text{ solves } \max_{(x', \psi') \in \Omega} u(x', \psi') \text{ subject to } q(x', \psi') \leq q(e(z))\}) = \nu(Z)$,
- (iii) $q(\phi) \leq q(\phi^*)$ for each $\phi \in \Phi_{\gamma^*}$,
- (iv) $\gamma^* \in \Phi_{\gamma^*}(\phi^*)$, and
- (v) $\int_{\Omega} i_{\Omega} d\tau_{\Omega} \leq \int_Z e d\nu + \phi^*$, where $i_{\Omega} : \Omega \rightarrow \Omega$ is the identity on Ω .

Theorem 8.1 below characterizes competitive equilibria. It requires all types to be productive in the following sense. For each $\gamma \in M(Y)$, let

$$\hat{Z}_{\gamma} = \{z \in Z : F(y, \gamma) = 0 \text{ for each } y \in Y \text{ such that } z_i(y) = z \text{ for each } i = 1, \dots, \ell(y)\}$$

be the set of knowledge types for which every feasible firm composed entirely of that type produces zero output; elements of the complement of $\hat{Z}_{\gamma}$ are called productive types. An economy has *productive types* if, for each $\gamma \in M(Y)$, $\nu(\hat{Z}_{\gamma}) = 0$. Thus, an economy has productive types if, for each $\gamma \in M(Y)$, almost every individual has a productive type.

Theorem 8.1 *In a knowledge economy with productive types, $(\tau, \gamma^*, \phi^*, (1, c))$ is a competitive equilibrium if and only if*

1. (γ^*, c) is stable,
2. $\tau = \nu \circ f^{-1}$, where $f : Z \rightarrow Z \times \Omega$ is such that $f(z) = (z, c(z), 0)$ for each $z \in Z$, and
3. $\phi^* = (x^*, \psi^*)$, where $x^* = \int_Y F(y, \gamma^*) d\gamma^*(y)$ and $\psi^*(B) = -\nu_{\gamma^*}(B) = -\nu(B)$ for each Borel $B \subseteq Z$.

8.1 Proof of Theorem 8.1

(Necessity) Let $(\tau, \gamma^*, \phi^*, q)$ be a competitive equilibrium. Note that $\phi \in \Phi_{\gamma^*}$ if and only if $\Phi_{\gamma^*}(\phi) \neq \emptyset$.

Lemma 8.1 $q(\phi^*) = 0$.

Proof. Note first that $0 \in \Phi_{\gamma^*}$ since $\gamma = 0 \in \Phi_{\gamma^*}(0)$; hence, $q(\phi^*) \geq q(0) = 0$. If $q(\phi^*) > 0$, then let $\lambda > 1$ and note that $\lambda\phi^* \in \Phi_{\gamma^*}$ since $\lambda\gamma^* \in \Phi_{\gamma^*}(\lambda\phi^*)$. Thus, $q(\lambda\phi^*) = \lambda q(\phi^*) > q(\phi^*)$, a contradiction. ■

If $y \in Y$, then $(F(y, \gamma^*), -\sum_{i=1}^{\ell(y)} n_i(y)1_{z_i(y)}) \in \Phi_{\gamma^*}$ since $1_y \in \Phi_{\gamma^*}(F(y, \gamma^*), -\sum_{i=1}^{\ell(y)} n_i(y)1_{z_i(y)})$. Hence,

$$q(F(y, \gamma^*), -\sum_{i=1}^{\ell(y)} n_i(y)1_{z_i(y)}) = F(y, \gamma^*) - \sum_{i=1}^{\ell(y)} n_i(y)c(z_i(y)) \leq 0.$$

Thus,

$$F(y, \gamma^*) \leq \sum_{i=1}^{\ell(y)} n_i(y)c(z_i(y)) \text{ for each } y \in Y \quad (8.1)$$

and

$$\begin{aligned} 0 = q(\phi^*) &= x^* + \int_Z c d\psi^* \leq \int_Y F(y, \gamma^*) d\gamma^*(y) - \int_Z c d\nu_{\gamma^*} \\ &= \int_Y \left(F(y, \gamma^*) - \sum_{i=1}^{\ell(y)} n_i(y)c(z_i(y)) \right) d\gamma^*(y) \leq 0.^{12} \end{aligned} \quad (8.2)$$

Then (8.1) and (8.2) imply that

$$x^* = \int_Y F(y, \gamma^*) d\gamma^*(y), \quad (8.3)$$

$$\int_Z c(z) d\psi^*(z) = - \int_Z c(z) d\nu_{\gamma^*}(z), \quad (8.4)$$

$$F(y, \gamma^*) - \sum_{i=1}^{\ell(y)} n_i(y)c(z_i(y)) = 0 \text{ for each } y \in \text{supp}(\gamma^*), \text{ and} \quad (8.5)$$

$$x^* = \int_Y F(y, \gamma^*) d\gamma^*(y) = \int_Z c(z) d\nu_{\gamma^*}(z). \quad (8.6)$$

Let $\hat{Z} = \hat{Z}_{\gamma^*}$ be as in the definition of productive types.

Lemma 8.2 $c(z) > 0$ for each $z \in \hat{Z}^c$.

¹²See the proof of Theorem 2 for a proof of the equality $\int_Z c d\nu_{\gamma^*} = \int_Y \sum_{i=1}^{\ell(y)} n_i(y)c(z_i(y)) d\gamma^*(y)$.

Proof. Let $z \in \hat{Z}^c$ and $y \in Y$ such that $z_i(y) = z$ for each $i = 1, \dots, \ell(y)$ and $F(y, \gamma^*) > 0$. Note that $(F(y, \gamma^*), -\sum_{i=1}^{\ell(y)} n_i(y)1_{z_i(y)}) \in \Phi_{\gamma^*}$ since we have $1_y \in \Phi_{\gamma^*}(F(y, \gamma^*), -\sum_{i=1}^{\ell(y)} n_i(y)1_{z_i(y)})$. Hence,

$$q((F(y, \gamma^*), -\sum_{i=1}^{\ell(y)} n_i(y)1_{z_i(y)})) = F(y, \gamma^*) - c(z) \sum_{i=1}^{\ell(y)} n_i(y) \leq 0$$

and $c(z) \geq \frac{F(y, \gamma^*)}{\sum_{i=1}^{\ell(y)} n_i(y)} > 0$. ■

It follows by Lemma 8.2 that

$$\begin{aligned} \{(z, x, \psi) : (x, \psi) \text{ solves } \max_{(x', \psi') \in \Omega} u(x', \psi') \text{ subject to } q(x', \psi') \leq q(e(z))\} = \\ \{(z, c(z)) : z \in Z\} \times M(\hat{Z}). \end{aligned}$$

Thus, $\text{supp}(\tau) \subseteq \{(z, c(z)) : z \in Z\} \times M(\hat{Z})$ since $\hat{Z}$ is closed and, hence, so is $\{(z, c(z)) : z \in Z\} \times M(\hat{Z})$.

It follows that

$$\int_{\Omega} i_{\Omega} d\tau_{\Omega} = \left(\int_Z c(z) d\nu(z), \pi \right)$$

where $\pi \in M(\hat{Z})$ is such that, for each Borel subset B of $\hat{Z}$, $\pi(B) = \int_{\Omega} \psi(B) d\tau_{\Omega}(x, \psi)$.

Then $\int_{\Omega} i_{\Omega} d\tau_{\Omega} \leq \int_Z e d\nu + \phi^*$ implies that

$$\int_Z c(z) d\nu(z) \leq x^* = \int_Z c(z) d\nu_{\gamma^*}(z) \text{ and} \quad (8.7)$$

$$\pi(B) \leq \nu(B) + \psi^*(B) \leq \nu(B) - \nu_{\gamma^*}(B) \quad (8.8)$$

for each Borel $B \subseteq Z$. Since $\nu(\hat{Z}) = 0$, it follows that $\pi = 0$. Hence, $\nu - \nu_{\gamma^*} \geq 0$. Since $\hat{Z}^c \subseteq \{z : c(z) > 0\}$, it follows that $[\nu - \nu_{\gamma^*}](Z) = [\nu - \nu_{\gamma^*}](\{z : c(z) > 0\})$. Since $c \geq 0$, $\nu - \nu_{\gamma^*} \geq 0$ and $\int_Z c d\nu \leq \int_Z c d\nu_{\gamma^*}$ (the latter by (8.7)), it follows that $\int_Z c d[\nu - \nu_{\gamma^*}] = 0$ and $[\nu - \nu_{\gamma^*}](\{z : c(z) > 0\}) = 0$. Hence, for each Borel $B \subseteq Z$, $0 \leq [\nu - \nu_{\gamma^*}](B) \leq [\nu - \nu_{\gamma^*}](Z) = [\nu - \nu_{\gamma^*}](\{z : c(z) > 0\}) = 0$. Thus, $\nu = \nu_{\gamma^*}$ and, by (8.8), $\nu = -\psi^*$.

It then follows from $\nu_{\gamma^*} = \nu$, (8.1) and (8.5) that (γ^*, c) is stable.

It follows from $\int_{Z \times \Omega} \psi(\hat{Z}) d\tau(z, x, \psi) = \pi(\hat{Z}) = 0$ that $\text{supp}(\tau) \subseteq \{(z, c(z), 0) : z \in Z\}$ since $\{(z, c(z), 0) : z \in Z\}$ is closed. Thus, for each bounded continuous function

$$g : Z \times \Omega \rightarrow \mathbb{R},$$

$$\begin{aligned} \int_{Z \times \Omega} g d\tau &= \int_{\text{supp}(\tau)} g d\tau = \int_{\text{supp}(\tau)} g \circ f(z) d\tau(z, x, \psi) = \int_{Z \times \Omega} g \circ f(z) d\tau(z, x, \psi) = \\ &= \int_Z g \circ f d\tau_Z = \int_Z g \circ f d\nu = \int_{Z \times \Omega} g d\nu \circ f^{-1}; \end{aligned}$$

hence, $\tau = \nu \circ f^{-1}$.

(Sufficiency) Let (γ^*, c) be stable and define $\tau = \nu \circ f^{-1}$, where $f : Z \rightarrow Z \times \Omega$ is such that $f(z) = (z, c(z), 0)$ for each $z \in Z$, and $\phi^* = (x^*, \psi^*)$, where $x^* = \int_Y F(y, \gamma^*) d\gamma^*(y)$, $\psi^*(B) = -\nu_{\gamma^*}(B) = -\nu(B)$ for each Borel $B \subseteq Z$.

Conditions (i), (ii) and (iv) of the definition of a competitive equilibrium hold by the definition of τ and ϕ^* . We have that

$$\begin{aligned} \int_{\Omega} i_{\Omega} d\tau_{\Omega} &= \left(\int_Z c(z) d\nu(z), 0 \right), \text{ and} \\ \int_Z e d\nu + \phi^* &= \left(\int_Y F(y, \gamma^*) d\gamma^*(y), 0 \right) \end{aligned}$$

and

$$\begin{aligned} &\int_Y F(y, \gamma^*) d\gamma^*(y) - \int_Z c(z) d\nu(z) \\ &= \int_Y F(y, \gamma^*) d\gamma^*(y) - \int_Z c(z) d\nu_{\gamma^*}(z) \\ &= \int_Y \left(F(y, \gamma^*) - \sum_{i=1}^{\ell(y)} n_i(y) c(z_i(y)) \right) d\gamma^*(y) = 0 \end{aligned}$$

since (γ^*, c) is stable. Thus, condition (v) holds.

The above argument shows that $q(\phi^*) = 0$. Since $F(y, \gamma^*) - \sum_{i=1}^{\ell(y)} n_i(y) c(z_i(y)) \leq 0$ for each $y \in Y$ by the stability of (γ^*, c) , it follows that $q(\phi) \leq 0$ for each $\phi \in \Phi_{\gamma^*}$. Thus, condition (iii) also holds.

9 Stable outcomes in the canonical economy

This section states and proves a characterization of the set of stable outcomes of the knowledge economy of Garicano and Rossi-Hansberg (2004) with multi-layered firms, as defined in Section 4 of the paper.

For each $\gamma \in M(Y^*)$, $L \in \{1, \dots, \bar{L}\}$ and $i \in \{1, \dots, L\}$, let $W_{L,i} = z_i(\text{supp}(\gamma) \cap Y_L^*)$ be the set of types who are employed in layer i of a firm with L layers, $M_L = W_{L,L}$ be the set of types who are managers of firms with L layers and $M = \cup_{L=1}^{\bar{L}} M_L$ be the set of manager types. If A and B are nonempty closed subsets of Z , $A < B$ means that $\max A \leq \min B$.

Theorem 9.1 *The knowledge economy E^* has a unique stable outcome (γ, u) , and (γ, u) satisfies the following property: There exists $K \subseteq \{1, \dots, \bar{L}\}$ and $\{\phi_{k,l}\}_{k \in K, l \in \{1, \dots, k-1\}}$ such that*

1. $\text{supp}(\gamma) \cap Y_k^* = \emptyset$ for each $k \notin K$, $\text{supp}(\gamma) \cap Y_k^* \neq \emptyset$ for each $k \in K$ and there is $L \in \{2, \dots, \bar{L}\}$ such that $K = \{L\}$ or $K = \{L-1, L\}$,
2. For each $k \in K$ and $l \in \{1, \dots, k\}$, $W_{k,l}$ is a closed interval of Z such that $\min W_{k,l} < \max W_{k,l}$, $Z = \cup_{k \in K} \cup_{l \leq k} W_{k,l}$ and the following holds:

$$W_{L,l} < W_{L,l+1}, W_{L,l} < W_{L-1,l} \text{ and } W_{L-1,l} < W_{L,l+1} \text{ for each } 1 \leq l \leq L-1,$$

with the convention that $W_{L-1,l} = \emptyset$ for each $1 \leq l \leq L-1$ when $K = \{L\}$,

3. For each $k \in K$ and $l \in \{1, \dots, k-1\}$, $\phi_{k,l} : M_k \rightarrow W_{k,l}$ is strictly increasing, differentiable and such that, for each $z \in M_k$,

$$\phi'_{k,l}(z) = \frac{n_l(\sigma_k(z))\theta(z)}{\theta(\phi_{k,l}(z))},$$

where, for each $k \in K$, $\sigma_k : M_k \rightarrow Y_k^*$ is defined by setting, for each $z \in M_k$,

$$\sigma_k(z) = (\phi_{k,1}(z), \dots, \phi_{k,k-1}(z), z),$$

4. $\gamma = \nu \circ \sigma^{-1}$, where $\sigma : M \rightarrow Y^*$ is defined by setting $\sigma(z) = \sigma_L(z)$ if $z \in M_L$ and $\sigma(z) = \sigma_{L-1}(z)$ if $z \in M_{L-1} \setminus M_L$,

5. u is continuous, strictly increasing and, for each $k \in K$ and $l \in \{1, \dots, k\}$, $u_{k,l} = u|_{W_{k,l}}$ is differentiable and such that, for each $z \in W_{k,l}$,

$$u'_{k,l}(z) = \frac{1}{n_l(y)} \left(\frac{\partial F(y)}{\partial z_l} - \sum_{i=1}^{k-1} u(z_i(y)) \frac{\partial n_i(y)}{\partial z_l} \right),$$

where $y = \sigma_k(\phi_{k,l}^{-1}(z))$ if $l < k$ and $y = \sigma_k(z)$ if $l = k$.

The proof consists mostly of appropriate manipulations of the conditions defining stability, which can be thought of as the first-order conditions of the surplus maximization problems. It is therefore quite standard and, apart from two technical lemmas (6.1 and 9.1), also elementary.

9.1 A Technical Lemma

Let (γ, u) be a stable outcome of E^* . We start with a technical lemma. It shows that each firm $y = (z_1, \dots, z_L)$ in the support of γ is the limit of firms, also in the support of γ , whose knowledge types differ from those of y in each coordinate, i.e. each of its knowledge types belongs to the complement of $\{z_1, \dots, z_L\}$. The statement is slightly more general in that any countable set Q can take the place of $\{z_1, \dots, z_L\}$. The lemma is a consequence of the nonatomicity of the knowledge distribution ν .

Lemma 9.1 *If $Q \subseteq Z$ is countable and $y \in \text{supp}(\gamma)$, then there exists a sequence $\{y_k\}_{k=1}^\infty$ such that $y_k \rightarrow y$ and, for each $k \in \mathbb{N}$, $\ell(y_k) = \ell(y)$, $y_k \in \text{supp}(\gamma)$ and $y_{k,i} \notin Q$ for each $1 \leq i \leq \ell(y)$.*

Proof. For each $1 \leq i \leq \ell(y)$, let $Q_i = \prod_{j=1}^{\ell(y)} A_j$, where $A_j = Q$ if $j = i$ and $A_j = Z$ otherwise. Assume that there is $\varepsilon > 0$ such that $\text{supp}(\gamma) \cap B_\varepsilon(y) \subseteq \bigcup_{i=1}^{\ell(y)} Q_i$. We have that, for each $1 \leq i \leq \ell(y)$,

$$\begin{aligned} 0 = \nu(Q) &\geq \int_{Y_{\ell(y)}} 1_Q(z_i(x)) n_i(x) d\gamma(x) \geq \\ &\underline{n} \int_{Y_{\ell(y)}} 1_Q(z_i(x)) d\gamma(x) = \underline{n} \gamma(Q_i) \geq 0, \end{aligned}$$

hence, $\gamma(Q_i) = 0$. This then implies that $\gamma(\text{supp}(\gamma) \cap B_\varepsilon(y)) = 0$. But this is a contradiction to the definition of $\text{supp}(\gamma)$ since $\text{supp}(\gamma) \setminus B_\varepsilon(y)$ is closed and strictly contained in $\text{supp}(\gamma)$.

Thus, it follows that, for each $\varepsilon > 0$, there exists $y_\varepsilon \in \text{supp}(\gamma) \cap B_\varepsilon(y) \cap \left(\bigcup_{i=1}^{\ell(y)} Q_i\right)^c$. There is $\bar{\varepsilon} > 0$ such that $B_\varepsilon(y) \subseteq Y_{\ell(y)}$ for each $0 < \varepsilon < \bar{\varepsilon}$ since $Y_{\ell(y)}$ is open, hence $B_\varepsilon(y) \cap \left(\bigcup_{i=1}^{\ell(y)} Q_i\right)^c = B_\varepsilon(y) \cap \left(\bigcap_{i=1}^{\ell(y)} (Y_{\ell(y)} \setminus Q_i)\right) = B_\varepsilon(y) \cap (Q^c \times \dots \times Q^c)$ for each $0 < \varepsilon < \bar{\varepsilon}$. ■

9.2 Increasing earnings, accordion property and $(L, L-1)$ result

This section contains the building blocks of our argument. It is convenient to express the stability conditions normalized by the number of workers (i.e. members of layer 1). For each $y \in Y^*$, let

$$S(y) = u(z_1(y)) + \sum_{i=2}^{\ell(y)} h(1 - z_{i-1}(y))u(z_i(y))$$

be the normalized wage bill. The stability of γ is then equivalent to

$$S(y) = z_{\ell(y)}(y) \text{ for each } y \in \text{supp}(\gamma) \text{ and} \quad (9.1)$$

$$S(y) \geq z_{\ell(y)}(y) \text{ for each } y \in Y^*. \quad (9.2)$$

Note that (9.2) with $y = z \in [\underline{z}, \bar{z}]$ implies that

$$u(z) \geq z > 0 \text{ for each } z \in Z. \quad (9.3)$$

The first lemma of this section shows that knowledge is increasing across layers. Indeed, if $z_{i-1} \geq z_i$ for some $2 \leq i \leq L$, then the firm obtained by removing layer i would have strictly positive profits.

Lemma 9.2 *If $(z_1, \dots, z_L) \in \text{supp}(\gamma)$, then $z_1 < \dots < z_L$.*

Proof. Suppose not; then there is $i \in \{2, \dots, L\}$ such that $z_{i-1} \geq z_i$. Let $z = (z_1, \dots, z_L)$ and $x \in Z^{L-1}$ be defined by setting, for each $j \in \{1, \dots, L-1\}$,

$$x_j = \begin{cases} z_j & \text{if } j < i, \\ z_{j+1} & \text{if } j \geq i. \end{cases}$$

If $i < L$, then (9.1) and (9.2) imply that $S(x) \geq z_L = S(z)$, hence, by (9.3) and

$$z_{i-1} \geq z_i,$$

$$\begin{aligned} 0 &\geq S(z) - S(x) = u(z_1) + \sum_{j=2}^{i-1} h(1 - z_{j-1})u(z_j) \\ &\quad + h(1 - z_{i-1})u(z_i) + h(1 - z_i)u(z_{i+1}) + \sum_{j=i+2}^L h(1 - z_{j-1})u(z_j) \\ &\quad - \left(u(z_1) + \sum_{j=2}^{i-1} h(1 - z_{j-1})u(z_j) + h(1 - z_{i-1})u(z_{i+1}) + \sum_{j=i+2}^L h(1 - z_{j-1})u(z_j) \right) \\ &= u(z_{i+1})h(z_{i-1} - z_i) + u(z_i)h(1 - z_{i-1}) > 0, \end{aligned}$$

a contradiction.

If $i = L$, then (9.1) and (9.2) imply that $S(x) \geq z_{L-1}$ and $S(z) = z_L$, hence, by (9.3) and $z_{L-1} \geq z_L$, $0 > -h(1 - z_{L-1})u(z_L) = S(x) - S(z) \geq z_{L-1} - z_L \geq 0$, a contradiction. ■

The next lemma shows earnings are strictly increasing in knowledge. Indeed, if $z > z'$ and $u(z) \leq u(z')$, then a firm employing individuals of type z' could profitably replace them with those with type z .

Lemma 9.3 *If $z, z' \in Z$ are such that $z > z'$, then $u(z) > u(z')$.*

Proof. Let, by Lemma 6.1, $y = (y_1, \dots, y_L) \in \text{supp}(\gamma)$ and $i \in \{1, \dots, L\}$ be such that $y_i = z'$. If $L = 1$, then stability implies that $u(z') = z'$ and $u(z) \geq z$, hence $u(z) \geq z > z' = u(z')$. Thus, we may assume that $L > 1$.

Let $x \in Z^L$ be defined by setting, for each $j \in \{1, \dots, L\}$, $x_j = z$ if $j = i$ and $x_j = y_j$ if $j \neq i$. Then, $S(x) - S(y) = n_i(y)(u(z) - u(z'))/n_1(y) + u(y_{i+1})h(z' - z)$, with the convention that $u(y_{L+1}) = 0$. Hence, (9.1) and (9.2) imply that

$$n_i(y)(u(z) - u(z')) + n_1(y)u(y_{i+1})h(z' - z) \geq (x_L - y_L)n_1(y).$$

If $i = L$, then we have $n_i(y)(u(z) - u(z')) \geq (z - z')n_1(y) > 0$; hence $u(z) > u(z')$. If $i < L$, then we have $n_i(y)(u(z) - u(z')) \geq n_1(y)u(y_{i+1})h(z - z') > 0$ by (9.3); hence $u(z) > u(z')$. ■

The following lemmas establish what Chen and Suen (2019) called the accordion property, which establishes a relationship between the knowledge levels of two active

firms (i.e. in the support of γ). We illustrate with an example: if $(z_1, \dots, z_4)$ and $(x_1, \dots, x_3)$ are in the support of γ and $z_4 > x_3$, then $z_4 > x_3 \geq z_3 \geq x_2 \geq z_2 \geq x_1 \geq z_1$. This means that the knowledge of someone in the top layer of an active firm is always at least as high as the knowledge of someone in the layer just below the top and so on.

Lemmas 9.4 and 9.5 show one part of the above statement, namely, if $(z_1, \dots, z_L)$ and $(x_1, \dots, x_K)$ are two active firms and the former has a more knowledgeable top manager (i.e. $z_L > x_K$), then $z_{L-l} \geq x_{K-l}$, i.e. the same holds when comparing layers in the two firms that have the same distance to the top. The idea is as follows: if $x_{K-l} > z_{L-l}$, then the firms $(x_1, \dots, x_{K-l}, z_{L-l+1}, \dots, z_L)$ and $(z_1, \dots, z_{L-l}, x_{K-l+1}, \dots, x_K)$ are such that the former has more knowledge both at the top and at distance l to the top, and at least one of them has strictly positive profits. The proof is by induction, with Lemma 9.4 dealing with the initial step and Lemma 9.5 with the inductive step.

Lemma 9.4 *If $(z_1, \dots, z_L), (x_1, \dots, x_K) \in \text{supp}(\gamma)$, $L > 1$, $K > 1$ and $z_L > x_K$, then $z_{L-1} \geq x_{K-1}$.*

Proof. Let $z = (z_1, \dots, z_L)$, $x = (x_1, \dots, x_K)$, $\tilde{z} = (x_1, \dots, x_{K-1}, z_L) \in Z^K$ and $\tilde{x} = (z_1, \dots, z_{L-1}, x_K) \in Z^L$. Then (9.1) and (9.2) imply that $S(z) = z_L$, $S(x) = x_K$, $S(\tilde{z}) \geq z_L$ and $S(\tilde{x}) \geq x_K$. Thus, $S(\tilde{z}) + S(\tilde{x}) \geq S(z) + S(x) \Leftrightarrow (S(\tilde{z}) - S(x)) - (S(z) - S(\tilde{x})) \geq 0$. Since $S(\tilde{z}) - S(x) = (u(z_L) - u(x_K))h(1 - x_{K-1})$ and $S(z) - S(\tilde{x}) = (u(z_L) - u(x_K))h(1 - z_{L-1})$, it follows that $(u(z_L) - u(x_K))h(z_{L-1} - x_{K-1}) \geq 0$. Since Lemma 9.3 implies that $u(z_L) - u(x_K) > 0$, it then follows that $z_{L-1} \geq x_{K-1}$. ■

Lemma 9.5 *If $(z_1, \dots, z_L), (x_1, \dots, x_K) \in \text{supp}(\gamma)$ and $z_L > x_K$, then $z_{L-l} \geq x_{K-l}$ for each $1 \leq l \leq J$, where $J = \min\{K, L\} - 1$.*

Proof. The conclusion holds when $l = 1$ by Lemma 9.4. Let $1 < l \leq J$ and assume that $z_{L-j} \geq x_{K-j}$ for each $1 \leq j < l$ and $(z_1, \dots, z_L), (x_1, \dots, x_K) \in \text{supp}(\gamma)$ such that $z_L > x_K$. We show that $z_{L-l} \geq x_{K-l}$ for each $(z_1, \dots, z_L), (x_1, \dots, x_K) \in \text{supp}(\gamma)$ such that $z_L > x_K$.

Let $(z_1, \dots, z_L), (x_1, \dots, x_K) \in \text{supp}(\gamma)$ be such that $z_L > x_K$. We have that $z_{L-l+1} \geq x_{K-l+1}$ by the induction hypothesis; we consider two cases: (i) $z_{L-l+1} > x_{K-l+1}$ and (ii) $z_{L-l+1} = x_{K-l+1}$.

Case (i): Let $z = (z_1, \dots, z_L), x = (x_1, \dots, x_K), \tilde{z} = (x_1, \dots, x_{K-l}, z_{L-l+1}, \dots, z_L) \in Z^K$ and $\tilde{x} = (z_1, \dots, z_{L-l}, x_{K-l+1}, \dots, x_K) \in Z^L$. Then (9.1) and (9.2) imply that $S(z) = z_L, S(x) = x_K, S(\tilde{z}) \geq z_L$ and $S(\tilde{x}) \geq x_K$. Thus, $S(\tilde{z}) + S(\tilde{x}) \geq S(z) + S(x) \Leftrightarrow S(\tilde{z}) + S(\tilde{x}) - S(z) - S(x) \geq 0$. Since

$$\begin{aligned}
& S(\tilde{z}) + S(\tilde{x}) - S(z) - S(x) = \\
& u(x_1) + \sum_{i=2}^{K-l} h(1 - x_{i-1})u(x_i) + h(1 - x_{K-l})u(z_{L-l+1}) + \sum_{i=L-l+2}^L h(1 - z_{i-1})u(z_i) \\
& + u(z_1) + \sum_{i=2}^{L-l} h(1 - z_{i-1})u(z_i) + h(1 - z_{L-l})u(x_{K-l+1}) + \sum_{i=K-l+2}^K h(1 - x_{i-1})u(x_i) \\
& - \left(u(z_1) + \sum_{i=2}^L h(1 - z_{i-1})u(z_i) + u(x_1) + \sum_{i=2}^K h(1 - x_{i-1})u(x_i) \right) \\
& = h(1 - z_{L-l})u(x_{K-l+1}) + h(1 - x_{K-l})u(z_{L-l+1}) \\
& - h(1 - z_{L-l})u(z_{L-l+1}) - h(1 - x_{K-l})u(x_{K-l+1}) \\
& = h(z_{L-l} - x_{K-l})(u(z_{L-l+1}) - u(x_{K-l+1})),
\end{aligned}$$

it follows that $h(z_{L-l} - x_{K-l})(u(z_{L-l+1}) - u(x_{K-l+1})) \geq 0$. Since Lemma 9.3 implies that $u(z_{L-l+1}) - u(x_{K-l+1}) > 0$, it then follows that $z_{L-l} \geq x_{K-l}$.

Case (ii): Letting $Q = \{x_1, \dots, x_K\}$ and $z = (z_1, \dots, z_L)$, Lemma 9.1 implies that there exists a sequence $\{z_m\}_{m=1}^\infty$ such that $z_m \rightarrow z$ and, for each $m \in \mathbb{N}$, $\ell(z_m) = L$, $z_m \in \text{supp}(\gamma)$ and $z_{m,i} \notin Q$ for each $1 \leq i \leq L$. Then, for each $m \in \mathbb{N}$ sufficiently large, $z_{m,L} > x_K$ and $z_{m,L-l+1} > x_{K-l+1}$, the latter inequality since $z_{m,L-l+1} \geq x_{K-l+1}$ and $z_{m,L-l+1} \notin Q$. It follows by case (i) that $z_{m,L-l} \geq x_{K-l}$ and, since $z_m \rightarrow z$, that $z_{L-l} \geq x_{K-l}$. ■

The following lemma shows that firms with better top managers have at least the same number of layers as those with worse managers. The idea is as follows: if $(z_1, \dots, z_L)$ and $(x_1, \dots, x_K)$ are two active firms, the former has a more knowledgeable top manager (i.e. $z_L > x_K$) but $L < K$, then the firms $(x_1, \dots, x_{K-L}, z_1, \dots, z_L)$

and $(x_{K-L+1}, \dots, x_K)$ are such that the former has more knowledge at the top and has the highest number of layers, and at least one of them has strictly positive profits.

Lemma 9.6 *If $(z_1, \dots, z_L), (x_1, \dots, x_K) \in \text{supp}(\gamma)$ and $z_L > x_K$, then $L \geq K$.*

Proof. Suppose not; then $K > L$. Lemma 9.5 implies that $z_1 \geq x_{K-L+1}$ (let $l = L - 1$). By Lemma 9.1, we may assume that $z_1 > x_{K-L+1}$.

Let $z = (z_1, \dots, z_L)$, $x = (x_1, \dots, x_K)$, $\tilde{z} = (x_1, \dots, x_{K-L}, z_1, \dots, z_L) \in Z^K$ and $\tilde{x} = (x_{K-L+1}, \dots, x_K) \in Z^L$. Then (9.1) and (9.2) imply that $S(z) = z_L$, $S(x) = x_K$, $S(\tilde{z}) \geq z_L$ and $S(\tilde{x}) \geq x_K$. Thus, $S(\tilde{z}) + S(\tilde{x}) \geq S(z) + S(x) \Leftrightarrow S(\tilde{z}) + S(\tilde{x}) - S(z) - S(x) \geq 0$. Since

$$\begin{aligned} S(\tilde{z}) + S(\tilde{x}) - S(z) - S(x) &= \\ u(x_1) + \sum_{i=2}^{K-L} h(1 - x_{i-1})u(x_i) + h(1 - x_{K-L})u(z_1) + \sum_{i=2}^L h(1 - z_{i-1})u(z_i) \\ &+ u(x_{K-L+1}) + \sum_{i=K-L+2}^K h(1 - x_{i-1})u(x_i) \\ &- \left(u(z_1) + \sum_{i=2}^L h(1 - z_{i-1})u(z_i) + u(x_1) + \sum_{i=2}^K h(1 - x_{i-1})u(x_i) \right) \\ &= h(1 - x_{K-L})u(z_1) + u(x_{K-L+1}) - u(z_1) - h(1 - x_{K-L})u(x_{K-L+1}) \\ &= -(1 - h(1 - x_{K-L}))(u(z_1) - u(x_{K-L+1})), \end{aligned}$$

it follows that $0 \geq (1 - h(1 - x_{K-L}))(u(z_1) - u(x_{K-L+1}))$. Since Lemma 9.3 implies that $u(z_1) - u(x_{K-L+1}) > 0$, it then follows that $0 \geq (1 - h(1 - x_{K-L}))(u(z_1) - u(x_{K-L+1})) > 0$, a contradiction. ■

Lemmas 9.8 and 9.9 show the remaining part of the accordion property, namely, if $(z_1, \dots, z_L)$ and $(x_1, \dots, x_K)$ are two active firms and the former has a more knowledgeable top manager (i.e. $z_L > x_K$), then $z_{L-l} \leq x_{K-l+1}$, i.e. those who are closer to the top are at least as knowledgeable as those who are further away when comparing any active firms. The idea is as follows: if $z_{L-l} > x_{K-l+1}$, then the firms $(x_1, \dots, x_{K-l+1}, z_{L-l+1}, \dots, z_L)$ and $(z_1, \dots, z_{L-l}, x_{K-l+2}, \dots, x_K)$ are such that the former has more knowledge at the top, those at distance l to the top have less knowledge (x_{K-l+1}) than those at distance $l + 1$ from the top in the latter firm (z_{L-l}), and

at least one of them has strictly positive profits. The proof is by induction, with Lemma 9.8 dealing with the initial step and Lemma 9.9 with the inductive step.

The case where $K = \bar{L}$ requires a different argument since, in this case, the firm $(x_1, \dots, x_{K-l+1}, z_{L-l+1}, \dots, z_L)$ considered above has $K + 1$ layers and, hence, is not feasible. To deal with this case, we consider, for each $x = (x_1, \dots, x_{\bar{L}})$ and $z = (z_1, \dots, z_{\bar{L}})$ in $\text{supp}(\gamma) \cap Z^{\bar{L}}$ and $1 \leq a \leq b \leq \bar{L} - 1$, the two firms obtained by exchanging the block $(z_a, \dots, z_b)$ of z with the block $(x_{a+1}, \dots, x_{b+1})$ of x :

$$P_{a,b} = (x_1, \dots, x_a, z_a, \dots, z_b, x_{b+2}, \dots, x_{\bar{L}}),$$

$$Q_{a,b} = (z_1, \dots, z_{a-1}, x_{a+1}, \dots, x_{b+1}, z_{b+1}, \dots, z_{\bar{L}}).$$

Then (9.1) and (9.2) imply that, if $b < \bar{L} - 1$, $S(P_{a,b}) + S(Q_{a,b}) \geq x_{\bar{L}} + z_{\bar{L}}$, $S(x) + S(z) = x_{\bar{L}} + z_{\bar{L}}$ and, hence,

$$S(P_{a,b}) + S(Q_{a,b}) - S(x) - S(z) \geq 0; \quad (9.4)$$

if $b = \bar{L} - 1$, they imply that $S(P_{a,b}) + S(Q_{a,b}) \geq z_{\bar{L}-1} + z_{\bar{L}}$, $S(x) + S(z) = x_{\bar{L}} + z_{\bar{L}}$ and, hence,

$$S(P_{a,b}) + S(Q_{a,b}) - S(x) - S(z) \geq z_{\bar{L}-1} - x_{\bar{L}}. \quad (9.5)$$

The following lemma characterizes (9.4) and (9.5). Define

$$\alpha_a = \begin{cases} h(u(z_a) - u(x_{a+1}))(z_{a-1} - x_a) & \text{if } 2 \leq a \leq \bar{L} - 1, \\ (1 - h(1 - x_1))(u(x_2) - u(z_1)) & \text{if } a = 1, \end{cases}$$

$$\beta_b = \begin{cases} h(z_b - x_{b+1})(u(z_{b+1}) - u(x_{b+2})) & \text{if } 1 \leq b \leq \bar{L} - 2, \\ -(z_{\bar{L}-1} - x_{\bar{L}})(1 - hu(z_{\bar{L}})) & \text{if } b = \bar{L} - 1. \end{cases}$$

Lemma 9.7 *For each $(z_1, \dots, z_{\bar{L}}), (x_1, \dots, x_{\bar{L}}) \in \text{supp}(\gamma)$ and $1 \leq a \leq b \leq \bar{L} - 1$, $\alpha_a + \beta_b \geq 0$ if and only if (9.4) holds if $b < \bar{L} - 1$ and (9.5) holds if $b = \bar{L} - 1$.*

Proof. Write $P = P_{a,b}$ and $Q = Q_{a,b}$ and suppose first that $b \leq \bar{L} - 2$. If $a \geq 2$,

then

$$\begin{aligned}
S(P) + S(Q) - S(x) - S(z) &= h(z_{a-1} - x_a)(u(z_a) - u(x_{a+1})) \\
&\quad + h(z_b - x_{b+1})(u(z_{b+1}) - u(x_{b+2})) \\
&= \alpha_a + \beta_b.
\end{aligned}$$

If $a = 1$, then

$$\begin{aligned}
S(P) + S(Q) - S(x) - S(z) &= (1 - h(1 - x_1))(u(x_2) - u(z_1)) \\
&\quad + h(z_b - x_{b+1})(u(z_{b+1}) - u(x_{b+2})) \\
&= \alpha_a + \beta_b.
\end{aligned}$$

Hence, (9.4) holds if and only if $\alpha_a + \beta_b \geq 0$ for each $1 \leq a \leq b \leq \bar{L} - 2$.

Suppose next that $b = \bar{L} - 1$. If $a \geq 2$, then

$$\begin{aligned}
S(P) + S(Q) - S(x) - S(z) &= h(z_{a-1} - x_a)(u(z_a) - u(x_{a+1})) \\
&\quad + h(z_{\bar{L}-1} - x_{\bar{L}})u(z_{\bar{L}}) \\
&= \alpha_a + h(z_{\bar{L}-1} - x_{\bar{L}})u(z_{\bar{L}}).
\end{aligned}$$

If $a = 1$, then

$$\begin{aligned}
S(P) + S(Q) - S(x) - S(z) &= (1 - h(1 - x_1))(u(x_2) - u(z_1)) \\
&\quad + h(z_{\bar{L}-1} - x_{\bar{L}})u(z_{\bar{L}}) \\
&= \alpha_a + h(z_{\bar{L}-1} - x_{\bar{L}})u(z_{\bar{L}}).
\end{aligned}$$

Hence, (9.5) holds if and only if $\alpha_a + h(z_{\bar{L}-1} - x_{\bar{L}})u(z_{\bar{L}}) \geq z_{\bar{L}-1} - x_{\bar{L}}$, i.e. $\alpha_a + \beta_b \geq 0$ for each $1 \leq a \leq b$ and $b = \bar{L} - 1$. ■

In the case where $K = \bar{L}$, we establish Lemma 9.8 by showing that if $z_{\bar{L}-1} > x_{\bar{L}}$, then, for some $1 \leq a \leq \bar{L} - 1$, $P_{a,\bar{L}-1}$ or $Q_{a,\bar{L}-1}$ have strictly positive profits.

Lemma 9.8 *If $L > 1$, $(z_1, \dots, z_L), (x_1, \dots, x_K) \in \text{supp}(\gamma)$ and $z_L > x_K$, then $x_K \geq z_{L-1}$.*

Proof. Assume first that $K < \bar{L}$ and let $z = (z_1, \dots, z_L)$, $x = (x_1, \dots, x_K)$, $\tilde{z} = (x_1, \dots, x_K, z_L) \in Z^{K+1}$ and $\tilde{x} = (z_1, \dots, z_{L-1}) \in Z^{L-1}$. Then (9.1) and (9.2) imply that $S(z) = z_L$, $S(x) = x_K$, $S(\tilde{z}) \geq z_L$ and $S(\tilde{x}) \geq z_{L-1}$. Since $S(\tilde{z}) = S(x) + h(1 - x_K)u(z_L)$ and $S(\tilde{x}) = S(z) - h(1 - z_{L-1})u(z_L)$, then

$$x_K + h(1 - x_K)u(z_L) \geq z_L \text{ and } z_L - h(1 - z_{L-1})u(z_L) \geq z_{L-1},$$

hence, $(z_L - x_K)(1 - z_{L-1}) \leq h(1 - x_K)(1 - z_{L-1})u(z_L) \leq (z_L - z_{L-1})(1 - x_K)$. It then follows that

$$(x_K - z_{L-1})(1 - z_L) = (z_L - z_{L-1})(1 - x_K) - (z_L - x_K)(1 - z_{L-1}) \geq 0$$

and, hence, $x_K \geq z_{L-1}$.

Next, assume that $K = \bar{L}$. By Lemma 9.6, we have $\bar{L} = K \leq L \leq \bar{L}$; hence, $K = L = \bar{L}$. Suppose for a contradiction that $z_{\bar{L}-1} > x_{\bar{L}}$. Let $z = (z_1, \dots, z_{\bar{L}})$ and $\tilde{z} = (z_1, \dots, z_{\bar{L}-1})$. Then $S(z) = z_{\bar{L}}$, $S(\tilde{z}) \geq z_{\bar{L}-1}$ and $S(z) - S(\tilde{z}) = h(1 - z_{\bar{L}-1})u(z_{\bar{L}})$ imply that $h(1 - z_{\bar{L}-1})u(z_{\bar{L}}) \leq z_{\bar{L}} - z_{\bar{L}-1}$. Since $z_{\bar{L}} - z_{\bar{L}-1} < 1 - z_{\bar{L}-1}$, it follows that $hu(z_{\bar{L}}) < 1$. Thus, $\beta_{\bar{L}-1} = -(z_{\bar{L}-1} - x_{\bar{L}})(1 - hu(z_{\bar{L}})) < 0$.

Then (9.5) and Lemma 9.7 imply that $\alpha_a > 0$ for each $1 \leq a \leq \bar{L} - 1$. This, together with Lemma 9.3, then implies that $x_{a+1} > z_a$ for each $1 \leq a \leq \bar{L} - 1$: this is clear if $a = 1$ and, when $a > 1$, it follows by the inductive hypothesis $x_a > z_{a-1}$. But then the case $a = \bar{L} - 1$ implies $x_{\bar{L}} > z_{\bar{L}-1}$, a contradiction. ■

Lemma 9.9 *If $(z_1, \dots, z_L), (x_1, \dots, x_K) \in \text{supp}(\gamma)$ and $z_L > x_K$, then $z_{L-l} \leq x_{K-l+1}$ for each $1 \leq l \leq J$, where*

$$J = \begin{cases} L - 1 & \text{if } K \geq L, \\ K & \text{if } L > K. \end{cases}$$

Proof. The conclusion holds for $l = 1$ by Lemma 9.8. Let $1 < l \leq J$ and assume that $z_{L-j} \leq x_{K-j+1}$ for each $1 \leq j < l$ and $(z_1, \dots, z_L), (x_1, \dots, x_K) \in \text{supp}(\gamma)$ such that $z_L > x_K$. We show that $z_{L-l} \leq x_{K-l+1}$ for each $(z_1, \dots, z_L), (x_1, \dots, x_K) \in \text{supp}(\gamma)$ such that $z_L > x_K$.

Assume first that $K < \bar{L}$. Let $(z_1, \dots, z_L), (x_1, \dots, x_K) \in \text{supp}(\gamma)$ be such that $z_L > x_K$. We have that $x_{K-l+2} \geq z_{L-l+1}$ by the induction hypothesis; we consider two cases: (i) $x_{K-l+2} > z_{L-l+1}$ and (ii) $x_{K-l+2} = z_{L-l+1}$.

Case (i): Let $z = (z_1, \dots, z_L)$, $x = (x_1, \dots, x_K)$, $\tilde{z} = (x_1, \dots, x_{K-l+1}, z_{L-l+1}, \dots, z_L) \in Z^{K+1}$ and $\tilde{x} = (z_1, \dots, z_{L-l}, x_{K-l+2}, \dots, x_K) \in Z^{L-1}$. Then (9.1) and (9.2) imply that $S(z) = z_L$, $S(x) = x_K$, $S(\tilde{z}) \geq z_L$ and $S(\tilde{x}) \geq x_K$. Thus, $S(\tilde{z}) + S(\tilde{x}) \geq S(z) + S(x) \Leftrightarrow S(\tilde{z}) + S(\tilde{x}) - S(z) - S(x) \geq 0$. Since

$$\begin{aligned} & S(\tilde{z}) + S(\tilde{x}) - S(z) - S(x) = \\ & u(x_1) + \sum_{i=2}^{K-l+1} h(1 - x_{i-1})u(x_i) + h(1 - x_{K-l+1})u(z_{L-l+1}) + \sum_{i=L-l+2}^L h(1 - z_{i-1})u(z_i) \\ & + u(z_1) + \sum_{i=2}^{L-l} h(1 - z_{i-1})u(z_i) + h(1 - z_{L-l})u(x_{K-l+2}) + \sum_{i=K-l+3}^K h(1 - x_{i-1})u(x_i) \\ & - \left(u(z_1) + \sum_{i=2}^L h(1 - z_{i-1})u(z_i) + u(x_1) + \sum_{i=2}^K h(1 - x_{i-1})u(x_i) \right) \\ & = h(1 - z_{L-l})(u(x_{K-l+2}) - u(z_{L-l+1})) + h(1 - x_{K-l+1})(u(z_{L-l+1}) - u(x_{K-l+2})) \\ & = h(x_{K-l+1} - z_{L-l})(u(x_{K-l+2}) - u(z_{L-l+1})), \end{aligned}$$

it follows that $h(x_{K-l+1} - z_{L-l})(u(x_{K-l+2}) - u(z_{L-l+1})) \geq 0$. Since Lemma 9.3 implies that $u(x_{K-l+2}) - u(z_{L-l+1}) > 0$, it then follows that $x_{K-l+1} \geq z_{L-l}$.

Case (ii): Letting $Q = \{x_1, \dots, x_K\}$ and $z = (z_1, \dots, z_L)$, Lemma 9.1 implies that there exists a sequence $\{z_m\}_{m=1}^\infty$ such that $z_m \rightarrow z$ and, for each $m \in \mathbb{N}$, $\ell(z_m) = L$, $z_m \in \text{supp}(\gamma)$ and $z_{m,i} \notin Q$ for each $1 \leq i \leq L$. Then, for each $m \in \mathbb{N}$ sufficiently large, $z_{m,L} > x_K$ and $x_{K-l+2} > z_{m,L-l+1}$, the latter inequality since $x_{K-l+2} \geq z_{m,L-l+1}$ and $z_{m,L-l+1} \notin Q$. It follows by case (i) that $z_{m,L-l} \leq x_{K-l+1}$ and, since $z_m \rightarrow z$, that $z_{L-l} \leq x_{K-l+1}$.

Finally, we consider the case $K = \bar{L}$. By Lemma 9.6, we have $\bar{L} = K \leq L \leq \bar{L}$; hence, $K = L = \bar{L}$. Suppose for a contradiction that there is $(z_1, \dots, z_{\bar{L}}) \in \text{supp}(\gamma)$ and $(x_1, \dots, x_{\bar{L}}) \in \text{supp}(\gamma)$ such that $z_{\bar{L}} > x_{\bar{L}}$ and $z_{\bar{L}-l} > x_{\bar{L}-l+1}$. Let $b = \bar{L} - l \in \{1, \dots, \bar{L} - 2\}$; then $z_b > x_{b+1}$ and, by the inductive hypothesis, $z_{b+1} \leq x_{b+2}$. By Lemma 9.1, there is a sequence $\{z_m\}_{m=1}^\infty$ such that $z_m \rightarrow z$ and, for each $m \in \mathbb{N}$,

$z_m \in \text{supp}(\gamma) \cap Z^{\bar{L}}$ and $z_{m,i} \notin \{x_1, \dots, x_{\bar{L}}\}$ for each $1 \leq i \leq \bar{L}$; for all m sufficiently large, $z_{m,\bar{L}} > x_{\bar{L}}$, $z_{m,b} > x_{b+1}$ and $z_{m,b+1} < x_{b+2}$. Replacing z by z_m , we may therefore assume $z_{b+1} < x_{b+2}$. Then $u(z_{b+1}) < u(x_{b+2})$ by Lemma 9.3, so

$$\beta_b = h(z_b - x_{b+1})(u(z_{b+1}) - u(x_{b+2})) < 0.$$

Then (9.4) and Lemma 9.7 imply that $\alpha_a > 0$ for each $1 \leq a \leq b$. This, together with Lemma 9.3, then implies that $x_{a+1} > z_a$ for each $1 \leq a \leq b$: this is clear if $a = 1$ and, when $a > 1$, it follows by the inductive hypothesis $x_a > z_{a-1}$. But then the case $a = b$ implies $x_{b+1} > z_b$, a contradiction. ■

Let $L = \max\{\ell(y) : y \in \text{supp}(\gamma)\}$. Thus, some active firms have L layers and all active firms have at most L layers. The following lemma is the so-called $(L, L - 1)$ result and shows that active firms have at least $L - 1$ layers. The logic is as follows: if $(z_1, \dots, z_L)$ and $(x_1, \dots, x_K)$ are active firms and $K < L - 1$, then the firms $(z_{L-K}, \dots, z_L)$ and $(z_1, \dots, z_{L-K-1}, x_1, \dots, x_K)$ satisfy the accordion property, the former has one more layer than $(x_1, \dots, x_K)$, the latter has one less layer than $(z_1, \dots, z_L)$, and at least one of them has strictly positive profits.

Lemma 9.10 $\ell(y) \geq L - 1$ for each $y \in \text{supp}(\gamma)$.

Proof. Suppose not; then there exists $(x_1, \dots, x_K) \in \text{supp}(\gamma)$ with $K < L - 1$. Let $(z_1, \dots, z_L) \in \text{supp}(\gamma)$, which exists given the definition of L . Lemma 9.6 implies that $z_L \geq x_K$ and Lemma 9.1 implies that we may assume that $z_L > x_K$. Then Lemma 9.9 implies that $x_1 \geq z_{L-K}$ (let $l = K$). By Lemma 9.1, we may assume that $x_1 > z_{L-K}$.

Let $z = (z_1, \dots, z_L)$, $x = (x_1, \dots, x_K)$, $\tilde{x} = (z_1, \dots, z_{L-K-1}, x_1, \dots, x_K) \in Z^{L-1}$ and $\tilde{z} = (z_{L-K}, \dots, z_L) \in Z^{K+1}$. Then (9.1) and (9.2) imply that $S(z) = z_L$, $S(x) = x_K$, $S(\tilde{z}) \geq z_L$ and $S(\tilde{x}) \geq x_K$. Thus, $S(\tilde{z}) + S(\tilde{x}) \geq S(z) + S(x) \Leftrightarrow S(\tilde{z}) + S(\tilde{x}) -$

$S(z) - S(x) \geq 0$. Since

$$\begin{aligned}
& S(\tilde{z}) + S(\tilde{x}) - S(z) - S(x) = \\
& u(z_{L-K}) + \sum_{i=L-K+1}^L h(1 - z_{i-1})u(z_i) \\
& + u(z_1) + \sum_{i=2}^{L-K-1} h(1 - z_{i-1})u(z_i) + h(1 - z_{L-K-1})u(x_1) + \sum_{i=2}^K h(1 - x_{i-1})u(x_i) \\
& - \left(u(z_1) + \sum_{i=2}^L h(1 - z_{i-1})u(z_i) + u(x_1) + \sum_{i=2}^K h(1 - x_{i-1})u(x_i) \right) \\
& = u(z_{L-K}) + h(1 - z_{L-K-1})u(x_1) - u(x_1) - h(1 - z_{L-K-1})u(z_{L-K}) \\
& = -(1 - h(1 - z_{L-K-1}))(u(x_1) - u(z_{L-K})),
\end{aligned}$$

it follows that $0 \geq (1 - h(1 - z_{L-K-1}))(u(x_1) - u(z_{L-K}))$. Since Lemma 9.3 implies that $u(x_1) > u(z_{L-K})$, it follows that $0 \geq (1 - h(1 - z_{L-K-1}))(u(x_1) - u(z_{L-K})) > 0$, a contradiction. ■

Define $K = \ell(\text{supp}(\gamma))$; Lemma 9.10 implies that $K = \{L\}$ or $K = \{L-1, L\}$, i.e. all active firms have L layers or some active firms have $L-1$ layers. The final lemma of this section shows that $L \neq 1$. Indeed, if $L = 1$, then $K = \{1\}$ and everyone would be self-employed; but then the 2-layer firm $y = (\underline{z}, \bar{z})$ would have strictly positive profits.

Lemma 9.11 $L \geq 2$.

Proof. If not, then $L = 1$ and $u(z) = z$ for each $z \in Z$ by stability and Lemma 6.1. Letting $y = (\underline{z}, \bar{z})$, (2) implies that

$$F(y) - \sum_{i=1}^2 n_i(y)u(z_i(y)) = \frac{\bar{z} - \underline{z}}{h(1 - \underline{z})} - \bar{z} > 0,$$

a contradiction to stability since $y \in Y^*$. ■

9.3 Occupational Stratification

In this section we establish occupational stratification, i.e. prove property 2 in the statement of Theorem 9.1.

Note first that $\text{supp}(\gamma)$ is compact and that z_i is continuous, hence $W_{k,i}$ is compact for each $k \in K$ and $1 \leq i \leq k$. We start with the case $K = \{L\}$.

Lemma 9.12 *If $K = \{L\}$, then $W_{L,1} < \cdots < W_{L,L-1} < M_L$.*

Proof. For each $1 \leq l \leq L$, we will write W_l instead of $W_{L,l}$. Let $l \in \{1, \dots, L-1\}$; we show that $W_l < W_{l+1}$. Let $z \in W_l$ and $y \in \text{supp}(\gamma)$ be such that $z_l(y) = z$. If $z_{l+1}(y) = \min W_{l+1}$, then $z = z_l(y) < z_{l+1}(y) = \min W_{l+1}$ by Lemma 9.2. Hence, $\max W_l \leq \min W_{l+1}$.

Thus, we may assume that $z_{l+1}(y) > \min W_{l+1}$. Let $x \in \text{supp}(\gamma)$ be such that $z_{l+1}(x) = \min W_{l+1}$; hence $z_{l+1}(y) > z_{l+1}(x)$. It then follows by Lemma 9.5 that $z_L(y) \geq z_L(x)$ since, otherwise, $z_L(x) > z_L(y)$ and, hence, $z_{l+1}(x) \geq z_{l+1}(y)$. By Lemma 9.1, there exists $\{x_k\}_{k=1}^\infty$ such that $x_k \rightarrow x$, $\ell(x_k) = \ell(x) = L$, $x_k \in \text{supp}(\gamma)$ and $x_{k,i} \notin \{y_1, \dots, y_L\}$ for each $k \in \mathbb{N}$. Since $x_k \rightarrow x$, we may assume that $z_{l+1}(x_k) < z_{l+1}(y)$ for each $k \in \mathbb{N}$. Hence, for each $k \in \mathbb{N}$, $z_L(y) \geq z_L(x_k)$ as above; since $z_L(x_k) \notin \{y_1, \dots, y_L\} = \{z_1(y), \dots, z_L(y)\}$, it follows that $z_L(y) > z_L(x_k)$. Lemma 9.9 implies that $z = z_l(y) \leq z_{l+1}(x_k)$ for each $k \in \mathbb{N}$. Since $z_{l+1}(x_k) \rightarrow z_{l+1}(x) = \min W_{l+1}$, it follows that $z \leq \min W_{l+1}$. Thus, $\max W_l \leq \min W_{l+1}$. ■

Corollary 9.1 *If $K = \{L\}$, then $\underline{z} = \min W_{L,1}$, $\bar{z} = \max M_L$ and $\max W_{L,l} = \min W_{L,l+1}$ for each $l \in \{1, \dots, L-1\}$.*

Proof. This follows by Lemma 6.1 and Lemma 9.12. ■

Consider next the case where $K = \{L-1, L\}$.

Lemma 9.13 *If $K = \{L-1, L\}$, then $W_{L,l} < W_{L-1,l} < W_{L,l+1}$ for each $1 \leq l \leq L-1$, i.e. $W_{L,1} < W_{L-1,1} < W_{L,2} < \cdots < W_{L-1,L-2} < W_{L,L-1} < M_{L-1} < M_L$.*

Proof. We start by showing that $M_{L-1} < M_L$. Let $z = \min M_L$, $x = \max M_{L-1}$, $y \in \text{supp}(\gamma)$ such that $z_L(y) = z$ and $y' \in \text{supp}(\gamma)$ such that $z_{L-1}(y') = x$. Suppose that $x > z$; then Lemma 9.6 implies that $L-1 = \ell(y') \geq \ell(y) = L$, a contradiction. Thus, it follows that $\max M_{L-1} = x \leq z = \min M_L$.

We next show that $W_{L,l} < W_{L-1,l}$ for each $1 \leq l \leq L-1$. Let $l \in \{1, \dots, L-1\}$, $z = \min W_{L-1,l}$, $x = \max W_{L,l}$, $y \in \text{supp}(\gamma)$ such that $z_l(y) = x$ and $y' \in \text{supp}(\gamma)$ such that $z_l(y') = z$. Suppose that $x > z$ and let, by Lemma 9.1, $\{y_k\}_{k=1}^\infty$ be such that $y_k \rightarrow y$ and, for each $k \in \mathbb{N}$, $\ell(y_k) = \ell(y) = L$, $y_k \in \text{supp}(\gamma)$ and $y_{k,i} \notin \{y'_1, \dots, y'_{L-1}\} = \{z_1(y'), \dots, z_{L-1}(y')\}$ for each $1 \leq i \leq L$. Since $y_k \rightarrow y$ and $z_l(y) > z_l(y')$, we may assume that $z_l(y_k) > z_l(y')$ for each $k \in \mathbb{N}$. Let $j \in \mathbb{N}$ be such that $L-j = l$; hence, $L-1-j+1 = l$, $z_{L-j}(y_k) = z_l(y_k)$ for each $k \in \mathbb{N}$ and $z_{L-1-j+1}(y') = z_l(y')$.

Let $k \in \mathbb{N}$. If $z_L(y_k) > z_{L-1}(y')$, then Lemma 9.9 implies that

$$z_l(y') = z_{L-1-j+1}(y') \geq z_{L-j}(y_k) = z_l(y_k),$$

a contradiction to $z_l(y_k) > z_l(y')$. Hence, $z_L(y_k) \leq z_{L-1}(y')$ and, in fact, $z_L(y_k) < z_{L-1}(y')$ since $z_L(y_k) = y_{k,L} \notin \{z_1(y'), \dots, z_{L-1}(y')\}$. But then $\min M_L \leq z_L(y_k) < z_{L-1}(y') \leq \max M_{L-1}$, a contradiction to what we have already shown. This contradiction then shows that $\max W_{L,l} = x \leq z = \min W_{L-1,l}$.

We conclude the proof by showing that $W_{L-1,l} < W_{L,l+1}$ for each $1 \leq l \leq L-1$. Note that the case $l = L-1$ has already been established, hence let $l \in \{1, \dots, L-2\}$, $z = \min W_{L,l+1}$, $x = \max W_{L-1,l}$, $y \in \text{supp}(\gamma)$ such that $z_{l+1}(y) = z$ and $y' \in \text{supp}(\gamma)$ such that $z_l(y') = x$. Suppose that $x > z$ and let, by Lemma 9.1, $\{y_k\}_{k=1}^\infty$ be such that $y_k \rightarrow y$ and, for each $k \in \mathbb{N}$, $\ell(y_k) = \ell(y) = L$, $y_k \in \text{supp}(\gamma)$ and $y_{k,i} \notin \{y'_1, \dots, y'_{L-1}\} = \{z_1(y'), \dots, z_{L-1}(y')\}$ for each $1 \leq i \leq L$. Since $y_k \rightarrow y$ and $z_l(y') > z_{l+1}(y)$, we may assume that $z_l(y') > z_{l+1}(y_k)$ for each $k \in \mathbb{N}$. Let $j \in \mathbb{N}$ be such that $L-j = l+1$; hence, $L-1-j = l$, $z_{L-j}(y_k) = z_{l+1}(y_k)$ for each $k \in \mathbb{N}$ and $z_{L-1-j}(y') = z_l(y')$.

Let $k \in \mathbb{N}$. If $z_L(y_k) > z_{L-1}(y')$, then Lemma 9.5 implies that

$$z_l(y') = z_{L-1-j}(y') \leq z_{L-j}(y_k) = z_{l+1}(y_k),$$

a contradiction to $z_l(y') > z_{l+1}(y_k)$. Hence, $z_L(y_k) \leq z_{L-1}(y')$ and, in fact, $z_L(y_k) < z_{L-1}(y')$ since $z_L(y_k) = y_{k,L} \notin \{z_1(y'), \dots, z_{L-1}(y')\}$. But then $\min M_L \leq z_L(y_k) < z_{L-1}(y') \leq \max M_{L-1}$, a contradiction to what we have already shown. This contradiction then shows that $\max W_{L-1,l} = x \leq z = \min W_{L,l+1}$. ■

Corollary 9.2 *If $K = \{L - 1, L\}$, then $\underline{z} = \min W_{L,1}$, $\bar{z} = \max M_L$, $\max W_{L,l} = \min W_{L-1,l}$ and $\max W_{L-1,l} = \min W_{L,l+1}$ for each $1 \leq l \leq L - 1$.*

Proof. This follows by Lemma 6.1 and Lemma 9.13. ■

9.4 Assignment Function

This section shows part of property 3 in Theorem 9.1, namely that for each manager type $z \in M_k$, with $k \in K$, there exists a unique type $\phi_{k,l}(z)$ that occupies layer $l \in \{1, \dots, k - 1\}$ of the firm of which z is the top manager.

For each $k \in K$ and $l \in \{1, \dots, k - 1\}$, define $\phi_{k,l} : M_k \rightrightarrows W_{k,l}$ by setting, for each $z \in M_k$,

$$\phi_{k,l}(z) = \{z' \in Z : \text{there exists } y \in \text{supp}(\gamma) \text{ such that } (z, z') = (z_k(y), z_l(y))\}.$$

Then $\phi_{k,l}$ is nonempty-valued by the definition of M , $\phi_{k,l}(M_k) = W_{k,l}$ by the definition of $W_{k,l}$ and $\phi_{k,l}$ has a closed graph since $\text{supp}(\gamma)$ is compact and z_k and z_l are continuous.

Let $Q_{k,l} = \{z \in M_k : \phi_{k,l}(z) \text{ is not a singleton}\}$ for each $k \in K$ and $l \in \{1, \dots, k - 1\}$. The correspondence $\phi_{k,l}$ is increasing and, hence, the set $Q_{k,l}$ of (lower hemi) discontinuities of $\phi_{k,l}$ is countable.

Lemma 9.14 *For each $k \in K$ and $l \in \{1, \dots, k - 1\}$, $Q_{k,l}$ is countable.*

Proof. For each $z \in Q_{k,l}$, let $r(z) \in \mathbb{Q}$ be such that $\min \phi_{k,l}(z) < r(z) < \max \phi_{k,l}(z)$. This defines a function $r : Q_{k,l} \rightarrow \mathbb{Q}$ which, as we now claim, is strictly increasing. Indeed, if $z, \hat{z} \in Q_{k,l}$ are such that $z < \hat{z}$, then let $y, \hat{y} \in \text{supp}(\gamma)$ be such that $z_k(y) = z$, $z_l(y) = \max \phi_{k,l}(z)$, $z_k(\hat{y}) = \hat{z}$ and $z_l(\hat{y}) = \min \phi_{k,l}(\hat{z})$. Lemma 9.5 implies that $\max \phi_{k,l}(z) \leq \min \phi_{k,l}(\hat{z})$. Thus, $r(z) < \max \phi_{k,l}(z) \leq \min \phi_{k,l}(\hat{z}) < r(\hat{z})$. Thus, r maps $Q_{k,l}$ in a one-to-one way to a subset of $\mathbb{Q}$, implying that $Q_{k,l}$ is countable. ■

We next show that each worker type in $W_{k,l}$ is matched with a unique manager type in M_k . The reason is roughly that if worker type z' were matched with manager

types z^* and $\tilde{z}$, then the accordion property implies that type z' is matched with all types in $[z^*, \tilde{z}]$; furthermore, those types in $[z^*, \tilde{z}] \setminus Q_{k,l}$ are only matched with z' , which means that a zero measure of workers (those with type z') is matched with a positive measure of managers (those with type in $[z^*, \tilde{z}] \setminus Q_{k,l}$).

Lemma 9.15 *For each $k \in K$, $l \in \{1, \dots, k-1\}$ and $z \in W_{k,l}$, there exists $z^* \in M_k$ such that $\{\hat{z} \in M_k : z \in \phi_{k,l}(\hat{z})\} = \{z^*\}$.*

Proof. The definition of $W_{k,l}$ implies that $\{\hat{z} \in M_k : z \in \phi_{k,l}(\hat{z})\}$ is nonempty. Suppose that the conclusion of the lemma fails; then let $z \in W_{k,l}$ and $z^*, \tilde{z} \in M_k$ be such that $z^*, \tilde{z} \in \{\hat{z} \in M_k : z \in \phi_{k,l}(\hat{z})\}$ and $z^* < \tilde{z}$. Then $z^*, \tilde{z} \in M_k$ and $[z^*, \tilde{z}] \subseteq M_k$ by Lemmas 9.12 and 9.13. Furthermore, $[z^*, \tilde{z}] \subseteq \{\hat{z} \in M_k : z \in \phi_{k,l}(\hat{z})\}$ since if $x \in (z^*, \tilde{z})$ and $y^*, \tilde{y}, y \in \text{supp}(\gamma)$ are such that $z_k(y^*) = z^*$, $z_k(\tilde{y}) = \tilde{z}$, $z_k(y) = x$ and $z_l(y^*) = z_l(\tilde{y}) = z$, then Lemma 9.5 implies that $z = z_l(y^*) \leq z_l(y) \leq z_l(\tilde{y}) = z$, hence $z_l(y) = z$.

We have that $(z^*, \tilde{z}) \setminus Q_{k,l} \subseteq M_k \setminus (\cup_{j \in K} \cup_{i=1}^{j-1} W_{j,i})$, $(z^*, \tilde{z}) \setminus Q_{k,l} \subseteq Z \setminus M_j$ for each $j \in K \setminus \{k\}$ by Lemmas 9.12 and 9.13 and $\phi_{k,l}(x) = \{z\}$ for each $x \in (z^*, \tilde{z}) \setminus Q_{k,l}$. Since $Q_{k,l}$ is countable by Lemma 9.14,

$$\begin{aligned} 0 < \nu([z^*, \tilde{z}]) &= \nu((z^*, \tilde{z}) \setminus Q_{k,l}) = \int_{Z^k} 1_{(z^*, \tilde{z}) \setminus Q_{k,l}}(z_k(y)) n_k(y) d\gamma(y) \\ &= \int_{\text{supp}(\gamma) \cap (z_k^{-1}((z^*, \tilde{z}) \setminus Q_{k,l}))} 1 d\gamma(y) = \gamma(\text{supp}(\gamma) \cap (z_k^{-1}((z^*, \tilde{z}) \setminus Q_{k,l}))). \end{aligned}$$

But then

$$\begin{aligned} 0 = \nu(\{z\}) &\geq \int_{Y_k} 1_{\{z\}}(z_l(y)) n_l(y) d\gamma(y) \geq \underline{n} \gamma(\text{supp}(\gamma) \cap z_l^{-1}(\{z\})) \\ &\geq \underline{n} \gamma(\text{supp}(\gamma) \cap (z_k^{-1}((z^*, \tilde{z}) \setminus Q_{k,l}))) > 0, \end{aligned}$$

a contradiction. ■

It then follows that matching is strictly positive assortative in the sense that better managers have better workers in every layer.

Lemma 9.16 *For each $k \in K$, $l \in \{1, \dots, k-1\}$ and $z, \hat{z} \in M_k$, if $z > \hat{z}$, then $\min \phi_{k,l}(z) > \max \phi_{k,l}(\hat{z})$.*

Proof. Let $y, \hat{y} \in \text{supp}(\gamma)$ be such that $z_k(y) = z$, $z_l(y) = \min \phi_{k,l}(z)$, $z_k(\hat{y}) = \hat{z}$ and $z_l(\hat{y}) = \max \phi_{k,l}(\hat{z})$. Lemma 9.5 implies that $\min \phi_{k,l}(z) \geq \max \phi_{k,l}(\hat{z})$ and Lemma 9.15 that $\min \phi_{k,l}(z) \neq \max \phi_{k,l}(\hat{z})$. Thus, $\min \phi_{k,l}(z) > \max \phi_{k,l}(\hat{z})$. ■

Lemma 9.16 then implies that $\phi_{k,l}$ is a function, i.e. $Q_{k,l} = \emptyset$. This happens because if manager type z were matched with worker types z^* and $\tilde{z}$, then strict positive assortativeness implies that type z is matched with all types in $[z^*, \tilde{z}]$ and that these types are not matched with any other manager type. But then a zero measure of managers (those with type z) is matched with a positive measure of workers (those with type in $[z^*, \tilde{z}]$).

Lemma 9.17 *For each $k \in K$ and $l \in \{1, \dots, k-1\}$, $\phi_{k,l}$ is a continuous and strictly increasing function. Furthermore, $\phi_{k,l}(\min M_k) = \min W_{k,l}$ and $\phi_{k,l}(\max M_k) = \max W_{k,l}$.*

Proof. We first show that $\phi_{k,l}(z)$ is a singleton for each $z \in M_k$, i.e. $Q_{k,l} = \emptyset$. Suppose not; then let $z \in M_k$ and $z^*, \tilde{z} \in \phi_{k,l}(z)$ be such that $z^* < \tilde{z}$. Then $[z^*, \tilde{z}] \subseteq W_{k,l}$ and $(z^*, \tilde{z}) \subseteq W_{k,l} \setminus \cup_{(j,i):(j,i) \neq (k,l)} W_{j,i}$ by Lemmas 9.12 and 9.13.

For each $z' \in [z^*, \tilde{z}]$ and $x \in M_k \setminus \{z\}$, $z' \notin \phi_{k,l}(x)$. Indeed, Lemma 9.16 implies that if $x > z$, then $\min \phi_{k,l}(x) > \max \phi_{k,l}(z) \geq \tilde{z}$ and, if $z > x$, then $z^* \geq \min \phi_{k,l}(z) > \max \phi_{k,l}(x)$.

It follows from $[z^*, \tilde{z}] \cap \phi_{k,l}(x) = \emptyset$ for each $x \in M_k \setminus \{z\}$ and from $[z^*, \tilde{z}] \subseteq W_{k,l}$ that $[z^*, \tilde{z}] \subseteq \phi_{k,l}(z)$. Hence, $\{y \in \text{supp}(\gamma) : z_l(y) \in [z^*, \tilde{z}]\} \subseteq \{y \in \text{supp}(\gamma) : z_k(y) = z\}$.

We have that

$$0 < \nu((z^*, \tilde{z})) = \int_{Y_k} 1_{(z^*, \tilde{z})}(z_l(y)) n_l(y) d\gamma(y) \leq \bar{n} \gamma(\text{supp}(\gamma) \cap z_l^{-1}((z^*, \tilde{z}))),$$

hence, $\gamma(\text{supp}(\gamma) \cap z_l^{-1}((z^*, \tilde{z}))) > 0$. Then,

$$\begin{aligned} 0 = \nu(\{z\}) &\geq \int_{Y_k} 1_{\{z\}}(z_k(y)) d\gamma(y) = \gamma(\text{supp}(\gamma) \cap z_k^{-1}(z)) \\ &\geq \gamma(\text{supp}(\gamma) \cap z_l^{-1}((z^*, \tilde{z}))) > 0, \end{aligned}$$

a contradiction. This contradiction shows that $\phi_{k,l}(z)$ is a singleton for each $z \in M_k$.

It then follows that $\phi_{k,l}$ is a function. Since the graph of $\phi_{k,l}$ is closed, it follows that $\phi_{k,l}$ is continuous. Lemma 9.16 implies that $\phi_{k,l}$ is strictly increasing.

It follows from $\phi_{k,l}(M_k) = W_{k,l}$ that $\phi_{k,l}$ is onto. This then implies that $\phi_{k,l}(\min M_k) = \min W_{k,l}$ and $\phi_{k,l}(\max M_k) = \max W_{k,l}$ since $\phi_{k,l}$ is strictly increasing. ■

For each $k \in K$, let σ_k and σ be defined as in the statement of Theorem 9.1.

Lemma 9.18 $\text{supp}(\gamma) = \cup_{k \in K} \sigma_k(M_k)$.

Proof. Let $y \in \text{supp}(\gamma)$ and $k = \ell(y)$. Then $y_k \in M_k$ and $y_l \in W_{k,l}$ for each $l \in \{1, \dots, k-1\}$. Thus, $y_l = \phi_{k,l}(y_k)$ for each $l \in \{1, \dots, k-1\}$ by Lemma 9.17 and, hence, $y = \sigma_k(y_k)$.

Conversely, let $y = \sigma_k(z)$ for some $z \in M_k$. Let $\tilde{y} \in \text{supp}(\gamma)$ be such that $\tilde{y}_k = z$. For each $l \in \{1, \dots, k-1\}$, we have that $\tilde{y}_l \in W_{k,l}$ and, hence, $\tilde{y}_l = \phi_{k,l}(z)$ by Lemma 9.17. Thus, $\tilde{y} = y$ and it follows that $y \in \text{supp}(\gamma)$. ■

Lemma 9.19 For each $k \in K$ and $l \in \{1, \dots, k\}$, $\min W_{k,l} < \max W_{k,l}$.

Proof. Suppose that there is $k \in K$ and $z \in Z$ such that $M_k = \{z\}$. Then, by Lemma 9.18, $\text{supp}(\gamma) \cap Y_k = \{\sigma_k(z)\}$. Thus, $\text{supp}(\gamma) \setminus \{\sigma_k(z)\} = \text{supp}(\gamma) \setminus Y_k$ is closed (since $\text{supp}(\gamma)$ is closed and Y_k is open) and

$$0 = \nu(\{z\}) = \int_Y \sum_{i=1}^{\ell(y)} 1_{\{z\}}(z_i(y)) n_i(y) d\gamma(y) \geq \gamma(\{\sigma_k(z)\}) \geq 0,$$

contradicting the definition of $\text{supp}(\gamma)$. This contradiction then shows that $\min W_{k,k} < \max W_{k,k}$ for each $k \in K$.

For each $k \in K$ and $l \in \{1, \dots, k-1\}$, $\min W_{k,l} = \phi_{k,l}(\min W_{k,k}) < \phi_{k,l}(\max W_{k,k}) = \max W_{k,l}$ since $\phi_{k,l}$ is strictly increasing by Lemma 9.17. ■

9.5 Differentiability and Representation

In this section we establish the differentiability of the restriction $u_{k,l}$ of the earnings function to each $W_{k,l}$ and that of the assignment functions $\phi_{k,l}$. We then show that the assignment γ is represented by the function σ that assigns to each manager the

unique firm managed by that individual (for the boundary manager $z \in M_L \cap M_{L-1}$, σ assigns to him the L -layer firm).

The following lemma considers the case of $u_{k,l}$ when $l < k$.

Lemma 9.20 *For each $k \in K$ and $l \in \{1, \dots, k-1\}$, $u_{k,l}$ is differentiable and, for each $z \in W_{k,l}$,*

$$u'_{k,l}(z) = \frac{1}{n_l(y)} \left(\frac{\partial F(y)}{\partial z_l} - \sum_{i=1}^{k-1} u(z_i(y)) \frac{\partial n_i(y)}{\partial z_l} \right),$$

where $y = \sigma_k(\phi_{k,l}^{-1}(z))$.

Proof. Let $z \in W_{k,l}$ and $\{z_n\}_{n=1}^\infty$ be such that, for each $n \in \mathbb{N}$, $z_n \in W_{k,l}$, $z_n > z$ and $z_n \rightarrow z$. For each $n \in \mathbb{N}$, let $\hat{z}_n = \phi_{k,l}^{-1}(z_n)$. We have that $\phi_{k,l}^{-1}$ exists and is continuous by Lemma 9.17 since M_k is compact. Hence, $\hat{z}_n \rightarrow \phi_{k,l}^{-1}(z)$.

Let $y = \sigma_k(\phi_{k,l}^{-1}(z))$ and, for each $n \in \mathbb{N}$, define $\tilde{y}_n$ by

$$\tilde{y}_{n,i} = \begin{cases} z_n & \text{if } i = l, \\ y_i & \text{otherwise.} \end{cases}$$

Since $z_i(y) = z_i(\tilde{y}_n)$ for each $i \neq l$, stability implies that

$$\begin{aligned} F(y) - n_l(y)u(z) - \sum_{i \in \{1, \dots, k-1\} \setminus \{l\}} n_i(y)u(z_i(y)) &\geq \\ F(\tilde{y}_n) - n_l(\tilde{y}_n)u(z_n) - \sum_{i \in \{1, \dots, k-1\} \setminus \{l\}} n_i(\tilde{y}_n)u(z_i(y)). \end{aligned}$$

A simple manipulation of this expression implies that

$$\begin{aligned} \frac{u(z_n) - u(z)}{z_n - z} &\geq \frac{1}{n_l(y)} \left(\frac{F(\tilde{y}_n) - F(y)}{z_n - z} - u(z_n) \frac{n_l(\tilde{y}_n) - n_l(y)}{z_n - z} \right. \\ &\quad \left. - \sum_{i \in \{1, \dots, k-1\} \setminus \{l\}} u(z_i(y)) \frac{n_i(\tilde{y}_n) - n_i(y)}{z_n - z} \right); \end{aligned}$$

hence,

$$\liminf_{n \rightarrow \infty} \frac{u(z_n) - u(z)}{z_n - z} \geq \frac{1}{n_l(y)} \left(\frac{\partial F(y)}{\partial z_l} - \sum_{i=1}^{k-1} u(z_i(y)) \frac{\partial n_i(y)}{\partial z_l} \right)$$

since u is continuous.

For each $n \in \mathbb{N}$, let $y_n = \sigma_k(\hat{z}_n)$ and define $\hat{y}_n$ by

$$\hat{y}_{n,i} = \begin{cases} z & \text{if } i = l, \\ y_{n,i} & \text{otherwise.} \end{cases}$$

Since $z_i(y_n) = z_i(\hat{y}_n)$ for each $i \neq l$, stability implies that

$$\begin{aligned} F(y_n) - n_l(y_n)u(z_n) - \sum_{i \in \{1, \dots, k-1\} \setminus \{l\}} n_i(y_n)u(z_i(y_n)) &\geq \\ F(\hat{y}_n) - n_l(\hat{y}_n)u(z) - \sum_{i \in \{1, \dots, k-1\} \setminus \{l\}} n_i(\hat{y}_n)u(z_i(y_n)). \end{aligned}$$

A simple manipulation of this expression implies that

$$\begin{aligned} \frac{u(z_n) - u(z)}{z_n - z} &\leq \frac{1}{n_l(y_n)} \left(\frac{F(y_n) - F(\hat{y}_n)}{z_n - z} - u(z) \frac{n_l(y_n) - n_l(\hat{y}_n)}{z_n - z} \right. \\ &\quad \left. - \sum_{i \in \{1, \dots, k-1\} \setminus \{l\}} u(z_i(y_n)) \frac{n_i(y_n) - n_i(\hat{y}_n)}{z_n - z} \right); \end{aligned}$$

hence,

$$\limsup_{n \rightarrow \infty} \frac{u(z_n) - u(z)}{z_n - z} \leq \frac{1}{n_l(y)} \left(\frac{\partial F(y)}{\partial z_l} - \sum_{i=1}^{k-1} u(z_i(y)) \frac{\partial n_i(y)}{\partial z_l} \right)$$

since F and n_i , for each $1 \leq i \leq k-1$, are continuously differentiable.

It thus follows that

$$\lim_{n \rightarrow \infty} \frac{u(z_n) - u(z)}{z_n - z} = \frac{1}{n_l(y)} \left(\frac{\partial F(y)}{\partial z_l} - \sum_{i=1}^{k-1} u(z_i(y)) \frac{\partial n_i(y)}{\partial z_l} \right).$$

An analogous argument establishes the same limit when $z_n < z$; hence, $u_{k,l}$ is differentiable and, for each $z \in W_{k,l}$, $u'_{k,l}(z) = \lim_{n \rightarrow \infty} \frac{u(z_n) - u(z)}{z_n - z}$. ■

The next lemma considers the case of $u_{k,k}$.

Lemma 9.21 *For each $k \in K$, $u_{k,k}$ is differentiable and, for each $z \in W_{k,k} = M_k$,*

$$u'_{k,k}(z) = n_1(y) = \frac{1}{n_k(y)} \left(\frac{\partial F(y)}{\partial z_k} - \sum_{i=1}^{k-1} u(z_i(y)) \frac{\partial n_i(y)}{\partial z_k} \right),$$

where $y = \sigma_k(z)$.

Proof. Let $z \in M_k$ and $\{z_n\}_{n=1}^\infty$ be such that, for each $n \in \mathbb{N}$, $z_n \in M_k$, $z_n > z$ and $z_n \rightarrow z$. Let $y = \sigma_k(z)$ and, for each $n \in \mathbb{N}$, let $y_n = \sigma_k(z_n)$, $\tilde{y}_n$ be defined by

$$\tilde{y}_{n,i} = \begin{cases} z_n & \text{if } i = k, \\ y_i & \text{otherwise} \end{cases}$$

and $\hat{y}_n$ be defined by

$$\hat{y}_{n,i} = \begin{cases} z & \text{if } i = k, \\ y_{n,i} & \text{otherwise} \end{cases}$$

Since $z_i(y) = z_i(\tilde{y}_n)$ and $n_i(y) = n_i(\tilde{y}_n)$ for each $1 \leq i \leq k-1$, stability implies that

$$F(y) - u(z) - \sum_{i=1}^{k-1} n_i(y)u(z_i(y)) \geq F(\tilde{y}_n) - u(z_n) - \sum_{i=1}^{k-1} n_i(y)u(z_i(y)).$$

A simple manipulation of this expression implies that $\frac{u(z_n) - u(z)}{z_n - z} \geq n_1(y)$; hence,

$$\liminf_{n \rightarrow \infty} \frac{u(z_n) - u(z)}{z_n - z} \geq n_1(y).$$

Since $z_i(y_n) = z_i(\hat{y}_n)$ and $n_i(y_n) = n_i(\hat{y}_n)$ for each $1 \leq i \leq k-1$, stability implies that

$$F(y_n) - u(z_n) - \sum_{i=1}^{k-1} n_i(y_n)u(z_i(y_n)) \geq F(\hat{y}_n) - u(z) - \sum_{i=1}^{k-1} n_i(y_n)u(z_i(y_n)).$$

A simple manipulation of this expression implies that

$$\limsup_{n \rightarrow \infty} \frac{u(z_n) - u(z)}{z_n - z} \leq \lim_{n \rightarrow \infty} n_1(y_n) = n_1(y)$$

since n_1 is continuous.

It thus follows that

$$\lim_{n \rightarrow \infty} \frac{u(z_n) - u(z)}{z_n - z} = n_1(y).$$

An analogous argument establishes the same limit when $z_n < z$; hence, $u_{k,k}$ is differentiable and, for each $z \in M_k = W_{k,k}$, $u'_{k,k}(z) = n_1(y)$. ■

The following lemma shows that the assignment γ is fully described by σ , which means that γ is the inverse measure of the type distribution ν with respect to σ .

Lemma 9.22 $\gamma = \nu \circ \sigma^{-1}$.

Proof. Note first that $\max M_L = \bar{z}$ and

$$0 = \nu(\{\bar{z}\}) = \int_Y \sum_{i=1}^{\ell(y)} 1_{\{\bar{z}\}}(z_i(y)) n_i(y) d\gamma(y) \geq \gamma(\{\sigma(\bar{z})\}) \geq 0,$$

hence, $\gamma(\{\sigma(\bar{z})\}) = 0$. If $K = \{L-1, L\}$, let $p = \max M_{L-1} = \min M_L$; an analogous argument shows that $\gamma(\{\sigma_{L-1}(p)\}) = 0$. If $K = \{L\}$, let $\{\sigma_{L-1}(p)\} = \emptyset$ by convention.

Let B be a Borel subset of Y . Then

$$\nu \circ \sigma^{-1}(B) = \int_{\text{supp}(\gamma) \setminus \{\sigma(\bar{z}), \sigma_{L-1}(p)\}} \sum_{i=1}^{\ell(y)} 1_{\sigma^{-1}(B)}(z_i(y)) n_i(y) d\gamma(y).$$

Let $f : \text{supp}(\gamma) \setminus \{\sigma(\bar{z}), \sigma_{L-1}(p)\} \rightarrow \mathbb{R}$ be defined by setting, for each $y \in \text{supp}(\gamma) \setminus \{\sigma(\bar{z}), \sigma_{L-1}(p)\}$,

$$f(y) = \sum_{i=1}^{\ell(y)} 1_{\sigma^{-1}(B)}(z_i(y)) n_i(y).$$

We have that

$$f(y) = \begin{cases} 1 & \text{if } y \in (\text{supp}(\gamma) \setminus \{\sigma(\bar{z}), \sigma_{L-1}(p)\}) \cap B, \\ 0 & \text{if } y \in (\text{supp}(\gamma) \setminus \{\sigma(\bar{z}), \sigma_{L-1}(p)\}) \cap B^c. \end{cases}$$

Indeed, if $y \in (\text{supp}(\gamma) \setminus \{\sigma(\bar{z}), \sigma_{L-1}(p)\})$, then $y = \sigma(z)$ for some $z \in M \setminus \{\bar{z}\}$. We have that $1_{\sigma^{-1}(B)}(z_i(y)) n_i(y) = 0$ if $i < \ell(y)$ since $z_i(y) \in M \cap (\cup_{k \in K} W_{k,i})$ implies $z = \bar{z}$. If $y \in B$, then $1_{\sigma^{-1}(B)}(z_{\ell(y)}(y)) n_{\ell(y)}(y) = 1$ and $f(y) = 1$; if $y \in B^c$, then $1_{\sigma^{-1}(B)}(z_{\ell(y)}(y)) n_{\ell(y)}(y) = 0$ and $f(y) = 0$.

Thus,

$$\int_{\text{supp}(\gamma) \setminus \{\sigma(\bar{z}), \sigma_{L-1}(p)\}} \sum_{i=1}^{\ell(y)} 1_{\sigma^{-1}(B)}(z_i(y)) n_i(y) d\gamma(y) = \gamma((\text{supp}(\gamma) \setminus \{\sigma(\bar{z}), \sigma_{L-1}(p)\}) \cap B) = \gamma(B).$$

Hence, $\nu \circ \sigma^{-1}(B) = \gamma(B)$ and since B is arbitrary, $\nu \circ \sigma^{-1} = \gamma$. ■

The following lemma computes the measure of $[\min W_{k,l}, \phi_{k,l}(z)]$ according to ν for each $z \in M_k$. It is a simple consequence of the previous lemma and it is useful to establish the differentiability of $\phi_{k,l}$. For each $k \in K$, let $z_k = \min M_k$ and, for each $l \in \{1, \dots, k-1\}$, let $\nu(z') = \nu([\min W_{k,l}, z'])$ for each $z' \in W_{k,l}$. Let also $z_{L+1} = \bar{z}$.

Lemma 9.23 For each $k \in K$, $l \in \{1, \dots, k-1\}$ and $z \in [z_k, z_{k+1}]$, $\nu(\phi_{k,l}(z)) = \int_{z_k}^z n_l(\sigma_k(x))\theta(x)dx$.

Proof. Let $k \in K$, $l \in \{1, \dots, k-1\}$ and $z \in [z_k, z_{k+1}]$. Then

$$\begin{aligned} \nu(\phi_{k,l}(z)) &= \nu([\min W_{k,l}, \phi_{k,l}(z)]) = \nu((\min W_{k,l}, \phi_{k,l}(z))) \\ &= \int_Y \sum_{i=1}^{\ell(y)} 1_{(\min W_{k,l}, \phi_{k,l}(z))}(z_i(y)) n_i(y) d\gamma(y) \\ &= \int_M \sum_{i=1}^{\ell(\sigma(x))} 1_{(\min W_{k,l}, \phi_{k,l}(z))}(z_i(\sigma(x))) n_i(\sigma(x)) d\nu(x) \\ &= \int_{(z_k, z)} n_l(\sigma_k(x)) d\nu(x) = \int_{[z_k, z]} n_l(\sigma_k(x)) d\nu(x) = \int_{z_k}^z n_l(\sigma_k(x))\theta(x)dx. \end{aligned}$$

■

The following lemma establishes the differentiability of $\phi_{k,l}$ and computes its derivative. It is largely a consequence of the previous lemma since it implies that $\phi_{k,l}$ is the composition of the inverse of $z' \mapsto \nu([\min W_{k,l}, z'])$ (with $z' \in W_{k,l}$) and $z \mapsto \int_{z_k}^z n_l(\sigma_k(x))\theta(x)dx$ (with $z \in M_k$).

Lemma 9.24 For each $k \in K$ and $l \in \{1, \dots, k-1\}$, $\phi_{k,l}$ is differentiable and, for each $z \in M_k$,

$$\phi'_{k,l}(z) = \frac{n_l(\sigma_k(z))\theta(z)}{\theta(\phi_{k,l}(z))}.$$

Proof. Let $k \in K$ and $l \in \{1, \dots, k-1\}$ be given. Define a function $\nu : W_{k,l} \rightarrow \mathbb{R}$ by setting, for each $z' \in W_{k,l}$, $\nu(z') = \nu([\min W_{k,l}, z'])$. Then, ν is strictly increasing; let $\lambda : [0, \nu(\max W_{k,l})] \rightarrow W_{k,l}$ be its inverse. It then follows by Lemma 9.23 that, for each $z \in M_k$,

$$\phi_{k,l}(z) = \lambda \left(\int_{z_k}^z n_l(\sigma_k(x))\theta(x)dx \right).$$

We have that $z \mapsto \nu(z)$ is differentiable and that its derivative at $z \in W_{k,l}$ is $\theta(z)$. Then λ is differentiable and $\lambda'(x) = \frac{1}{\theta(\lambda(x))}$ for each $x \in [0, \nu(\max W_{k,l})]$. Let $\zeta : M_k \rightarrow \mathbb{R}$ be defined by setting, for each $z \in M_k$, $\zeta(z) = \int_{z_k}^z n_l(\sigma_k(x))\theta(x)dx$. Then ζ is differentiable with $\zeta'(z) = n_l(\sigma_k(z))\theta(z)$ for each $z \in M_k$. Since $\phi_{k,l} = \lambda \circ \zeta$, it follows that $\phi_{k,l}$ is differentiable and that, for each $z \in M_k$, $\phi'_{k,l}(z) = \frac{n_l(\sigma_k(z))\theta(z)}{\theta(\lambda(\zeta(z)))}$.

Since $\zeta(z) = \nu(\phi_{k,l}(z))$ by Lemma 9.23, we obtain that $\lambda(\zeta(z)) = \phi_{k,l}(z)$ and, hence, $\phi'_{k,l}(z) = \frac{n_l(\sigma_k(z))\theta(z)}{\theta(\phi_{k,l}(z))}$. ■

To complete the proof of the necessity part of Theorem 9.1, note that property 1 follows from Lemmas 9.10 and 9.11, 2 from Lemmas 9.12 and 9.13, 3 from Lemmas 9.17 and 9.24, 4 from Lemma 9.22, and 5 from Lemmas 9.3, 9.20 and 9.21.

9.6 Uniqueness

We conclude the proof of Theorem 9.1 in this section by establishing that E^* has a unique stable outcome.

Let (γ, u) and $(\hat{\gamma}, \hat{u})$ be stable outcomes of E^* ; we will show that $\gamma = \hat{\gamma}$ and that $u = \hat{u}$. The following lemma establishes the uniqueness of the assignment.

Lemma 9.25 $\gamma = \hat{\gamma}$.

Proof. Note that γ and $\hat{\gamma}$ solve the surplus maximization problem and that u and $\hat{u}$ solve its dual. Let $\tilde{\gamma} = (\gamma + \hat{\gamma})/2$. It follows that $(\tilde{\gamma}, u)$ is a stable outcome by Theorem 2 since $\tilde{\gamma}$ also solves the surplus maximization problem, and that $\text{supp}(\tilde{\gamma}) = \text{supp}(\gamma) \cup \text{supp}(\hat{\gamma})$. We show that $\gamma = \nu \circ \tilde{\sigma}^{-1}$.

Define $\Lambda : \text{supp}(\tilde{\gamma}) \rightarrow \tilde{M}$ by setting, for each $y \in \text{supp}(\tilde{\gamma})$, $\Lambda(y) = z_{\ell(y)}(y)$ and note that Λ is continuous and onto. If $\tilde{K} = \{\tilde{L} - 1, \tilde{L}\}$, let $p = \max \tilde{M}_{\tilde{L}-1} = \min \tilde{M}_{\tilde{L}}$ and $G = \text{supp}(\tilde{\gamma}) \setminus \Lambda^{-1}(p)$; if $\tilde{K} = \{\tilde{L}\}$, let $G = \text{supp}(\tilde{\gamma})$ and (by convention) $\{p\} = \emptyset$. In either case, Lemma 9.18 yields

$$y = \tilde{\sigma}(\Lambda(y)) \text{ for each } y \in G. \quad (9.6)$$

Moreover $\tilde{\sigma}$ is measurable since it is continuous on $\tilde{M}_{\tilde{L}}$ and on $\tilde{M}_{\tilde{L}-1} \setminus \{p\}$ by Lemma 9.17.

Since $\text{supp}(\gamma) \subseteq \text{supp}(\tilde{\gamma})$, the pushforward $m_\gamma = \gamma \circ \Lambda^{-1}$ is concentrated on $\Lambda(\text{supp}(\tilde{\gamma})) = \tilde{M}$. We claim that $m_\gamma = \nu|_{\tilde{M}}$.

For each Borel subset B of $\tilde{M}$, let $R(B) = \int_Y \sum_{i=1}^{\ell(y)-1} 1_B(z_i(y)) n_i(y) d\gamma(y)$. Let B be a Borel subset of $\tilde{M}$. We have that $R(B) = 0$. Indeed, let $y \in \text{supp}(\gamma)$ and

$i < \ell(y)$. Then $y \in \text{supp}(\tilde{\gamma})$, so $z_i(y) \in \tilde{W}_{\ell(y),i} < \tilde{M}$ by occupational stratification. Thus $z_i(y) \in B \subseteq \tilde{M}$ implies $z_i(y) = \min \tilde{M}$, and

$$0 \leq R(B) \leq \int_Y \sum_{i=1}^{\ell(y)} 1_{\{\min \tilde{M}\}}(z_i(y)) n_i(y) d\gamma(y) = \nu(\{\min \tilde{M}\}) = 0.$$

Hence,

$$\nu(B) = \int_Y 1_B(z_{\ell(y)}(y)) d\gamma(y) + \int_Y \sum_{i=1}^{\ell(y)-1} 1_B(z_i(y)) n_i(y) d\gamma(y) = m_\gamma(B) + R(B) = m_\gamma(B).$$

Thus, $m_\gamma = \nu|_{\tilde{M}}$.

Note that it follows from $m_\gamma = \nu|_{\tilde{M}}$ that $\gamma(\Lambda^{-1}(p)) = m_\gamma(\{p\}) = \nu(\{p\}) = 0$, so $\gamma(\text{supp}(\tilde{\gamma}) \setminus G) = 0$. It also follows that $\gamma = \nu \circ \tilde{\sigma}^{-1}$. To see this, let $B \subseteq Y$ be Borel. By (9.6), $1_B(y) = 1_{\tilde{\sigma}^{-1}(B)}(\Lambda(y))$ for each $y \in G$. Using $\text{supp}(\gamma) \subseteq \text{supp}(\tilde{\gamma})$, $\gamma(\text{supp}(\tilde{\gamma}) \setminus G) = 0$ and $m_\gamma = \nu|_{\tilde{M}}$,

$$\begin{aligned} \gamma(B) &= \int_G 1_{\tilde{\sigma}^{-1}(B)}(\Lambda(y)) d\gamma(y) = \int_Y 1_{\tilde{\sigma}^{-1}(B) \setminus \{p\}}(\Lambda(y)) d\gamma(y) \\ &= \int_{\tilde{M}} 1_{\tilde{\sigma}^{-1}(B) \setminus \{p\}}(z) d\gamma \circ \Lambda^{-1}(z) = m_\gamma(\tilde{\sigma}^{-1}(B) \setminus \{p\}) = \nu(\tilde{\sigma}^{-1}(B) \setminus \{p\}) = \nu(\tilde{\sigma}^{-1}(B)). \end{aligned}$$

Hence, $\gamma = \nu \circ \tilde{\sigma}^{-1}$.

Since γ and $\hat{\gamma}$ play a symmetric role, the above arguments also imply that $\hat{\gamma} = \nu \circ \tilde{\sigma}^{-1}$, i.e. $\gamma = \hat{\gamma}$. ■

The following lemma establishes the uniqueness of the earnings function and concludes the proof of Theorem 9.1.

Lemma 9.26 $u = \hat{u}$.

Proof. Lemma 9.25 implies that (γ, u) and $(\gamma, \hat{u})$ are stable outcomes. Let $W_{k,l}$, $\phi_{k,l}$, and σ_k be as in the statement of Theorem 9.1 and

$$d = \hat{u} - u.$$

By Theorem 9.1,

$$u'_{k,k} = \hat{u}'_{k,k} = n_1 \circ \sigma_k$$

on each M_k . Thus d is constant on each M_k . If $K = \{L-1, L\}$, then $\max M_{L-1} = \min M_L$, so continuity of d implies that the two constants coincide. Hence there is $c \in \mathbb{R}$ such that

$$d(z) = c \text{ for each } z \in M. \quad (9.7)$$

Without loss of generality (interchanging the roles of u and $\hat{u}$ if necessary), we may assume that $c \geq 0$.

Claim 11 *For each $k \in K$, $l < k$, and $x \in W_{k,l}$, let*

$$\begin{aligned} y(x) &= \sigma_k(\phi_{k,l}^{-1}(x)), \\ \pi(x) &= z_{l+1}(y(x)), \text{ and} \\ w_l(y) &= \begin{cases} h, & l = 1, \\ \frac{1}{1 - z_{l-1}(y)}, & l \geq 2. \end{cases} \end{aligned}$$

Then:

$$\begin{aligned} u'_{k,l}(x) &= w_l(y(x))u(\pi(x)), \text{ and} \\ \hat{u}'_{k,l}(x) &= w_l(y(x))\hat{u}(\pi(x)). \end{aligned}$$

Proof. Let $k \in K$, $l < k$, $x \in W_{k,l}$ and $y = y(x)$. If $l \leq k-2$, then $\partial F(y)/\partial z_l = 0$ and $\partial n_i(y)/\partial z_l = -1/(1 - z_{k-1}(y))$ if $i = l+1$ and $\partial n_i(y)/\partial z_l = 0$ otherwise, so Theorem 9.1 gives

$$u'_{k,l}(x) = \frac{u(z_{l+1}(y))}{n_l(y)(1 - z_{k-1}(y))} = w_l(y)u(\pi(x)).$$

If $l = k-1$, then $\partial F(y)/\partial z_{k-1} = F(y)/(1 - z_{k-1}(y))$ and $\partial n_i(y)/\partial z_{k-1} = n_i(y)/(1 - z_{k-1}(y))$ for each $i < k$, so

$$u'_{k,l}(x) = \frac{F(y) - \sum_{i < k} n_i(y)u(z_i(y))}{n_{k-1}(y)(1 - z_{k-1}(y))} = \frac{u(z_k(y))}{n_{k-1}(y)(1 - z_{k-1}(y))} = w_{k-1}(y)u(\pi(x)),$$

where the second equality uses $y \in \text{supp}(\gamma)$ and stability. The same holds for $\hat{u}$. ■

Define $r = d/u$, which is well-defined and continuous since $u(z) \geq z > 0$ for each $z \in Z$. Then, for each $k \in K$, $l < k$, and $x \in W_{k,l}$,

$$r'_{k,l}(x) = \kappa(x)(r(\pi(x)) - r(x)), \quad (9.8)$$

where $r_{k,l} = r|_{W_{k,l}}$ and

$$\kappa(x) = \frac{w_l(y(x))u(\pi(x))}{u(x)} > 0.$$

Claim 12 r is nonincreasing.

Proof. Order the occupational sets from top to bottom (if $K = \{L\}$ then drop the $L - 1$ sets):

$$M_L > M_{L-1} > W_{L,L-1} > W_{L-1,L-2} > W_{L,L-2} > \dots > W_{L,1}.$$

The proof is by induction on these sets. By (9.7),

$$r(z) = \frac{c}{u(z)} \text{ for } z \in M.$$

Since u is strictly increasing, r is nonincreasing on M .

Let $k \in K$ and $l < k$ be given, let $a' = \min W_{k,l}$ and $a = \max W_{k,l}$ so that $W_{k,l} = [a', a]$ and assume by the inductive hypothesis that r is nonincreasing on $[a, \bar{z}]$. Let $x \in W_{k,l}$ and note that $\pi(x) \in [a, \bar{z}]$; thus $r(\pi(x)) \leq r(a)$. Let

$$s(x) = r(x) - r(a).$$

Then $s(a) = 0$ and (9.8) gives

$$s'(x) = \kappa(x)(r(\pi(x)) - r(a) - s(x)) \leq -\kappa(x)s(x).$$

Letting

$$H(x) = \int_{a'}^x \kappa(t) dt,$$

we have $(s(x)e^{H(x)})' = e^{H(x)}(s'(x) + \kappa(x)s(x)) \leq 0$ and hence $s(x)e^{H(x)} \geq s(a)e^{H(a)} = 0$. Thus, $s(x) \geq 0 \Leftrightarrow r(x) \geq r(a)$ for all $x \in W_{k,l}$. Since $r(\pi(x)) \leq r(a)$, (9.8) implies $r'_{k,l}(x) = \kappa(x)(r(\pi(x)) - r(x)) \leq \kappa(x)(r(a) - r(x)) \leq 0$; hence r is nonincreasing on $[a', \bar{z}]$. The conclusion then follows by induction. ■

Thus $r(z) \geq r(\bar{z}) = \frac{c}{u(\bar{z})} \geq 0$ for each $z \in Z$. Since $u > 0$, $d(z) \geq 0$ for each $z \in Z$. For each $y \in \text{supp}(\gamma)$,

$$0 = \sum_{i=1}^{\ell(y)} n_i(y)(\hat{u}(z_i(y)) - u(z_i(y))) = \sum_{i=1}^{\ell(y)} n_i(y)d(z_i(y)).$$

Since $n_i(y) > 0$ and $d(z_i(y)) \geq 0$ for each i , it follows that $d(z_i(y)) = 0$ for each i .

Lemma 6.1 then implies that $d(z) = 0$ for each $z \in Z$. ■

10 Technical Difficulties in Chen and Suen (2019)

We explain in this section the technical difficulties in the formal arguments in Chen and Suen (2019); this prevents us from using them in the characterization of stable outcomes in the multi-layer version of Garicano and Rossi-Hansberg's (2004) setting.

Such characterization relies on the following monotonicity property: Let E^* be the multi-layer version of Garicano and Rossi-Hansberg's (2004) economy. If γ is a stable assignment of E^* , $(z_1, \dots, z_L), (x_1, \dots, x_K) \in \text{supp}(\gamma)$ and $z_L > x_K$, then $x_{K-l} \leq z_{L-l} \leq x_{K-l+1}$ for each $1 \leq l \leq J$, where

$$J = \begin{cases} L - 1 & \text{if } K \geq L, \\ K & \text{if } L > K. \end{cases}$$

This property is analogous to what Chen and Suen (2019) call the accordion property and could (in principle) be established using the arguments in the proofs of their Propositions 1–4. We now show that their argument is incorrect. The proof of their Proposition 3 claims that Proposition 1 implies the following: If $(z_1, \dots, z_L) \in \text{supp}(\gamma)$ and $(\tilde{z}_{j+1}, \dots, \tilde{z}_L) \in \text{supp}(\gamma)$ for some $0 \leq j < L$, then $(z_{j+1}, \dots, z_L) \geq (\tilde{z}_{j+1}, \dots, \tilde{z}_L)$.¹³

In the case where $j = 0$, i.e. the two organizations have length L , one can assume that $z_L \geq \tilde{z}_L$ (i.e. let $(z_1, \dots, z_L)$ denote the one with the highest value in its last coordinate); the proof of their Proposition 1 then asserts that $(z_1, \dots, z_{L-1}) \geq (\tilde{z}_1, \dots, \tilde{z}_{L-1})$ “follows directly from Milgrom and Shannon (1994).” The case where $j > 0$ uses a similar argument, namely, it concludes from (the correct argument that) $z_{j+1} \geq \tilde{z}_{j+1}$ that $(z_{j+2}, \dots, z_L) \geq (\tilde{z}_{j+2}, \dots, \tilde{z}_L)$, presumably also from Milgrom and Shannon (1994).

Theorem 4 of Milgrom and Shannon (1994), which is their monotonicity result, does not allow such conclusion. In the latter case, it implies that a set A containing $(z_{j+2}, \dots, z_L)$ is higher than a set $\tilde{A}$ containing $(\tilde{z}_{j+2}, \dots, \tilde{z}_L)$; the set A , resp. $\tilde{A}$, is the set of maximizers of a specific function treating z_{j+1} , resp. $\tilde{z}_{j+1}$, as a parameter. This

¹³Note that $(z_1, \dots, z_L) \in \text{supp}(\gamma)$ implies that $(z_1, \dots, z_L)$ is an optimal organization in the terminology of Chen and Suen (2019) and likewise for $(\tilde{z}_{j+1}, \dots, \tilde{z}_L)$.

means that, endowing Z^{L-j-1} with the coordinate order, for each $(a_{j+2}, \dots, a_L) \in A$ and $(\tilde{a}_{j+2}, \dots, \tilde{a}_L) \in \tilde{A}$, then $(a_{j+2}, \dots, a_L) \wedge (\tilde{a}_{j+2}, \dots, \tilde{a}_L) \in \tilde{A}$ and $(a_{j+2}, \dots, a_L) \vee (\tilde{a}_{j+2}, \dots, \tilde{a}_L) \in A$. This clearly does not imply that $(a_{j+2}, \dots, a_L) \geq (\tilde{a}_{j+2}, \dots, \tilde{a}_L)$. It shows that there is a solution, $(z_{j+2}, \dots, z_L) \vee (\tilde{z}_{j+2}, \dots, \tilde{z}_L)$, of the optimization problem with z_{j+1} as a parameter that is bigger than one solution, $(z_{j+2}, \dots, z_L) \wedge (\tilde{z}_{j+2}, \dots, \tilde{z}_L)$, of the optimization problem with $\tilde{z}_{j+1}$ as a parameter; of course, if those solutions were unique, then, indeed, $(z_{j+2}, \dots, z_L) \geq (\tilde{z}_{j+2}, \dots, \tilde{z}_L)$ but not otherwise.

11 Equivalent knowledge economies

The definition of knowledge economies without externalities is sufficiently general to allow for some redundancy, namely if the cumulative distribution function of the knowledge distribution is continuous and strictly increasing (e.g. if ν has a continuous and strictly positive density), then we may assume that the knowledge distribution is uniform on the unit interval by adjusting the other elements of the knowledge economy.

The formal argument is as follows. Let $E = (Z, \nu, Y, F)$ be a knowledge economy without externalities (i.e. $F : Y \rightarrow \mathbb{R}$), X be a compact metric space and $\psi : Z \rightarrow X$ be a homeomorphism. Define $\hat{\psi} : Y \rightarrow \cup_{L=1}^{\bar{L}} (X \times N)^L$ by setting, for each $y \in Y$, $\hat{\psi}(y) = (\psi(z_1(y)), n_1, \dots, \psi(z_{\ell(y)}(y)), n_{\ell(y)})$ and let $\Psi = \hat{\psi}(Y)$. Then consider the knowledge economy

$$E_\psi = (X, \nu \circ \psi^{-1}, \Psi, F \circ \hat{\psi}^{-1}).$$

In the concrete case where ν has a continuous and strictly positive density on $Z = [\underline{z}, \bar{z}]$, where $\underline{z}, \bar{z} \in \mathbb{R}$ and $\underline{z} < \bar{z}$, let $X = [0, \nu(Z)]$ and $\psi(z) = \nu([\underline{z}, z])$ for each $z \in Z$, i.e. ψ is the cumulative distribution function of ν and is continuous, strictly increasing and satisfies $\psi(Z) = X$. Thus, ψ is a homeomorphism between Z and X . The knowledge economy E_ψ has a uniform knowledge distribution since, for each

$x, x' \in X$ with $x < x'$,

$$\begin{aligned}\nu \circ \psi^{-1}([x, x']) &= \nu(\psi^{-1}([x, x'])) = \nu([\psi^{-1}(x), \psi^{-1}(x')]) \\ &= \nu([\underline{z}, \psi^{-1}(x')]) - \nu([\underline{z}, \psi^{-1}(x)]) \\ &= \psi(\psi^{-1}(x')) - \psi(\psi^{-1}(x)) = x' - x.\end{aligned}$$

The knowledge economies E and E_ψ have the same stable outcomes up to homeomorphisms.

Theorem 11.1 *An outcome (γ, v) is stable in E if and only if $(\gamma \circ \hat{\psi}^{-1}, v \circ \psi^{-1})$ is a stable outcome in E_ψ .*

Proof. (Necessity) Let (γ, v) be a stable outcome of E . Note that, for each $y \in Y$ and $1 \leq i \leq \ell(y)$,

$$\ell(\hat{\psi}(y)) = \ell(y), n_i(\hat{\psi}(y)) = n_i(y) \text{ and } z_i(\hat{\psi}(y)) = \psi(z_i(y)). \quad (11.1)$$

The latter implies that, for each Borel B of X , we have that $z_i(\hat{\psi}(y)) \in B$ if and only if $\psi(z_i(y)) \in B$, if and only if $z_i(y) \in \psi^{-1}(B)$. Hence,

$$\begin{aligned}\int_{\Psi} \sum_{i=1}^{\ell(w)} 1_B(z_i(w)) n_i(w) d\gamma \circ \hat{\psi}^{-1}(w) &= \int_Y \sum_{i=1}^{\ell(\hat{\psi}(y))} 1_B(z_i(\hat{\psi}(y))) n_i(\hat{\psi}(y)) d\gamma(y) \\ &= \int_Y \sum_{i=1}^{\ell(y)} 1_{\psi^{-1}(B)}(z_i(y)) n_i(y) d\gamma(y) = \nu(\psi^{-1}(B)) = \nu \circ \psi^{-1}(B).\end{aligned}$$

Thus, $\gamma \circ \hat{\psi}^{-1}$ is an assignment in E_ψ .

Note that (11.1) implies that, for each $w \in \Psi$, and $1 \leq i \leq \ell(w)$, $\ell(\hat{\psi}^{-1}(w)) = \ell(w)$, $n_i(\hat{\psi}^{-1}(w)) = n_i(w)$ and $z_i(\hat{\psi}^{-1}(w)) = \psi^{-1}(z_i(w))$. For each $w \in \text{supp}(\gamma \circ \hat{\psi}^{-1})$, it follows that $\hat{\psi}^{-1}(w) \in \text{supp}(\gamma)$ since $\hat{\psi}$ is an homeomorphism. Hence,

$$\begin{aligned}\sum_{i=1}^{\ell(w)} n_i(w) v \circ \psi^{-1}(z_i(w)) &= \sum_{i=1}^{\ell(w)} n_i(w) v(\psi^{-1}(z_i(w))) \\ &= \sum_{i=1}^{\ell(\hat{\psi}^{-1}(w))} n_i(\hat{\psi}^{-1}(w)) v(z_i(\hat{\psi}^{-1}(w))) = F(\hat{\psi}^{-1}(w)) = F \circ \hat{\psi}^{-1}(w).\end{aligned}$$

Furthermore, for each $w \in \Psi$,

$$\begin{aligned} \sum_{i=1}^{\ell(w)} n_i(w) v \circ \psi^{-1}(z_i(w)) &= \sum_{i=1}^{\ell(\hat{\psi}^{-1}(w))} n_i(\hat{\psi}^{-1}(w)) v(z_i(\hat{\psi}^{-1}(w))) \\ &\geq F(\hat{\psi}^{-1}(w)) = F \circ \hat{\psi}^{-1}(w). \end{aligned}$$

Thus, $(\gamma \circ \hat{\psi}^{-1}, v \circ \psi^{-1})$ is a stable outcome of E_ψ .

(Sufficiency) Let $\lambda : X \rightarrow Z$ be the inverse of ψ and note that $\hat{\lambda} : \Psi \rightarrow \cup_{L=1}^{\bar{L}} (Z \times N)^L$ defined by setting, for each $w \in \Psi$, $\hat{\lambda}(w) = (\lambda(w_1), n_1, \dots, \lambda(w_{\ell(w)}), n_{\ell(w)})$ is the inverse of $\hat{\psi}$; hence, $\hat{\lambda}(\Psi) = Y$. Thus, the economy

$$E_\lambda = (Z, (\nu \circ \psi^{-1}) \circ \lambda^{-1}, Y, (F \circ \hat{\psi}^{-1}) \circ \hat{\lambda}^{-1})$$

equals E . Since λ is a homeomorphism, the necessity part above implies that if $(\gamma \circ \hat{\psi}^{-1}, v \circ \psi^{-1})$ is a stable outcome in E_ψ , then $\gamma = (\gamma \circ \hat{\psi}^{-1}) \circ \hat{\lambda}^{-1}$ and $v = (v \circ \psi^{-1}) \circ \lambda^{-1}$ form a stable outcome in $E = E_\lambda$. ■

We give two illustrations of Theorem 11.1 in the context of Garicano and Rossi-Hansberg (2004). The first one was already described in Section 5.2 and consists in transforming the economy so that it has a uniform knowledge distribution. Let E be defined by $Z = [0, \bar{z}]$ with $0 < \bar{z} < 1$, ν such that it has a continuous and strictly positive density, and $\nu(Z) = 1$, $\bar{L} = 2$,

$$Y = (Z \times \{1\}) \cup \left\{ (z', n, z, 1) : z', z \in Z \text{ and } n = \frac{1}{h(1 - z')} \right\},$$

$F(z, 1) = z$ for each $z \in Z$ and $F(z', n, z, 1) = zn$ for each $z', z \in Z$ and $n = (h(1 - z'))^{-1}$. Let $X = [0, 1]$ and $\psi : Z \rightarrow X$ be defined by setting, for each $z \in Z$, $\psi(z) = \nu([0, z])$. Then ψ is a homeomorphism between Z and X ; letting G denote the inverse of ψ , it follows that $E_\psi = ([0, 1], \text{unif}, \Psi, F_\psi)$ where unif is the uniform distribution on $[0, 1]$,

$$\Psi = (X \times \{1\}) \cup \left\{ (x', n, x, 1) : x', x \in X \text{ and } n = \frac{1}{h(1 - G(x'))} \right\},$$

$F_\psi(x, 1) = G(x)$ for each $x \in X$ and $F_\psi(x', n, x, 1) = G(x)n$ for each $x', x \in X$ and $n = (h(1 - G(x')))^{-1}$. Theorem 11.1 implies that the function $(\gamma, v) \mapsto (\gamma \circ \hat{\psi}^{-1}, v \circ \psi^{-1})$

is a homeomorphism between the stable pairs of E and E_ψ . Note that the stable pairs of E_ψ are described in terms of the percentiles of ν , e.g. if $x = \frac{k}{100}$ with $k \in \{0, \dots, 100\}$, $v \circ \psi^{-1}(x)$ is the earnings of the k th percentile of the distribution ν .

Another application of Theorem 11.1 shows that assuming that the distribution of problems is uniform in Garicano and Rossi-Hansberg (2004) is without loss of generality, in the following sense. Let $Z = [0, \bar{z}]$ with $\bar{z} > 0$ and $G : Z \rightarrow \mathbb{R}_+$ be a continuous and strictly increasing cumulative distribution function with $G(0) = 0$ and $G(\bar{z}) < 1$ (e.g. the cumulative distribution function of a distribution with a continuous density $g : Z \rightarrow \mathbb{R}_+$ such that $g(z) > 0$ for each $z \in Z \setminus \{\bar{z}\}$). Then define an economy E by letting

$$Y = (Z \times \{1\}) \cup \left\{ (z', n, z, 1) : z', z \in Z \text{ and } n = \frac{1}{h(1 - G(z'))} \right\},$$

$F(z, 1) = G(z)$ for each $z \in Z$ and $F(z', n, z, 1) = G(z)n$ for each $z', z \in Z$ and $n = (h(1 - G(z')))^{-1}$. The interpretation is that each worker draws a problem $w \in \mathbb{R}_+$ from G and production equals 1 if the problem is solved, which happens if and only if $w \leq z$, and zero otherwise.

Let $X = [0, G(\bar{z})]$ and $\psi : Z \rightarrow X$ be defined by setting, for each $z \in Z$, $\psi(z) = G(z)$. Then ψ is a homeomorphism between Z and X . It then follows that $E_\psi = ([0, G(\bar{z})], \nu \circ G^{-1}, \Psi, F_\psi)$ where

$$\Psi = (X \times \{1\}) \cup \left\{ (x', n, x, 1) : x', x \in X \text{ and } n = \frac{1}{h(1 - x')} \right\},$$

$F_\psi(x, 1) = x$ for each $x \in X$ and $F_\psi(x', n, x, 1) = xn$ for each $x', x \in X$ and $n = (h(1 - x'))^{-1}$. Theorem 11.1 implies that the function $(\gamma, v) \mapsto (\gamma \circ \hat{\psi}^{-1}, v \circ \psi^{-1})$ is a homeomorphism between the stable pairs of E and E_ψ . Note that the stable pairs of E_ψ are described in terms of the distribution G , i.e. $v \circ \psi^{-1}(x)$ is the earnings of the individuals who can solve a fraction x of problems.

12 A variation of the Garicano and Rossi-Hansberg (2004) economy

We show that the two specifications of the Garicano and Rossi-Hansberg's (2004) economy in Section 5.4 yields the same stable outcomes.

Assume that $Z = [\underline{z}, \bar{z}]$ with $0 < \underline{z} < \bar{z} < 1$ and let $N = [(1 - \bar{z})(1 - \underline{z})^{-1}, (h(1 - \bar{z}))^{-1}]$. Let E be defined as follows: $Y = \cup_{L=1}^{\bar{L}} Y_L$, $Y_1 = Z \times \{1\}$ and, for each $1 < L \leq \bar{L}$,

$$\begin{aligned} Y_L &= \{(z_1, n_1, \dots, z_L, n_L) \in (Z \times N)^L : n_L = 1, n_1 = (h(1 - \max\{z_1, \dots, z_{L-1}\}))^{-1} \\ &\text{and } n_i = (1 - \max\{z_1, \dots, z_{i-1}\})(1 - \max\{z_1, \dots, z_{L-1}\})^{-1} \text{ for each } 1 < i < L\}, \\ F(z, 1) &= z, \text{ and, for each } L > 1, \\ F(z_1, n_1, \dots, z_L, n_L) &= \max\{z_1, \dots, z_L\}n_1. \end{aligned}$$

Let E^* be defined as follows: $Y^* = \cup_{L=1}^{\bar{L}} Y_L^*$, $Y_1^* = Z \times \{1\}$ and, for each $1 < L \leq \bar{L}$,

$$\begin{aligned} Y_L^* &= \{(z_1, n_1, \dots, z_L, n_L) \in (Z \times N)^L : n_L = 1, n_1 = (h(1 - z_{L-1}))^{-1} \\ &\text{and } n_i = (1 - z_{i-1})(1 - z_{L-1})^{-1} \text{ for each } 1 < i < L\}, \\ F^*(z, 1) &= z, \text{ and, for each } L > 1, \\ F^*(z_1, n_1, \dots, z_L, n_L) &= z_L n_1. \end{aligned}$$

Theorem 12.1 *An outcome (γ, u) is stable in E if and only if it is stable in E^* .*

The following lemma, which is needed for the proof of Theorem 12.1, shows that knowledge is increasing across layers in E and is analogous to the same result for E^* , Lemma 9.2 in Section 9 of the appendix.

In this section, we let $F(\tau) = \int_Y F d\tau$ for each $\tau \in M(Y)$ and, likewise, $F^*(\tau) = \int_{Y^*} F^* d\tau$ for each $\tau \in M(Y^*)$. Also, from now on, we regard Y and Y^* as $\cup_{L=1}^{\bar{L}} Z^L$.

Lemma 12.1 *If γ is a stable assignment of E and $(z_1, \dots, z_L) \in \text{supp}(\gamma)$, then $z_1 < \dots < z_L$.*

Proof. We first show that $z_1 < \dots < z_{L-1}$. Suppose not; then there is $i \in \{1, \dots, L-2\}$ such that $z_i \geq z_{i+1}$. Consider first the case where $i < L-2$. Consider $\tau = 1_{(z_1, \dots, z_L)}$ and

$$\xi = 1_{(z_1, \dots, z_i, z_{i+2}, \dots, z_L)} + n_{i+1} 1_{z_{i+1}} + (n_{i+2} - \tilde{n}_{i+2}) 1_{z_{i+2}},$$

where $n_j = n_j(z_1, \dots, z_L)$ for each $1 \leq j \leq L-1$ and $\tilde{n}_j = n_j(z_1, \dots, z_i, z_{i+2}, \dots, z_L)$ for each $j \in \{1, \dots, L-1\} \setminus \{i+1\}$. We have that $n_{i+1} > 0$, $n_j = \tilde{n}_j$ for each $j \in \{1, \dots, L-1\} \setminus \{i+1, i+2\}$, and $n_{i+2} = \tilde{n}_{i+2} = \frac{1 - \max\{z_1, \dots, z_i\}}{1 - \max\{z_1, \dots, z_{L-1}\}}$. Hence, $\nu_\tau = \nu_\xi$. Then $\max\{z_1, \dots, z_i, z_{i+2}, \dots, z_L\} = \max\{z_1, \dots, z_L\}$,

$$F(\xi) = \max\{z_1, \dots, z_L\} n_1 + n_{i+1} z_{i+1} > \max\{z_1, \dots, z_L\} n_1 = F(\tau),$$

a contradiction to Theorem 3.

Consider next the case where $i = L-2$. Let $\tau = 1_{(z_1, \dots, z_L)}$ and

$$\xi = 1_{(z_1, \dots, z_{L-2}, z_L)} + n_{L-1} 1_{z_{L-1}},$$

where $n_j = n_j(z_1, \dots, z_L)$ for each $1 \leq j \leq L-1$. Letting $\tilde{n}_j = n_j(z_1, \dots, z_{L-2}, z_L)$ for each $1 \leq j \leq L-2$, we have that $\tilde{n}_j = n_j$ for each $j \in \{1, \dots, L-2\}$ and $n_{L-1} > 0$. Hence, $\nu_\tau = \nu_\xi$. Then $\max\{z_1, \dots, z_{L-2}, z_L\} = \max\{z_1, \dots, z_L\}$, $\max\{z_1, \dots, z_{L-2}\} = \max\{z_1, \dots, z_{L-1}\}$,

$$F(\xi) = \max\{z_1, \dots, z_L\} n_1 + n_{L-1} z_{L-1} > \max\{z_1, \dots, z_L\} n_1 = F(\tau),$$

a contradiction to Theorem 3.

We next show that $z_L > \max\{z_1, \dots, z_{L-1}\}$. Since $z_1 < \dots < z_{L-1}$, it suffices to show that $z_{L-1} < z_L$. Suppose not; then $z_{L-1} \geq z_L$. Let $\tau = 1_{(z_1, \dots, z_L)}$ and

$$\xi = 1_{z_L} + n_{L-1} 1_{(z_1, \dots, z_{L-1})}.$$

Then $\nu_\tau = \nu_\xi$ and, letting $\tilde{n}_1 = n_1(z_1, \dots, z_{L-1})$,

$$F(\xi) = z_L + n_{L-1} \tilde{n}_1 z_{L-1} = z_L + n_1 z_{L-1} > n_1 z_{L-1} = F(\tau),$$

a contradiction to Theorem 3. ■

We now turn to the proof of Theorem 12.1.

Proof of Theorem 12.1. (Necessity) Let (γ, u) be a stable outcome of E . To show that (γ, u) is stable in E^* , it suffices to show condition (ii) of the definition of a stable outcome by Lemma 12.1. Let $(z_1, \dots, z_l, z) \in Y^*$. If $l = 0$, then condition (ii) holds since $u(z) \geq F(z) = F^*(z)$; hence, assume that $l \geq 1$. Let $n_i^* = n_i^*(z_1, \dots, z_l, z)$ and $n_i = n_i(z_1, \dots, z_l, z)$.

Suppose that

$$u(z) + \sum_{i=1}^l n_i^* u(z_i) < F^*(z_1, \dots, z_l, z). \quad (12.1)$$

We may assume in addition that $z_1 \leq \dots \leq z_l$. Formally, (12.1) implies that there exists $\tilde{y} = (\tilde{z}_1, \dots, \tilde{z}_{\tilde{l}}, z) \in Y^*$ such that $\tilde{z}_1 \leq \dots \leq \tilde{z}_{\tilde{l}}$ and $u(z) + \sum_{i=1}^{\tilde{l}} n_i^*(\tilde{y}) u(\tilde{z}_i) < F^*(\tilde{y})$.

To see the above, note first that $u(z_i) \geq z_i > 0$ for each $1 \leq i \leq l$ (due to the stability of (γ, u) in E). Let $i = \min\{j \in \{1, \dots, l-1\} : z_j > z_{j+1}\}$, and consider first the case where $i < l-1$. Let $y_1 = (z_1, \dots, z_i, z_{i+2}, \dots, z_l, z)$ and $\tilde{n}_j = n_j^*(z_1, \dots, z_i, z_{i+2}, \dots, z_l, z)$ for each $j \in \{1, \dots, l\} \setminus \{i+1\}$. We have that $n_{i+1}^* > 0$, $n_j^* = \tilde{n}_j$ for each $j \in \{1, \dots, l\} \setminus \{i+1, i+2\}$ and $n_{i+2}^* = \frac{1-z_{i+1}}{1-z_l} > \frac{1-z_i}{1-z_l} = \tilde{n}_{i+2}$. Hence,

$$\begin{aligned} u(z) + \sum_{i \in \{1, \dots, l\} \setminus \{i+1\}} \tilde{n}_i u(z_i) &= u(z) + \sum_{i=1}^l n_i^* u(z_i) - n_{i+1}^* u(z_{i+1}) - (n_{i+2}^* - \tilde{n}_{i+2}) u(z_{i+2}) \\ &< u(z) + \sum_{i=1}^l n_i^* u(z_i) < F^*(z_1, \dots, z_l, z) = \frac{z}{h(1-z_l)} = F^*(z_1, \dots, z_i, z_{i+2}, \dots, z_l, z). \end{aligned}$$

In the case $i = l-1$, let $y_1 = (z_1, \dots, z_{l-1}, z)$ and multiply both sides of

$$u(z) + \frac{1}{h(1-z_l)} u(z_1) + \sum_{i=2}^l \frac{1-z_{i-1}}{1-z_l} u(z_i) < \frac{z}{h(1-z_l)}$$

by $\frac{1-z_l}{1-z_{l-1}} > 1$ to obtain

$$\frac{1-z_l}{1-z_{l-1}} u(z) + \frac{1}{h(1-z_{l-1})} u(z_1) + \sum_{i=2}^{l-1} \frac{1-z_{i-1}}{1-z_{l-1}} u(z_i) + u(z_l) < \frac{z}{h(1-z_{l-1})};$$

hence,

$$\begin{aligned}
u(z) + \sum_{i \in \{1, \dots, l\} \setminus \{i+1\}} n_i^*(y_1) u(z_i) &= \\
u(z) + \frac{1}{h(1 - z_{l-1})} u(z_1) + \sum_{l=2}^{l-1} \frac{1 - z_{i-1}}{1 - z_{l-1}} u(z_i) &< \\
\frac{1 - z_l}{1 - z_{l-1}} u(z) + \frac{1}{h(1 - z_{l-1})} u(z_1) + \sum_{l=2}^{l-1} \frac{1 - z_{i-1}}{1 - z_{l-1}} u(z_i) + u(z_l) &< \\
\frac{z}{h(1 - z_{l-1})} &= F^*(y_1).
\end{aligned}$$

Thus, by induction, obtain $y_n = (z_{n,1}, \dots, z_{n,l_n})$ such that, if $i_n = \min\{j \in \{1, \dots, l_n - 1\} : z_{n,j} > z_{n,j+1}\}$, then $i_n \geq i_{n-1}$ and, if $i_n = i_{n-1}$, then $|\{j \in \{1, \dots, l_n\} : j > i_n\}| < |\{j \in \{1, \dots, l_{n-1}\} : j > i_{n-1}\}|$. Since $l \in \mathbb{N}$, there is n such that $\{j \in \{1, \dots, l_n - 1\} : z_{n,j} > z_{n,j+1}\} = \emptyset$; hence, let $\tilde{y} = y_n$.

It then follows from $z_1 \leq \dots \leq z_l$ that $n_i^* = n_i$ for each $1 \leq i \leq l$ and, hence,

$$\begin{aligned}
u(z) + \sum_{i=1}^l n_i u(z_i) &< \frac{z}{h(1 - z_l)} = \frac{z}{h(1 - \max\{z_1, \dots, z_l\})} \\
&\leq \frac{\max\{z_1, \dots, z_l, z\}}{h(1 - \max\{z_1, \dots, z_l\})} = F(z_1, \dots, z_l, z).
\end{aligned}$$

But this contradicts the stability of (γ, u) in E . This contradiction then shows that condition (ii) of the definition of a stable outcome holds and that (γ, u) is a stable outcome of E^* .

(Sufficiency) Let (γ, u) be a stable outcome of E^* . To show that (γ, u) is stable in E , it suffices to show condition (ii) of the definition of a stable outcome by Lemma 9.2. Let $(z_1, \dots, z_l, z) \in Y$. If $l = 0$, then condition (ii) holds since $u(z) \geq F^*(z) = F(z)$; hence, assume that $l \geq 1$. Let $n_i^* = n_i^*(z_1, \dots, z_l, z)$ and $n_i = n_i(z_1, \dots, z_l, z)$.

Suppose that

$$u(z) + \sum_{i=1}^l n_i u(z_i) < F(z_1, \dots, z_l, z).$$

Since $u(z_i) \geq z_i > 0$ for each $1 \leq i \leq l$ (due to the stability of (γ, u) in E^*), it follows that we may assume that $z_1 \leq \dots \leq z_l$. Indeed, if not, then $z_i > z_{i+1}$ for some $1 \leq i < l$ and replace $(z_1, \dots, z_l, z)$ with y , where $y = (z_1, \dots, z_i, z_{i+2}, \dots, z_l, z)$

if $i + 1 < l$ and $y = (z_1, \dots, z_{l-1}, z)$ if $i + 1 = l$, while still obtaining

$$u(z) + \sum_{i \in \{1, \dots, l\} \setminus \{i+1\}} n_i(y) u(z_i) < F(y);$$

thus, after at most $l - 1$ steps, obtain $(\tilde{z}_1, \dots, \tilde{z}_{\tilde{l}}, z)$ such that $\tilde{z}_1 \leq \dots \leq \tilde{z}_{\tilde{l}}$ and

$$u(z) + \sum_{i=1}^{\tilde{l}} n_i(\tilde{z}_1, \dots, \tilde{z}_{\tilde{l}}, z) u(\tilde{z}_i) < F(\tilde{z}_1, \dots, \tilde{z}_{\tilde{l}}, z).$$

To see the above, assume that $z_i > z_{i+1}$ for some $1 \leq i < l$. Let $\tilde{n}_j = n_j(y)$ for each $j \in \{1, \dots, l\} \setminus \{i + 1\}$. We have that $n_{i+1} > 0$ and $n_j = \tilde{n}_j$ for each $j \in \{1, \dots, l\} \setminus \{i + 1\}$. Hence,

$$\begin{aligned} u(z) + \sum_{i \in \{1, \dots, l\} \setminus \{i+1\}} \tilde{n}_i u(z_i) &= u(z) + \sum_{i=1}^l n_i u(z_i) - n_{i+1} u(z_{i+1}) \\ &< u(z) + \sum_{i=1}^l n_i u(z_i) < F(z_1, \dots, z_l, z) = \frac{\max\{z_1, \dots, z_l, z\}}{h(1 - \max\{z_1, \dots, z_l\})} = F(y). \end{aligned}$$

It then follows from $z_1 \leq \dots \leq z_l$ that $n_i^* = n_i$ for each $1 \leq i \leq l$ and, hence,

$$u(z) + \sum_{i=1}^l n_i^* u(z_i) < F(z_1, \dots, z_l, z) = \frac{\max\{z, z_l\}}{h(1 - z_l)}.$$

This contradicts the stability of (γ, u) in E^* if $z \geq z_l$ since then $F(z_1, \dots, z_l, z) = F^*(z_1, \dots, z_l, z)$.

Thus, assume that $z < z_l$; hence,

$$u(z) + \sum_{i=1}^l n_i^* u(z_i) < \frac{z_l}{h(1 - z_l)}.$$

Since $u(z) \geq z > 0$, it follows that

$$\frac{1}{h(1 - z_l)} u(z_1) + \sum_{i=2}^l \frac{1 - z_{i-1}}{1 - z_l} u(z_i) < \frac{z_l}{h(1 - z_l)}.$$

Multiplying both sides by $\frac{1 - z_l}{1 - z_{l-1}}$ if $l > 1$, and by $h(1 - z_l)$ if $l = 1$, and letting $y = (z_1, \dots, z_l) \in Y$ yields

$$\begin{aligned} u(z_l) + \sum_{i=1}^{l-1} n_i^*(y) u(z_i) &= u(z_l) + \frac{1}{h(1 - z_{l-1})} u(z_1) + \sum_{i=2}^{l-1} \frac{1 - z_{i-1}}{1 - z_{l-1}} u(z_i) \\ &< \frac{z_l}{h(1 - z_{l-1})} = F^*(y) \end{aligned}$$

if $l > 1$ and $u(z_1) < z_1$ if $l = 1$. But this contradicts the stability of (γ, u) in E^* . This contradiction then shows that condition (ii) of the definition of a stable outcome holds and that (γ, u) is a stable outcome of E . ■

13 The topological space $\cup_{L=1}^{\bar{L}} (Z \times N)^L$

In this section, we establish some basic properties of the topological space $\cup_{L=1}^{\bar{L}} (Z \times N)^L$. Let $X = \cup_{L=1}^{\bar{L}} (Z \times N)^L$ and, for each $L = 1, \dots, \bar{L}$, $X_L = (Z \times N)^L$. We endow X_L with the product topology and X with the sum topology, i.e. $V \subseteq X$ is open if and only if $V \cap X_L$ is open in X_L for all $1 \leq L \leq \bar{L}$.

We first note that X is metrizable. For each $1 \leq L \leq \bar{L}$, X_L is a metric space and we may assume that its metric d_L is bounded above by 1. Then define a metric d in X by setting, for each $x, y \in X$,

$$d(x, y) = \begin{cases} d_L(x, y), & \text{if both } x \text{ and } y \text{ belong to } X_L, \\ 1, & \text{otherwise.} \end{cases}$$

One easily checks that d is indeed a metric on X . It induces the sum topology on X . Indeed, for each $x \in X$ and $\varepsilon > 0$, $B_\varepsilon(x)$ is open in the sum topology since, letting $1 \leq J \leq \bar{L}$ be such that $x \in X_J$, $B_\varepsilon(x) \cap X_J = \{y \in X_J : d_J(x, y) < \varepsilon\}$, while for $L \neq J$ this intersection is empty if $\varepsilon \leq 1$ and all of X_L if $\varepsilon > 1$. Conversely, if V is open in the sum topology and $x \in V$, then there is $1 \leq L \leq \bar{L}$ such that $x \in X_L$ and $V \cap X_L$ is open in X_L . Thus, there is $\varepsilon \in (0, 1)$ such that $x \in \{y \in X_L : d_L(x, y) < \varepsilon\} \subseteq V \cap X_L$. Since $\varepsilon < 1$, it follows that $B_\varepsilon(x) = \{y \in X_L : d_L(x, y) < \varepsilon\}$ and, hence, $x \in B_\varepsilon(x) \subseteq V \cap X_L \subseteq V$.

Next, note that X is compact. This follows because if $\{O_\alpha\}_\alpha$ is an open cover of X , then, for each $1 \leq L \leq \bar{L}$, $\{O_\alpha \cap X_L\}_\alpha$ is an open cover X_L . Thus, for some $\{O_{\alpha_{L,1}}, \dots, O_{\alpha_{L,J_L}}\}$, $X_L \subseteq \cup_{j=1}^{J_L} O_{\alpha_{L,j}}$ and, hence, $X \subseteq \cup_{(L,j)} O_{\alpha_{L,j}}$.

We also note that $S \subseteq X$ is closed if and only if $S \cap X_L$ is closed in X_L for each $1 \leq L \leq \bar{L}$. Indeed, S is closed if and only if S^c is open, if and only if $S^c \cap X_L$ is open in X_L for each L , if and only if $X_L \setminus (S^c \cap X_L) = S \cap X_L$ is closed in X_L for each L .

Finally, note that if a sequence converges to a point in some component X_L , then the sequence must eventually be contained in X_L . Indeed, let $\{x_n\}_{n=1}^\infty$ be a sequence in X converging to some $x \in X_L$, with $1 \leq L \leq \bar{L}$. Since X_L is an open neighborhood of x , it follows that there exists $N \in \mathbb{N}$ such that $x_n \in X_L$ for each $n \geq N$.

14 A network-flow formulation

We give a sufficient condition on Y and F such that the surplus maximization problem has an equivalent network-flow formulation for a knowledge economy with a finite type space Z containing K elements and no externalities.

Assumption 14.1 *For each $1 \leq i < \bar{L}$, there exist $R_i \subseteq Z \times Z$ and $c_i : R_i \rightarrow \mathbb{R}_{++}$, and for each $1 \leq L \leq \bar{L}$, there exists $g_L : Z \rightarrow \mathbb{R}_+$ such that:*

- (i) *For $2 \leq L \leq \bar{L}$, the feasible firms are exactly the tuples $y = (z_i, n_i)_{i=1}^L$ satisfying $n_L = 1$ and, for each $1 \leq i < L$,*

$$\begin{aligned} (z_i, z_{i+1}) &\in R_i, \\ \frac{n_{i+1}}{n_1} &= c_i(z_i, z_{i+1}), \end{aligned} \tag{14.1}$$

- (ii) *For every feasible firm $y = (z_i, n_i)_{i=1}^L$,*

$$F(y) = n_1 g_L(z_L).$$

The flexible technology considered in Sections 4 and 5 of the paper satisfies Assumption 14.1 with $R_i = Z \times Z$, $c_i(a, b) = h(1 - a)^\sigma$ and $g_L(a) = a^\alpha$ for each $a, b \in Z$, $1 \leq i < \bar{L}$ and $1 \leq L \leq \bar{L}$.

Since each feasible knowledge profile $z = (z_1, \dots, z_L)$ uniquely determines all layer sizes, we identify a firm with $z = (z_1, \dots, z_L)$ and (abusing notation) continue to denote the set of feasible firms by Y . For each $z \in Y$ and $q \in Z$, write $\nu_q = \nu(\{q\})$, $\gamma_z = \gamma(\{z\})$, and let

$$a_z(q) = \sum_{i=1}^{\ell(z)} n_i(z) 1_{\{q\}}(z_i)$$

be the total amount of type q employed in firm z . The surplus maximization problem is

$$\begin{aligned} & \max_{\gamma \in \mathbb{R}_+^Y} \sum_{z \in Y} F(z) \gamma_z \\ & \text{subject to } \sum_{z \in Y} a_z(q) \gamma_z = \nu_q \text{ for each } q \in Z. \end{aligned} \tag{14.2}$$

For each $1 \leq i \leq \bar{L}$, let $Z_i = \{z_i : z \in Y \text{ and } \ell(z) \geq i\}$ be the set of types that belong to layer i of some feasible firm. We construct a directed graph $(V, \mathcal{E})$ with vertices s, t and (i, a) , for each $1 \leq i \leq \bar{L}$ and $a \in Z_i$, i.e.

$$V = \{s, t\} \cup \{(i, a) : 1 \leq i \leq \bar{L} \text{ and } a \in Z_i\}.$$

For $w, v \in V$, we write $w \rightarrow v$ to refer to the ordered pair (w, v) . The set of arcs $\mathcal{E}$ consists of the following ordered pairs of vertices:

- (i) For each $a \in Z_1$, an entry arc $s \rightarrow (1, a)$,
- (ii) For each $1 \leq i < \bar{L}$, $a \in Z_i$ and b such that $(a, b) \in R_i$, a transition arc $(i, a) \rightarrow (i+1, b)$,
- (iii) For each $(i, a) \in V \setminus \{s, t\}$, an exit arc $(i, a) \rightarrow t$.

An entry arc $s \rightarrow (1, a)$ uses one unit of a . A transition arc $(i, a) \rightarrow (i+1, b)$ uses $c_i(a, b)$ units of b . An exit arc $(i, a) \rightarrow t$ uses no labor and yields output $g_i(a)$; all other arcs yield zero output. For each $e \in \mathcal{E}$, denote its labor input coefficients by $(c_e(q))_{q \in Z}$ and its output coefficient by r_e , i.e. for each $q \in Z$,

$$c_e(q) = \begin{cases} 1_{\{q\}}(a) & \text{if } e = s \rightarrow (1, a), \\ c_i(a, b) 1_{\{q\}}(b) & \text{if } e = (i, a) \rightarrow (i+1, b), \\ 0 & \text{if } e = (i, a) \rightarrow t, \end{cases}$$

and

$$r_e = \begin{cases} g_i(a) & \text{if } e = (i, a) \rightarrow t, \\ 0 & \text{otherwise.} \end{cases}$$

For each $v_0, \dots, v_m \in V$ such that $v_{k-1} \rightarrow v_k \in \mathcal{E}$ for each $1 \leq k \leq m$, $p = v_0 \rightarrow \dots \rightarrow v_m$ is a path from v_0 to v_m . We identify each path with its set of arcs, i.e.

$p = \{v_{k-1} \rightarrow v_k : 1 \leq k \leq m\}$. For each $z = (z_1, \dots, z_L) \in Y$, let p_z be the path $s \rightarrow (1, z_1) \rightarrow \dots \rightarrow (L, z_L) \rightarrow t$. By Assumption 14.1, $z \mapsto p_z$ is a bijection between Y and the set of s - t paths. Moreover, for each $z \in Y$,

$$\frac{a_z(q)}{n_1(z)} = \sum_{e \in p_z} c_e(q) \text{ for each } q \in Z \text{ and } \frac{F(z)}{n_1(z)} = \sum_{e \in p_z} r_e. \quad (14.3)$$

For each $v \in V$, let $\mathcal{E}^+(v) = \{(v, w) \in \mathcal{E} : w \in V\}$ and $\mathcal{E}^-(v) = \{(w, v) \in \mathcal{E} : w \in V\}$ be the sets of arcs leaving and entering v , respectively. Consider the linear program of choosing a flow through the network to maximize output:

$$\begin{aligned} & \max_{f \in \mathbb{R}_+^{\mathcal{E}}} \sum_{e \in \mathcal{E}} r_e f_e \\ & \text{subject to } \sum_{e \in \mathcal{E}^+(v)} f_e = \sum_{e \in \mathcal{E}^-(v)} f_e \text{ for each } v \in V \setminus \{s, t\}, \\ & \sum_{e \in \mathcal{E}} c_e(q) f_e = \nu_q \quad \text{for each } q \in Z. \end{aligned} \quad (14.4)$$

The first set of constraints, flow conservation, requires equality of flows in and out of each $v \in V \setminus \{s, t\}$ and does not have an explicit counterpart in (14.2) because every flow induced by an assignment satisfies it. Conversely, as we show, it implies that every feasible flow is induced by some assignment. Note that the program has $O(\bar{L}K^2)$ variables and $O(\bar{L}K)$ constraints. The next proposition shows that solving the network-flow program is equivalent to solving the surplus maximization problem.

Proposition 14.1 *Under Assumption 14.1, if γ solves (14.2), then the flow $f = (f_e)_{e \in \mathcal{E}}$ defined by setting, for each $e \in \mathcal{E}$,*

$$f_e = \sum_{z \in Y: e \in p_z} n_1(z) \gamma_z$$

solves (14.4). Conversely, if f solves (14.4), it admits a nonnegative path decomposition, i.e. some $x \in \mathbb{R}_+^Y$ such that $f_e = \sum_{z \in Y: e \in p_z} x_z$ for each $e \in \mathcal{E}$, and any such decomposition yields a solution to (14.2) by setting $\gamma_z = x_z/n_1(z)$ for each $z \in Y$.

Proof. Given a feasible assignment γ , define f as in the statement. Clearly, $f \geq 0$. Each path p_z visits each vertex at most once. Hence, for each $v \in V \setminus \{s, t\}$,

p_z contains exactly one arc entering v and exactly one arc leaving v if it visits v , and no such arcs otherwise, i.e.

$$\sum_{e \in \mathcal{E}^-(v)} 1_{p_z}(e) = \sum_{e \in \mathcal{E}^+(v)} 1_{p_z}(e) \quad \text{for each } z \in Y \text{ and } v \in V \setminus \{s, t\}. \quad (14.5)$$

Since $f_e = \sum_{z \in Y} n_1(z) \gamma_z 1_{p_z}(e)$ for each $e \in \mathcal{E}$, it follows that, for each $v \in V \setminus \{s, t\}$,

$$\sum_{e \in \mathcal{E}^-(v)} f_e = \sum_{z \in Y} n_1(z) \gamma_z \sum_{e \in \mathcal{E}^-(v)} 1_{p_z}(e) = \sum_{z \in Y} n_1(z) \gamma_z \sum_{e \in \mathcal{E}^+(v)} 1_{p_z}(e) = \sum_{e \in \mathcal{E}^+(v)} f_e.$$

Moreover, for each $q \in Z$,

$$\begin{aligned} \sum_{e \in \mathcal{E}} c_e(q) f_e &= \sum_{e \in \mathcal{E}} c_e(q) \sum_{z \in Y} n_1(z) \gamma_z 1_{p_z}(e) = \sum_{z \in Y} n_1(z) \gamma_z \sum_{e \in \mathcal{E}} c_e(q) 1_{p_z}(e) \\ &= \sum_{z \in Y} n_1(z) \gamma_z \sum_{e \in p_z} c_e(q) = \sum_{z \in Y} a_z(q) \gamma_z = \nu_q \end{aligned}$$

and

$$\begin{aligned} \sum_{e \in \mathcal{E}} r_e f_e &= \sum_{e \in \mathcal{E}} r_e \sum_{z \in Y} n_1(z) \gamma_z 1_{p_z}(e) = \sum_{z \in Y} n_1(z) \gamma_z \sum_{e \in \mathcal{E}} r_e 1_{p_z}(e) \\ &= \sum_{z \in Y} n_1(z) \gamma_z \sum_{e \in p_z} r_e = \sum_{z \in Y} F(z) \gamma_z, \end{aligned}$$

where, in both cases, the fourth equality follows from (14.3); the last equality in the first case holds since γ is feasible for (14.2). Hence, f is feasible for (14.4) and has the same output as γ .

Next, we show that each $f \in \mathbb{R}_+^{\mathcal{E}}$ satisfying the flow conservation constraints can be written as $f_e = \sum_{z \in Y: e \in p_z} x_z$ for each $e \in \mathcal{E}$, for some $x \in \mathbb{R}_+^Y$. The proof is by induction on the number $m(f)$ of arcs e such that $f_e > 0$. If $m(f) = 0$, let $x = 0$. Suppose that $m(f) > 0$ and that the claim holds for each flow f' with $m(f') < m(f)$. Let $e_0 = (v, w) \in \mathcal{E}$ be such that $f_{e_0} > 0$. If $w \neq t$, then the flow conservation constraint at w implies that

$$\sum_{e \in \mathcal{E}^+(w)} f_e = \sum_{e \in \mathcal{E}^-(w)} f_e \geq f_{e_0} > 0,$$

so some arc leaving w has positive flow; iterate until we reach t . Similarly, if $v \neq s$, then the flow conservation constraint at v implies that some arc entering v has positive

flow; iterate until we reach s . This yields an s - t path p_{z^*} such that $f_e > 0$ for each $e \in p_{z^*}$. Let $x^* = \min_{e \in p_{z^*}} f_e$ and $f' = f - x^* 1_{p_{z^*}}$. Then $0 \leq f' \leq f$, f' satisfies the flow conservation constraints since f does and using (14.5), and $m(f') < m(f)$ since $f'_e = 0$ for some $e \in p_{z^*}$. By the induction hypothesis, there exists $x' \in \mathbb{R}_+^Y$ such that $f'_e = \sum_{z \in Y: e \in p_z} x'_z$ for each $e \in \mathcal{E}$. Letting $x_{z^*} = x'_{z^*} + x^*$ and $x_z = x'_z$ for each $z \neq z^*$, we obtain $f_e = \sum_{z \in Y: e \in p_z} x_z$ for each $e \in \mathcal{E}$.

Let f be feasible for (14.4), let $x \in \mathbb{R}_+^Y$ be such that $f_e = \sum_{z \in Y: e \in p_z} x_z$ for each $e \in \mathcal{E}$, and let $\gamma_z = x_z/n_1(z)$ for each $z \in Y$, so that $\gamma \geq 0$. Since $f_e = \sum_{z \in Y} x_z 1_{p_z}(e)$ for each $e \in \mathcal{E}$, we have, for each $q \in Z$,

$$\begin{aligned} \sum_{z \in Y} a_z(q) \gamma_z &= \sum_{z \in Y} x_z \sum_{e \in p_z} c_e(q) = \sum_{z \in Y} x_z \sum_{e \in \mathcal{E}} c_e(q) 1_{p_z}(e) \\ &= \sum_{e \in \mathcal{E}} c_e(q) \sum_{z \in Y} x_z 1_{p_z}(e) = \sum_{e \in \mathcal{E}} c_e(q) f_e = \nu_q \end{aligned}$$

and

$$\begin{aligned} \sum_{z \in Y} F(z) \gamma_z &= \sum_{z \in Y} x_z \sum_{e \in p_z} r_e = \sum_{z \in Y} x_z \sum_{e \in \mathcal{E}} r_e 1_{p_z}(e) \\ &= \sum_{e \in \mathcal{E}} r_e \sum_{z \in Y} x_z 1_{p_z}(e) = \sum_{e \in \mathcal{E}} r_e f_e, \end{aligned}$$

where, in both cases, the first equality follows from (14.3) and $\gamma_z = x_z/n_1(z)$; the last equality in the first case holds since f is feasible for (14.4). Hence, γ is feasible for (14.2) and has the same output as f .

Let γ solve (14.2) and let f be defined as in the statement. If f' is feasible for (14.4) and γ' is obtained from a path decomposition of f' as above, then γ' is feasible for (14.2) and, hence,

$$\sum_{e \in \mathcal{E}} r_e f'_e = \sum_{z \in Y} F(z) \gamma'_z \leq \sum_{z \in Y} F(z) \gamma_z = \sum_{e \in \mathcal{E}} r_e f_e.$$

Thus, f solves (14.4). Conversely, let f solve (14.4) and let γ be obtained from a path decomposition of f . If γ' is feasible for (14.2) and f' is defined from γ' as in the statement, then f' is feasible for (14.4) and, hence,

$$\sum_{z \in Y} F(z) \gamma'_z = \sum_{e \in \mathcal{E}} r_e f'_e \leq \sum_{e \in \mathcal{E}} r_e f_e = \sum_{z \in Y} F(z) \gamma_z.$$

Thus, γ solves (14.2). ■

Finally, we show equivalence of the duals. The dual of (14.2) is

$$\begin{aligned} & \min_{u \in \mathbb{R}^Z} \sum_{q \in Z} u_q \nu_q \\ & \text{subject to } \sum_{q \in Z} a_z(q) u_q \geq F(z) \text{ for each } z \in Y. \end{aligned} \tag{14.6}$$

The dual of (14.4) is

$$\begin{aligned} & \min_{u \in \mathbb{R}^Z, \eta \in \mathbb{R}^V} \sum_{q \in Z} u_q \nu_q \\ & \text{subject to } \sum_{q \in Z} c_e(q) u_q + \eta_v - \eta_w \geq r_e \text{ for each } e = (v, w) \in \mathcal{E}, \\ & \eta_s = \eta_t = 0. \end{aligned} \tag{14.7}$$

For $u \in \mathbb{R}^Z$, define the cost of arc e by

$$d_e(u) = \sum_{q \in Z} c_e(q) u_q - r_e.$$

Set $\eta_s(u) = \eta_t(u) = 0$ and, for each $v \in V \setminus \{s, t\}$, let

$$\eta_v(u) = \min_{p \in P(s, v)} \sum_{e \in p} d_e(u), \tag{14.8}$$

where $P(s, v)$ is the set of all paths from s to v .

Proposition 14.2 *Under Assumption 14.1, if u solves (14.6), then $(u, \eta(u))$ solves (14.7). Conversely, if (u, η) solves (14.7), then u solves (14.6).*

Proof. Let u be feasible for (14.6). By (14.3), the cost of every s - t path p_z satisfies

$$\sum_{e \in p_z} d_e(u) = \frac{\sum_{q \in Z} a_z(q) u_q - F(z)}{n_1(z)} \geq 0.$$

Consider an arc $e = (v, w)$ with $w \neq t$. A minimum-cost path from s to v , followed by e , is a path from s to w . If $v = s$, take the empty path, whose cost is zero. Hence

$$\eta_w(u) \leq \eta_v(u) + d_e(u).$$

For an exit arc $e = (v, t)$, a minimum-cost path from s to v followed by e is an s - t path. Its cost is nonnegative, so

$$\eta_v(u) + d_e(u) \geq 0 = \eta_t(u).$$

Thus every arc $e = (v, w)$ satisfies $d_e(u) + \eta_v(u) - \eta_w(u) \geq 0$ and hence the constraints in (14.7) are satisfied. Therefore $(u, \eta(u))$ is feasible for that problem.

Conversely, let (u, η) be feasible for (14.7). Let $z \in Y$ and write p_z as $s = v_0 \rightarrow v_1 \rightarrow \dots \rightarrow v_m = t$, so that the arcs of p_z are $v_{k-1} \rightarrow v_k$, for each $k \in \{1, \dots, m\}$. Summing the constraints of (14.7) over these arcs and using $\sum_{k=1}^m (\eta_{v_{k-1}} - \eta_{v_k}) = \eta_s - \eta_t$ gives

$$\sum_{e \in p_z} \sum_{q \in Z} c_e(q) u_q + \eta_s - \eta_t \geq \sum_{e \in p_z} r_e.$$

Since $\eta_s = \eta_t = 0$, multiplying by $n_1(z) > 0$ and applying (14.3) yields

$$\sum_{q \in Z} a_z(q) u_q \geq F(z).$$

This holds for every $z \in Y$, so u is feasible for (14.6).

Let u solve (14.6). If (u', η') is feasible for (14.7), then u' is feasible for (14.6) and, hence, $\sum_{q \in Z} u'_q \nu_q \geq \sum_{q \in Z} u_q \nu_q$. Since $(u, \eta(u))$ is feasible for (14.7), it solves (14.7). Conversely, let (u, η) solve (14.7). If u' is feasible for (14.6), then $(u', \eta(u'))$ is feasible for (14.7) and, hence, $\sum_{q \in Z} u'_q \nu_q \geq \sum_{q \in Z} u_q \nu_q$. Since u is feasible for (14.6), it solves (14.6). ■

15 Numerical Appendix

This appendix describes the discretization of knowledge, the computation of stable outcomes, and the calibration procedure.

Knowledge grid. We discretize $Z = [0.001, 0.999]$ using the knowledge grid $Z_J = \{z_1, \dots, z_J\}$, where $z_1 < \dots < z_J$. The grid is fixed across parameter values. It combines three grids with approximately equal numbers of points: one uniform in z , one uniform in $\log(1 - z)$ to refine the upper end, and one uniform in $\log z$ to refine

the lower end. Taking their union and removing duplicates gives J points.¹⁴ We use $J = 401$ for the organizational comparative statics in Section 5.1 and $J = 301$ for the calibrations in Section 5.2.

For given (β_1, β_2) , let $B(z)$ be the CDF of a $\text{Beta}(\beta_1, \beta_2)$ variable. Holding the grid points fixed as these parameters vary, we assign each interval $(z_{j-1}, z_j]$ its exact probability conditional on knowledge lying in Z :

$$\nu_1 = 0, \quad \nu_j = \frac{B(z_j) - B(z_{j-1})}{B(0.999) - B(0.001)}, \quad j = 2, \dots, J.$$

Each interval's population mass ν_j is assigned to its right endpoint z_j . The smallest grid point $z_1 = 0.001$ is assigned zero mass.

Equilibrium computation. For the technologies used in the numerical exercises, a firm's knowledge profile uniquely determines its layer sizes. We therefore identify a firm q with $(z_1(q), \dots, z_{\ell(q)}(q))$ and write

$$Y_J = \bigcup_{L=1}^{\bar{L}} Z_J^L$$

for the finite set of feasible firms in the discretized economy. Layer i of firm q contains $n_i(q)$ agents of type $z_i(q)$, and the firm's output is $F(q)$. Firms with $\ell(q) = 1$ represent self-employment.

Let γ_q denote the mass of firms of type q . The surplus-maximization problem is

¹⁴For each component grid with m points, evaluate its map at $t_r = (r-1)/(m-1)$ for $r = 1, \dots, m$, so that the t_r are equally spaced on $[0, 1]$. The three maps are

$$\begin{aligned} z_{\text{uniform}}(t) &= z_1 + (z_J - z_1)t, \\ z_{\text{upper}}(t) &= 1 - \exp\{(1-t)\log(1-z_1) + t\log(1-z_J)\}, \\ z_{\text{lower}}(t) &= \exp\{(1-t)\log z_1 + t\log z_J\}. \end{aligned}$$

The component sizes (uniform, upper, lower) are $(102, 102, 101)$ for $J = 301$ and $(135, 135, 135)$ for $J = 401$. Each component includes both endpoints; removing the duplicate endpoints from their union yields exactly J points.

the linear program

$$\begin{aligned}
& \max_{\{\gamma_q\}_{q \in Y_J}} \quad \sum_{q \in Y_J} F(q) \gamma_q \\
& \text{subject to} \quad \sum_{q \in Y_J} \gamma_q \sum_{i=1}^{\ell(q)} 1_{\{z_j\}}(z_i(q)) n_i(q) = \nu_j, \quad j = 1, \dots, J, \\
& \quad \gamma_q \geq 0, \quad q \in Y_J.
\end{aligned}$$

The corresponding dual problem is

$$\begin{aligned}
& \min_{\{u(z_j)\}_{j=1}^J} \quad \sum_{j=1}^J u(z_j) \nu_j \\
& \text{subject to} \quad \sum_{i=1}^{\ell(q)} n_i(q) u(z_i(q)) \geq F(q), \quad q \in Y_J.
\end{aligned}$$

The dual variables $u(z_j)$, also written u_j , are equilibrium earnings. For every firm used with positive mass, complementary slackness implies that its output equals the sum of the earnings of its members.

In the multi-layer economy, we solve an equivalent sparse network-flow formulation (see Section 14). This replaces the enumeration of $O(J^{\bar{L}})$ firm types by $O(\bar{L}J^2)$ arc variables. The formulation also uses ordered knowledge types and a nonbinding numerical firm-size cap. Establishment counts for firms with several layers are recovered from a path decomposition of the optimal flow.

Calibration targets. To form model earnings percentiles, we sort the u_j with their population masses ν_j and take the first earnings value at which the cumulative mass reaches percentile p . These percentiles include all agents, so self-employed agents enter with their equilibrium earnings and population masses whenever self-employment occurs. Denote this weighted percentile by $u_p(\theta)$, where θ is the parameter vector.¹⁵ For $p = 5, \dots, 95$, define

$$y_p(\theta) = \log u_p(\theta) - \log u_{50}(\theta).$$

¹⁵Reordering Z_L if necessary, assume that $u(z_i) \leq u(z_j)$ whenever $i < j$. Then $u_p(\theta) = u(z_j)$ where $j \in \{1, \dots, J\}$ is such that $\sum_{l=1}^{j-1} \nu_l < p/100 \leq \sum_{l=1}^j \nu_l$.

Let x_p denote the corresponding empirical log wage relative to the median. The wage-distribution loss is

$$Q_w(\theta) = \frac{1}{91} \sum_{p=5}^{95} [x_p - y_p(\theta)]^2.$$

We compute model establishment shares in the four employment categories used in the data, excluding self-employment. Employment counts workers and middle managers but not the top manager. If $m_k(\theta)$ and $\hat{m}_k$ denote the model and empirical shares in category k , respectively, the establishment-size loss is

$$Q_s(\theta) = \frac{1}{4} \sum_{k=1}^4 [\hat{m}_k - m_k(\theta)]^2.^{16}$$

The main calibration objective is

$$Q(\theta) = \max\{Q_w(\theta), Q_s(\theta)\}.$$

For the exercise that also targets the top 1% income-share benchmark $\hat{t}$, we rank types by earnings and include exactly the highest 1% of population mass, splitting a type's mass if the cutoff falls within it. If $t(\theta)$ is this group's share of total model earnings, the additional loss is $Q_t(\theta) = [t(\theta) - \hat{t}]^2$ and the objective becomes $\max\{Q_w(\theta), Q_s(\theta), Q_t(\theta)\}.$ ¹⁷

Parameter search. We minimize the calibration objective using a TikTak-style multistart procedure (see Arnoud, Guvenen, and Kleineberg (2019)) that combines Nelder–Mead local search with quasi-random Sobol draws. Let N denote the number of local searches. We proceed as follows:

Step 1: Choose an initial parameter vector θ_1 and set $i = 1$.

Step 2: Starting from θ_i , apply the Nelder–Mead simplex method to search for a local minimum of Q , and retain the parameter vector $\tilde{\theta}_i$ with the lowest objective value found during the search.

¹⁶Let $m(q) = \sum_{i=1}^{\ell(q)-1} n_i(q)$ for each $q \in Y_J$, and let I_k denote the k th employment category. Then $m_k(\theta) = \frac{\sum_{q \in Y_J: \ell(q) \geq 2, m(q) \in I_k} \gamma_q}{\sum_{q \in Y_J: \ell(q) \geq 2} \gamma_q}.$

¹⁷In the notation of Footnote 15, let $j \in \{1, \dots, J\}$ be such that $\sum_{l=j+1}^J \nu_l < 0.01 \leq \sum_{l=j}^J \nu_l$, and set $x = 0.01 - \sum_{l=j+1}^J \nu_l$. Then $t(\theta) = (\sum_{l=j+1}^J \nu_l u(z_l) + x u(z_j)) / (\sum_{l=1}^J \nu_l u(z_l)).$

Step 3: Let $\tilde{\theta}_i^*$ denote the best candidate obtained up to iteration i , so that

$$Q(\tilde{\theta}_i^*) = \min_{1 \leq j \leq i} Q(\tilde{\theta}_j).$$

Step 4: Obtain the next starting value θ_{i+1} as a convex combination of $\tilde{\theta}_i^*$ and a new parameter vector $\hat{\theta}_i$:

$$\theta_{i+1} = f(i)\tilde{\theta}_i^* + (1 - f(i))\hat{\theta}_i,$$

where $f(i) = \min\{0.95, i/N\}$. Here $\hat{\theta}_i$ is obtained by linearly rescaling each coordinate of a scrambled Sobol point from $[0, 1]$ to the corresponding parameter interval. Early iterations therefore explore the parameter space broadly, while later iterations place greater weight on the best candidate found so far.

Step 5: If $i < N$, increase i by one and return to Step 2. Otherwise, report the candidate with the lowest objective value.

We use $N = 20$. For the GRH models, the search region is $(h, \beta_1, \beta_2) \in [0.05, 1] \times [0.5, 30]^2$. For the flexible model, it is $(\alpha, \sigma, h, \beta_1, \beta_2) \in [0.5, 3] \times [0.25, 50] \times [0.05, 1] \times [0.25, 3] \times [0.25, 50]$.

16 Wage Polarization in the Flexible Model

The flexible model is one of the alternative organizational technologies compared in the paper. Here we use it for a further numerical illustration. As documented by Acemoglu and Autor (2011), U.S. wages rose at both the lower and upper tails relative to the median between 1988 and 2008. We ask whether two calibrations of the flexible model, with the knowledge distribution held fixed and only technology parameters varying, can generate a similar pattern. Because wage percentiles in both years are calibration targets, this is a possibility result within the model: it neither tests the model out of sample nor identifies the historical causes of wage polarization.

The 2008 cross-sectional comparison shows how the alternative technologies fit one wage distribution, but leaves open what changes in its shape they can accommodate. Polarization is a useful case because relative gains at both tails require more

than a simple increase in dispersion. The two-date exercise and parameter counterfactuals therefore reveal whether, and through which margin, the flexible technology can generate such a change.

The flexible model allows self-employment and manager–worker firms. Knowledge z is distributed on $Z = [0.001, 0.999]$ according to a $\text{Beta}(\beta_1, \beta_2)$ distribution. A self-employed individual of type z produces z^α . In a manager–worker firm employing workers with knowledge z_1 and a manager with knowledge z_2 , span of control and output are

$$n_1(z_1, z_2) = \frac{1}{h(1 - z_1)^\sigma} \quad \text{and} \quad F(z_1, z_2) = z_2^\alpha n_1(z_1, z_2).$$

The parameter α governs the elasticity of output with respect to manager knowledge, h is the communication cost, and σ governs the sensitivity of span of control to worker knowledge. Given these parameters and the knowledge distribution, we compute the stable allocation by solving the surplus-maximization problem described in the paper; the solution to its dual gives the equilibrium earnings schedule.

The wage targets are the 5th through 95th percentiles of hourly wages constructed by Acemoglu and Autor (2011) from the Current Population Survey, expressed in logs relative to the median. The establishment-size targets are from the Business Dynamics Statistics and give the shares of establishments in four employment categories, excluding self-employment: 1–9, 10–99, 100–999, and 1,000 or more employees. We use three-year averages for both sets of targets. The 2008 calibration jointly targets these two distributions and yields $(\alpha, \sigma, h, \beta_1, \beta_2) = (0.969, 28.873, 0.228, 1.063, 18.643)$. To study changes between 1988 and 2008, we hold the knowledge-distribution parameters (β_1, β_2) fixed at their 2008 values and recalibrate (α, σ, h) to the 1988 wage and establishment-size distributions.

The 1988 calibration yields $(\alpha, \sigma, h) = (0.816, 33.683, 0.260)$. Between the two calibrated economies, α increases by approximately 18.8%, while σ and h decrease by approximately 14.3% and 12.1%, respectively. The two fitted equilibria imply untargeted top 1% earnings shares of 10.7% in 1988 and 12.5% in 2008, and manager shares of 8.3% and 8.1%, respectively. Panel A of Figure 1 shows that the two fitted

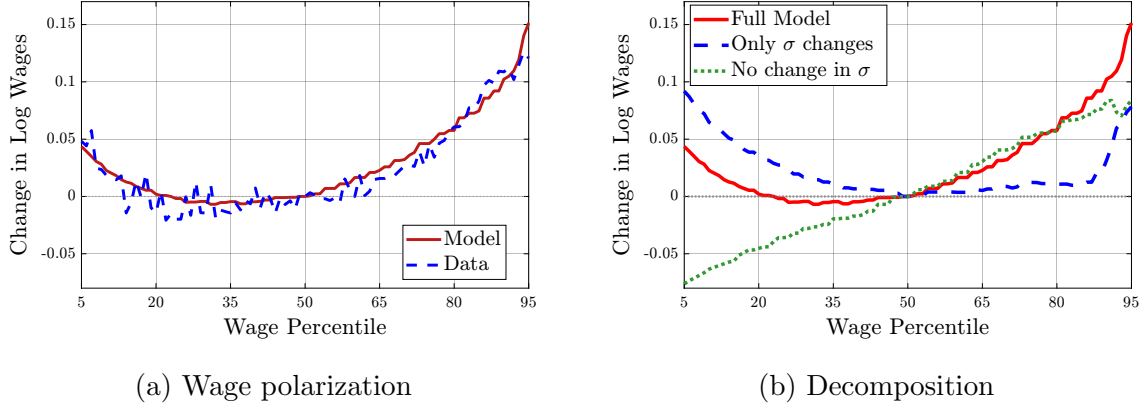


Figure 1: **Wage Polarization in the Flexible Knowledge Economy.** Panel A reports changes in log wages relative to the median between 1988 and 2008 in the model and the data. Panel B compares the full calibrated change with two counterfactuals. In the first, only σ changes from its 1988 value to its 2008 value. In the second, α and h change to their 2008 values while σ remains fixed at its 1988 value.

flexible economies generate a broad U-shaped change in relative wages.

The parameter changes affect distinct margins of the model's production function. Raising α steepens relative output across manager types, even though z^α falls for each fixed $0 < z < 1$. A lower h increases span of control at fixed worker knowledge. For σ , the relevant derivatives are

$$\frac{\partial \log n_1(z)}{\partial z} = \frac{\sigma}{1-z} \quad \text{and} \quad \frac{\partial \log n_1(z)}{\partial \sigma} = -\log(1-z) > 0 \quad (z > 0).$$

Thus a lower σ both weakens the scale advantage of employing more knowledgeable workers and reduces span at each fixed positive worker type, holding h constant.

Panel B of Figure 1 separates these effects. When α and h move to their 2008 values while σ remains at its 1988 value, wage changes rise across the distribution: the lower tail falls relative to the median and the upper tail rises. Changing only σ instead produces a U-shaped profile, with a relative gain at the lower tail and a smaller gain at the upper tail. Combining the three parameter changes yields the fitted profile in Panel A. For these fitted values, the change in σ turns the wage-change profile from monotone to U-shaped.

The parameter movements also admit familiar economic interpretations. A higher α is reminiscent of skill-biased technical change (SBTC) because it raises the relative output of more knowledgeable managers, while a lower h relaxes the communication constraint. The weaker dependence of organizational scale on worker knowledge under a lower σ may be read as a stylized analogue of task codification or routinization, as discussed by Autor, Levy, and Murnane (2003) and Acemoglu and Autor (2011). These are interpretations of how the fitted model generates the result, not evidence that these particular changes occurred historically; the model does not explicitly represent tasks.

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